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# TIME SPENT BELOW A RANDOM THRESHOLD BY A POISSON DRIVEN SEQUENCE OF OBSERVATIONS

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**Abstract**

The mean and the variance of the time $S(t)$ spent by a system below a random threshold until $t$ are obtained when the system level is modelled by the current value of a sequence of independent and identically distributed random variables appearing at the epochs of a nonhomogeneous Poisson process. In the case of the homogeneous Poisson process, the asymptotic distribution of $S(t)/t$ as $t \to \infty$ is derived.

*Keywords:* Exceedance statistics; limit theorems, nonhomogeneous Poisson process; Poisson driven sequence of observations; random threshold; order statistics

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## 1. Introduction

If shocks occurring in time affect the level of an economic, financial, environmental, biological or engineering system, then the proportion of time spent by the system below a threshold is frequently of interest. If $Y_i$ denotes the system level during the period between the $(i-1)$th and $i$th shocks and if $N(t)$ counts the number of shocks during $[0, t]$ for $t \geq 0$, then the process $Z_t = \sum_{n=1}^{\infty} Y_n \mathbf{1}(N(t) = n-1)$, $t \geq 0$, keeps track of the level of the system. Given any $t > 0$, we are interested in the proportion of time during $[0, t]$ when the process $\boldsymbol{Z} = (Z_t)_{t \geq 0}$ is below the system's threshold. However, there are situations in which there is no practical way to identify the system's threshold with certainty. Therefore, the threshold will be considered a random variable $X$ and interest will center on $S(t)/t$, where $S(t)$ denotes the total time during $[0, t]$ that the process $\boldsymbol{Z}$ falls below $X$. Assuming that the process $\boldsymbol{N} = (N(t))_{t \geq 0}$ is a nonhomogeneous Poisson process with $\boldsymbol{N}$, $\boldsymbol{Y} = (Y_i)_{i \geq 1}$, and $X$ independent, we obtain the mean and variance of $S(t)/t$ in Section 2. The choice of the nonhomogeneous Poisson process to model the time epochs of shocks was first proposed by Esary *et al.* [4]. We also prove, in Section 3, that if the shock process $\boldsymbol{N}$ is homogeneous Poisson, then $S(t)/t$ converges in distribution to $G(X)$, where $G$ denotes the common distribution function of the $Y_i$.

A related but easier scheme of exceedance was proposed by Wesołowski and Ahsanullah [8]. They investigated the exact and asymptotic distributions of three statistics connected with exceeding an independent random threshold in a sequence of independent and identically distributed (i.i.d.) observations. In particular, they considered a discrete analogue of our $S(t)$. Some additional distributional properties related to the exceedance scheme of [8] have been recently studied by Bairamov and Eryilmaz [1], Bairamov and Kotz [2] and Eryilmaz [3].

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## 2. Mean and variance

Assuming that $\boldsymbol{Y} = (Y_i)_{i=1,2,\ldots}$ is a sequence of i.i.d. observations with common distribution function $G$, one of the statistics considered in [8] was the number $S^*(n)$ of observations in a sample of size $n$ falling below the random level $X$, where $X$ is independent of $\boldsymbol{Y}$. It was proved there that the conditional distribution of $S^*(n)$ given $X$ is binomial with parameters $n$ and $G(X)$. Consequently,

$$\mathrm{E}(S^*(n)) = n\,\mathrm{E}(G(X))$$

and

$$\operatorname{var}(S^*(n)) = n\,\mathrm{E}(G(X)\bar{G}(X)) + n^2 \operatorname{var}(G(X)),$$

where $\bar{G} = 1 - G$.

Here, we study similar characteristics in the case when observations arrive at the epochs of a nonhomogeneous Poisson process $\boldsymbol{N} = (N(t))_{t\geq 0}$ which is independent of $(\boldsymbol{Y}, X)$.

The main object of our interest is the process $\boldsymbol{Z} = (Z(t))_{t\geq 0}$ which keeps track of the current $Y_j$ as follows:

$$Z(t) = \sum_{n=1}^{\infty} Y_n \mathbf{1}(N(t) = n-1), \qquad t \geq 0.$$

Denote by $S(t)$ the time spent by the process $\boldsymbol{Z}$ below the level $X$ up to the time $t$, and by $W_i$, $i = 0, 1, \ldots$, the interarrival times of the process $\boldsymbol{N}$, that is, $W_i = V_{i+1} - V_i$, where $V_i = \inf\{t \geq 0 : N(t) = i\}$ is the $i$th epoch of $\boldsymbol{N}$, $i = 0, 1, 2, \ldots$. Then $S(t)$ has the form

$$\begin{aligned} S(t) &= \left[\sum_{i=0}^{N(t)-1} W_i \mathbf{1}(Y_{i+1} \leq X) + \left(t - \sum_{i=0}^{N(t)-1} W_i\right) \mathbf{1}(Y_{N(t)+1} \leq X)\right] \mathbf{1}(N(t) \geq 1) \\ &\quad + t\,\mathbf{1}(Y_{N(t)+1} \leq X)\,\mathbf{1}(N(t) = 0) \\ &= \left(\sum_{i=0}^{N(t)-1} W_i[\mathbf{1}(Y_{i+1} \leq X) - \mathbf{1}(Y_{N(t)+1} \leq X)]\right) \mathbf{1}(N(t) \geq 1) \\ &\quad + t\,\mathbf{1}(Y_{N(t)+1} \leq X). \end{aligned} \tag{1}$$

Though we are unable to derive the exact distribution of $S(t)$, the first two moments are computable and similar in form to the corresponding results in [8].

**Proposition 1.** *In the model defined above, for any $t \geq 0$,*

$$\mathrm{E}(S(t)) = t\,\mathrm{E}(G(X)), \tag{2}$$

$$\operatorname{var}(S(t)) = \chi(t)\,\mathrm{E}(G(X)\bar{G}(X)) + t^2 \operatorname{var}(G(X)), \tag{3}$$

*where*

$$\chi(t) = 2 \iint_{0<x<y<t} \mathrm{P}(N(y) = N(x))\,\mathrm{d}x\,\mathrm{d}y \leq t^2.$$

Before proving the above proposition, we present a result on order statistics which will be used later on in the proof.

**Lemma 1.** *Let $X_{1:n}, \ldots, X_{n:n}$ be order statistics from an i.i.d. sample with distribution function $F$ having support* $[0, a]$. *Then*

$$\mathrm{E}\left[X_{1:n}^2 + \sum_{i=2}^{n}(X_{i:n} - X_{i-1:n})^2 + (a - X_{n:n})^2\right] = 2\iint_{0<x<y<a} (F(x) + \bar{F}(y))^n \,\mathrm{d}x\,\mathrm{d}y. \quad (4)$$

*Proof.* Observe first that, for any square integrable random variable $X$ with distribution function $F$ having support $[0, a]$,

$$\mathrm{E}(X^2) = 2\int_0^a y\bar{F}(y)\,\mathrm{d}y = 2\iint_{0<x<y<a} \bar{F}(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Consequently,

$$\begin{aligned}\mathrm{E}((a - X)^2) &= 2\iint_{0<x<y<a} F(a - y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\iint_{0<x<y<a} F(x)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We proceed by induction with respect to $n$. For $n = 1$, the result has just been proved. Denote the left-hand side of (4) by $L_n$. Then

$$L_n = \mathrm{E}\left[\mathrm{E}\left(X_{1:n}^2 + \sum_{i=2}^{n}(X_{i:n} - X_{i-1:n})^2 \;\middle|\; X_{n:n}\right) + (a - X_{n:n})^2\right].$$

It is known (see, for instance, [6, Chapter 4]) that the conditional distribution of $(X_{1:n}, \ldots, X_{n-1:n})$ given $X_{n:n} = x$ is the same as the joint distribution of order statistics from an i.i.d. sample of size $n - 1$ based on the distribution function

$$G_x(u) = \begin{cases} \dfrac{F(u)}{F(x)} & \text{for } u < x, \\ 1 & \text{for } u \geq x. \end{cases}$$

Then, by the induction assumption, it follows that

$$\begin{aligned}L_n &= \mathrm{E}\left(2\iint_{0<x<y<X_{n:n}} [G_{X_{n:n}}(x) + \bar{G}_{X_{n:n}}(y)]^{n-1}\,\mathrm{d}x\,\mathrm{d}y + (a - X_{n:n})^2\right) \\ &= 2\iint_{0<x<y<a}\left(\int_y^a [G_u(x) + \bar{G}_u(y)]^{n-1}\,\mathrm{d}F_{n:n}(u) + F_{n:n}(x)\right)\mathrm{d}x\,\mathrm{d}y \\ &= 2\iint_{0<x<y<a}\left(n\int_{F(y)}^1 [t + F(x) - F(y)]^{n-1}\,\mathrm{d}t + F^n(x)\right)\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which immediately implies the result.

*Proof of Proposition 1.* By the independence properties, we have

$$
\begin{aligned}
\mathrm{E}(S(t)) &= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} \mathrm{E}(W_i[\mathbf{1}(Y_{i+1} \le X) - \mathbf{1}(Y_{n+1} \le X)] \mid N(t) = n)\,\mathrm{P}(N(t) = n) \\
&\quad + t\sum_{n=0}^{\infty} \mathrm{E}(\mathbf{1}(Y_{n+1} \le X) \mid N(t) = n)\,\mathrm{P}(N(t) = n) \\
&= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} \mathrm{E}(W_i \mid N(t) = n)[\mathrm{E}(\mathbf{1}(Y_{i+1} \le X)) - \mathrm{E}(\mathbf{1}(Y_{n+1} \le X))]\,\mathrm{P}(N(t) = n) \\
&\quad + t\,\mathrm{E}(G(X)).
\end{aligned}
$$

The formula (2) follows since $\mathrm{E}(\mathbf{1}(Y_{i+1} \le X)) - \mathrm{E}(\mathbf{1}(Y_{n+1} \le X)) = \mathrm{E}(G(X)) - \mathrm{E}(G(X)) = 0$.

In order to find the variance of $S(t)$, we first compute its second conditional moment given $N(t) = n > 0$:

$$
\begin{aligned}
\mathrm{E}(S^2(t) \mid N(t) = n) &= \sum_{i=0}^{n-1} \mathrm{E}(W_i^2 \mid N(t) = n)\,\mathrm{E}((I_{i+1} - I_{n+1})^2) \\
&\quad + \sum_{i \ne j}^{n-1} \mathrm{E}(W_i W_j \mid N(t) = n)\,\mathrm{E}((I_{i+1} - I_{n+1})(I_{j+1} - I_{n+1})) \\
&\quad + 2t\sum_{i=0}^{n-1} \mathrm{E}(W_i \mid N(t) = n)\,\mathrm{E}(I_{n+1}(I_{i+1} - I_{n+1})) + t^2\,\mathrm{E}(I_{n+1}),
\end{aligned}
$$

where $I_j = \mathbf{1}(Y_j \le X)$ for $j = 1, 2, \dots$. Observe that

$$
\begin{aligned}
\mathrm{E}((I_{i+1} - I_{n+1})^2) &= 2\,\mathrm{E}(G(X)\bar{G}(X)), && 0 \le i < n, \\
\mathrm{E}((I_{i+1} - I_{n+1})(I_{j+1} - I_{n+1})) &= \mathrm{E}(G(X)\bar{G}(X)), && 0 \le i, j < n, i \ne j, \\
\mathrm{E}(I_{n+1}(I_{i+1} - I_{n+1})) &= -\,\mathrm{E}(G(X)\bar{G}(X)), && 0 \le i < n.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\mathrm{E}(S^2(t) \mid N(t) = n) \\
&\quad = \mathrm{E}(G(X)\bar{G}(X))\left[\mathrm{E}((t - V_n)^2 \mid N(t) = n) + \mathrm{E}\left(\sum_{i=0}^{n-1} W_i^2 \,\middle|\, N(t) = n\right)\right] \\
&\qquad + t^2\,\mathrm{E}(G^2(X))
\end{aligned}
$$

for $n > 0$. Setting $\sum_{i=0}^{-1} = 0$ and recalling that $V_0 = 0$ almost surely, we find that the above formula also holds for $n = 0$ since, by direct computation, $\mathrm{E}(S^2(t) \mid N(t) = 0) = t^2\,\mathrm{E}(G(X))$. Hence, for any $t \ge 0$,

$$
\begin{aligned}
\mathrm{var}(S(t)) &= \mathrm{E}(G(X)\bar{G}(X))\left[\mathrm{E}((t - V_{N(t)})^2) + \mathrm{E}\left(\sum_{i=0}^{N(t)-1} W_i^2\right)\right] + t^2\,\mathrm{var}(G(X)) \\
&= \mathrm{E}(G(X)\bar{G}(X))\,\mathrm{E}\left[V_1^2 + \sum_{i=2}^{N(t)} (V_i - V_{i-1})^2 + (t - V_{N(t)})^2\right] + t^2\,\mathrm{var}(G(X)).
\end{aligned}
$$

Now to get (3) it suffices to compute $\chi(t)$. In order to do that, we recall (see, for instance, [7, Theorem 12.2.1]) that the conditional distribution of the random vector $(V_1, \dots, V_{N(t)})$ given $N(t) = n$ is equal to the joint distribution of the order statistics of a random sample of size $n$ from the distribution

$$H(x) = \begin{cases} 0, & x < 0, \\ \dfrac{\Lambda(x)}{\Lambda(t)}, & 0 \le x < t, \\ 1, & t \le x, \end{cases}$$

where $\Lambda$ is the mean value function of the Poisson process. Now applying Lemma 1 and changing the order of integration, we obtain that

$$\begin{aligned} &\mathrm{E}\bigg(\mathrm{E}\bigg[V_1^2 + \sum_{i=2}^{N(t)} (V_i - V_{i-1})^2 + (t - V_n)^2 \;\bigg|\; N(t)\bigg]\bigg) \\ &= 2\iint_{0<x<y<t} \mathrm{E}([H(x) + \bar{H}(y)]^{N(t)})\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\iint_{0<x<y<t} \exp(\Lambda(t)[H(x) + \bar{H}(y) - 1])\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\iint_{0<x<y<t} \exp(-[\Lambda(y) - \Lambda(x)])\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

**Remark 1.** Observe that $\mathrm{E}(S(t))$ does not depend on the parameters of the driving process $N$. On the other hand, $\mathrm{var}(S(t))$ depends on the intensity $\Lambda$ of the Poisson process and increases in $t$ at most as a quadratic function.

**Remark 2.** If $N$ is a homogeneous Poisson process with mean value function $\Lambda(t) = \lambda t$, where $\lambda$ is a positive constant, then

$$\chi(t) = 2\int_0^t \int_0^{t-y} \mathrm{e}^{-\lambda w}\,\mathrm{d}w\,\mathrm{d}y = \frac{2}{\lambda^2}(\mathrm{e}^{-\lambda t} - 1 + \lambda t).$$

In case $N$ is a nonhomogeneous Poisson process with $\Lambda(t) = \log(1 + t)$,

$$\chi(t) = 2\int_0^t \int_0^{t-y} \frac{1+y}{1+y+w}\,\mathrm{d}w\,\mathrm{d}y = t + \frac{t^2}{2} - \log(1+t).$$

Note that $\chi(t)/t^2 \to 0$ as $t \to \infty$ when $N$ is homogeneous, but not when $\Lambda(t) = \log(1 + t)$.

## 3. Asymptotic distribution

It was shown in [8] that $S^*(n)/n$ converges in distribution to $G(X)$, $S^*(n)/n \xrightarrow{\mathrm{D}} G(X)$ as $n \to \infty$. Here the formulae for $\mathrm{E}(S(t))$ and $\mathrm{var}(S(t))$ suggest that a similar result cannot be true unless $\chi(t)/t^2$ converges to 0 as $t \to \infty$. At the same time it is reasonable to conjecture that $S(t)/t \xrightarrow{\mathrm{D}} G(X)$ if $\chi(t)/t^2 \to 0$. We are not able to answer this question in full generality; however, in the case of the homogeneous Poisson process, the answer is affirmative.

**Proposition 2.** *If $N$ is a homogeneous Poisson process, then*

$$\frac{S(t)}{t} \xrightarrow{\mathrm{D}} G(X) \quad \text{as } t \to \infty.$$

*Proof.* Let $I_j = \mathbf{1}(Y_j \le X)$ for $j = 1, 2, \dots$ and rewrite (1) as $S(t) = S_1(t) + S_2(t) + S_3(t)$, where

$$S_1(t) = \mathbf{1}(N(t) \ge 1) \sum_{j=0}^{N(t)-1} W_j I_{j+1},$$

$$S_2(t) = (t - V_{N(t)}) I_{N(t)+1}\, \mathbf{1}(N(t) \ge 1)$$

and

$$S_3(t) = t I_{N(t)+1}\, \mathbf{1}(N(t) = 0).$$

Clearly, $S_2(t) \le t - V_{N(t)}$. But the distribution of $t - V_{N(t)}$ is known; see, for instance, [5]. Thus, for any $\varepsilon > 0$ and $t$ sufficiently large,

$$\mathrm{P}\biggl(\frac{t - V_{N(t)}}{t} > \varepsilon\biggr) = \mathrm{e}^{-\lambda \varepsilon t}.$$

Consequently, $S_2(t)/t \xrightarrow{\mathrm{P}} 0$ as $t \to \infty$.

Further, for any $\varepsilon > 0$,

$$\begin{aligned}\mathrm{P}\biggl(\frac{S_3(t)}{t} > \varepsilon\biggr) &= \mathrm{P}(I_{N(t)+1}\, \mathbf{1}(N(t) = 0) > \varepsilon)\\ &\le \mathrm{P}(N(t) = 0)\\ &= \mathrm{e}^{-\lambda t}\\ &\to 0 \quad \text{as } t \to \infty;\end{aligned}$$

hence $S_3(t)/t \xrightarrow{\mathrm{P}} 0$.

Observe now that $\mathbf{1}(N(t) \ge 1) \xrightarrow{\mathrm{P}} 1$ as $t \to \infty$. Consequently, to prove that $S_1(t)/t \xrightarrow{\mathrm{D}} G(X)$, it suffices to prove that

$$\frac{1}{t} \sum_{j=0}^{N(t)-1} W_j I_{j+1} \xrightarrow{\mathrm{D}} G(X).$$

This will be done in two steps.

Denoting by $\lambda$ the intensity of the Poisson process, we will first prove that

$$\frac{1}{t} \sum_{j=0}^{\lfloor \lambda t \rfloor - 1} W_j I_{j+1} \xrightarrow{\mathrm{D}} G(X)$$

where $\lfloor \cdot \rfloor$ is the integer-part function. Note that the sequence $(W_j I_{j+1})_{j=0,1\dots}$ is conditionally i.i.d. given $X$ and $\mathrm{E}(W_j I_{j+1} \mid X) = G(X)/\lambda$. Then, by the law of large numbers, for any real $z$,

$$\begin{aligned}&\mathrm{E}\biggl[\exp\biggl(\frac{\mathrm{i}z}{t} \sum_{j=0}^{\lfloor \lambda t \rfloor - 1} W_j I_{j+1}\biggr)\biggr]\\ &\quad = \mathrm{E}\biggl[\mathrm{E}\biggl(\exp\biggl(\frac{\lfloor \lambda t \rfloor - 1}{\lambda t} \frac{\mathrm{i}z\lambda}{\lfloor \lambda t \rfloor - 1} \sum_{j=0}^{\lfloor \lambda t \rfloor - 1} W_j I_{j+1}\biggr) \biggm| X\biggr)\biggr] \to \mathrm{E}(\mathrm{e}^{\mathrm{i}zG(X)}) \quad \text{as } t \to \infty.\end{aligned}$$

Secondly, we will show that, as $t \to \infty$,

$$\frac{1}{t}\left(\sum_{j=0}^{N(t)-1} W_j I_{j+1} - \sum_{j=0}^{\lfloor \lambda t \rfloor -1} W_j I_{j+1}\right) \xrightarrow{\mathrm{P}} 0.$$

To this end, observe that

$$\frac{N(t)}{\lfloor \lambda t \rfloor} = \frac{N(t)}{\lambda t}\frac{\lambda t}{\lfloor \lambda t \rfloor} \xrightarrow{\mathrm{P}} 1 \quad \text{as } t \to \infty.$$

Let

$$A_\varepsilon = \{(1-\varepsilon)\lfloor \lambda t \rfloor < N(t) < (1+\varepsilon)\lfloor \lambda t \rfloor\} \quad \text{for any } \varepsilon > 0$$

and

$$\sigma^2 = \operatorname{var}(W_0 I_1) = \mathrm{E}(G(X))\frac{2-\mathrm{E}(G(X))}{\lambda^2}.$$

Then, for sufficiently large $t$, $\mathrm{P}(A_\varepsilon) > 1-\varepsilon$. Thus,

$$\mathrm{P}\left(\frac{1}{t}\left|\sum_{j=0}^{N(t)-1} W_j I_{j+1} - \sum_{j=0}^{\lfloor \lambda t \rfloor -1} W_j I_{j+1}\right| > \varepsilon_1\right)$$
$$\leq \mathrm{P}\left(\left\{\left|\sum_{j=0}^{N(t)-1} W_j I_{j+1} - \sum_{j=0}^{\lfloor \lambda t \rfloor -1} W_j I_{j+1}\right| > t\varepsilon_1\right\} \cap A_\varepsilon\right) + \varepsilon.$$

But

$$\mathrm{P}\left(\left\{\left|\sum_{j=0}^{N(t)-1} W_j I_{j+1} - \sum_{j=0}^{\lfloor \lambda t \rfloor -1} W_j I_{j+1}\right| > t\varepsilon_1\right\} \cap A_\varepsilon\right)$$
$$= \mathrm{P}\left(\left\{\sum_{j=\min\{N(t),\lfloor \lambda t \rfloor\}}^{\max\{N(t),\lfloor \lambda t \rfloor\}-1} W_j I_{j+1} > t\varepsilon_1\right\} \cap A_\varepsilon\right)$$
$$\leq \mathrm{P}\left(\left\{\sum_{j=\lfloor(1-\varepsilon)\lfloor \lambda t \rfloor\rfloor}^{\lfloor(1+\varepsilon)\lfloor \lambda t \rfloor\rfloor-1} W_j I_{j+1} > t\varepsilon_1\right\} \cap A_\varepsilon\right)$$
$$\leq \mathrm{P}\left(\sum_{j=0}^{2\lfloor \varepsilon \lfloor \lambda t \rfloor\rfloor} W_j I_{j+1} > t\varepsilon_1\right)$$
$$\leq \frac{2\varepsilon\lambda t\sigma^2}{\varepsilon_1^2 t^2} \to 0 \quad \text{as } t \to \infty.$$

This completes the proof of the theorem.

**Remark 3.** Observe that, in the case of nonrandom threshold, the convergence in Proposition 2 holds in probability.

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