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CONSTANCY OF REGRESSIONS FOR BETA DISTRIBUTIONS

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SUMMARY. Properties of preserving independence under some transformations are known to characterize such important families of distributions as normal, exponential, gamma, Cauchy, uniform, (generalized) inverse Gaussian. Recently, a similar type of result of such a shape for the beta distribution has been proved in Wesołowski (2002a). In the present paper related characterizations of beta laws are obtained under weaker conditions of constancy of regressions.

1. Introduction

Matsumoto and Yor (2001) have recently discovered that the map $(x, y) \to ((x+y)^{-1}, x^{-1} - (x+y)^{-1})$, acting on $(0,\infty)^2$, preserves a bivariate probability measure which is a product of the generalized inverse Gaussian (GIG) and the gamma distributions. This result was extended to matrix variate distributions in Letac and Wesołowski (2000), where also a complete converse in the univariate case and a partial converse, with an assumption of smooth densities, in the matrix variate case, were given. Related questions have been recently studied in Seshadri and Wesołowski (2001) and Wesołowski (2002b). In particular in Wesołowski (2002b) a complete characterization was obtained under the assumption of constancy of regressions, while in an earlier paper: Seshadri and Wesołowski (2001), single constancy of regression condition led only to mutual characterizations of gamma and GIG distributions.

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A phenomenon, similar to the Matsumoto-Yor property, holds for beta distributions. Recall that the beta distribution (of the first kind) $\beta_{p,q}$ is defined by

$$\beta_{p,q}(dx) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} I_{(0,1)}(x)\, dx,$$

where the parameters p and q are positive numbers, and $B(p,q)$ is the beta Euler function.

Consider a map $\psi:\ (0,1)^2 \to (0,1)^2$ defined by

$$\psi(x,y) = \left(\frac{1-y}{1-xy},\ 1-xy\right), \quad (x,y) \in (0,1)^2.$$

Observe that $\psi \circ \psi = id$, where id denotes the identity map. Exactly the same property holds for the map introduced in Matsumoto and Yor (2001).

For a random vector (X,Y) with the distribution $\beta_{p,q} \otimes \beta_{p+q,r}$ define a new random vector $(U,V) = \psi(X,Y)$, i.e. $V = 1-XY$ and $U = (1-Y)/V$. Then it follows from the classical algebra of beta, gamma and Dirichlet distributions that (U,V) has the distribution $\beta_{r,q} \otimes \beta_{r+q,p}$. In Wesołowski (2002a) it was proved that the independence of components for both (X,Y) and (U,V) implies that the parent distributions are beta, provided the random variables (rv's) X and Y have strictly positive densities on $(0,1)$ with locally integrable logarithms. It is worth mentioning that the version of the independence property discussed above for the matrix variate beta distribution has been derived recently in Letac and Wesołowski (2001) however no converse in the matrix variate case is known at present.

In the next section we gather several rather elementary, but nice, properties of n-variate version of the transformation ψ applied to random vectors with independent beta components of specially related parameters. In Section 3 it is shown that constancy of regressions, instead of the independence property, leads to characterizations of beta distributions. Two results of such a form are obtained and one of them is a straightforward generalization of the characterization given in Wesołowski (2002a).

2. Decomposition and Martingale Properties

First we define a map, an n-dimensional version of ψ given above, which preserves the independence property for a collection of n independent beta rv's.

PROPOSITION 1 *Let $(X_1,\dots,X_n)$ be a random vector with independent beta components $X_k \sim \beta_{\sum_{i=k}^n p_i, p_{k-1}}$, $k=1,\dots,n$. Define a map $\psi:(0,1)^n \to (0,1)^n$*

by

$$\psi(x_1,\dots,x_n)=\left(\frac{1-x_1}{1-x_1x_2},\dots,\frac{1-x_1\dots x_{n-1}}{1-x_1\dots x_n},1-x_1\dots x_n\right)$$

for any $(x_1,\dots,x_n)\in(0,1)^n$. *Then the random vector* $(U_1,\dots,U_n)=\psi(X_1,\dots,X_n)$ *has independent beta components,* $U_k\sim\beta_{\sum_{i=0}^{k-1}p_i,p_k}$, $k=1,\dots,n$.

The above result is an easy consequence of the representation of the beta distribution in terms of ratios of subsequent sums of independent gamma random variables with the same scale parameter. Immediate consequences of the above proposition are listed below:

REMARK 1. Let X_1 be a beta β_{p,p_0} rv. Then for any $n>1$ and any collection of positive numbers p_k, $k=1,\dots,n$, such that $\sum_{k=1}^n p_k=p$ there exist independent beta rv's $X_2,\dots,X_n$, independent also of X_1, such that $X_k\sim\beta_{\sum_{i=k}^n p_i,p_{k-1}}$, $k=1,\dots,n$, and

$$1-X_1=\left[\prod_{k=1}^{n-1}\frac{1-X_1\dots X_k}{1-X_1\dots X_{k+1}}\right](1-X_1\dots X_n)=\prod_{k=1}^n U_k\ ,$$

where $U_1,\dots,U_n$ are independent beta rv's such that $U_k\sim\beta_{\sum_{i=0}^{k-1}p_i,p_k}$, $k=1,\dots,n$.

REMARK 2. Let $(p_k)_{k\geq 0}$ be a summable sequence of positive numbers, and let (X_k) bé a sequence of independent beta rv's, such that $X_k\sim\beta_{\sum_{i=k}^\infty p_i,p_{k-1}}$, $k=1,2,\dots$. Define $Z_n=1-X_1\dots X_n$, $n=1,2,\dots$. Then for any $r>0$ the sequence $(Z_n^r/E(Z_n^r))_{n\geq 1}$ is a backward martingale with respect to the natural filtration.

REMARK 3. The random vector $(Z_1,\dots,Z_n)$, as defined in Remark 2, has an ordered Dirichlet distribution with the following density function

$$f(z_1,\dots,z_n)=\frac{\Gamma\left(\sum_{i=0}^n q_i\right)}{\prod_{i=0}^n\Gamma(q_i)}\prod_{i=0}^n(z_{i+1}-z_i)^{q_i-1},0=z_0<z_1<\dots<z_n<z_{n+1}=1,$$

where $q_i=p_i$, $i=0,\dots,n-1$, and $q_n=\sum_{j=n}^\infty p_j$.

3. Regression Characterizations

As mentioned in the introduction, Wesołowski (2002a), under some smoothness conditions imposed on densities, proved that if X and Y are independent rv's with values in $(0,1)$ and also $U=(1-Y)/(1-XY)$ and $V=1-XY$ are independent then X and Y, and consequently, U and V, are beta rv's with suitably related parameters. The proof was based on solving of the functional equation

$$f_U(u)f_V(v)=\frac{v}{1-uv}f_X\left(\frac{1-v}{1-uv}\right)f_Y(1-uv)\ ,\quad u,v\in(0,1).$$

In this section we present two characterizations of beta distributions based on constancy of regressions conditions. Since X and Y are bounded then the first result is a straight forward generalization of the characterization obtained in Wesołowski (2002a) since no assumption about densities is used. In both cases the proof is based on the application of the method of moments. A similar approach has been proved recently to be useful for characterizing the gamma distribution through dual regression versions of the Lukacs independence property - see Bobecka and Wesołowski (2002).

THEOREM 1. *Let X and Y be independent non-degenerate rv's valued in $(0,1)$. Denote $U=(1-Y)/(1-XY)$ and $V=1-XY$. Assume that*

$$E(U|V)=c \tag{1}$$

and

$$E(U^2|V)=d \tag{2}$$

for some real constants c, d.

Then $q=\frac{(1-c)(d-c)}{c^2-d}>0$, $r=\frac{c(d-c)}{c^2-d}>0$ and there exists $p>0$ such that $(X,Y)\sim\beta_{p,q}\otimes\beta_{p+q,r}$ and, consequently, $(U,V)\sim\beta_{r,q}\otimes\beta_{r+q,p}$.

PROOF. Observe that (1) and (2) can be rewritten, respectively, as

$$E(1-Y|XY)=c(1-XY), \tag{3}$$

$$E((1-Y)^2|XY)=d(1-XY)^2. \tag{4}$$

Moreover since the rv's are bounded all the moments of X and Y are finite and uniquely determine both the distributions. Consequently (3) implies that for any $k=0,1,\ldots$

$$E\left[(1-Y)(XY)^k\right]=cE\left[(1-XY)(XY)^k\right]\ ,$$

which can be written in the following form

$$h(k)[1-cg(k)] = 1-c\,, \quad k=0,1,\dots, \tag{5}$$

where

$$g(k) = \frac{E(X^{k+1})}{E(X^k)}\,, \qquad h(k) = \frac{E(Y^{k+1})}{E(Y^k)}\,.$$

On the other hand (4) leads to

$$E\left[(1-Y)^2(XY)^k\right] = dE\left[(1-XY)^2(XY)^k\right],$$

which takes the form

$$1-2h(k)+h(k)h(k+1)=d-2dg(k)h(k)+dg(k)g(k+1)h(k)h(k+1),\ k=0,1,\dots \tag{6}$$

Now find $g(k)h(k)$ and $g(k+1)h(k+1)$ from (5) and substitute into the rhs of (6) to arrive, after some elementary algebra, at

$$h(k+1)[(c^2-d)h(k)+d(1-c)] = [d(1-c)+2(c^2-d)]h(k)-(c^2-d).$$

Note that since $c=E(U)$, $d=E(U^2)$ and the rv's are non-degenerate then $1>c>d>c^2$. Denote

$$r = \frac{c(d-c)}{c^2-d} > 0.$$

Then the above equation takes the form

$$h(k+1)[r+1-h(k)] = 1+(r-1)h(k).$$

Consequently $h(k) \neq r+1$ for any $k=0,1,\dots$ and thus

$$h(k+1) = \frac{1+(r-1)h(k)}{r+1-h(k)}\,, \quad k=0,1,\dots.$$

Define now a by $h(0)=E(Y)=a/(a+r)$. Observe that $a+r>0$. Since otherwise $a<0$ and then $1>a/(a+r)$ implies $0>a>a+r$, yielding $r<0$ which is contradictory.

Then the above equation leads to

$$E(Y^{k+1}) = \frac{k+a}{k+a+r}E(Y^k),$$

for any k=0, 1, …, where a and r are positve real constants. Hence $Y \sim \beta_{a,r}$.

Using now (5) we get

$$g(k) = \frac{h(k) - 1 + c}{ch(k)} = \frac{k + a - \frac{1-c}{c} r}{k + a} < 1 \, , \quad k = 0, 1, \ldots$$

Consequently,

$$p = a - \frac{1-c}{c} r > 0$$

and $X \sim \beta_{p,q}$ with $q = a - p = r(1-c)/c > 0$. □

The next regression characterization needs an additional assumption of the existence of a reciprocal moment of $1 - Y$.

THEOREM 2. *Let X and Y be independent non-degenerate rv's with values in $(0,1)$. Assume that $E((1-Y)^{-1}) < \infty$. If (1) holds and also*

$$E(U^{-1}|V) = b, \tag{7}$$

then $q = \frac{(b-1)(1-c)}{bc-1} > 0$, $r = \frac{c(b-1)}{bc-1} > 0$ and there exists $p > 0$ such that $(X, Y) \sim \beta_{p,q} \otimes \beta_{p+q,r}$ and, consequently, $(U, V) \sim \beta_{r,q} \otimes \beta_{r+q,p}$.

PROOF. We first note that (7) can be rewritten as

$$E((1-Y)^{-1}|XY) = b(1-XY)^{-1} \, .$$

Since, by the integrability assumption, $E((1-Y)^{-1}Y^k) < \infty$ for any $k = 0, 1, \ldots$, then the above identity yields

$$E(X^k) E\left(\frac{Y^k}{1-Y}\right) = bE\left(\frac{(XY)^k}{1-XY}\right) \, , \quad k = 0, 1, \ldots$$

Note that $(1-Y)^{-1} > Y^k + Y^{k+1} + \ldots$ a.s. and also $(1-Y)^{-1} > (1-XY)^{-1}$ a.s. Hence it follows that

$$E(X^k) \sum_{l=k}^{\infty} E(Y^l) = b \sum_{l=k}^{\infty} E(X^l) E(Y^l) \, .$$

Then taking in the above equality succesive differences for k and $k+1$ we obtain

$$H(k) - g(k) H(k+1) = b[H(k) - H(k+1)] \, , \tag{8}$$

where $H(k) = \sum_{l=k}^{\infty} E(Y^l)$ and $g(k) = E(X^{k+1})/E(X^k)$, $k = 0, 1, \ldots$.

On the other hand (5) can be written as

$$(1-c)[H(k)-H(k+1)] = (1-cg(k))[H(k+1)-H(k+2)], \quad k=0,1,\dots \tag{9}$$

Now denoting $P(k) = H(k+1)/H(k)$ we have for (8) and (9), respectively

$$1-b = [g(k)-b]P(k) \tag{10}$$

and

$$(1-c)(1-P(k)) = (1-cg(k))P(k)(1-P(k+1)). \tag{11}$$

Substituting now $P(k)$ and $P(k+1)$ from (10) into (11) it follows that

$$g(k+1) = \frac{1+(q-1)g(k)}{q+1-g(k)}, \quad k=0,1,\dots,$$

where $q = \frac{(b-1)(1-c)}{bc-1}$. Observe that $q > 0$ since $b > 1$, $c < 1$ and $bc = E(U)E(U^{-1}) > 1$ (X and Y are non-degenerate). Then as in the proof of Theorem 1 we obtain

$$E(X^{k+1}) = \frac{p+k}{p+q+k}E(X^k), \quad k=0,1,\dots,$$

where p is defined by $g(0) = E(X) = \frac{p}{p+q}$. And thus $X \sim \beta_{p,q}$.

Now (5) implies that

$$E(Y^{k+1}) = \frac{p+q+k}{p+\frac{q}{1-c}+k}E(Y^k), \quad k=0,1,\dots$$

Hence $Y \sim \beta_{p+q,r}$, where $r = \frac{qc}{1-c}$. □

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