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# On a functional equation related to an independence property for beta distributions

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**Summary.** A study of a transformation preserving independence for beta distributions leads to a new functional equation which is solved under the assumption of local integrability of the functions involved.

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## 1. Introduction

Denote the beta probability distribution on $(0,1)$ by $\beta_{p,q}$, where $p$ and $q$ are positive numbers. Recall that it is defined by the density of the form

$$\beta_{p,q}(dx) = B(p,q)^{-1}x^{p-1}(1-x)^{q-1}I_{(0,1)}(x)\,dx,$$

where the normalizing constant $B(p,q)$ is the Euler beta function. Define also a transformation $\psi : (0,1)^2 \to (0,1)^2$ by

$$\psi(x,y) = \left(\frac{1-y}{1-xy}, 1-xy\right)$$

and observe that it is bijective and $\psi = \psi^{-1}$.

It is a simple exercise to check that a random vector $(X,Y)$ has independent components with distributions $\beta_{p,q}$ and $\beta_{p+q,r}$ (we write it as $(X,Y) \sim \beta_{p,q} \otimes \beta_{p+q,r}$) iff the random vector $(U,V) = \psi(X,Y) \sim \beta_{r,q} \otimes \beta_{r+q,p}$, $p,q,r > 0$. For instance, it suffices to plug the respective densities into the identity, which is equivalent to the independence condition

$$f_U(u)f_V(v) = \frac{v}{1-uv} f_X\left(\frac{1-v}{1-uv}\right) f_Y(1-uv), \qquad u,v \in (0,1), \tag{1}$$

where $f_U$, $f_V$, $f_X$ and $f_Y$ are the probability density functions of the random variables $U$, $V$, $X$ and $Y$, respectively – notice that $v/(1-uv)$ is the Jacobian of the transformation $\psi$. Another possibility is to use representations of beta variables in terms of quotients of suitable sums of independent gamma random variables with a common scale parameter.

Similar independence properties are known for other important probability distributions involving, obviously, different functions $\psi$. For instance, for the standard normal distribution one can take $\psi(x, y) = (x + y, x - y)$, $x, y \in (-\infty, +\infty)$; in the case of the gamma distribution $\psi(x, y) = (x/y, x + y)$, $x, y \in (0, +\infty)$. Rather recently a new example of a transformation preserving independence for so called generalized inverse Gaussian and gamma distributions was discovered in Matsumoto and Yor [12]: $\psi(x, y) = (1/(x + y), 1/x - 1/(x + y))$, $x, y \in (0, \infty)$. Consider the converse question: Assume that the distributions of $(X, Y)$ and $(U, V) = \psi(X, Y)$ are product measures. Does this property characterize respective distributions? Such a problem for the normal law was solved in Bernstein [2], and for the gamma law in Lukacs [11]. The Matsumoto–Yor property was treated through Laplace transform techniques in Letac and Wesołowski [10] and Wesołowski [17] and via the density approach in Wesołowski [18]. We would like to stress that, though assuming (smooth) densities is somewhat restrictive, however such an approach while bringing to the attention interesting new functional equations, at the same time may give an insight into more general settings and/or even lead to stronger conclusions than other methods – see for instance the version of the Lukacs theorem in the cone of positive definite symmetric matrices obtained in Bobecka and Wesołowski [3], where the traditional assumption of invariance of the "quotient" was relaxed due to the approach through densities, while necessarily it is kept in theorems proved via the standard Laplace transform methods – see Olkin and Rubin [13] or, much more recently, Casalis and Letac [4] or Letac and Massam [9].

Here we would like to answer the same type of a converse question for the independence property of the beta distributions, observed above. Again, as in the case of the Matsumoto–Yor property the problem can be treated without the assumption that densities exist – in such a setting the moment method was successfully applied in Seshadri and Wesołowski [16] for a related problem of constancy of regression. However again, such an approach seems to be hard to adopt for instance in the matrix variate case. Following the path used earlier for the Matsumoto–Yor property, in this paper we study the univariate situation under the assumption that the densities exist. Then the problem reduces to describing all probabilistic solutions of the equation (1) with unknown $f_U$, $f_V$, $f_X$ and $f_Y$ – observe that we can assume that (1) holds for all $u, v \in (0, 1)$ if only some smoothness conditions are imposed on densities.

While attacking the question along these lines (see below) we are faced up with an interesting functional equation:

$$g\left(\frac{1-x}{1-xy}\right) - g\left(\frac{1-y}{1-xy}\right) = \alpha(x) - \alpha(y), \quad x, y \in (0, 1), \tag{2}$$

with unknown functions $g$ and $\alpha$. A search through literature including Aczél [1], Kuczma, Choczewski and Ger [8], Ramachandran and Lau [14], Sahoo and Riedel [15] suggests that such an equation has not been yet studied. Somewhat related, but different equations, related to properties of the Dirichlet distributions, have been recently investigated by Geiger and Heckerman [5] and Járai [7]. The present paper is devoted to a solution of a generalized version of the equation (2). It is done under the assumption that unknown functions are locally integrable.

But first we will explain how to arrive at the equation (2) starting from the independence property described above.

Since the random vectors $(X, Y)$ and $(U, V)$ have independent components, the identity (1) holds true. Changing now the role of $u$ and $v$ in (1) one gets

$$f_U(v)f_V(u) = \frac{u}{1-uv} f_X\left(\frac{1-u}{1-uv}\right) f_Y(1-uv), \qquad u, v \in (0,1), \tag{3}$$

Now combining (1) and (3), under the assumption that the densities are always strictly positive, it follows that

$$\frac{h(u)}{h(v)} = \frac{f_X\left(\frac{1-v}{1-uv}\right)}{f_X\left(\frac{1-u}{1-uv}\right)}, \quad \forall\, u, v \in (0,1),$$

where the function $h$ is defined by $h(x) = x f_U(x)/f_V(x)$, $x \in (0,1)$. Then (2) follows with $g(x) = \log(f_X(x))$, $\alpha(x) = \log(h(x))$, $x \in (0,1)$, just by taking logarithms in the above equation.

## 2. Solution of the functional equation

Now we are ready to formulate the main result of the paper which gives the solution to a more general version of equation (2). It has to be emphasized that in our result we assume that the unknown functions are locally integrable. This assumption leads then to smoothness properties of the functions involved and is crucial for the proof we offer here. The question of solutions without any smoothness properties assumed, or even with a, natural in the probabilistic context, Borel measurability assumption remains open – similarly as in the case of the functional equation related to the Matsumoto–Yor property, which was studied in Wesołowski [17], again for locally integrable unknown functions.

**Theorem 1.** *Let $g_1$, $g_2$, $\alpha_1$ and $\alpha_2$ be locally integrable real functions defined on $(0,1)$ satisfying the equation*

$$g_1\left(\frac{1-x}{1-xy}\right) + g_2\left(\frac{1-y}{1-xy}\right) = \alpha_1(x) + \alpha_2(y), \quad \forall\, x, y \in (0,1). \tag{4}$$

*Then there exist real numbers $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$, $A+B+C+D=0$, $E+F=G+H$, such that for all $x \in (0,1)$*

$$g_1(x) = A\log(x) + B\log(1-x) + E,$$
$$g_2(x) = C\log(x) + D\log(1-x) + F,$$

$$\alpha_1(x) = B\log(x) + (A+D)\log(1-x) + G,$$
$$\alpha_2(x) = D\log(x) + (B+C)\log(1-x) + H.$$

*Proof.* Since the functions are locally integrable we can take any $x_0,\ x_1 \in (0,1)$ such that $x_0 < x_1$, and integrate both sides of equation (4) with respect to $x$:

$$\int_{x_0}^{x_1} g_1\left(\frac{1-x}{1-xy}\right) dx + \int_{x_0}^{x_1} g_2\left(\frac{1-y}{1-xy}\right) dx = \int_{x_0}^{x_1} \alpha_1(x)dx + (x_1-x_0)\alpha_2(y), \quad \forall y \in (0,1).$$

Then substituting in the first integral on the left-hand side (lhs) $s = \frac{1-x}{1-xy}$ and in the second $t = \frac{1-y}{1-xy}$ one gets

$$(1-y)\int_{\frac{1-x_1}{1-x_1y}}^{\frac{1-x_0}{1-x_0y}} \frac{g_1(s)}{(1-sy)^2}\,ds + \frac{1-y}{y}\int_{\frac{1-y}{1-x_0y}}^{\frac{1-y}{1-x_1y}} \frac{g_2(t)}{t^2}\,dt$$
$$= \int_{x_0}^{x_1} \alpha_1(x)\,dx + (x_1-x_0)\alpha_2(y), \quad \forall\, y \in (0,1). \tag{5}$$

Dually, integrating (4) with respect to $y$ from $y_0$ to $y_1$, $0 < y_0 < y_1$, we get

$$\frac{1-x}{x}\int_{\frac{1-x}{1-y_0x}}^{\frac{1-x}{1-y_1x}} \frac{g_1(s)}{s^2}\,ds + (1-x)\int_{\frac{1-y_1}{1-y_1x}}^{\frac{1-y_0}{1-y_0x}} \frac{g_2(t)}{(1-tx)^2}\,dt$$
$$= (y_1-y_0)\alpha_1(x) + \int_{y_0}^{y_1} \alpha_2(y)\,dy, \quad \forall\, x \in (0,1). \tag{6}$$

Observe that the lhs of (5) is a continuous function in $y$. Consequently $\alpha_2$ is a continuous function. Similarly, by (6) it follows that $\alpha_1$ is continuous.

Now insert in (4) $u = \frac{1-x}{1-xy}$ and $v = \frac{1-y}{1-xy}$. Consequently $x = (1-u)/v$, $y = (1-v)/u$, $u, v \in (0,1)$, $u+v > 1$ and (4) takes the form

$$g_1(u) + g_2(v) = \alpha_1((1-u)/v) + \alpha_2((1-v)/u), \quad \forall\, u, v \in (0,1),\ u+v > 1. \tag{7}$$

Since $\alpha_1$ and $\alpha_2$ are continuous it follows by (7) that $g_1$ and $g_2$ are also continuous in $(0,1)$. But for continuous $g_i$'s the lhs of (5) is a $C^1$-function in $y$. Hence $\alpha_2$ is also $C^1$. Also (6) implies, analogously, that $\alpha_1$ is a $C^1$-function. Using again (7) we conclude that $g_1$ and $g_2$ are also $C^1$-functions.

Let us now differentiate (4) with respect to $x$. Then

$$-\frac{1-y}{(1-xy)^2}g_1'\left(\frac{1-x}{1-xy}\right)+\frac{(1-y)y}{(1-xy)^2}g_2'\left(\frac{1-y}{1-xy}\right)=\alpha_1'(x),\quad \forall\, x,y\in(0,1). \quad (8)$$

Inserting in the above equation $x=y$ we get immediately that

$$\alpha_1'(x)=\frac{1}{(1-x)(1+x)^2}\left[xg_2'\left(\frac{1}{1+x}\right)-g_1'\left(\frac{1}{1+x}\right)\right],\quad x\in(0,1).$$

Plugging it back into (8) and then returning to the variables $u$ and $v$, we get after simplification

$$-ug_1'(u)+(1-v)g_2'(v)=\frac{v}{(v+1-u)^2}\left[\frac{1-u}{v}g_2'\left(\frac{v}{v+1-u}\right)-g_1'\left(\frac{v}{v+1-u}\right)\right],\quad (9)$$

$\forall\, u,v\in(0,1),\ u+v>1$. Observe that $\lim_{u\to 0} ug_1'(u)=A$ exists since the rhs of (9) has a limit as $u\to 0$. Similarly to (9), but this time differentiating (4) with respect to $y$ we get

$$(1-u)g_1'(u)-vg_2'(v)=\frac{u}{(u+1-v)^2}\left[\frac{1-v}{u}g_1'\left(\frac{u}{u+1-v}\right)-g_2'\left(\frac{u}{u+1-v}\right)\right],$$

$\forall\, u,v\in(0,1),\ u+v>1$. Consequently the limit $\lim_{u\to 1}(1-u)g_1'(u)=-B$ exists since the rhs of the above identity has a limit as $u\to 1$.

Changing in (9) the variable $u=1-cv$, $c\in(0,1)$, yields

$$-(1-cv)g_1'(1-cv)+(1-v)g_2'(v)=\frac{1}{v(1+c)^2}\left[g_2'\left(\frac{1}{1+c}\right)-g_1'\left(\frac{1}{1+c}\right)\right]$$

for any $c,v\in(0,1)$. Denoting for any $t\in(0,1)$: $m(t)=-(1-t)g_1'(1-t)$, $n(t)=(1-t)g_2'(t)$,

$$K(t)=\frac{1}{(1+t)^2}\left[g_2'\left(\frac{1}{1+t}\right)-g_1'\left(\frac{1}{1+t}\right)\right],$$

we can rewrite the above equation as

$$m(cv)+n(v)=\frac{1}{v}K(c),\quad c,v\in(0,1). \quad (10)$$

Observe that $\lim_{c\to 1} m(cv)$ exists and equals $m(v)$. Consequently, passing to the limit as $c\to 1$ in (10) we obtain

$$v[m(v)+n(v)]=K_1=\lim_{c\to 1}K(c)=\frac{g_2'\left(\frac{1}{2}\right)-g_1'\left(\frac{1}{2}\right)}{4}. \quad (11)$$

Hence it follows that for any $c,v\in(0,1)$

$$v[m(cv)-m(v)]=K(c)-K_1=L(c).$$

Observe that $\lim_{v\to 1} m(v) = -A$. Then $L(c) = m(c) + A$ and thus

$$cvm(cv) - cm(v) = cm(c) + cA, \quad c, v \in (0, 1).$$

Hence for the function $z$ defined on $(0, 1)$ by $z(x) = xm(x) + A$ it follows that

$$z(xy) = yz(x) + z(y), \quad x, y \in (0, 1).$$

Taking now $x = 1/2$ we get $z(y) = z(y/2) - yz(1/2)$ which by iteration leads to

$$z(y) = z(y/2^n) - yz(1/2)(1 + 1/2 + \ldots + 1/2^{n-1}).$$

Observe that $\lim_{x\to 0} z(x) = \lim_{x\to 0}[-x(1-x)g_1'(1-x)+A] = A+B$. Consequently, the above relation yields, by taking $n \to \infty$ that $z(y) = A + B - ay$, where $a = 2z(1/2)$. Since $z(1) = \lim_{x\to 1} z(x)$ exists then necessarily $z(1) = 0$ and thus $z(y) = (A + B)(1 - y)$, $y \in (0, 1)$. Returning to $g_1'$ we see that $g_1'(x) = A/x + B/(1 - x)$ which immediately yields the desired form of $g_1$. Due to the symmetry of the problem $g_2$ is identified in the same way and the relation between constants $A$, $B$, $C$, and $D$ follows now directly from (4). Knowing $g_1$ and $g_2$, both $\alpha_1$ and $\alpha_2$ can be immediately derived from (4), together with the relation involving the constants $E$, $F$, $G$ and $H$. □

**Remark 1.** Observe that without any conditions on the behaviour of $g_i$'s and $\alpha_i$'s a possible solution could be of the form: $\forall\, x \in (0, 1)$

$$g_1(x) = A(x) + B(1 - x) + E, \qquad g_2(x) = C(x) + D(1 - x) + F,$$

$$\alpha_1(x) = B(x)+A(1-x)+D(1-x)+G, \qquad \alpha_2(x) = D(x)+B(1-x)+C(1-x)+H,$$

where $A$, $B$, $C$ and $D$ are generalized logarithmic functions, i.e. they satisfy the logarithmic equation: $f(xy) = f(x) + f(y)$, $x, y \in (0, 1)$, and as such may not have any smoothness properties. It is an open question if these are all possible solutions to (4) if no restrictions on the unknown functions are imposed.

**Remark 2.** Observe that in the problem of characterizing the beta distributions by the independence property, as described in Section 1, we arrived at equation (2) which is a version of (4) with $g_1 = -g_2 = g$ and $\alpha_1 = -\alpha_2 = \alpha$. In such a setting the results of Theorem 1 hold true even under the assumption that at least one of the functions $g$ or $\alpha$ is locally integrable. Hence if we assume that $\log(f_X)$ is locally integrable on $(0, 1)$, then it follows from Theorem 1 that the density function $f_X$ has the form

$$f_X(x) = \exp(g(x)) = e^E x^A (1 - x)^B, \quad x \in (0, 1).$$

Consequently $f_X$ is the density of a beta distribution, say $\beta_{p_1,q_1}$. Observe now that since $\psi = \psi^{-1}$ we can write down the equation dual to (1) by simply changing $(U, V)$ into $(X, Y)$. Again using the result of Theorem 1, this time with $f_U(x) = \exp(g(x))$, $x \in (0, 1)$, we conclude that $U$ is also a beta random variable, say $\beta_{p_3,q_3}$. On the other hand in the original setting, again by Theorem 1, we have

$$xf_U(x)/f_V(x) = \exp(\alpha(x)) = e^G x^B (1 - x)^{A-B}, \quad x \in (0, 1).$$

Since $U$ has the beta distribution it follows that

$$f_V(x) = Kx^{p_4-1}(1-x)^{q_4-1}, \quad x \in (0,1),$$

which again is the density of a beta distribution. Dually $Y$ is also a beta random variable $\beta_{p_2,q_2}$. Inserting now all these densities into (1) we conclude that $p_2 = p+q$, $p_3 = r$, $q_3 = p$, $p_4 = q+r$, $q_4 = p$ where $p = p_1$, $q = q_1$, $r = q_2$.

Thus the independence property characterizes the beta distributions if it is assumed that the densities exist, are strictly positive on $(0,1)$ and their logarithms are locally integrable, which is far more restrictive than the nonnegativeness, Borel measurability and integrability over $(0,1)$ to 1, which are natural properties of density functions. Possibly a way to get rid of these, seemingly, technical assumptions lies in adopting regularization methods developed in Járai [6] and then, as mentioned above, successfully applied in Járai [7] for the functional equation considered in Geiger and Heckerman [5]. However, at the present moment we are not aware how to implement these techniques for the equation considered in the present paper.

Summing up the observations given in Remark 2 we have the following partial answer to the characterization problem posed in the beginning of the paper.

**Theorem 2.** *Let $X$ and $Y$ be independent random variables having strictly positive densities on $(0,1)$ and such that logarithms of these densities are locally integrable. If the random vector $(U,V) = \psi(X,Y)$ has independent components, then $(X,Y) \sim \beta_{p,q} \otimes \beta_{p+q,r}$, and consequently $(U,V) \sim \beta_{r,q} \otimes \beta_{r+q,p}$, for some positive constants $p,q,r$.*

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