

## SWITCHING ORDER STATISTICS THROUGH RANDOM POWER CONTRACTIONS

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### Summary

This paper investigates a new random contraction scheme which complements the length-biasing and convolution contraction schemes considered in the literature. A random power contraction is used with order statistics, leading to new and elegant characterizations of the power distribution. In view of Rossberg's counter-example of a non-exponential law with exponentially distributed spacings of order statistics, possibly the most appealing consequence of the result is a characterization of the exponential distribution via an independent exponential shift of order statistics.

*Key words:* exponential distribution; order statistics; power distribution; random contraction; random shifting.

### 1. Introduction

Let  $U_1, \dots, U_n$  be independent identically distributed (iid) random variables (rvs) with a common distribution uniform on  $[0, 1]$ . Let  $(V_1, \dots, V_n)$  and  $(U_1, \dots, U_n)$  be iid random vectors. Then, as observed by Nevzorov (2001 Chapter 3),

$$U_{k:n} \stackrel{d}{=} U_{k:m} V_{m+1:n}, \quad (1)$$

where  $X_{k:n}$  denotes the  $k$ th order statistic from the sample  $(X_1, \dots, X_n)$ .

This observation falls in the general random contraction setting, which can be described in the following way: let  $U$  be an rv with a distribution concentrated on  $[0, 1]$ , and let  $X$  be a positive rv which is independent of  $U$ . Then the distribution of  $XU$  is a random contraction of the distribution of  $X$ . Assume that  $Y$  is an rv such that the distributions of  $X$  and  $Y$  are somehow related, and consider the equation

$$Y \stackrel{d}{=} XU. \quad (2)$$

Such schemes have been studied in the literature mainly in the context of identifiability and identification of the distribution of  $X$  at least in two cases:

- (1) length-biasing, i.e. the distribution functions (cdfs)  $F_X$  of  $X$  and  $F_Y$  of  $Y$  are related by  $E(Y^r)F_X(x) = \int_0^x y^r dF_Y(y)$ , for any  $x > 0$ , while it is assumed that  $X$  and  $Y$  are positive and  $E(Y^r) < \infty$  for some real  $r$ . Recent contributions in this area include Pakes (1996, 1997) and Pakes, Sapatinas & Fosam (1996).

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- (2) convolutions, i.e.  $X \stackrel{d}{=} Y_1 + \dots + Y_m$ , where  $Y_1, \dots, Y_m$  are iid rvs with the same distribution as  $Y$ . Within such a scheme, given the distribution of  $U$  the law of the  $Y_j$  has been characterized for instance by Kotz & Steutel (1988), Milne & Yeo (1989), Yeo & Milne (1991) and Pakes (1994, 1995).

Often in these schemes the contraction distribution of  $U$  has been chosen to be the  $\alpha$ -power distribution  $\text{pow}(1, \alpha)$ , where in general the  $\text{pow}(a, \alpha)$  distribution, with positive parameters  $a$  and  $\alpha$  is defined by the probability density function (pdf)

$$f(x) = \frac{\alpha}{a} \left(\frac{x}{a}\right)^{\alpha-1} I_{(0,a)}(x).$$

In the order statistics scheme defined by (1) with  $m + 1 = n$  and  $U = V_{n:n}$  we have that  $U \stackrel{d}{=} \text{pow}(1, n)$ . Observe also that if  $X_1, \dots, X_n$  are iid rvs with  $\text{pow}(1, \alpha)$  distribution, and  $(Z_1, \dots, Z_n)$  and  $(X_1, \dots, X_n)$  are iid, then (1) can be extended to

$$X_{k:n} \stackrel{d}{=} X_{k:m} Z_{m+1:n},$$

which follows from (1) by the representation  $X_{k:n} \stackrel{d}{=} U_{k:n}^{1/\alpha}$ . If  $m + 1 = n$  then the contracting variable  $Z_{n:n} \stackrel{d}{=} \text{pow}(1, n\alpha)$ .

In this paper we are interested in the following version of (2)

$$X_{k:n} \stackrel{d}{=} X_{k:n-1} Z, \tag{3}$$

where  $Z \stackrel{d}{=} \text{pow}(1, \alpha)$  is independent of the  $X_j$ . If  $X_1 \stackrel{d}{=} \text{pow}(a, \alpha/n)$  for an arbitrary fixed  $a > 0$  then (3) holds. We also consider

$$X_{k:n} \stackrel{d}{=} X_{k+1:n} Z. \tag{4}$$

If  $X_1 \stackrel{d}{=} \text{pow}(a, \alpha/k)$  then (4) holds. Converses of both results are treated in Section 2.

Section 3 is devoted to a related problem connected with the equation

$$X_{k:n-1} Z_1 \stackrel{d}{=} X_{k+1:n} Z_2, \tag{5}$$

where  $Z_i \stackrel{d}{=} \text{pow}(1, \alpha_i)$ ,  $i = 1, 2$ , are independent of the  $X_j$ . Again it follows that if  $\alpha_1/n = \alpha_2/k = \alpha$ , say, and  $X_1 \stackrel{d}{=} \text{pow}(a, \alpha)$ , for an arbitrary  $a > 0$  then (5) holds true.

It is well known that if  $X_1, \dots, X_n$  are iid exponential rvs, then  $X_{k+1:n} - X_{k:n} \stackrel{d}{=} W$ , where  $W$  has an exponential distribution, for any  $k = 1, \dots, n - 1$ . However Rossberg (1972) gives an example showing this property alone for a single value  $k$  does not characterize the parent distribution as exponential. The characterization holds true under some additional technical and rather unfriendly conditions; see e.g. Riedel & Rossberg (1994) or Rossberg, Riedel & Ramachandran (1997). It appears that our results on power distributions lead to characterizing the exponential law by equidistribution of  $X_{k+1:n}$  and  $X_{k:n} + W$ , where  $W$  is an independent exponential rv — which looks rather unexpected in view of Rossberg’s counter-example. For a recent discussion of related characterizations via equidistribution conditions for order statistics consult Gather, Kamps & Schweitzer (1998).

The approach we develop in this paper is applicable to characterizations of the logistic distribution based on exponential or Laplace random shifts of order statistics; see e.g. George

& Mudholkar (1981) or George & Rousseau (1987). It appears that most of rather technical conditions used in these papers can be avoided; details are given by Markiewicz (2002).

If  $Z \stackrel{d}{=} \text{pow}(1, \alpha)$  then  $W = -\log(Z) \stackrel{d}{=} \exp(\alpha)$ . Since  $-\log$  is a decreasing function the following analogues of (3), (4) and (5) can be immediately formulated. Assume that  $W \stackrel{d}{=} \exp(\alpha)$  is independent of the rvs  $Y_1, \dots, Y_n$  which are iid. If  $Y_1 \stackrel{d}{=} \exp(\alpha/n)$  then  $Y_{k+1:n} \stackrel{d}{=} Y_{k:n-1} + W$ . If  $Y_1 \stackrel{d}{=} \exp(\alpha/(n-k))$  then  $Y_{k+1:n} \stackrel{d}{=} Y_{k:n} + W$ . Let  $W_1 \stackrel{d}{=} \exp(n\alpha)$  and  $W_2 \stackrel{d}{=} \exp((n-k)\alpha)$  and  $W_1, W_2$  be independent of  $Y_1, \dots, Y_n$ . If  $Y_1 \stackrel{d}{=} \exp(\alpha)$  then  $Y_{k:n-1} + W_1 \stackrel{d}{=} Y_{k:n} + W_2$ . The converse statements follow directly from our discussion of the power contraction scheme in Section 2.

We denote by  $\{X_{k:n}, k = 1, \dots, n\}$  order statistics from the iid rvs  $X_1, \dots, X_n$  with cdf  $F$  ( $f$  denotes the pdf if it exists). Similarly,  $F_{k:n}$  ( $f_{k:n}$ ) denotes the cdf (pdf if it exists) of respective order statistics,  $k = 1, \dots, n$ .

The recurrence relations connecting cdfs and pdfs of order statistics gathered below, which are known (see e.g. David & Shu, 1978; David, 1981 p.25), are used in the proofs of main results in Sections 2 and 3.

For any  $k = 1, \dots, n - 1$  and any  $n = 2, 3, \dots$

$$(i) \quad F_{k:n} - F_{k:n-1} = \binom{n-1}{k-1} F^k (1 - F)^{n-k}, \tag{6}$$

and if the pdf exists it has the form

$$n(F_{k:n} - F_{k:n-1})f = Ff_{k:n}. \tag{7}$$

$$(ii) \text{ Also, } F_{k:n} - F_{k+1:n} = \binom{n}{k} F^k (1 - F)^{n-k}, \tag{8}$$

and if the pdf exists it has the form

$$k(F_{k:n} - F_{k+1:n})f = Ff_{k:n}. \tag{9}$$

(iii) If the pdf exists then

$$n(1 - F)f_{k:n-1} = (n - k)f_{k:n}, \tag{10}$$

$$k(1 - F)f_{k+1:n} = (n - k)Ff_{k:n}. \tag{11}$$

### 2. One-sided contractions

This section gives two characterizations of the power distribution. Both are based on properties of switching order statistics by one-sided contraction, while the contracting rv has a power distribution in the interval  $[0, 1]$ . Such results, while giving a new insight into the structure of the order statistics distribution, also complement characterizations within other random contraction schemes considered in the literature, where random contraction is applied to length-biased distributions and convolutions.

**Theorem 1.** Let  $U \stackrel{d}{=} \text{pow}(1, \alpha)$  for some  $\alpha > 0$  be independent of  $X_1, \dots, X_n$ , which are positive iid rvs. If

$$X_{k:n} \stackrel{d}{=} X_{k:n-1}U \tag{12}$$

for an arbitrary but fixed  $k \in \{1, \dots, n - 1\}$  then  $a > 0$  exists such that  $X_1 \stackrel{d}{=} \text{pow}(a, \alpha/n)$ .

**Proof.** By (12), for any  $x > 0$ ,

$$F_{k:n}(x) = \int_0^1 F_{k:n-1}(x/u) \alpha u^{\alpha-1} du .$$

Denote  $a = \sup\{x > 0: F(x) < 1\} \leq \infty$ . Then (12) implies  $\inf\{x > 0: F(x) > 0\} = 0$ . Substituting  $t = x/u$  in the above integral we obtain

$$\begin{aligned} F_{k:n}(x) &= \alpha \int_0^{x/a} u^{\alpha-1} du + \alpha \int_{x/a}^1 F_{k:n-1}(x/u) u^{\alpha-1} du \\ &= (x/a)^\alpha + \alpha x^\alpha \int_x^a F_{k:n-1}(t) t^{-\alpha-1} dt , \end{aligned}$$

where it is understood that  $(x/a)^\alpha = 0$  in the case  $a = \infty$ . Observe that the last expression is differentiable in  $x$ . Hence it follows that  $F_{k:n}$  is also differentiable. Consequently the density  $f$  of  $X_1$  and the densities of all order statistics exist. Upon differentiation we get

$$f_{k:n}(x) = \frac{\alpha x^{\alpha-1}}{a^\alpha} + \alpha^2 x^{\alpha-1} \int_x^a F_{k:n-1}(t) t^{-\alpha-1} dt - \alpha x^{-1} F_{k:n-1}(x) .$$

Substituting for the integral from the previous equation we arrive at

$$x f_{k:n}(x) = \alpha (F_{k:n}(x) - F_{k:n-1}(x)) , \tag{13}$$

holding for any  $x \in (0, a)$ . By (6) for any  $x \in (0, a)$ , the difference  $F_{k:n}(x) - F_{k:n-1}(x)$  is non-zero, since  $\inf\{x > 0: F(x) > 0\} = 0$ . Consequently the density  $f$  is always positive in  $(0, a)$  and (13) via (7) leads to

$$\frac{F'(x)}{F(x)} = \frac{\alpha}{nx} ,$$

whence  $F(x) = Kx^{\alpha/n}$  for any  $x \in (0, a)$ , which yields  $a < \infty$  and  $K = a^{-\alpha/n}$ .

**Remark 1.** For  $U$  in the above result we can take  $U_{n:n}$ , where  $U_1, \dots, U_n$  is a random sample from the uniform (or, more generally,  $\text{pow}(1, \alpha)$ ) distribution. Then (12) implies that  $X_1$  is uniform on  $(0, a)$  (or, more generally,  $\text{pow}(a, \alpha)$ ).

By the duality between the power and exponential distribution mentioned in Section 1 we have the following consequence of Theorem 1.

**Corollary 1.** Let  $W \stackrel{d}{=} \exp(\alpha)$  for some  $\alpha > 0$  be independent of  $Y_1, \dots, Y_n$ , which are iid rvs. If  $Y_{k+1:n} \stackrel{d}{=} Y_{k:n-1} + W$  for an arbitrary and fixed  $k \in \{1, \dots, n - 1\}$  then there exists  $\gamma \in \mathbb{R}$  such that  $Y_1 + \gamma \stackrel{d}{=} \exp(\alpha/n)$ .

Now we are ready to consider the condition dual to (12).

**Theorem 2.** Let  $U \stackrel{d}{=} \text{pow}(1, \alpha)$  for some  $\alpha > 0$  be independent of  $X_1, \dots, X_n$ , which are positive, iid rvs. If

$$X_{k:n} \stackrel{d}{=} X_{k+1:n} U \tag{14}$$

for an arbitrary but fixed  $k \in \{1, \dots, n - 1\}$  then  $a > 0$  exists such that  $X_1 \stackrel{d}{=} \text{pow}(a, \alpha/k)$ .

**Proof.** As in the proof of Theorem 1 denote  $a = \sup\{x > 0: F(x) < 1\} \leq \infty$  and observe that  $\inf\{x > 0: F(x) > 0\} = 0$ . By (14) we arrive at

$$F_{k:n}(x) = \int_0^1 F_{k+1:n}\left(\frac{x}{u}\right)\alpha u^{\alpha-1} du = \left(\frac{x}{a}\right)^\alpha + \alpha x^\alpha \int_x^a F_{k+1:n}(t)t^{-\alpha-1} dt$$

for any  $x \in (0, a)$ . Again the form of the above equation implies existence of densities. Then

$$xf_{k:n}(x) = \alpha(F_{k:n}(x) - F_{k+1:n}(x)) \quad (x \in (0, a)). \tag{15}$$

By (8),  $F_{k:n}(x) - F_{k+1:n}$  and, consequently,  $f_{k:n}(x)$  are non-zero, and thus  $f(x)$  is also non-zero on  $(0, a)$ . Thus by (9) we get from (15) for any  $x \in (0, a)$

$$\frac{F'(x)}{F(x)} = \frac{\alpha}{kx},$$

leading to  $F(x) = Kx^{\alpha/k}$  in  $(0, a)$ , implying  $a < \infty$  and  $K = a^{-\alpha/k}$  and the hypothesis.

**Remark 2.** In Theorem 2, taking  $U_{k:k}$  for  $U$  as a special case where  $U_1, \dots, U_k$  is a random sample from the  $U(0, 1)$  (or  $\text{pow}(1, \alpha)$ ) distribution, results in the  $U(0, a)$  (or  $\text{pow}(a, \alpha)$ ) distribution for  $X_1$ .

As for Theorem 1, an exponential analogue of Theorem 2 follows directly by the standard transformation leading from exponential to power distributions.

**Corollary 2.** Let  $W \stackrel{d}{=} \exp(\alpha)$  for some  $\alpha > 0$  be independent of  $Y_1, \dots, Y_n$  which are iid rvs. If  $Y_{k+1:n} \stackrel{d}{=} Y_{k:n} + W$  for an arbitrary but fixed  $k \in \{1, \dots, n - 1\}$  then there exists  $\gamma \in \mathbb{R}$  such that  $Y_1 + \gamma \stackrel{d}{=} \exp(\alpha/(n - k))$ .

We say again that the above result looks somewhat unexpected in light of Rossberg’s (1972) counter-example — it is known that a closely related condition,  $Y_{k+1:n} - Y_{k:n} \stackrel{d}{=} W$  does not characterize the exponential law.

### 3. Two-sided contractions

In this section we consider equality in distribution of random power contractions of two order statistics, i.e. we equate in distribution right-hand sides of (12) and (14) with different contraction variables. However, here we have to restrict our considerations to absolutely continuous distributions. An exponential analogue follows along the same lines as the respective results in the previous section.

**Theorem 3.** Let  $U_i$  be an rv with the power distribution  $\text{pow}(1, \alpha_i)$ ,  $i = 1, 2$ , and let  $U_1, U_2$  be independent of  $X_1, \dots, X_n$ , which are positive absolutely continuous iid rvs such that  $\inf\{x > 0: F(x) > 0\} = 0$ . Assume that

$$X_{k:n-1}U_1 \stackrel{d}{=} X_{k+1:n}U_2 \tag{16}$$

for an arbitrary but fixed  $k \in \{1, \dots, n - 1\}$ . Then  $\alpha_1/n = \alpha_2/k = \alpha$ , say, and there exists  $a > 0$  such that  $X_1 \stackrel{d}{=} \text{pow}(a, \alpha)$ .

**Proof.** Denote, as earlier,  $a = \sup\{x > 0: F(x) < 1\} \leq \infty$ . Then (16) is equivalent to

$$\int_0^1 F_{k:n-1}\left(\frac{x}{u}\right)\alpha_1 u^{\alpha_1-1} du = \int_0^1 F_{k+1:n}\left(\frac{x}{u}\right)\alpha_2 u^{\alpha_2-1} du \quad (x \in (0, a)),$$

and with similarity to the previous section, it follows that for any  $x \in (0, a)$

$$\left(\frac{x}{a}\right)^{\alpha_1} + \alpha_1 x^{\alpha_1} \int_x^a F_{k:n-1}(t)t^{-\alpha_1-1} dt = \left(\frac{x}{a}\right)^{\alpha_2} + \alpha_2 x^{\alpha_2} \int_x^a F_{k+1:n}(t)t^{-\alpha_2-1} dt.$$

Consider first the case  $\alpha_1 = \alpha_2$ . Then the above equation implies  $F_{k:n-1}(x) = F_{k+1:n}(x)$ , for all  $x \in (0, a)$ . But this is impossible in view of (6) and (8). So,  $\alpha_1 \neq \alpha_2$ .

Taking the derivative with respect to  $x$  we get

$$x^{-\alpha_2}(\alpha_2 F_{k+1:n}(x) - \alpha_1 F_{k:n-1}(x)) = (\alpha_2 - \alpha_1) \left( a^{-\alpha_2} + \alpha_2 \int_x^a F_{k+1:n}(t)t^{-\alpha_2-1} dt \right).$$

Differentiating again we obtain, after some simple algebra,

$$x(\alpha_2 f_{k+1:n}(x) - \alpha_1 f_{k:n-1}(x)) = \alpha_1 \alpha_2 (F_{k+1:n}(x) - F_{k:n-1}(x)) \quad (x \in (0, a)),$$

and by (6) and (8) the expressions on both sides of this equation are non-zero. But, by (10) and (11), the expression on the left-hand side equals

$$x \frac{n-k}{1-F(x)} \left( \frac{\alpha_2 F(x)}{k} - \frac{\alpha_1}{n} \right) f_{k:n}(x),$$

so  $f_{k:n}$  is non-zero, and hence  $f$  also is non-zero. Then using (7) and (9) we arrive at

$$\frac{f(x)}{F(x)(1-F(x))} (c_1 - c_2 F(x)) = \frac{c_1 c_2}{x} \quad (x \in (0, a)),$$

where  $c_1 = \alpha_1/n$  and  $c_2 = \alpha_2/k$ . The above equation implies  $c_1 \geq c_2$ . Also it can be rewritten as a simple differential equation in  $(0, a)$ ,

$$c_1 \frac{F'(x)}{F(x)} + (c_1 - c_2) \frac{F'(x)}{1-F(x)} = \frac{c_1 c_2}{x}.$$

Hence it follows that

$$F(x)^{c_1} (1-F(x))^{c_1-c_2} = Kx^{c_1 c_2} \tag{17}$$

for any  $x \in (0, a)$ , where  $K$  is a non-zero real constant. This implies  $a < \infty$ . Observe now that for  $x \rightarrow a$  the left-hand side of (17) tends to zero if  $c_1 > c_2$  while the right-hand side tends to  $Ka^{c_1 c_2} > 0$ . Hence  $c_1 = c_2 = \alpha$ , say, and (17) then has the form  $F(x) = K_1 x^\alpha$ ,  $x \in (0, a)$ , with  $K_1 = a^{-\alpha}$ .

**Remark 3.** Similarly to the results of Section 2, there is an interesting case in Theorem 3 if we take  $U_{n:n}$  and  $U_{k:k}$  for  $U_1$  and  $U_2$ , respectively, where  $U_{1:d}, \dots, U_n$  is a sample from the  $U(0, 1)$  (or  $\text{pow}(1, \alpha)$ ) distribution. Then it follows that  $X_1 = U(0, a)$  (or  $\text{pow}(a, \alpha)$ ).

Finally we have the exponential version of Theorem 3.

**Corollary 3.** Let  $W_i \stackrel{d}{=} \exp(\alpha_i)$ ,  $i = 1, 2$ , and let  $W_1, W_2$  be independent of  $Y_1, \dots, Y_n$ , which are absolutely continuous iid rvs with  $\sup\{x: F(x) < 1\} = \infty$ . Assume that

$$Y_{k:n-1} + W_1 \stackrel{d}{=} Y_{k:n} + W_2$$

for an arbitrary but fixed  $k \in \{1, \dots, n-1\}$ . Then  $\alpha_1/n = \alpha_2/(n-k) = \alpha$ , say, and there exists  $\gamma \in \mathbb{R}$  such that  $Y_1 + \gamma \stackrel{d}{=} \exp(\alpha)$ .

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