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Multivariate Lukacs theorem

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Abstract

According to the celebrated Lukacs theorem, independence of quotient and sum of two independent positive random variables characterizes the gamma distribution. Rather unexpectedly, it appears that in the multivariate setting, the analogous independence condition does not characterize the multivariate gamma distribution in general, but is far more restrictive: it implies that the respective random vectors have independent or linearly dependent components. Our basic tool is a solution of a related functional equation of a quite general nature. As a side effect the form of the multivariate distribution with univariate Pareto conditionals is derived.

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1. Introduction

In [19] the following celebrated theorem is proved: *let X and Y be nondegenerate, positive, independent random variables* (*rv's*). *If X/Y and $X+Y$ are also independent, then the distributions of X and Y are gamma with the common scale parameter*. This result was a starting point of numerous investigations which led to discoveries of further characteristic properties of the gamma distribution of a similar nature. Mostly, these studies were concerned with related regression schemes and/or probabilistic measures on more abstract structures—see, for instance, [6–11, 14,15,17,18,22,24–26].

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However, among different results related to the Lukacs theorem it is rather difficult to find anything in the n-variate setting. At first, it seems to be rather strange. However, there is a reason: it lies in problems connected with infinite divisibility of the multivariate gamma distribution—see for instance [12,13,21,23]. It appears that only the bivariate gamma distribution happens to be infinitely divisible without any additional restrictions. On the other hand the matrix variate versions of the Lukacs theorem were studied thoroughly, for instance in [8,10,22]. So a gap arouse between univariate and matrix variate approaches. The only early result falling into the gap was given in [24] and was concerned with regression conditions in the bivariate case. There exists also another characterization of the bivariate gamma distribution, given in [20], but by a quite different regression property. Both these results suffer from inaccuracies related to the proper definition of the bivariate gamma distribution—until the paper [5] the set of admissible parameter values has not been specified precisely. These problems are carefully explained in [6], where the bivariate version of the Lukacs theorem is given—leading to random vectors with independent or linearly dependent gamma components. On the other hand, it is shown there that different pairs of constancy of regression conditions lead to characterizations of the general bivariate gamma distribution.

Here, we are concerned with the independence property in the general multivariate setting. Again, as in the bivariate case, it appears that the parent random vectors have to have elements grouped into independent subvectors, each subvector having linearly dependent components. This is the main result of the present paper and it is given in Section 3. Let us stress that, rather unexpectedly, the n-variate independence condition is more restrictive than its matrix variate analogue, in which case it characterizes the general matrix variate gamma (Wishart) distribution. Earlier in Section 2 a general version of a functional equation related to the characterization problem is considered. It appears that its solution is also useful in describing multivariate distributions with Pareto univariate conditional distributions—see Section 4.

2. Functional equation

In this section, we consider a functional equation related to the characterization problem, which is the main subject of the paper. Its bivariate version was considered in [6] and is presented below, since it will be used in the sequel.

Lemma 1. *Let $b, d : (-\infty, 0] \to (-\infty, 0]$ and a, c are some real functions. Suppose that*

$$a(x)(1 + b(x)y)^p = c(y)(1 + d(y)x)^q \tag{2.1}$$

holds for every $x, y \leqslant 0$ and some real numbers p, q.

Then only two cases are possible:

1. *$a = c = 0$ and $b(x), d(y)$ are optional*;
2. *$a = c \neq 0$ and*

$$a(x) = a(1 + dx)^q, \quad c(y) = a(1 + by)^p$$

and either $p \neq q$ *and*

$$b(x) = b, \quad d(y) = d$$

or $p = q$ *and*

$$b(x) = \frac{Mx}{1 + dx} + b, \quad d(y) = \frac{My}{1 + by} + d,$$

where $a = a(0),\ b = b(0),\ c = c(0),\ d = d(0)$ *and* M *is a constant.*

To formulate n-variate version of (2.1) denote for any $i = 1, \ldots, n$

$$\bar{s}_{(i)} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n), \quad \bar{s}^{(i)} = (s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n).$$

Then the functional equation (2.1) in n dimensions reads

$$f(s_1, \ldots, s_n) = A_i(\bar{s}_{(i)})(1 + B_i(\bar{s}_{(i)})s_i)^{p_i} \quad \forall i = 1, \ldots, n. \tag{2.2}$$

Its solution, being the main result of this section, is given below.

Theorem 1. *Let* $A_i : (-\infty, 0]^{n-1} \to [0, +\infty)$ *and* $B_i : (-\infty, 0]^{n-1} \to (-\infty, 0]$, $i = 1, \ldots, n$. *Suppose that* (2.2) *holds for every* $s_1, \ldots, s_n \leqslant 0$.

Define $q_k = p_{U(k)},\ k = 1, \ldots, r$, *where* $U(1) = 1$,

$$U(k) = \min\{j > U(k-1) : p_j \notin \{q_1, \ldots, q_{k-1}\}\}, \quad r = \max_k U(k) \leqslant n.$$

Moreover, let $W_k = \{j \in \{1, \ldots, n\} : p_j = q_k\}, \# W_k = m_k$.

Then

$$f(s_1, \ldots, s_n) = A \prod_{k=1}^{r} \left[1 + \sum_{i=1}^{m_k} \sum_{\{j_1, \ldots, j_i\} \subset W_k} a_{j_1 \ldots j_i} s_{j_1} \ldots s_{j_i} \right]^{q_k}, \tag{2.3}$$

where $A = A_l(\bar{0}),\ l = 1, \ldots, n$, *and* $a_{j_1 \ldots j_i}$*'s are some constants.*

Proof. From (2.2), for $i \neq j$, it follows that

$$A_i(\bar{s}_{(i)})(1 + B_i(\bar{s}_{(i)})s_i)^{p_i} = A_j(\bar{s}_{(j)})(1 + B_j(\bar{s}_{(j)})s_j)^{p_j}. \tag{2.4}$$

Inserting $\bar{s} = 0$ into (2.4) we obtain

$$A_i(\bar{0}) = A_j(\bar{0}) = A \quad \forall i, j = 1, \ldots, n, \quad i \neq j. \tag{2.5}$$

First, it will be observed that $A = 0$ implies $f \equiv 0$.

Inserting $s_i = 0$ into (2.2) gives

$$\begin{aligned} f(\bar{s}^{(i)}) &= A_i(\bar{s}_{(i)}) \\ &= A_j(\bar{s}^{(i)}_{(j)})(1 + B_j(\bar{s}^{(i)}_{(j)})s_j)^{p_j} \quad \forall i, j = 1, \ldots, n, \quad i \neq j. \end{aligned} \tag{2.6}$$

Hence, for $i = 1$, we get

$$A_1(\bar{s}_{(1)}) = A_j(\bar{s}^{(1)}_{(j)})(1 + B_j(\bar{s}^{(1)}_{(j)})s_j)^{p_j} \quad \forall j = 2, \ldots, n.$$

Inserting in the above equation $s_j = 0$ leads to

$$A_j(\bar{s}^{(1)}_{(j)}) = A_k(\bar{s}^{(1,j)}_{(k)})(1 + B_k(\bar{s}^{(1,j)}_{(k)})s_k)^{p_k} \quad \forall k = 2, \ldots, n, \quad k \neq j,$$

where the numbers in the superscript denote zero entries of the $\bar{s}$ vector. Particularly, for $j = 2$, we have

$$A_2(\bar{s}^{(1)}_{(2)}) = A_k(\bar{s}^{(1,2)}_{(k)})(1 + B_k(\bar{s}^{(1,2)}_{(k)})s_k)^{p_k} \quad \forall k = 3, \ldots, n.$$

Repeating this argument n times we arrive at

$$\begin{aligned} A_1(\bar{s}_{(1)}) &= A_n(\bar{s}^{(1,\ldots,n-1)}_{(n)}) \prod_{k=2}^{n} (1 + B_k(\bar{s}^{(1,\ldots,k-1)}_{(k)})s_k)^{p_k} \\ &= A \prod_{k=2}^{n} (1 + B_k(\bar{s}^{(1,\ldots,k-1)}_{(k)})s_k)^{p_k}, \end{aligned}$$

which is due to the fact that $A_n(\bar{s}^{(1,\ldots,n-1)}_{(n)}) = A_n(\bar{0}) = A$. Similarly, we can show that

$$A_i(\bar{s}_{(i)}) = A \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 + B_k(\bar{s}^{(1,\ldots,k-1)}_{(k)})s_k)^{p_k} \quad \forall i = 1, \ldots, n. \tag{2.7}$$

Hence, if $A = 0$ then $A_i(\bar{s}_{(i)}) = 0 \ \forall \bar{s}_{(i)} \ \forall i = 1, \ldots, n$, and $f(s_1, \ldots, s_n) = 0 \ \forall s_1, \ldots, s_n \leqslant 0$.

Assume that $A \neq 0$. Then, by (2.7), $A_i(\bar{s}_{(i)}) > 0 \ \forall \bar{s}_{(i)} \ \forall i = 1, \ldots, n$.

First, by induction on n, we give a proof for $r = 1$ and any n. Then $p_i = p \ \forall i = 1, \ldots, n$, and (2.2) takes the form

$$f(s_1, \ldots, s_n) = A_i(\bar{s}_{(i)})(1 + B_i(\bar{s}_{(i)})s_i)^p \quad \forall i = 1, \ldots, n. \tag{2.8}$$

We need to show

$$f(s_1, \ldots, s_n) = A \left[1 + \sum_{k=1}^{n} \sum_{1 \leqslant l_1 < \cdots < l_k \leqslant n} a_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k} \right]^p. \tag{2.9}$$

Let us rewrite (2.8) as

$$\tilde{f}(s_1, \ldots, s_n) = \tilde{A}_i(\bar{s}_{(i)}) + C_i(\bar{s}_{(i)})s_i \quad \forall i = 1, \ldots, n, \tag{2.10}$$

where

$$\begin{aligned} &\tilde{f}(s_1, \ldots, s_n) = [f(s_1, \ldots, s_n)]^{\frac{1}{p}}, \\ &\tilde{A}_i(\bar{s}_{(i)}) = [A_i(\bar{s}_{(i)})]^{\frac{1}{p}}, \\ &C_i(\bar{s}_{(i)}) = \tilde{A}_i(\bar{s}_{(i)})B_i(\bar{s}_{(i)}). \end{aligned}$$

Then (2.9) takes the form

$$\tilde{f}(s_1, \ldots, s_n) = \tilde{A}\left[1 + \sum_{k=1}^{n} \sum_{1 \leqslant l_1 < \cdots < l_k \leqslant n} a_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k}\right], \tag{2.11}$$

where $\tilde{A} = A^{\frac{1}{p}}$.

For $n = 2$ (2.11) is satisfied by Lemma 1. We now proceed by induction on n. Assume that (2.11) holds for $n - 1$. Inserting $s_i = 0$ into (2.10) we get

$$\begin{aligned}\tilde{f}(\bar{s}^{(i)}) &= \tilde{A}_i(\bar{s}_{(i)}) \\ &= \tilde{A}_j(\bar{s}^{(i)}_{(j)}) + C_j(\bar{s}^{(i)}_{(j)}) s_j \quad \forall i, j = 1, \ldots, n, \ i \neq j.\end{aligned}$$

Hence, by the assumption,

$$\tilde{A}_i(\bar{s}_{(i)}) = \tilde{A}^{(i)}\left[1 + \sum_{k=1}^{n-1} \sum_{\substack{1 \leqslant l_1 < \cdots < l_k \leqslant n \\ l_m \neq i \ \forall m = 1, \ldots, k}} a^{(i)}_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k}\right] \quad \forall i = 1, \ldots, n. \tag{2.12}$$

From (2.12) we conclude that $\tilde{A}_i(\bar{0}) = \tilde{A}^{(i)}$ $\forall i = 1, \ldots, n$. Therefore, by (2.5), $\tilde{A}^{(i)} = \tilde{A}$ $\forall i = 1, \ldots, n$. Moreover, let us note that $a^{(i)}_{l_1, \ldots, l_k} = a_{l_1, \ldots, l_k}$ $\forall i = 1, \ldots, n$ (it follows from substituting (2.12) into (2.10) and then plugging zeros for suitable s_i's). Hence (2.10) can be rewritten as

$$\begin{aligned}\tilde{f}(s_1, \ldots, s_n) = \tilde{A}&\left[1 + \sum_{k=1}^{n-1} \sum_{\substack{1 \leqslant l_1 < \cdots < l_k \leqslant n \\ l_m \neq i \ \forall m = 1, \ldots, k}} a^{(i)}_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k}\right] \\ &+ C_i(\bar{s}_{(i)}) s_i \quad \forall i = 1, \ldots, n.\end{aligned} \tag{2.13}$$

From (2.13), for $i \neq j$, it follows that

$$\begin{aligned}&\tilde{A} \sum_{k=1}^{n-1} \sum_{\substack{1 \leqslant l_1 < \cdots < l_k \leqslant n \\ l_m \neq i \ \forall m = 1, \ldots, k}} a_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k} + C_i(\bar{s}_{(i)}) s_i \\ &\quad = \tilde{A} \sum_{k=1}^{n-1} \sum_{\substack{1 \leqslant l_1 < \cdots < l_k \leqslant n \\ l_m \neq j \ \forall m = 1, \ldots, k}} a_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k} + C_j(\bar{s}_{(j)}) s_j.\end{aligned} \tag{2.14}$$

Let $i = 1$. Then (2.14) implies

$$\begin{aligned}&\tilde{A} \sum_{k=1}^{n-1} \sum_{\substack{2 \leqslant l_1 < \cdots < l_k \leqslant n \\ \exists m \ l_m = j}} a_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k} + C_1(\bar{s}_{(1)}) s_1 \\ &\quad = \tilde{A} \sum_{k=1}^{n-1} \sum_{\substack{1 = l_1 < \cdots < l_k \leqslant n \\ l_m \neq j \ \forall m = 1, \ldots, k}} a_{l_1 \ldots l_k} s_{l_1} \ldots s_{l_k} + C_j(\bar{s}_{(j)}) s_j.\end{aligned} \tag{2.15}$$

Dividing both sides of (2.15) by $s_1 s_j$ gives

$$\frac{1}{s_j}\left[C_1(\bar{s}_{(1)}) - \tilde{A}\frac{1}{s_1}\sum_{k=1}^{n-1}\sum_{\substack{1=l_1<\cdots<l_k\leqslant n\\ l_m\neq j\ \forall m=1,\ldots,k}} a_{l_1\ldots l_k}s_{l_1}\ldots s_{l_k}\right]$$

$$=\frac{1}{s_1}\left[C_j(\bar{s}_{(j)}) - \tilde{A}\frac{1}{s_j}\sum_{k=1}^{n-1}\sum_{\substack{2\leqslant l_1<\cdots<l_k\leqslant n\\ \exists m\ \ l_m=j}} a_{l_1\ldots l_k}s_{l_1}\ldots s_{l_k}\right] = C_{1j}(\bar{s}_{(1,j)}).$$

Consequently,

$$C_1(\bar{s}_{(1)}) = \tilde{A}\sum_{k=1}^{n-1}\sum_{\substack{1=l_1<\cdots<l_k\leqslant n\\ l_m\neq j\ \forall m=1,\ldots,k}} a_{l_1\ldots l_k}s_{l_2}\ldots s_{l_k} + C_{1j}(\bar{s}_{(1,j)})s_j \quad \forall j=2,\ldots,n. \tag{2.16}$$

Therefore, by the induction assumption, we obtain

$$C_1(\bar{s}_{(1)}) = \tilde{C}\left[1+\sum_{k=1}^{n-1}\sum_{2\leqslant l_1<\cdots<l_k\leqslant n} c_{l_1\ldots l_k}s_{l_1}\ldots s_{l_k}\right], \tag{2.17}$$

where $\tilde{C}$ and $c_{l_1\ldots l_k}$, $k=1,\ldots,n-1$, are some constants.

Combining (2.17) with (2.13) gives us

$$\begin{aligned}\tilde{f}(s_1,\ldots,s_n) &= \tilde{A}\left[1+\sum_{k=1}^{n-1}\sum_{2\leqslant l_1<\cdots<l_k\leqslant n} a_{l_1\ldots l_k}s_{l_1}\ldots s_{l_k}\right]\\ &\quad + \tilde{C}\left[1+\sum_{k=1}^{n-1}\sum_{2\leqslant l_1<\cdots<l_k\leqslant n} c_{l_1\ldots l_k}s_{l_1}\ldots s_{l_k}\right]s_1\\ &= \tilde{A}\left[1+\sum_{k=1}^{n}\sum_{1\leqslant l_1<\cdots<l_k\leqslant n} \tilde{a}_{l_1\ldots l_k}s_{l_1}\ldots s_{l_k}\right],\end{aligned}$$

where

$$\tilde{a}_1 = \frac{\tilde{C}}{\tilde{A}}, \quad \tilde{a}_{l_1\ldots l_k} = \begin{cases} \frac{\tilde{C}}{\tilde{A}}c_{l_2\ldots l_k} & \text{for} \quad 1=l_1<\cdots<l_k\leqslant n,\\ a_{l_1\ldots l_k} & \text{for} \quad 2\leqslant l_1<\cdots<l_k\leqslant n.\end{cases}$$

Now we proceed by induction on n. We have just proved the result for $r=1$ and any $n\geqslant 1$. Also, by Lemma 1, it holds true for $r=n=2$. Assume now that the result is true for some $n-1\geqslant 2$ and any $1\leqslant r\leqslant n-1$. We will prove that it also holds for n and any $1\leqslant r\leqslant n$.

Let $\bar{s}(k) = (s_{j_1},\ldots,s_{j_{m_k}})$, where $\{j_1,\ldots,j_{m_k}\} = W_k$, and let

$$g_k(\bar{s}(k)) = \left[1+\sum_{i=1}^{m_k}\sum_{\{j_1,\ldots,j_i\}\subset W_k} a_{j_1\ldots j_i}s_{j_1}\ldots s_{j_i}\right]^{q_k}. \tag{2.18}$$

Then we can rewrite (2.3) as

$$f(s_1,\dots,s_n)=A\prod_{k=1}^{r} g_k(\bar{s}(k)). \tag{2.19}$$

Then from (2.6) and the induction assumption we have

$$\begin{aligned} A_i(\bar{s}_{(i)}) &= A^{(i)}\prod_{k=1}^{r} g_k^{(i)}(\bar{s}_{(i)}(k)) \\ &= A^{(i)}\prod_{\substack{k=1\\ k\neq t(i)}}^{r} g_k^{(i)}(\bar{s}(k))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i)) \quad \forall i=1,\dots,n, \end{aligned} \tag{2.20}$$

where $t:\{1,\dots,n\}\to\{1,\dots,r\}$ is such that $t_i=t(i)=k\Leftrightarrow i\in W_k$ (if $W_{t_i}=\{i\}$ then we write $g_{t_i}^{(i)}(\bar{S}_{(i)}(t_i))\equiv 1$) for $k=1,\dots r$).

Let us note that for any $1\leqslant l\leqslant r$ such that $l\neq t_i$ and $l\neq t_j$

$$g_l^{(i)}(\bar{s}(l))=g_l^{(j)}(\bar{s}(l))=g_l(\bar{s}(l)) \quad \forall i,j=1,\dots,n. \tag{2.21}$$

Indeed, using (2.20), we can rewrite (2.4) as follows:

$$\begin{aligned} &\prod_{\substack{k=1\\ k\neq t_i}}^{r} g_k^{(i)}(\bar{s}(k))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i))(1+B_i(\bar{s}_{(i)})s_i)^{p_i} \\ &\quad=\prod_{\substack{k=1\\ k\neq t_j}}^{r} g_k^{(j)}(\bar{s}(k))g_{t_j}^{(j)}(\bar{s}_{(j)}(t_j))(1+B_j(\bar{s}_{(j)})s_j)^{p_j} \end{aligned} \tag{2.22}$$

and then take $\bar{s}(k)=\bar{0}$ for $k\neq l$ (note that by the assumption we have $g_k^{(i)}(\bar{0})=1$ $\forall i=1,\dots,n,\ k=1,\dots,r$).

Moreover, (2.20) implies $A_i(\bar{0})=A^{(i)}$. Thus, by (2.5), we have $A^{(i)}=A$ $\forall i=1,\dots,n$.

Hence (2.20) takes the form

$$A_i(\bar{s}_{(i)})=A\prod_{\substack{k=1\\ k\neq t_i}}^{r} g_k(\bar{s}(k))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i)) \quad \forall i=1,\dots,n, \tag{2.23}$$

which, by (2.2), implies

$$f(s_1,\dots,s_n)=A\prod_{\substack{k=1\\ k\neq t_i}}^{r} g_k(\bar{s}(k))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i))(1+B_i(\bar{s}_{(i)})s_i)^{p_i} \quad \forall i=1,\dots,n. \tag{2.24}$$

Combining (2.21) with (2.22) gives

$$\prod_{\substack{k=1\\k\neq t_i}}^{r} g_k(\bar{s}(k))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i))(1+B_i(\bar{s}_{(i)})s_i)^{p_i} = \prod_{\substack{k=1\\k\neq t_j}}^{r} g_k(\bar{s}(k))g_{t_j}^{(j)}(\bar{s}_{(j)}(t_j))(1+B_j(\bar{s}_{(j)})s_j)^{p_j}$$

and hence (recall that we have $A_i(\bar{s}_{(i)})>0$)

$$g_{t_j}(\bar{s}(t_j))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i))(1+B_i(\bar{s}_{(i)})s_i)^{p_i} = g_{t_i}(\bar{s}(t_i))g_{t_j}^{(j)}(\bar{s}_{(j)}(t_j))(1+B_j(\bar{s}_{(j)})s_j)^{p_j} \tag{2.25}$$

From Lemma 1 it follows that

$$B_i(\bar{s}_{(i)}) = b_i(\bar{s}_{(i,j)}), \quad B_j(\bar{s}_{(j)}) = b_j(\bar{s}_{(i,j)}).$$

Hence, inserting $s_j=0$ into (2.25) we get

$$g_{t_j}(\bar{s}^{(j)}(t_j))g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i))(1+B_i(\bar{s}_{(i)})s_i)^{p_i} = g_{t_i}(\bar{s}(t_i))g_{t_j}^{(j)}(\bar{s}_{(j)}(t_j)). \tag{2.26}$$

Taking $\bar{s}(t_i)=\bar{0}$ in (2.26) leads to

$$g_{t_j}(\bar{s}^{(j)}(t_j)) = g_{t_j}^{(j)}(\bar{s}_{(j)}(t_j)) \quad \forall j=1,\ldots,n. \tag{2.27}$$

By (2.27) and (2.26) we have

$$(1+B_i(\bar{s}_{(i)}s_i)^{p_i} = [g_{t_i}^{(i)}(\bar{s}_{(i)}(t_i))]^{-1}g_{t_i}(\bar{s}(t_i))$$

and hence

$$f(s_1,\ldots,s_n) = A\prod_{\substack{k=1\\k\neq t_i}}^{r} g_k(\bar{s}(k))g_{t_i}(\bar{s}(t_i)) = A\prod_{k=1}^{r} g_k(\bar{s}(k)). \qquad \square$$

Remark 1. The above theorem holds true for $A_i, B_i : [0,+\infty)^{n-1}\to[0,+\infty)$, $i=1,\ldots,n$.

3. *n*-Variate analogue of the Lukacs theorem

Recall first a version of the classical result (see [16, Chapter 1]) relating linearity of regression to properties of Laplace transforms. It will be used in the proof of the main result of the paper.

Lemma 2. *Let Z be a positive random variable and $\bar{X} = (X_1, \dots, X_n)$ a random vector with positive components. Suppose that* $E(Z)$ and $E(\bar{X})$ *exist. Then Z has linear regression on $\bar{X}$,*

$$E(Z|\bar{X}) = \alpha + \sum_{j=1}^{n} \beta_j X_j,$$

iff the relation

$$E\left(Z \exp\left(\sum_{j=1}^{n} s_j X_j\right)\right) = \alpha E\left(\exp\left(\sum_{j=1}^{n} s_j X_j\right)\right) + \sum_{j=1}^{n} \beta_j E\left(X_j \exp\left(\sum_{j=1}^{n} s_j X_j\right)\right),$$

holds for all vectors $\bar{s} = (s_1, \dots, s_n)$, $s_j \leqslant 0$, $j = 1, \dots, n$, where α, β_j, $j = 1, \dots, n$, are real constants.

Our main result is a multivariate version of the Lukacs characterization theorem. It appears that n-variate analogue of the independence condition looks somewhat stronger than in the univariate case since it does not characterize the n-variate gamma distribution in general but additionally enforces independence of components or groups of linearly dependent components of the involved random vectors. The bivariate version of this result has been obtained recently in [6] and it is a starting point of the induction argument which is the core of the proof of the theorem below.

Theorem 2. *Let $\bar{X} = (X_1, \dots, X_n)$ and $\bar{Y} = (Y_1, \dots, Y_n)$ be independent non-degenerate n-variate positive random vectors. If random vectors $\bar{U} = (U_1, \dots, U_n) = (\frac{X_1}{X_1+Y_1}, \dots, \frac{X_n}{X_n+Y_n})$ and $\bar{V} = (V_1, \dots, V_n) = (X_1 + Y_1, \dots, X_n + Y_n)$ are independent then $\exists W_1, \dots, W_r$ such that $\bigcup_{i=1}^{r} W_i = \{1, \dots, n\}, W_i \cap W_j = \emptyset, i \neq j$ and the Laplace transform of $\bar{X}$ is of the form*

$$L_{\bar{X}}(s_1, \dots, s_n) = \prod_{k=1}^{r} \left(1 - \sum_{j \in W_k} \lambda_j s_j\right)^{-p_k},$$

where $\lambda_1, \dots, \lambda_n, p_1, \dots, p_r$ are positive constants, i.e. vectors $\bar{Z}_1 = (X_i)_{i \in W_1}, \dots, \bar{Z}_r = (X_i)_{i \in W_r}$ are independent and $\forall k\ \exists i \in W_k$ such that $X_j = \frac{\lambda_j}{\lambda_i} X_i\ \forall j \in W_k$, where X_i has the gamma distribution with the shape p_k and the scale $1/\lambda_i$. *Similarly the Laplace transform of $\bar{Y}$ is of the form*

$$L_{\bar{Y}}(s_1, \dots, s_n) = \prod_{k=1}^{r} \left(1 - \sum_{j \in W_k} \lambda_j s_j\right)^{-q_k},$$

where $q_1, \dots, q_r$ are positive constants, i.e. vectors $\bar{T}_1 = (Y_i)_{i \in W_1}, \dots, \bar{T}_r = (Y_i)_{i \in W_r}$ are independent and $\forall k\ \exists i \in W_k$ such that $Y_j = \frac{\lambda_j}{\lambda_i} Y_i\ \forall j \in W_k$, where Y_i has the gamma distribution with the shape q_k and the scale $1/\lambda_i$.

Proof. The proof is by induction on n.

For $n=2$ theorem holds true (see [6] for the proof).

Assume that it is true for $n-1$, $n \geqslant 3$. We will show that it is true for n.

Note that since $U_j \in (0,1)$ we have $E(U_j^k) < \infty$ for $k=1,2,\dots, j=1,\dots,n$. Let

$$EU_j = a_j, \quad EU_j^2 = b_j, \quad j=1,\dots,n.$$

It follows easily that

$$0 < a_j^2 < b_j < a_j < 1, \quad j=1,\dots,n.$$

Since $\bar{U}$ and $\bar{V}$ are independent we have

$$E\left(\frac{X_j}{X_j+Y_j}\middle|\bar{X}+\bar{Y}\right) = a_j, \tag{3.1}$$

$$E\left(\left(\frac{X_j}{X_j+Y_j}\right)^2\middle|\bar{X}+\bar{Y}\right) = b_j, \quad j=1,\dots,n. \tag{3.2}$$

For $s_1,\dots,s_n \leqslant 0$ let

$$f(s_1,\dots,s_n) = E(\exp(s_1X_1+\cdots+s_nX_n)),$$
$$g(s_1,\dots,s_n) = E(\exp(s_1Y_1+\cdots+s_nY_n)).$$

We will consider f and g on $(-\infty,0)^n$. Let us note that f,g and their derivatives are strictly positive on $(-\infty,0)^n$. By Lemma 2 Eqs. (3.1) and (3.2) imply

$$(1-a_j)\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)g(s_1,\dots,s_n) = a_jf(s_1,\dots,s_n)\frac{\partial g}{\partial s_j}(s_1,\dots,s_n), \tag{3.3}$$

$$\begin{aligned}(1-b_j)&\frac{\partial^2 f}{\partial s_j^2}(s_1,\dots,s_n)g(s_1,\dots,s_n)\\ &= b_jf(s_1,\dots,s_n)\frac{\partial^2 g}{\partial s_j^2}(s_1,\dots,s_n) + 2b_j\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)\frac{\partial g}{\partial s_j}(s_1,\dots,s_n),\end{aligned} \tag{3.4}$$

$j=1,\dots,n$. Dividing both sides of (3.3) by fg yields

$$\frac{1-a_j}{a_j}\frac{\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)}{f(s_1,\dots,s_n)} = \frac{\frac{\partial g}{\partial s_j}(s_1,\dots,s_n)}{g(s_1,\dots,s_n)}, \quad j=1,\dots,n. \tag{3.5}$$

Therefore, we get

$$g(s_1,\dots,s_n) = c_i(\bar{s}_{(i)})[f(s_1,\dots,s_n)]^{\frac{1-a_i}{a_i}} \quad \forall i=1,\dots,n. \tag{3.6}$$

Now differentiate (3.3) wrt s_j and substitute it into (3.4) to get

$$\left(1-\frac{b_j}{a_j}\right)\frac{\partial^2 f}{\partial s_j^2}(s_1,\dots,s_n)g(s_1,\dots,s_n) = \frac{b_j}{a_j}\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)\frac{\partial g}{\partial s_j}(s_1,\dots,s_n),$$

which implies

$$\frac{a_j - b_j}{b_j}\frac{\frac{\partial^2 f}{\partial s_j^2}(s_1,\dots,s_n)}{\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)} = \frac{\frac{\partial g}{\partial s_j}(s_1,\dots,s_n)}{g(s_1,\dots,s_n)}, \quad j=1,\dots,n. \tag{3.7}$$

Using (3.5) and (3.7) we have

$$c_j\frac{\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)}{f(s_1,\dots,s_n)} = \frac{\frac{\partial^2 f}{\partial s_j^2}(s_1,\dots,s_n)}{\frac{\partial f}{\partial s_j}(s_1,\dots,s_n)},$$

where $c_j = \frac{(1-a_j)b_j}{a_j(a_j-b_j)}$, $j=1,\dots,n$. Consequently,

$$f(s_1,\dots,s_n) = A_i(\bar{s}_{(i)})(1+B_i(\bar{s}_{(i)})s_i)^{-p_i} \quad \forall i=1,\dots,n, \tag{3.8}$$

where $p_j = -\frac{1}{1-c_j} = \frac{a_j(a_j-b_j)}{b_j-a_j^2} > 0,\ j=1,\dots,n.$

Let us note that there are two possible cases:

I. For any $\{i_1,\dots,i_{n-1}\}\subset\{1,\dots,n\}$ the components of $(X_{i_1},\dots,X_{i_{n-1}})$ are independent

or

II. there exists $\{j_1,\dots,j_{n-1}\}\subset\{1,\dots,n\}$ such that the components of the vector $(X_{j_1},\dots,X_{j_{n-1}})$ are not independent.

First, let us consider case I.

If $X_{i_1},\dots,X_{i_{n-1}}$ are independent for any $\{i_1,\dots,i_{n-1}\}\subset\{1,\dots,n\}$ then by the induction assumption we obtain

$$f(s_1,\dots,s_n) = \prod_{k=1,\,k\neq i}^{n}(1-\lambda_k s_k)^{-p_k}(1+B_i(\bar{s}_{(i)})s_i)^{-p_i} \quad \forall i=1,\dots,n, \tag{3.9}$$

where

$\lambda_k = -B_k(0,\dots,0) \quad \forall k=1,\dots,n$ are positive constants.

Case I.1: If $\exists i,j$ such that $p_i \neq p_j$ then by (3.9) we get

$$(1-\lambda_i s_i)^{-p_i}(1+B_j(\bar{s}_{(j)})s_j)^{-p_j} = (1-\lambda_j s_j)^{-p_j}(1+B_i(\bar{s}_{(i)})s_i)^{-p_i}. \tag{3.10}$$

Lemma 1 implies

$$B_i(\bar{s}_{(i)}) = b_i(\bar{s}_{(i,j)}), \quad B_j(\bar{s}_{(j)}) = b_j(\bar{s}_{(i,j)}).$$

Thus inserting separately $s_i=0$ or $s_j=0$ in (3.10) we obtain

$$B_i(\bar{s}_{(i)}) = -\lambda_i, \quad B_j(\bar{s}_{(j)}) = -\lambda_j.$$

Hence $B_k(\bar{s}_{(k)}) = -\lambda_k$ for any $k=1,\dots,n$ and we have

$$f(s_1,\dots,s_n) = \prod_{k=1}^{n}(1-\lambda_k s_k)^{-p_k}$$

which means that $\bar{X}=(X_1,\dots,X_n)$ has independent components: X_k having the gamma distribution with the shape p_k and the scale $1/\lambda_k$ $\forall k=1,\dots,n$. Thus, by

(3.6), we obtain

$$g(s_1, \ldots, s_n) = c_l(\bar{s}_{(l)}) \left[\prod_{k=1}^{n} (1 - \lambda_k s_k)^{-p_k} \right]^{\frac{1-a_l}{a_l}} \quad \forall l = 1, \ldots, n. \tag{3.11}$$

Inserting $s_l = 0$ in (3.11) we get

$$g(\bar{s}^{(l)}) = c_l(\bar{s}_{(l)}) \left[\prod_{\substack{k=1 \\ k \neq l}}^{n} (1 - \lambda_k s_k)^{-p_k} \right]^{\frac{1-a_l}{a_l}}.$$

On the other hand, by the induction assumption

$$g(\bar{s}^{(l)}) = \prod_{\substack{k=1 \\ k \neq 1}}^{n} (1 - \lambda_k s_k)^{-q_k},$$

where $q_k = p_k \frac{1-a_k}{a_k}$, $k = 1, \ldots, n$. Hence

$$g(s_1, \ldots, s_n) = \prod_{k=1}^{n} (1 - \lambda_k s_k)^{-q_k}, \quad k = 1, \ldots, n,$$

which implies that $\bar{Y} = (Y_1, \ldots, Y_n)$ has independent components: Y_k having the gamma distribution with the scale $1/\lambda_k$ and the shape $q_k = p_k \frac{1-a_k}{a_k}$ $\forall k = 1, \ldots, n$.

Case I.2: If $p_k = p$ and $q_k = q$ (that is $a_k = a$) $\forall k = 1, \ldots, n$, then by Theorem 1 we have

$$f(s_1, \ldots, s_n) = A \left[1 + \sum_{k=1}^{n} \sum_{1 \leqslant l_1 < \cdots < l_k \leqslant n} a_{l_1 \ldots l_k} s_{l_1 \ldots l_k} \right]^{p}. \tag{3.12}$$

Observe that $A = 1$. Moreover, combining (3.12) with (3.9) and inserting, separately, $s_i = 0$, $i = 1, \ldots, n$, leads to

$$a_{l_1 \ldots l_k} = (-1)^k \lambda_{l_1} \ldots \lambda_{l_k} \quad \text{for } 1 \leqslant k \leqslant n-1$$

and

$$a_{1 \ldots n} = (-1)^n \prod_{k=1}^{n} \lambda_k + M,$$

where M is a constant. Hence

$$f(s_1, \ldots, s_n) = \left[\prod_{k=1}^{n} (1 - \lambda_k s_k) + M \prod_{k=1}^{n} s_k \right]^{-p}.$$

From (3.6) it follows that

$$g(s_1, \ldots, s_n) = c_i(\bar{s}_{(i)}) \left[\prod_{k=1}^{n} (1 - \lambda_k s_k) + M \prod_{k=1}^{n} s_k \right]^{-p \frac{1-a}{a}} \quad \forall i = 1, \ldots, n,$$

which implies

$$c_i(\bar{s}_{(i)}) = c_j(\bar{s}_{(j)}) \quad \forall i,j = 1, \dots, n, \quad i \neq j. \tag{3.13}$$

Taking $i = 1$ and inserting sequentially $s_1 = 0, \dots, s_{n-1} = 0$ in (3.13) we obtain

$$c_1(\bar{s}_{(1)}) = c_2(\bar{s}_{(2)}^{(1)}) = c_3(\bar{s}_{(3)}^{(1,2)}) = \cdots = c_n(\bar{s}_{(n)}^{(1,\dots,n-1)}) = c_n(\bar{0}) = g(\bar{0}) = 1.$$

Hence

$$c_i(\bar{s}_{(i)}) = 1 \quad \forall i = 1, \dots, n$$

and we have

$$g(s_1, \dots, s_n) = \left[\prod_{k=1}^{n} (1 - \lambda_k s_k) + M \prod_{k=1}^{n} s_k \right]^{-q},$$

where $q = p\frac{1-a}{a} > 0$.

We claim that $M = 0$, which means that $\bar{X} = (X_1, \dots, X_n)$ and $\bar{Y} = (Y_1, \dots, Y_n)$ have independent components: X_k having the gamma distribution with the shape p and the scale $1/\lambda_k$ and Y_k having the gamma distribution with the shape q and the scale $1/\lambda_k$ $k = 1, \dots, n$. Indeed, from the independency of $\bar{U}$ and $\bar{V}$ it follows that

$$E(U_1 \dots U_n | \bar{V}) = c,$$

where c is a positive constant. Hence

$$E(X_1 \dots X_n | \bar{V}) = c V_1 \dots V_n,$$

and we have

$$\begin{aligned} &E(X_1 \dots X_n \exp(s_1 X_1 + \cdots + s_n X_n)) E(\exp(s_1 Y_1 + \cdots + s_n Y_n)) \\ &\quad = c E(V_1 \dots V_n \exp(s_1 V_1 + \cdots + s_n V_n)). \end{aligned} \tag{3.14}$$

Inserting in (3.14) $s_1 = \cdots = s_{n-2} = 0$ we obtain

$$\begin{aligned} &E(X_1 \dots X_n \exp(s_{n-1} X_{n-1} + s_n X_n)) E(\exp(s_{n-1} Y_{n-1} + s_n Y_n)) \\ &\quad = c E(V_1 \dots V_n \exp(s_{n-1} V_{n-1} + s_n V_n)). \end{aligned} \tag{3.15}$$

Since $\bar{X}$ and $\bar{Y}$ are independent, the Laplace transform of $\bar{V} = \bar{X} + \bar{Y}$ is of the form

$$h(s_1, \dots, s_n) = E(\exp(s_1 V_1 + \cdots + s_n V_n)) = \left[\prod_{k=1}^{n} (1 - \lambda_k s_k) + M \prod_{k=1}^{n} s_k \right]^{-(p+q)}.$$

Differentiating f and h we can rewrite (3.15) as follows:

$$p^n \prod_{i=1}^{n} \lambda_i (1-\lambda_{n-1}s_{n-1})(1-\lambda_n s_n) - pM(1+p\lambda_{n-1}s_{n-1})(1+p\lambda_n s_n)$$
$$= c\Bigg\{ (p+q)^n \prod_{i=1}^{n} \lambda_i (1-\lambda_{n-1}s_{n-1})(1-\lambda_n s_n)$$
$$- (p+q)M[1+(p+q)\lambda_{n-1}s_{n-1}][1+(p+q)\lambda_n s_n] \Bigg\},$$

which leads to

$$p^n \prod_{i=1}^{n} \lambda_i - pM = c\left\{ (p+q)^n \prod_{i=1}^{n} \lambda_i - (p+q)M \right\}, \tag{3.16}$$

$$p^n \prod_{i=1}^{n} \lambda_i + p^2 M = c\left\{ (p+q)^n \prod_{i=1}^{n} \lambda_i + (p+q)^2 M \right\}, \tag{3.17}$$

$$p^n \prod_{i=1}^{n} \lambda_i - p^3 M = c\left\{ (p+q)^n \prod_{i=1}^{n} \lambda_i - (p+q)^3 M \right\}. \tag{3.18}$$

If $M = 0$ then the above equations are satisfied with $c = p/(p+q)$. Suppose that $M \neq 0$. Then subtracting (3.16) from (3.17) and then dividing by M we get

$$p(p+1) = c(p+q)(p+q+1)$$

which implies

$$c = \frac{p(p+1)}{(p+q)(p+q+1)}.$$

On the other hand, subtracting (3.18) from (3.17) and dividing the obtained equation by M gives

$$p^2(p+1) = c(p+q)^2(p+q+1),$$

which implies

$$c = \frac{p^2(p+1)}{(p+q)^2(p+q+1)}.$$

Hence

$$\frac{p(p+1)}{(p+q)(p+q+1)} = \frac{p^2(p+1)}{(p+q)^2(p+q+1)}$$

and we have

$$\frac{p}{p+q} = 1$$

which is impossible. This proves that $M = 0$.

Case I.3: If $p_k = p,\ \forall k = 1, \dots, n$ and $\exists i, j$ such that $q_i \neq q_j$ $\left(q_k = p\frac{1-a_k}{a_k},\ k = 1, \dots, n\right)$ then, by Theorem 1 and (3.6) we have

$$f(s_1, \dots, s_n) = \left[\prod_{k=1}^{n} (1 - \lambda_k s_k) + M \prod_{k=1}^{n} s_k \right]^{-p},$$

$$g(s_1, \dots, s_n) = c_i(\bar{s}_{(i)}) \left[\prod_{k=1}^{n} (1 - \lambda_k s_k) + M \prod_{k=1}^{n} s_k \right]^{-q_i} \quad \forall i = 1, \dots, n, \tag{3.19}$$

where M is a constant. Inserting $s_i = 0$ in (3.19) we get

$$g(\bar{s}^{(i)}) = c_i(\bar{s}_{(i)}) \left[\prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - \lambda_k s_k)^{-p_k} \right]^{-q_i}.$$

On the other hand, by the induction assumption

$$g(\bar{s}^{(i)}) = \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - \lambda_k s_k)^{-q_i}.$$

Hence

$$\begin{aligned} g(s_1, \dots, s_n) &= \prod_{k=1, k \neq i}^{n} (1 - \lambda_k s_k)^{-q_k} \left[1 + \left(M \prod_{k=1, k \neq i}^{n} \frac{s_k}{1 - \lambda_k s_k} - \lambda_i \right) s_i \right]^{-q_i} \\ &= \prod_{k=1, k \neq i}^{n} (1 - \lambda_k s_k)^{-q_k} (1 + D_i(\bar{s}_{(i)}) s_i)^{-q_i} \quad \forall i = 1, \dots, n, \end{aligned}$$

where

$$D_i(\bar{s}_{(i)}) = M \prod_{k=1, k \neq i}^{n} \frac{s_k}{1 - \lambda_k s_k} - \lambda_i.$$

Thus, similarly as in case I.1, we obtain

$$g(s_1, \dots, s_n) = \prod_{k=1}^{n} (1 - \lambda_k s_k)^{-q_k},$$

which means that $M = 0$ and we have

$$f(s_1, \dots, s_n) = \prod_{k=1}^{n} (1 - \lambda_k s_k)^{-p}.$$

Hence $\bar{X}$ and $\bar{Y}$ have independent components: X_k having the gamma distribution with the shape p and the scale $1/\lambda_k$ and Y_k having the gamma distribution with the shape q_k and the scale $1/\lambda_k$ $k = 1, \dots, n$.

Case II: There exists $\{j_1, \dots, j_{n-1}\} \subset \{1, \dots, n\}$ such that the components of the vector $(X_{j_1}, \dots, X_{j_{n-1}})$ are not independent. Without loss of generality we can assume that this is the vector $(X_1, \dots, X_{n-1})$. Then, by the induction assumption, $\exists \tilde{W}_1, \dots, \tilde{W}_r$: $\bigcup_{i=1}^r \tilde{W}_i = \{1, \dots, n-1\}$, $\tilde{W}_i \cap \tilde{W}_j = \emptyset$, $i \neq j$ and $\exists i \ \#\tilde{W}_i > 1$, such that the Laplace transform of $(X_1, \dots, X_{n-1})$ is of the form

$$f(s_1, \dots, s_{n-1}) = \prod_{k=1}^{r} \left(1 - \sum_{j \in \tilde{W}_k} \lambda_j s_j\right)^{-p_k}.$$

Now take $X_{i_1} \in \tilde{W}_1, \dots, X_{i_r} \in \tilde{W}_r$ (note that we have $r \leqslant n-2$) and consider the vector $(X_{i_1}, \dots, X_{i_r}, X_n)$. Since the dimension of $(X_{i_1}, \dots, X_{i_r}, X_n)$ is not greater than $n-1$, its Laplace transform has the desired form by the induction assumption, that is either X_n is independent of $X_{i_1}, \dots, X_{i_r}$ or there exists $k \in \{1, \dots, r\}$ such that $X_n = \frac{\lambda_n}{\lambda_{i_k}}$. □

Remark 2. Let us note that instead of independency of $\bar{U}$ and $\bar{V}$ in the above theorem if suffices to assume the constancy of regressions:

$$E(U_j|\bar{V}) = a_j, \quad E(U_j^2|\bar{V}) = b_j, \ j = 1, \dots, n$$

and

$$E(U_1 \dots U_n|\bar{V}) = c.$$

4. Multivariate distribution with univariate Pareto conditionals

The results obtained in Section 2 can be used for deriving the form of the density of a multivariate distribution having all univariate conditional distributions of the Pareto type. Such problems were considered first in the bivariate case in [1]. Multivariate extensions can be found in [2,4].

The problem lies in identification of all n-variate distributions of the random vector $\bar{X}$ valued in the positive orthant with univariate conditional densities of the form

$$f_{X_i|\bar{X}_{(i)}=\bar{x}_{(i)}}(x_i) = \frac{\alpha_i}{\sigma_i(\bar{x}_{(i)})}\left(1 + \frac{x_i}{\sigma_i(\bar{x}_{(i)})}\right)^{-(\alpha_i+1)}, \quad \bar{x} \in [0, \infty)^n,$$

where $\sigma_i : [0, \infty)^{n-1} \to (0, \infty)$ is a measurable function and α_i is a positive number, $i = 1, \dots, n$ (recall that $f(x) = \frac{\alpha}{\sigma}\left(1 + \frac{x}{\sigma}\right)^{-(\alpha+1)}$, $x > 0$, is a density of the standard Pareto distribution). Thus the joint density f of the random vector $\bar{X}$ has to satisfy the following system of equations:

$$f(\bar{x}) = \frac{\alpha_i}{\sigma_i(\bar{x}_{(i)})}\left(1 + \frac{x_i}{\sigma_i(\bar{x}_{(i)})}\right)^{-(\alpha_i+1)} f_i(\bar{x}_{(i)}), \quad \bar{x} \in [0, \infty)^n, \tag{4.1}$$

where f_i is a density of $\bar{X}_{(i)}$, $i = 1, \ldots, n$.

Observe that (4.1) is of the form (2.2) with

$$A_i(\bar{x}_{(i)}) = \frac{\alpha_i}{\sigma_i(\bar{x}_{(i)})} f_i(\bar{x}_{(i)}), \quad B_i(\bar{x}_{(i)}) = \frac{1}{\sigma_i(\bar{x}_{(i)})}, \quad p_i = -(\alpha_i + 1), \quad i = 1, \ldots, n.$$

Then it follows from Theorem 1 that the joint density of $\bar{X}$ has the form

$$f(\bar{x}) = A \prod_{k=1}^{r} \left[1 + \sum_{i=1}^{m_k} \sum_{\{j_1, \ldots, j_i\} \subset W_k} a_{j_1 \ldots j_i} x_{j_1} \ldots x_{j_i} \right]^{-(\alpha_{U(k)}+1)}, \quad \bar{x} \in (0, \infty)^n, \tag{4.2}$$

where A and $a_{j_1 \ldots j_i}$'s are real constants such that f is positive and integrable.

In [4] (see also [3]) a simplified version of the problem was considered due to the assumption that all α_i's are equal. The formula for the joint density provided there agrees with (4.2) in this special case but instead of the proof, only a suggestion to adopt the argument used in the bivariate case is given. However, such an approach does not seem to be that straightforward, even for all α_i's equal, as can be seen in Section 2 above.

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