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BIVARIATE LUKACS TYPE REGRESSION CHARACTERIZATIONS

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ABSTRACT. Wang (1980) and recently Bobecka (2003) studied bivariate regression versions of the Lukacs characterization. They identified in such a way families of bivariate distributions with gamma marginals. The main result of the present paper is a characterization of a wider family of bivariate distributions (without having necessarily gamma marginals) but still having some Lukacs type regression properties.

1. INTRODUCTION

Let $\gamma_{\lambda,p}$ denote the gamma distribution defined by the density

$$\gamma_{\lambda,p}(dx) = \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x} I_{(0,\infty)}(x)dx,$$

where λ, p are positive numbers (scale and shape parameters, respectively). The corresponding Laplace transform is of the form

$$f(s) = (1-\lambda s)^{-p}, \quad s < \frac{1}{\lambda}.$$

Lukacs (1955) proved that if two positive, nondegenerate random variables (rv's) X, Y are independent and the rv's $U = \frac{X}{X+Y}, V = X+Y$ are also independent then X and Y have gamma distributions with the same scale parameter. Olkin and Rubin (1962) extended this result to matrix valued rv's, with further developments in Casalis and Letac (1996), Letac and Massam (1998), Bobecka and Wesołowski (2002a). Other authors considered similar characterizations of the gamma law by weakening the independence condition to constancy of regressions, mostly in the univariate setting. Such regression versions of the Lukacs theorem were obtained for instance in Bolger and Harkness (1965), Hall and Simons (1969), Wesołowski (1990), Li, Huang and Huang (1994), Huang and Su (1997), Bobecka and Wesołowski (2002b), Chao and Huang (2002).

Here we are concerned with the bivariate situation. We say that a positive random vector $\bar{X} = (X_1, X_2)$ has a bivariate gamma distribution $BG(\bar{\lambda}, p)$, where $\bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, if it has a Laplace transform of the form

$$E(\exp(s_1X_1 + s_2X_2)) = (1 - \lambda_1 s_1 - \lambda_2 s_2 + \lambda_3 s_1 s_2)^{-p}, \qquad (1)$$
$$\lambda_1 s_1 + \lambda_2 s_2 - \lambda_3 s_1 s_2 < 1,$$

where $p > 0,\ \lambda_1 > 0, \lambda_2 > 0$ and $\lambda_1\lambda_2 \geq \lambda_3 \geq 0$. Let us note that the set $\{(s_1, s_2) : \lambda_1 s_1 + \lambda_2 s_2 - \lambda_3 s_1 s_2 < 1\}$ contains $\{(s_1, s_2) : s_1, s_2 \leq 0\}$. Thus the Laplace transform (1), due to the analytic extension principle, is uniquely determined by its values for $s_1, s_2 \leq 0$.

Characterizations of the bivariate gamma distribution were first considered by Lukacs (1977), Wang (1981) and more recently in Bar-Lev, Bshouty, Enis, Letac, Lu and Richards (1994). They were mostly concerned with bivariate regression versions of the Lukacs theorem. In a recent paper by Bobecka (2003) the problem of bivariate versions of the Lukacs

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characterization was thoroughly studied. It was shown there that, rather unexpectedly, for independent random vectors $\bar{X} = (X_1, X_2), \bar{Y} = (Y_1, Y_2)$ having bivariate gamma distributions $BG(\bar{\lambda}, p)$, $BG(\bar{\lambda}, q)$, respectively, the rv's $\bar{U} = (U_1, U_2) = \left(\frac{X_1}{X_1+Y_1}, \frac{X_2}{X_2+Y_2}\right)$ and $\bar{V} = (V_1, V_2) = (X_1 + Y_1, X_2 + Y_2)$ may not be independent. They are independent iff $\lambda_3 = \lambda_1\lambda_2$ or $\lambda_3 = 0$. Let us note that if $\lambda_3 = \lambda_1\lambda_2$ then $\bar{X}, \bar{Y}$ have independent components and if $\lambda_3 = 0$ then the components of $\bar{X}, \bar{Y}$ are linearly dependent. Analyzing the proof of the bivariate Lukacs theorem given in Bobecka (2003) it is visible that assuming, instead of independence, five constancy of regression conditions:

$$E(U_j|\bar{V}) = a_j, \quad j = 1, 2, \tag{2}$$

$$E(U_j^2|\bar{V}) = b_j, \quad j = 1, 2, \tag{3}$$

$$E(U_1U_2|\bar{V}) = c\,, \tag{4}$$

it follows that (as in the case of independence) either

1^o $\bar{X}$ and $\bar{Y}$ have independent gamma components, i.e. $X_1 \sim \gamma_{\lambda_1,p_1}$, $X_2 \sim \gamma_{\lambda_2,p_2}$, $Y_1 \sim \gamma_{\lambda_1,q_1}$, $Y_2 \sim \gamma_{\lambda_2,q_2}$, where $p_1, p_2, q_1, q_2, \lambda_1, \lambda_2$ are some positive numbers; or

2^o the components of $\bar{X}$ and $\bar{Y}$ are linearly dependent: $X_1 = aX_2$, $Y_1 = aY_2$, where $X_2 \sim \gamma_{\lambda,p}$, $Y_2 \sim \gamma_{\lambda,q}$ and a, p, q, λ are some positive numbers.

On the other hand Wang (1981) proved that if the conditions (2) and (3) hold with $a_1 = a_2$ and $b_1 = b_2$ then $\bar{X}$ and $\bar{Y}$ have general bivariate gamma distributions $BG(\bar{\lambda}, p)$ and $BG(\bar{\lambda}, q)$. However it was observed in Bobecka (2003) that for $a_1 \neq a_2$ or $b_1 \neq b_2$ the conditions (2) and (3) characterize $\bar{X}$ and $\bar{Y}$ as in the case of independence (see 1^o or 2^o above). Also other regression characterizations were treated in that paper. Recently the study has been developed further by Chao and Huang (2003).

The present paper is concerned with a more thorough analysis of regression conditions (2-4), with an emphasis on the condition (4), which is complementary to existing results. First we show a non-gamma example satisfying conditions (2) and (4). Then we identify a family of bivariate distributions for which (2), (4) and one of the conditions (3) hold. This family of distributions, which is characterized in Section 2 below, is different from the general bivariate gamma family characterized in Wang (1981). It includes, for instance, distributions of (X, Z_X), where X is a gamma rv independent of a Lévy process (Z_t).

It is worth to mention that there are only few results concerning the Lukacs type characterizations in the n-variate situation. It was shown in Bobecka and Wesołowski (2003a) that for independent random vectors $\bar{X} = (X_1, \ldots, X_n), \bar{Y} = (Y_1, \ldots, Y_n)$, independence of $\bar{U} = (U_1, \ldots, U_n) = \left(\frac{X_1}{X_1+Y_1}, \ldots, \frac{X_n}{X_n+Y_n}\right)$ and $\bar{V} = (V_1, \ldots, V_n) = (X_1 + Y_1, \ldots, X_n + Y_n)$ characterizes random vectors with independent sub-vectors of linearly dependent gamma components. It was observed there also that the independence condition can be relaxed to analogues of (2-4), that is, to

$$E(U_j|\bar{V}) = a_j, \quad j = 1, \ldots, n,$$
$$E(U_j^2|\bar{V}) = b_j, \quad j = 1, \ldots, n,$$
$$E(U_1 \cdots U_n|\bar{V}) = c\,.$$

The same family of distributions has been recently characterized by dual regression conditions, i.e. regressions of powers of $\bar{X}$ on $\bar{Y}$ in Bobecka and Wesołowski (2003b).

2. Bivariate regressions

To analyze regression conditions (2-4) we will use the following bivariate Laplace transform version of Lemma 1.1. from Kagan, Linnik, Rao (1973).

Lemma 1. *Let f and g be the Laplace transforms of $\bar{X}$ and $\bar{Y}$, respectively, i.e.*

$$f(s_1, s_2) = E\left(e^{s_1 X_1 + s_2 X_2}\right), \quad g(s_1, s_2) = E\left(e^{s_1 Y_1 + s_2 Y_2}\right),$$

$(s_1, s_2) \in \Delta$, $(-\infty, 0]^2 \subset \Delta$. *The regression conditions (2), (3) and (4) are (under appropriate moment conditions), respectively, equivalent to the following differential equations for f and g:*

$$(1 - a_j)\frac{\partial f}{\partial s_j} g = a_j f \frac{\partial g}{\partial s_j}, \quad j = 1, 2, \tag{5}$$

$$(1 - b_j)\frac{\partial^2 f}{\partial s_j^2} g = 2b_j \frac{\partial f}{\partial s_j}\frac{\partial g}{\partial s_j} + b_j f \frac{\partial^2 g}{\partial s_j^2}, \quad j = 1, 2, \tag{6}$$

$$(1 - c)\frac{\partial^2 f}{\partial s_1 \partial s_2} g = c\frac{\partial f}{\partial s_1}\frac{\partial g}{\partial s_2} + c\frac{\partial f}{\partial s_2}\frac{\partial g}{\partial s_1} + cf\frac{\partial^2 g}{\partial s_1 \partial s_2}. \tag{7}$$

Consider first the regression conditions (2) and (3), which in terms of $\bar{X}$ and $\bar{Y}$ can be written as

$$E(X_j | \bar{X} + \bar{Y}) = a_j (X_j + Y_j), \qquad j = 1, 2, \tag{8}$$

$$E(X_1 X_2 | \bar{X} + \bar{Y}) = c(X_1 + Y_1)(X_2 + Y_2). \tag{9}$$

It appears that these conditions are rather far away from being characteristic for bivariate gamma distributions. As the example given below shows there exist even discrete distributions satisfying those three regression properties.

Example 2. *Consider independent random vectors $\bar{X}$ and $\bar{Y}$ with a common bivariate negative binomial distribution of the form*

$$P(X_1 = i, X_2 = j) = \frac{(i + j + r - 1)!}{i! j! (r - 1)!} p_1^i p_2^j (1 - p_1 - p_2)^r,$$

where

$$p_1, p_2 > 0, \quad p_1 + p_2 < 1, \quad r > 0, \quad i, j = 0, 1, 2, \ldots$$

Then their Laplace transforms are

$$f(s_1, s_2) = g(s_1, s_2) = \frac{(1 - p_1 - p_2)^r}{(1 - p_1 \exp s_1 - p_2 \exp s_2)^r} =$$

$$= \left(1 + \frac{p_1}{1 - p_1 - p_2}(1 - \exp s_1) + \frac{p_2}{1 - p_1 - p_2}(1 - \exp s_2)\right)^{-r}.$$

Thus it is easily seen that

$$\frac{\partial f(s_1, s_2)}{\partial s_j} = r\frac{p_j}{1 - p_1 - p_2} e^{s_j} [f(s_1, s_2)]^{\frac{r+1}{r}} = \frac{\partial g(s_1, s_2)}{\partial s_j}, \quad j = 1, 2,$$

$$\frac{\partial^2 f(s_1, s_2)}{\partial s_1 \partial s_2} = r(r + 1)\frac{p_1 p_2}{(1 - p_1 - p_2)^2} e^{s_1 + s_2} [f(s_1, s_2)]^{\frac{r+2}{r}} = \frac{\partial^2 g(s_1, s_2)}{\partial s_1 \partial s_2}.$$

Now it is immediately seen that the equations (5) and (7) are satisfied with $a_1 = a_2 = \frac{1}{2}$ and $c = \frac{r+1}{2(2r+1)}$. Consequently the regression conditions (8) and (9) also hold.

Observe that the above example can be extended to any bivariate distributions with Laplace transforms of the form

$$f(s_1, s_2) = g(s_1, s_2) = (\phi_1(s_1) + \phi_2(s_2))^p ,$$

with suitable functions ϕ_1, ϕ_2 and a real number p.

Let $\bar{X}$ and $\bar{Y}$ be independent random vectors with non-degenerate positive components. Now we complement conditions (2) and (4) with (3) for $j = 1$. It can be easily checked that this set of conditions is satisfied for the following families of distributions:

Case I: $\bar{X}, \bar{Y}$ have independent components: $X_1 \sim \gamma_{\lambda,p}$, $Y_1 \sim \gamma_{\lambda,q}$, X_2, Y_2 are integrable and the Laplace transforms of $\bar{X}, \bar{Y}$ are, respectively, of the form:

$$f(s_1, s_2) = (1 - \lambda s_1)^{-p} A(s_2), \quad g(s_1, s_2) = (1 - \lambda s_1)^{-q} [A(s_2)]^{\frac{1-a_2}{a_2}} ,$$

where A is a Laplace transform of X_2, and thus $A^{\frac{1-a_2}{a_2}}$ has to be the Laplace transform of Y_2.

Then it can be checked directly by taking dervatives that (2), (3) for $j = 1$ and (4) are satisfied with

$$a_1 = a_2 = \frac{p}{p+q}, \quad b_1 = \frac{p(p+1)}{(p+q)(p+q+1)}, \quad c = \left(\frac{p}{p+q}\right)^2 ,$$

or

Case II: $\bar{X}, \bar{Y}$ are such, that $X_1 \sim \gamma_{\lambda,p}$, $Y_1 \sim \gamma_{\lambda,q}$, X_2, Y_2 are integrable and the Laplace transforms $\bar{X}, \bar{Y}$ are of the form:

$$f(s_1, s_2) = (1 - \lambda s_1 + M(s_2))^{-p}, \quad g(s_1, s_2) = (1 - \lambda s_1 + M(s_2))^{-q},$$

where M is a suitable function, such that $(1 - M)^{-p}$ is a Laplace transform of X_2 and $(1 - M)^{-q}$ is a Laplace transform of Y_2.

Then again it can be checked by differentiating that (2), (3) for $j = 1$ and (4) are satisfied with

$$a_1 = a_2 = \frac{p}{p+q}, \quad b_1 = c = \frac{p(p+1)}{(p+q)(p+q+1)}.$$

Moreover it appears that the above families of distributions are characterized by the regression conditions (2), (3) for $j = 1$ and (4):

Theorem 3. *Let $\bar{X}$ and $\bar{Y}$ be independent random vectors with non-degenerate positive components. If the conditions (2), (3) for $j = 1$ and (4) are satisfied for some real numbers a_1, a_2, b_1, c, then either*

(1) *$c = a_1 a_2$ and $\bar{X}, \bar{Y}$ are as in the Case I,*

or

(2) *$c \neq a_1 a_2$, $a_1 = a_2$ and $\bar{X}$, $\bar{Y}$ are as in the Case II,*

where

$$p = \frac{a_1(a_1 - b_1)}{b_1 - a_1^2} > 0, \qquad q = p\frac{1 - a_1}{a_1} > 0.$$

The proof of the above theorem is given in Section 3.

We end this section with an example of random vectors $\bar{X}$ and $\bar{Y}$ having distributions as in the *Case II*, i.e. such random vectors with dependent components for which the conditions of constancy of regressions (2), (3) for $j = 1$ and (4) are satisfied

Example 4. *Let $(Z_t, t \geq 0)$ be a Lévy stochastic process with $E(\exp(sZ_t)) = \exp(t\Psi(s))$, $s \leq 0$, and let $\bar{X}$ and $\bar{Y}$ be defined as: $X_1 \sim \gamma_{\lambda,p}$ and $X_2 = Z_{X_1}$ and $Y_1 \sim \gamma_{\lambda,q}$ and $Y_2 = Z_{Y_1}$, where X_1 and Y_1 are independent of the process (Z_t). Then the Laplace transform of $\bar{X}$ is of the form*

$$\begin{aligned} f(s_1, s_2) = E(\exp(s_1X_1 + s_2X_2)) &= E(\exp(s_1X_1)\exp(X_1\Psi(s_2))) = \\ &= E(\exp[s_1 + \Psi(s_2)]X_1) = (1 - \lambda[s_1 + \Psi(s_2)])^{-p} = \\ &= (1 - \lambda s_1 + M(s_2))^{-p}, \end{aligned}$$

where

$$M(s_2) = -\lambda\Psi(s_2).$$

Similarly, the Laplace transform of $\bar{Y}$ is of the form

$$g(s_1, s_2) = (1 - \lambda s_1 + M(s_2))^{-q}.$$

3. Proof of the main result

Proof. From (2), (3) for $j = 1$ and (4) it follows that $0 < a_1^2 < b_1 < a_1 < 1$ and $0 < c < a_1, a_2 < 1$.

We will consider the Laplace transforms f, g of $\bar{X}$, $\bar{Y}$, respectively, on $(-\infty, 0) \times (-\infty, 0)$ (since there f and g are analytic we do not need to have any moment conditions imposed). Observe that f, g and their derivatives are strictly positive on $(-\infty, 0) \times (-\infty, 0)$. Then the equations (5), (6) for $j = 1$ and (7) of Lemma 1 are satisfied for $s_1 < 0$ and $s_2 < 0$.

Differentiating (5) with respect to s_i we get

$$(1 - a_j)\left(\frac{\partial^2 f}{\partial s_i \partial s_j} g + \frac{\partial f}{\partial s_j}\frac{\partial g}{\partial s_i}\right) = a_j\left(\frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_j} + f\frac{\partial^2 g}{\partial s_i \partial s_j}\right), \quad i, j = 1, 2. \tag{10}$$

From (6) for $j = 1$ and (10) for $i = j = 1$ we obtain

$$\frac{a_1 - b_1}{b_1}\frac{\frac{\partial^2 f}{\partial^2 s_1}}{\frac{\partial f}{\partial s_1}} = \frac{\frac{\partial g}{\partial s_1}}{g},$$

but (5) implies that

$$\frac{1 - a_1}{a_1}\frac{\frac{\partial f}{\partial s_1}}{f} = \frac{\frac{\partial g}{\partial s_1}}{g}. \tag{11}$$

Thus

$$\frac{\frac{\partial^2 f}{\partial^2 s_1}}{\frac{\partial f}{\partial s_1}} = \frac{(1 - a_1)b_1}{(a_1 - b_1)a_1}\frac{\frac{\partial f}{\partial s_1}}{f},$$

which yields

$$f(s_1, s_2) = A(s_2)(1 + B(s_2)s_1)^{-p}, \quad \text{for } p = \frac{a_1(a_1 - b_1)}{b_1 - a_1^2} > 0. \tag{12}$$

Then form (11) it follows that

$$g(s_1, s_2) = C(s_2)\left[f(s_1, s_2)\right]^{\frac{1-a_1}{a_1}}. \tag{13}$$

Inserting (7) into (10) with $i = 2$, $j = 1$, gives

$$\frac{a_1 - c}{c}\frac{\frac{\partial^2 f}{\partial s_1 \partial s_2}}{\frac{\partial f}{\partial s_1}} = \frac{\frac{\partial g}{\partial s_2}}{g}, \tag{14}$$

and inserting (7) into (10) with $i = 1$, $j = 2$ leads to

$$\frac{a_2 - c}{c}\frac{\frac{\partial^2 f}{\partial s_1 \partial s_2}}{\frac{\partial f}{\partial s_2}} = \frac{\frac{\partial g}{\partial s_1}}{g}. \tag{15}$$

From (14) it follows that

$$g(s_1, s_2) = E(s_1)\left[\frac{\partial f}{\partial s_1}(s_1, s_2)\right]^{\frac{a_1-c}{c}}. \tag{16}$$

From (5) for $j = 2$ and (14) we get

$$\frac{a_1 - c}{c}\frac{\frac{\partial^2 f}{\partial s_1 \partial s_2}}{\frac{\partial f}{\partial s_1}} = \frac{1 - a_2}{a_2}\frac{\frac{\partial f}{\partial s_2}}{f},$$

which can be written as

$$\frac{\partial^2 f}{\partial s_1 \partial s_2} f = \frac{c(1-a_2)}{a_2(a_1-c)}\frac{\partial f}{\partial s_1}\frac{\partial f}{\partial s_2}. \tag{17}$$

Similarly from (5) for $j = 1$ and (15) we get

$$\frac{\partial^2 f}{\partial s_1 \partial s_2} f = \frac{c(1-a_1)}{a_1(a_2-c)}\frac{\partial f}{\partial s_1}\frac{\partial f}{\partial s_2}.$$

Thus we arrive at

$$\frac{c(1-a_2)}{a_2(a_1-c)} = \frac{c(1-a_1)}{a_1(a_2-c)},$$

yielding

$$c(c - a_1 a_2)(a_1 - a_2) = 0.$$

Consequently there are the following two cases possible: either *Case I*: $c = a_1 a_2$ or *Case II*: $c \neq a_1 a_2$ and $a_1 = a_2$.

Denote

$$d = \frac{c(1-a_2)}{a_2(a_1-c)}.$$

Then (17) can be written as

$$\frac{\frac{\partial^2 f}{\partial s_1 \partial s_2}}{\frac{\partial f}{\partial s_1}} = d\frac{\frac{\partial f}{\partial s_2}}{f},$$

which yields

$$\frac{\partial f}{\partial s_1}(s_1, s_2) = D_1(s_1)\left[f(s_1, s_2)\right]^d. \tag{18}$$

Now we differentiate (12) with respect to s_1 getting

$$\frac{\partial f}{\partial s_1}(s_1, s_2) = -pA(s_2)B(s_2)\left(1 + B(s_2)s_1\right)^{-p-1}. \tag{19}$$

Inserting (12) and (19) into (18) we obtain

$$-pA(s_2)B(s_2)\left(1 + B(s_2)s_1\right)^{-p-1} = D_1(s_1)\left[A(s_2)(1 + B(s_2)s_1)^{-p}\right]^d.$$

Hence

$$D_1(s_1) = -p\left[A(s_2)\right]^{1-d} B(s_2)\left[1 + B(s_2)s_1\right]^{-p-1+pd}. \tag{20}$$

Case I. If $c = a_1 a_2$ then $d = 1$ and (20) takes the form

$$D_1(s_1) = -\frac{pB(s_2)}{1 + B(s_2)s_1}. \tag{21}$$

We insert in (21) $s_2 = 0$ and we put it back into (21). Thus

$$\frac{B(s_2)}{1+B(s_2)s_1} = \frac{B(0)}{1+B(0)s_1},$$

which yields $B(s_2) = B(0) = -\lambda$ for some $\lambda = const.$ Consequently

$$f(s_1, s_2) = A(s_2)(1-\lambda s_1)^{-p}. \tag{22}$$

Since the limit of f for $(s_1, s_2) \nearrow (0,0)$ is 1, then by the above equation it follows that $A(0) = 1$. Plugging (22) into (13) we have

$$g(s_1, s_2) = C(s_2)\,[A(s_2)]^{\frac{1-a_1}{a_1}}\,(1-\lambda s_1)^{-q}, \tag{23}$$

where $q = p\frac{1-a_1}{a_1} > 0$. Differentiating (22) with respect to s_1 and putting it into (16) we have

$$g(s_1, s_2) = E(s_1)\,[\lambda p]^{\frac{a_1-c}{c}}\,[A(s_2)]^{\frac{a_1-c}{c}}\,(1-\lambda s_1)^{-(p+1)\frac{a_1-c}{c}}. \tag{24}$$

Since the limit of g for $(s_1, s_2) \nearrow (0,0)$ is 1 then from (24) it follows that

$$E(0)\,[\lambda p]^{\frac{a_1-c}{c}} = 1 \tag{25}$$

Observe that $\frac{a_1-c}{c} = \frac{1-a_2}{a_2}$. On comparing (23) and (24) we arrive at

$$C(s_2)\,[A(s_2)]^{\frac{1-a_1}{a_1}}\,(1-\lambda s_1)^{-q} = E(s_1)\,[\lambda p]^{\frac{1-a_2}{a_2}}\,[A(s_2)]^{\frac{1-a_2}{a_2}}\,(1-\lambda s_1)^{-(p+1)\frac{1-a_2}{a_2}}.$$

Taking now $s_1 \nearrow 0$ and using (25) we get

$$C(s_2) = [A(s_2)]^{\frac{1-a_2}{a_2}-\frac{1-a_1}{a_1}}.$$

Hence

$$g(s_1, s_2) = (1-\lambda s_1)^{-q}\,[A(s_2)]^{\frac{1-a_2}{a_2}}.$$

Case II. If $c \neq a_1a_2$ and $a_1 = a_2 = a$ then $d = \frac{c(1-a)}{a(a-c)} \neq 1$.

a) First we consider $d = \frac{p+1}{p} = \frac{b_1(1-a)}{a(a-b_1)}$ (then $b_1 = c$), thus (20) has the form

$$D_1(s_1) = -p\,[A(s_2)]^{-\frac{1}{p}}\,B(s_2).$$

For $s_2 \nearrow 0$ it gives us

$$D_1(s_1) = -p\,[A(0)]^{-\frac{1}{p}}\,B(0) = D,$$

where $D = const.$ Hence (18) can be written as

$$\frac{\partial f}{\partial s_1}(s_1, s_2) = D\,[f(s_1, s_2)]^d,$$

which yields

$$f(s_1, s_2) = [(1-d)Ds_1 + m(s_2)]^{-p}, \tag{26}$$

for some function m. On comparing the above equation with (12) we get

$$A(s_2)^{-\frac{1}{p}}\,[1 + B(s_2)s_1] = (1-d)Ds_1 + m(s_2). \tag{27}$$

For $s_1 \nearrow 0$ it yields

$$A(s_2)^{-\frac{1}{p}} = m(s_2).$$

Plugging it back into (27) we have

$$m(s_2)B(s_2)s_1 = (1-d)Ds_1.$$

In the above equation we take $s_2 \nearrow 0$ which leads to

$$m(0)B(0)s_1 = (1-d)Ds_1.$$

Now we put it back into (26) and observe that $m(0) = 1$. We get

(28) $$f(s_1, s_2) = (1 - \lambda s_1 + M(s_2))^{-p},$$

where $\lambda = -B(0)$ and $M(s_2) = m(s_2) - 1$ (in particular it yields $M(0) = 0$). Inserting (28) into (13) we obtain

(29) $$g(s_1, s_2) = C(s_2)\,(1 - \lambda s_1 + M(s_2))^{-q},$$

where $q = p\frac{1-a}{a} > 0$. In particular, for $(s_1, s_2) \nearrow (0,0)$ it implies that $C(0) = 1$. Differentiate now (28) with respect to s_1 and then insert it into (16). Thus we get

(30) $$g(s_1, s_2) = E(s_1)\,[\lambda p]^{\frac{a-c}{c}}\,(1 - \lambda s_1 + M(s_2))^{-(p+1)\frac{a-c}{c}}.$$

Observe that $(p+1)\frac{a-c}{c} = q$. Comparing (29) and (30) we obtain

$$C(s_2) = E(s_1)\,[\lambda p]^{\frac{a-c}{c}} = C,$$

where $C = const$. But $C(0) = 1$, meaning $C = 1$ and thus (29) yields

$$g(s_1, s_2) = (1 - \lambda s_1 + M(s_2))^{-q}.$$

b) If $d \neq \frac{p+1}{p}$ then taking $s_1 \nearrow 0$ in (20) we get

(31) $$D_1(0) = -p\,[A(s_2)]^{1-d}\,B\,(s_2).$$

Thus (20) takes on the form

$$D_1(s_1) = D_1(0)\,[1 + B(s_2)s_1]^{-p-1+pd},$$

which implies $B(s_2) = B = const$. Hence (31) implies $A(s_2) = A = const$, and thus

$$f(s_1, s_2) = A(1 + Bs_1)^{-p},$$

which contradicts the assumption that $\bar{X}$ has non-degenerate components. □

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