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Journal of Multivariate Analysis 91 (2004) 224–239
Journal of Multivariate Analysis
http://www.elsevier.com/locate/jmva

Limit theorems for random permanents with exchangeable structure

Grzegorz A. Rempała[a,b,*] and Jacek Wesołowski[c]

[a] *Institute of Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, MN 55455-0436, USA*
[b] *Department of Mathematics, University of Louisville, USA*
[c] *Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, Warszawa, Poland*

Received 28 November 2001

Abstract

Permanents of random matrices extend the concept of U-statistics with product kernels. In this paper, we study limiting behavior of permanents of random matrices with independent columns of exchangeable components. Our main results provide a general framework which unifies already existing asymptotic theory for projection matrices as well as matrices of all-iid entries. The method of the proofs is based on a Hoeffding-type orthogonal decomposition of a random permanent function. The decomposition allows us to relate asymptotic behavior of permanents to that of elementary symmetric polynomials based on triangular arrays of rowwise independent rv's.

AMS 2000 subject classifications: primary 60F05; secondary 15A15; 15A52

Keywords: Random permanent; Orthogonal expansion; Central limit theorem

1. Introduction

Denote by $\mathbf{A} = [a_{ij}]$ an $m \times n$ real matrix with $m \leqslant n$. Then a permanent of the matrix $\mathbf{A}$ is defined by

$$\operatorname{Per}(\mathbf{A}) = \sum_{(i_1,\dots,i_m):\ \{i_1,\dots,i_m\} \subset \{1,\dots,n\}} a_{1\, i_1} \dots a_{m\, i_m}.$$

*Corresponding author. Institute of Mathematics and its Applications (IMA), University of Minnesota, 400 Lind Hall, 207 Church Street S.E., Minneapolis, MN 55455-0436, USA. Fax: +502-852-7132.
E-mail address: rempala@ima.umn.edu (G.A. Rempała).

0047-259X/$ - see front matter
doi:10.1016/j.jmva.2004.01.003

The permanent function has a long history, having been first introduced by Cauchy in 1812 in his celebrated memoir on determinants and, almost simultaneously, by Binet [3]. More recently several problems in statistical mechanics, quantum field theory, and chemistry as well as enumeration problems in combinatorics and linear algebra have been reduced to the computation of a permanent. Unfortunately, the fastest known algorithm for computing a permanent of an $n \times n$ matrix runs, as shown by Ryser [19], in $O(n2^n)$ time. Moreover, strong evidence for the apparent complexity of the problem was provided by Valiant [24], who showed that evaluating a permanent is $\#P$-complete, even when restricted to 0–1 matrix. In view of these results, the approximation theory for a permanent function in case of a large dimension of the matrix $\mathbf{A}$ has become a very active area of research over the past 20 years (for most recent results for both random and non-random settings cf. e.g., [6,8,13,17]).

In this work we shall be concerned with the asymptotic behavior of *a random permanent* function, that is we shall consider the case of a random matrix $\mathbf{A}$—the situation which often arises naturally in statistical physics or statistical mechanics problems, when the investigated physical phenomenon is driven by some random process, and hence stochastic in nature. The asymptotic theory of random permanents has been also of major interest since it was shown that, as in the deterministic case, their direct calculation can be also computationally very difficult (cf. e.g., [7] and references therein).

The investigation of the asymptotic behavior of random permanents has been until now mostly limited to two special settings. In the first one, which, due to its natural connection with the theory of *U-statistics of infinite order*, has been also receiving the most attention, only very special types of matrices $\mathbf{A}$ were considered, namely, the so-called finite dimensional projection matrices (i.e., of independent, identically distributed columns and a finite number of independent blocks of identical rows). In this setting, the limiting theory for permanents was considered for instance in [20,25,4,11,21]. In the second, perhaps somewhat more natural setting introduced in the early works of Girko (see e.g., [9, Chapter 2 and 7] and references therein), the limiting theory for random permanents has been concerned simply with the matrices of all independent identically distributed (iid) entries. As pointed out in [17], the iid setting is of interest to statisticians since then the permanent function is an example of a *U-statistic of permanent design*, an object belonging to a certain sub-class of *incomplete U-statistics* (cf. e.g., [12]) which enjoys some optimal properties in the sense of the statistical design theory. The results for the matrices of iid entries have been also obtained for the special case of Bernoulli 0–1 entries in the context of perfect matchings problem by Janson [10], and later still under some restrictive assumptions on the entries of $\mathbf{A}$ by Rempała [14]. More recently, by reducing the problem of the asymptotic behavior of a permanents of an iid-entries matrix to the one of the asymptotic behavior of the permanent of some one-dimensional projection matrix, Rempała and Wesołowski [15] have obtained a general limiting result solely under the assumptions of square integreability of the entries of $\mathbf{A}$. Although the approach taken in [15] was quite effective, it was limited to the particular case of all independent entries and, unfortunately, could not be extended to handle a more general setting.

The purpose of current paper is to present a quite general and thus unifying approach to the problem of investigating the asymptotic behavior of random permanents for matrices of independent columns. The primary tool of our investigation is the orthogonal decomposition device introduced elsewhere (cf. [16]) and similar to the one known for U-statistics. Indeed, the subtle connection between the representation of a permanent function in terms of its orthogonal decomposition components allows us to reduce the problem of describing limiting behavior of random permanents to that of describing a limiting behavior of certain statistics which turn out to be closely related to linear combinations of elementary symmetric polynomials (i.e., U-statistics based on product kernels, see next section for a definition). The method allows us to state a general limit theorem for random permanents of matrices of independent columns of exchangeable components with inter-component correlation coefficient ρ. The result puts on the common ground the asymptotic results of both settings described above, that is, for one-dimensional projection matrices (since then $\rho = 1$), as well as all-iid-entries matrices (since then $\rho = 0$). In both those cases our results specialize to the ones obtained earlier.

The paper is organized as follows. In the next section, we provide some preliminary results on elementary symmetric polynomial statistics based on random arrays. In Section 3, we briefly discuss the martingale-type orthogonal decomposition of a random permanent function of Rempała and Wesolowski [16]. This decomposition, which for the case of a one-dimensional projection matrix coincides with the well-known Hoefding decomposition for an elementary symmetric polynomial is our main tool in obtaining the limiting results. In Section 3, we also prove a key result on the weak convergence of the components of the permanent's orthogonal decomposition. The result is based on a limit theorem for elementary symmetric polynomials statistics obtained in Section 2. Once the result of Section 3 is established, the theorems for random permanents follow. They are the main focus of this paper and are discussed in detail in Section 4. Some additional technical results on weak law of large numbers for random arrays are briefly outlined in the Appendix.

2. A limit theorem for elementary symmetric polynomials

We shall start with a preliminary result on the weak convergence of elementary symmetric polynomials based on triangular arrays of real random variables (rv's). For an arbitrary triangular array $\{Y_{lk}\}$ $(1 \leqslant k \leqslant l)$, of rv's let us define a corresponding elementary symmetric polynomial statistic of order $c \geqslant 1$ as follows:

$$S_l(c) = \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant l} Y_{l,i_1} \ldots Y_{l,i_c}. \tag{2.1}$$

The result on the weak convergence of $S_l(c)$ is a consequence of the following version of Waring formula which has been reported e.g., in [1]. For the sake of readers' convenience, however, we give here its elementary derivation.

Lemma 1. *Let $S_l(0) = 1$, then for any $1 \leqslant c \leqslant l$ we have*

$$cS_l(c) = \sum_{d=0}^{c-1} (-1)^d S_l(c-d-1)\left(\sum_{k=1}^{l} Y_{lk}^{d+1}\right). \tag{2.2}$$

Proof. Let us introduce an auxiliary statistic $Q(c,d)$ defined as

$$\begin{aligned}
&Q(c,d)\\
&\quad = \sum_{1\leqslant k_1<\cdots<k_c\leqslant l} Y_{l,k_1}^d \ldots Y_{l,k_c} + \sum_{1\leqslant k_1<\cdots<k_c\leqslant l} Y_{l,k_1} Y_{l,k_2}^d \ldots Y_{l,k_c}\\
&\qquad + \cdots + \sum_{1\leqslant k_1<\cdots<k_c\leqslant l} Y_{l,k_1} \ldots Y_{l,k_c}^d\\
&\quad = \sum_{k=1}^{l} Y_{lk}^d \left(\sum_{k_i\neq k} Y_{l,k_1} \ldots Y_{l,k_{c-1}}\right),
\end{aligned}$$

where in the last expression the inner summation is taken over all possible choices of $c-1$ distinct indices $k_1, \ldots, k_{c-1}$ out of the set of indices $\{1, 2, \ldots, l\}\backslash\{k\}$. From the definition of $Q(c,d)$ it follows that for any positive integers $1 \leqslant c \leqslant l$ and d

$$Q(c,1) = cS_l(c) \quad \text{and} \quad Q(1,d) = \sum_{k=1}^{l} Y_{lk}^d, \tag{2.3}$$

as well as that

$$Q(c,d) + Q(c-1,d+1) = S_l(c-1)\left(\sum_{k=1}^{l} Y_{lk}^d\right). \tag{2.4}$$

Now, solving for $Q(c,1)$ with the help of (2.3) and (2.4) we arrive at (2.2). □

With the above lemma, as well as the result of Proposition A.1 of the Appendix, we are in position to prove our main result of this section.

Theorem 1. *Let $\{Y_{lk}\}$ be a triangular array of square integrable, rowwise independent, real rv's with $EY_{lk} = 0$ and $\lim_{l\to\infty} Var(\sum_{k=1}^{l} Y_{lk})/l = \sigma^2 > 0$ satisfying additionally the Lindeberg condition*

(LC) $\forall_{\varepsilon>0}\ \frac{1}{l}\sum_{k=1}^{l} EY_{lk}^2 I\{|Y_{lk}| > \sigma\sqrt{l}\varepsilon\} \to 0 \quad$ *as $l \to \infty$.*

Then, for any integer $c \geqslant 1$,

$$\left[\frac{S_l(1)}{l^{1/2}}, \frac{S_l(2)}{l^{2/2}}, \ldots, \frac{S_l(c)}{l^{c/2}}\right]^T \xrightarrow{d} \left[\frac{\sigma H_1(\mathcal{N})}{1!}, \frac{\sigma^2 H_2(\mathcal{N})}{2!}, \ldots, \frac{\sigma^c H_c(\mathcal{N})}{c!}\right]^T, \tag{2.5}$$

where H_c is a Hermite polynomial of order c with the leading coefficient equal to one and $\mathcal{N}$ is a standard normal rv.

Proof. Let us assume (without loosing generality) that $\sigma^2 = 1$. We shall prove the above result by induction with respect to c. For $c = 1$ the result is obvious since we have assumed the required condition (LC) and since $H_1(\mathcal{N}) = \mathcal{N}$. Hence

$$\frac{S_l(1)}{l^{1/2}} \xrightarrow{d} H_1(\mathcal{N}). \tag{2.6}$$

Let us assume thus that (2.5) holds true for all integers $i = 1, \dots, c-1$ and define a $(c-1)$-dimensional vector

$$\mathbb{Y}_c^{(l)} = \left[\frac{\sum Y_{lk}^2}{l^{2/2}}, \frac{\sum Y_{lk}^3}{l^{3/2}}, \dots, \frac{\sum Y_{lk}^c}{l^{c/2}}\right]^T.$$

First, let us note that by taking $X_{lk} = Y_{lk}^2$ we obtain the rowwise independent random array satisfying conditions (A1) and (A2) of Proposition A.1 of the Appendix with $\beta = 1$ and $c_l = \sqrt{l}$. Thus,

$$\mathbb{Y}_c^{(l)} \xrightarrow{P} [1, 0, \dots, 0]^T \quad \text{as } l \to \infty. \tag{2.7}$$

But (2.6) along with (2.7) imply that (cf. e.g., [2, Chapter 1]) the vector $\left[S_l(1)/\sqrt{l}, \mathbb{Y}_c^{(l)}\right]^T$ converges weakly to the corresponding limit. On the other hand, by (2.2) we have that for $i = 1, \dots, c-1$,

$$l^{-i/2} S_l(i) = G_i\left(\frac{S_l(1)}{l^{1/2}}, \mathbb{Y}_c^{(l)}\right),$$

where the continuous function G_i (known sometimes as a Waring function) depends only upon i but not upon l. This and the induction hypothesis imply

$$\begin{bmatrix} \dfrac{S_l(1)}{l^{1/2}} \\ \vdots \\ \dfrac{S_l(c-1)}{l^{(c-1)/2}} \\ \dfrac{S_l(c)}{l^{c/2}} \end{bmatrix} = \begin{bmatrix} G_1\left(\dfrac{S(1)}{l^{1/2}}, \mathbb{Y}_c^{(l)}\right) \\ \vdots \\ G_{c-1}\left(\dfrac{S(1)}{l^{1/2}}, \mathbb{Y}_c^{(l)}\right) \\ G_c\left(\dfrac{S_l(1)}{l^{1/2}}, \mathbb{Y}_c^{(l)}\right) \end{bmatrix} \xrightarrow{D} \begin{bmatrix} H_1(\mathcal{N})/1! \\ \vdots \\ H_{c-1}(\mathcal{N})/(c-1)! \\ B_c \end{bmatrix} \tag{2.8}$$

for some rv B_c. Hence, in order to arrive at (2.5), we need only to identify B_c. To this end, using formula (2.2) we write

$$\begin{aligned} &\frac{S_l(c)}{l^{c/2}} \\ &\quad = \frac{1}{c}\left[\frac{S_l(c-1)}{l^{(c-1)/2}}\frac{S_l(1)}{l^{1/2}} - \frac{S_l(c-2)}{l^{(c-2)/2}}\frac{\sum_{k=1}^{l} Y_{lk}^2}{l}\right] \\ &\qquad + \frac{A_c}{c}. \end{aligned}$$

By (2.7) and (2.8), the first part of the above right-hand side converges to

$$\frac{H_{c-1}(\mathcal{N})\mathcal{N}-(c-1)H_{c-2}(\mathcal{N})}{c!}=\frac{H_c(\mathcal{N})}{c\,!},$$

in view of the well-known recurrence relation for Hermite polynomials. For the reminder term A_c, we have

$$A_c=\frac{1}{c}\sum_{d=2}^{c-1}(-1)^d\frac{S_l(c-1-d)}{l^{(c-1-d)/2}}\left(\sum_{k=1}^{l}\frac{Y_{lk}^{2(d+1)/2}}{l^{(d+1)/2}}\right)\xrightarrow{P}0.$$

The above implies that $B_c=H_c(\mathcal{N})/c\,!$, which entails (2.5). $\square$

Remark 1. The results above implies in particular that

$$\frac{S(c)}{l^{c/2}}\xrightarrow{d}\frac{H_c(\mathcal{N})}{c!},$$

which is a result derived in [23] and a special case of the quite general limit theorem for U-statistics based on arrays of rv's of Rubin and Vitale [18].

3. Orthogonal decomposition of a permanent function

Having established the results on elementary symmetric polynomials in the last section, let us turn to the problem of weak convergence of random permanents. Throughout the paper we shall assume that $\mathbf{X}=[X_{ij}]$ is an $m\times n$ ($m\leqslant n$) real random matrix of square integrable components and such that its columns are build from the first m terms of iid sequences $(X_{i,1})_{i\geqslant 1},(X_{i,2})_{i\geqslant 1},\ldots,(X_{i,n})_{i\geqslant 1}$ of exchangeable rv's. Clearly, under these assumptions all entries of the matrix $\mathbf{X}$ are identically distributed although not necessarily independent. For $i,k=1,\ldots,m$ and $j=1,\ldots,n$ we denote $\mu=EX_{ij}$, $\sigma^2=Var\,X_{ij}$ and $\rho=Corr(X_{kj},X_{ij})$. Observe that, necessarily, we must have $\rho\geqslant 0$. In what follows, we shall always assume that $\mu\neq 0$ and we shall also denote by $\gamma=\sigma/\mu$ the coefficient of variation.

In the sequel the major tool of our investigation will be the orthogonal decomposition result of Rempała and Wesołowski [16] which states that

$$\frac{\operatorname{Per}\mathbf{X}}{\binom{n}{m}m!\mu^m}=1+\sum_{c=1}^{m}\binom{m}{c}U_c^{(m,n)},\tag{3.1}$$

where

$$U_c^{(m,n)}=\binom{n}{c}^{-1}\binom{m}{c}^{-1}c!^{-1}\sum_{1\leqslant i_1<\cdots<i_c\leqslant m}\;\sum_{1\leqslant j_1<\cdots<j_c\leqslant n}\operatorname{Per}\,[\tilde{X}_{i_uj_v}]_{\substack{u=1,\ldots,c\\ v=1,\ldots,c}}$$

for $\tilde{X}_{ij}=X_{ij}/\mu-1$, $i=1,\ldots,m$, $j=1,\ldots,n$. Moreover, under our assumptions on the entries of $\mathbf{X}$ the rv's $U_c^{(m,n)}$ for $c=1,2,\ldots,m$ are orthogonal, i.e.,

$$Cov(U_{c_1}^{(m,n)},U_{c_2}^{(m,n)})=0\quad\text{for }c_1\neq c_2\tag{3.2}$$

with the variance

$$Var\, U_c^{(m,n)} = \binom{n}{c}^{-1}\binom{m}{c}^{-1}\gamma^{2c}\sum_{r=0}^{c}\binom{m-r}{c-r}\frac{\rho^{c-r}(1-\rho)^r}{r!}. \tag{3.3}$$

Let us note that when $\rho = 1$ then (3.1) becomes the well-known Hoeffding decomposition of a normalized elementary symmetric polynomial (2.1), i.e., the U-statistic based on the product kernel. It is well know that in this special case the rv's $U_c^{(m,n)}$ are backward martingales. In fact, it is not difficult to see that this property carries over to $0 \leqslant \rho < 1$ and thus throughout the paper we often refer to (3.1) as the "martingale decomposition".

For any c and suitably large m and n define

$$W_c(n) = \binom{n}{c}\binom{m}{c}c!\,U_c^{(m,n)} = \sum_{1\leqslant i_1<\cdots<i_c\leqslant m}\;\sum_{1\leqslant j_1<\cdots<j_c\leqslant n} \mathrm{Per}\,[\tilde{X}_{i_u j_v}]_{\substack{u=1,\dots,c\\ v=1,\dots,c}}.$$

The following results describes the asymptotic behavior of $W_c(n)$ which is the key to investigating the asymptotic behavior of decomposition (3.1).

Proposition 1. *Let c be an arbitrary positive integer. Assume that $m = m_n \to \infty$ as $n \to \infty$.*

If $\rho = 0$ then

$$\left[\frac{W_1(n)}{(\sqrt{nm})}, \dots, \frac{W_c(n)}{(\sqrt{nm})^c}\right]^T \xrightarrow{d} \left[\frac{\gamma}{1}H_1(\mathcal{N}), \dots, \frac{\gamma^c}{c!}H_c(\mathcal{N})\right]^T.$$

If $\rho > 0$ then

$$\left[\frac{W_1(n)}{\sqrt{nm}}, \dots, \frac{W_c(n)}{(\sqrt{nm})^c}\right]^T \xrightarrow{d} \left[\frac{\gamma\rho^{1/2}}{1!}H_1(\mathcal{N}), \dots, \frac{\gamma^c\rho^{c/2}}{c!}H_c(\mathcal{N})\right]^T.$$

Proof. Consider first the case $\rho = 0$. Define for an arbitrary fixed positive integer c

$$V_c(n) = n^{-c/2}\sum_{1\leqslant j_1<\cdots<j_c\leqslant n} Y_{n,j_1}\dots Y_{n,j_c},$$

where

$$Y_{n,j} = \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\tilde{X}_{ij},$$

$j = 1, 2, \dots, n$. Here the first subscript indicates the dependence on n through $m = m(n)$. Observe that $V_c(n)$ is just an elementary symmetric polynomial in variables which are column sums of the matrix $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(n)$ normalized by $\sqrt{m}$. Thus, $E(Y_{n,1}^2) = \gamma^2$ and we may use the limit theorem for elementary symmetric polynomials for rowwise iid double arrays of square integrable rv's of Section 2 to conclude that

$$[V_1(n), \dots, V_c(n)]^T \xrightarrow{d} \left[\frac{\gamma}{1!}H_1(\mathcal{N}), \dots, \frac{\gamma^c}{c!}H_c(\mathcal{N})\right]^T$$

as long as the following version of the Lindeberg condition holds:

$$E(Y_{n,1}^2 I(|Y_{n,1}| > \sqrt{n}\varepsilon)) \to 0$$

for any $\varepsilon > 0$ as $n \to \infty$. To prove that the condition is satisfied observe first that

$$E(Y_{n,1}^2 I(|Y_{n,1}| > \sqrt{n}\varepsilon)) \leqslant \sup_{k \geqslant 1} E(Y_{k,1}^2 I(Y_{k,1}^2 > n\varepsilon^2)).$$

Consequently, it suffices to show that the sequence of rv's $(Y_{k,1}^2)_{k \geqslant 1}$ is uniformly integrable. To this end, let us observe the following:

(i) By a central limit theorem for exchangeable sequences—see e.g., [22, Chapter 2]—it follows that $Y_{k,1}^2$ converges in distribution to $E(\tilde{X}_{1,1}^2|\mathscr{F})\mathscr{N}^2$, where $\mathscr{F}$ is the σ-algebra of permutable events for the exchangeable sequence $(\tilde{X}_{i,1})_{i \geqslant 1}$, and $\mathscr{N}$ is a standard normal rv independent of $\mathscr{F}$, and

(ii) $E(E(\tilde{X}_{1,1}^2|\mathscr{F})\mathscr{N}^2) = E(\tilde{X}_{1,1}^2) = \gamma^2$ which, on the other hand equals $E(Y_{k,1}^2)$ for any $k \geqslant 1$.

Finally, we conclude that the sequence $\{Y_{k,1}^2\}_{k \geqslant 1}$ is uniformly integrable since it converges in distribution and the corresponding sequence of expectations also converges (all being equal) to the suitable limit.

Observe that for any $k = 1, \ldots, c$,

$$\frac{W_k(n)}{(\sqrt{nm})^k} = V_k(n) + \frac{R_k(n)}{(\sqrt{mn})^k},$$

where $R_k(n)$ is a sum of different products $\tilde{X}_{i_1,j_1} \ldots \tilde{X}_{i_k,j_k}$ such that $1 \leqslant j_1 < \cdots j_k \leqslant n$, $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$ and at least one of $i_1, \ldots, i_k$ in the sequence $(i_1, \ldots, i_k)$ repeats.

Using the fact that $R_k(n)$ is a sum of orthogonal products (observe that the covariance of any two of such different products equals zero since the columns are independent and elements in each column have zero correlation) and that the variance of any of such single product equals γ^{2k} while the number of products in $R_k(n)$ equals $\binom{n}{k}\left(m^k - \binom{m}{k}k!\right)$ we obtain

$$Var\left(\frac{R_k(n)}{(\sqrt{nm})^k}\right) = \gamma^2 \frac{\binom{n}{k}\left(m^k - \binom{m}{k}k!\right)}{n^k m^k} \leqslant \frac{\gamma^2}{k!} \frac{m^k - \binom{m}{k}k!}{m^k} = O(1/m) \to 0$$

as $n \to \infty$ (since the numerator is of the order m^{k-1} while the denominator is of the order m^k).

Consequently, the first assertation of the proposition follows.

Now, let us consider the case $\rho > 0$. Similarly as above, let us define

$$V_c(n) = n^{-c/2} \sum_{1 \leqslant j_1 < \cdots < j_c \leqslant n} Y_{n,j_1} \ldots Y_{n,j_c},$$

where

$$Y_{n,j} = \frac{1}{m} \sum_{i=1}^{m} \tilde{X}_{ij},$$

$j = 1, 2, \ldots, n.$

Similar to the first case, we will show that the sequence $\{Y_{k,1}^2\}_{k\geqslant 1}$ is uniformly integrable. To this end we observe first that by a version of law of large numbers for exchangeable sequences (see, e.g., [5, Chapter 7]) it follows that as $k\to\infty$,

$$Y_{k,1}^2 \xrightarrow{d} E^2(\tilde{X}_{1,1}|\mathscr{F}),$$

where $\mathscr{F}$ is the σ-algebra of permutable events. Further, by the de Finetti theorem we have

$$\begin{aligned} E(E^2(\tilde{X}_{1,1}|\mathscr{F})) &= E(E(\tilde{X}_{1,1}|\mathscr{F})E(\tilde{X}_{2,1}|\mathscr{F})) = E(E(\tilde{X}_{1,2}\tilde{X}_{2,1}|\mathscr{F})) \\ &= E(\tilde{X}_{1,1}\tilde{X}_{1,2}) = \rho\gamma^2. \end{aligned}$$

Also,

$$E(Y_{k,1}^2) = \frac{1}{m^2}[m\gamma^2 + m(m-1)\rho\gamma^2] \to \rho\gamma^2, \quad k\to\infty$$

since $m=m(k)\to\infty$ Thus, the sequence $\{Y_{k,1}^2\}$ is uniformly integrable and the Lindeberg condition of Theorem 1 is satisfied. This allows us to conclude that

$$[V_1(n),\dots,V_c(n)]^T \xrightarrow{d} \left[\frac{(\sqrt{\rho}\gamma)^1}{1!}H_1(\mathscr{N}),\dots,\frac{(\sqrt{\rho}\gamma)^c}{c!}H_c(\mathscr{N})\right]^T.$$

Similar to the first case, for any $k=1,\dots,c$, we have

$$\frac{W_k(n)}{m^k(\sqrt{n})^k} = V_k(n) + \frac{R_k(n)}{m^k(\sqrt{n})^k}.$$

However, this time some of the elements of $R_k(n)$ are correlated—this is true for pairs of products originating from exactly the same columns; if at least one column in the pair of products is different then their correlation is zero. Consequently,

$$Var\left(\frac{R_k(n)}{m^k\sqrt{n}^k}\right) = \frac{1}{m^{2k}n^k}\sum_{1\leqslant j_1<\cdots j_k\leqslant n} Var(R_{j_1,\dots,j_k}(n) = \frac{\binom{n}{k}}{m^{2k}n^k}Var(R_{1,\dots,k}(n),$$

where $R_{j_1,\dots,j_k}(n)$ denotes sum of respective products arising from the columns $j_1,\dots,j_k$. Since

$$|Cov(\tilde{X}_{i_1,j_1}\dots\tilde{X}_{i_k,j_k},\tilde{X}_{l_1,j_1}\dots\tilde{X}_{l_k,j_k})|\leqslant\gamma^{2k}$$

for any choices of rows $(i_1,\dots,i_k)$ and $(l_1,\dots,l_k)$, we conclude that

$$Var(R_{1,\dots,k}(n)<\left(m^k-\binom{m}{k}k!\right)^2\gamma^{2k}.$$

Hence, it follows that

$$Var\left(\frac{R_k(n)}{m^k(\sqrt{n})^k}\right) < \frac{\binom{n}{k}\left(m^k-\binom{m}{k}k!\right)^2\gamma^{2k}}{m^{2k}n^k} < \frac{\gamma^2}{k!}\frac{\left(m^k-\binom{m}{k}k!\right)^2}{m^{2k}}\to 0$$

as $n \to \infty$. Consequently, $R_k(n)n^{-c/2}m^{-c} \xrightarrow{P} 0$ and the second assertation of the proposition follows. $\square$

4. Main results

The application of the martingale decomposition (3.1) along with Proposition 1 from the previous section allow us to finally formulate our main results for the asymptotic behavior of random permanents. We state them in Theorems 2 and 3 below, covering the cases $\rho = 0$ and $\rho > 0$, respectively. Our first result extends that of Rempala and Wesolowski [15].

Theorem 2. *Assume that* $\rho = 0$.

If $m/n \to \lambda > 0$ *as* $n \to \infty$ *then*

$$\frac{1}{\binom{n}{m} m! \mu^m} \mathrm{Per}(\mathbf{X}) \xrightarrow{d} \exp\left(\sqrt{\lambda}\gamma\mathcal{N} - \lambda\gamma^2/2\right). \tag{4.1}$$

If $m/n \to \lambda = 0$ *and* $m = m_n \to \infty$ *as* $n \to \infty$ *then*

$$\sqrt{\frac{n}{m}}\left(\frac{\mathrm{Per}(\mathbf{X})}{\binom{n}{m} m! \mu^m} - 1\right) \xrightarrow{d} \gamma\mathcal{N}. \tag{4.2}$$

Proof. Consider first the case $\lambda > 0$. For any n and any N such that $N < m_n$ denote

$$S_{N,n} = 1 + \sum_{c=1}^{N} \binom{m}{c} U_c^{(m,n)} = 1 + \sum_{c=1}^{N} \frac{W_c(n)}{\binom{n}{c} c!} = 1 + \sum_{c=1}^{N} \frac{(\sqrt{nm})^c}{\binom{n}{c} c!} \frac{W_c(n)}{(\sqrt{nm})^c}.$$

Observe that by the first assertation of Proposition 1 we have that

$$S_{N,n} \xrightarrow{d} G_N = \sum_{c=0}^{N} \frac{(\lambda\gamma^2)^{c/2}}{c!} H_c(\mathcal{N}),$$

as $n \to \infty$, since for any $c = 1, 2, \ldots,$

$$\frac{(\sqrt{nm})^c}{\binom{n}{c} c!} \to \sqrt{\lambda}.$$

Let us define also

$$T_{N,n} = \sum_{c=N+1}^{m_n} \binom{m}{c} U_c^{(m,n)}$$

and observe that since $U_c^{(m,n)}$ are orthogonal then

$$Var(T_{N,n}) = \sum_{c=N+1}^{m_n} \frac{\binom{m}{c}}{\binom{n}{c} c!} \gamma^{2c} \leqslant \sum_{c=N+1}^{\infty} \frac{\gamma^{2c}}{c!} = a_N$$

and $a_N \to 0$ as $N \to \infty$. Consequently, for $Z_n = \frac{\operatorname{Per}(\mathbf{X})}{\binom{n}{m} m! \mu^m}$ we have for any $\varepsilon > 0$

$$\begin{aligned} P(Z_n \leqslant x) &= P(S_{N,n} + T_{N,n} \leqslant x) \leqslant P(S_{N,n} \leqslant x + \varepsilon,\ |T_{N,n}| \leqslant \varepsilon) + P(|T_{N,n}| > \varepsilon) \\ &\leqslant P(S_{N,n} \leqslant x + \varepsilon) + P(|T_{N,n}| > \varepsilon). \end{aligned}$$

On the other hand,

$$P(Z_n \leqslant x) \geqslant P(S_{N,n} \leqslant x - \varepsilon,\ |T_{N,n}| \leqslant \varepsilon) \geqslant P(S_{N,n} \leqslant x - \varepsilon) - P(|T_{N,n}| > \varepsilon).$$

Thus, for any $x \in \mathbf{R}$ and any $\varepsilon > 0$ we obtain the double inequality

$$P(S_{N,n} \leqslant x - \varepsilon) - P(|T_{N,n}| > \varepsilon) \leqslant P(Z_n \leqslant x) \leqslant P(S_{N,n} \leqslant x + \varepsilon) + P(|T_{N,n}| > \varepsilon).$$

Hence, by the Tchebyshev inequality it follows that

$$P(S_{N,n} \leqslant x - \varepsilon) - a_N/\varepsilon^2 \leqslant P(Z_n \leqslant x) \leqslant P(S_{N,n} \leqslant x + \varepsilon) + a_N/\varepsilon^2.$$

Taking the limit as $n \to \infty$ we obtain

$$P(G_N \leqslant x - \varepsilon) \leqslant \lim_{n \to \infty} P(Z_n \leqslant x) \leqslant P(G_N \leqslant x + \varepsilon).$$

But now G_N converges almost surely to $G_\infty = \exp\left(\sqrt{\lambda}\gamma\mathcal{N} - \lambda\gamma^2/2\right)$ as $N \to \infty$ since by the well-known property of Hermite polynomials (see, e.g., [23])

$$\exp\left(\sqrt{\lambda}\gamma\mathcal{N} - \lambda\gamma^2/2\right) = \sum_{c=0}^{\infty} \frac{(\lambda\gamma^2)^{c/2}}{c!} H_c(\mathcal{N}).$$

Hence, for any $\varepsilon > 0$

$$P(G_\infty \leqslant x - \varepsilon) \leqslant \lim_{n \to \infty} P(Z_n \leqslant x) \leqslant P(G_\infty \leqslant x + \varepsilon).$$

Consequently, Z_n converges in distribution to G_∞.

For the proof in the case $\lambda = 0$, let us write

$$\sqrt{\frac{n}{m}}\left(\frac{\operatorname{Per}(\mathbf{X})}{\binom{n}{m} m! \mu^m} - 1\right) = \sqrt{\frac{n}{m}}\binom{m}{1} U_1^{(m,n)} + R_{m,n} = \frac{W_1(n)}{\sqrt{nm}} + R_{m,n},$$

where

$$R_{m,n} = \sqrt{\frac{n}{m}} \sum_{c=2}^{m} \binom{m}{c} U_c^{(m,n)}.$$

Observe that $R_{m,n}$ converges in probability to zero, since by (2) it follows that

$$Var\ R_{m,n} = \frac{n}{m} \sum_{c=2}^{m} \frac{\binom{m}{c}\gamma^2}{\binom{n}{c} c!} \leqslant \frac{m}{n} \exp(\gamma^2) \to 0.$$

Hence, the result follows by Proposition 1 for $m = m_n \to \infty$. $\square$

Our second result treats the case $\rho > 0$ and, in particular, for $\rho = 1$ specializes to the theorems in [25,11].

Theorem 3. *Assume that* $\rho > 0$.

If $m^2/n \to \lambda > 0$ *as* $n \to \infty$ *then*

$$\frac{1}{\binom{n}{m} m! \mu^m} \operatorname{Per}(\mathbf{X}) \xrightarrow{d} \exp\left(\sqrt{\lambda\rho}\gamma\mathcal{N} - \lambda\rho\gamma^2/2\right). \tag{4.3}$$

If $m^2/n \to \lambda = 0$ *and* $m = m_n \to \infty$ *as* $n \to \infty$ *then*

$$\frac{\sqrt{n}}{m}\left(\frac{\operatorname{Per}(\mathbf{X})}{\binom{n}{m} m! \mu^m} - 1\right) \xrightarrow{d} \sqrt{\rho}\gamma\mathcal{N}. \tag{4.4}$$

Proof. As before, let us first consider the case $\lambda > 0$. Then for any n and any N such that $N < m_n$ denote

$$S_{N,n} = 1 + \sum_{c=1}^{N} \binom{m}{c} U_c^{(m,n)} = 1 + \sum_{c=1}^{N} \frac{(\sqrt{n}m)^c}{\binom{n}{c} c!} \frac{W_c(n)}{(\sqrt{n}m)^c}.$$

Observe that by the second assertion of Proposition 1 we have that

$$S_{N,n} \xrightarrow{d} G_N = \sum_{c=0}^{N} \frac{(\lambda\rho\gamma^2)^{c/2}}{c!} H_c(\mathcal{N}),$$

as $n \to \infty$, since for any $c = 1, 2, \ldots,$

$$\frac{(\sqrt{n}m)^c}{\binom{n}{c} c!} \to \sqrt{\lambda}.$$

Let us define also

$$T_{N,n} = \sum_{c=N+1}^{m_n} \binom{m}{c} U_c^{(m,n)}.$$

Since $U_c^{(m,n)}$ are orthogonal, then by (2)

$$Var(T_{N,n}) = \sum_{c=N+1}^{m_n} \frac{\binom{m}{c}^2}{\binom{n}{c} c!} \gamma^{2c} e \leqslant 2\lambda e \sum_{c=N+1}^{\infty} \frac{\gamma^{2c}}{c!} = a_N$$

and $a_N \to 0$ as $N \to \infty$.

The final part of the proof follows now exactly along the lines of the proof of Theorem 2 with γ^2 replaced by $\rho\gamma^2$.

For the case $\lambda = 0$, let us write

$$\frac{\sqrt{n}}{m}\left(\frac{\operatorname{Per}(\mathbf{X})}{\binom{n}{m} m! \mu^m} - 1\right) = \frac{\sqrt{n}}{m}\binom{m}{1} U_1^{(m,n)} + R_{m,n} = \frac{W_1(n)}{\sqrt{n}m} + R_{m,n},$$

where

$$R_{m,n} = \sqrt{\frac{n}{m}} \sum_{c=2}^{m} \binom{m}{c} U_c^{(m,n)}.$$

Observe that $R_{m,n}$ converges in probability to zero, since by (2) it follows that

$$
\begin{aligned}
Var\, R_{m,n} &= \frac{n}{m^2} \sum_{c=2}^{m} \frac{\binom{m}{c}\gamma^{2c}}{\binom{n}{c}} \sum_{r=0}^{c} \frac{1}{r!}\binom{m-r}{c-r}(1-\rho)^r \rho^{c-r} \\
&\leqslant \exp(1)\,\frac{n}{m^2} \sum_{c=2}^{m} \left(\frac{m^2}{n}\right)^{c} \frac{\gamma^{2c}}{c!} \\
&\leqslant \exp(1)\,\frac{m^2}{n} \sum_{c=2}^{m} \left(\frac{m^2}{n}\right)^{c-2} \frac{\gamma^{2c}}{c!} \leqslant \exp(1)\,\frac{m^2}{n} \sum_{c=2}^{m} \frac{\gamma^{2c}}{c!} \leqslant \frac{m^2}{n}\exp(1+\gamma^2)
\end{aligned}
$$

if only n is large enough to have $m^2/n < 1$. Hence, $Var\, R_{m,n} \to 0$ as $m^2/n \to 0$ and consequently $R_{m,n}$ converges in probability to zero. Thus, the final result follows by the second part of Proposition 1 if $m = m_n \to \infty$ and by Remark 1 if m is a constant. $\square$

Remark 2. For $c = 1$, the conclusions of Proposition 1 remain true also for a constant m. It follows from two observations (see the proof above): (i) the Lindeberg condition is then trivially satisfied; (ii) the remainder term R_n vanishes, i.e. properly normalized $W_1(n)$ simply equals $V_1(n)$. Consequently, the reasoning similar to the above one above gives then

$$
\frac{\sqrt{n}}{m}\left(\frac{\mathrm{Per}\,\mathbf{X}}{\binom{n}{m} m!\mu^m} - 1\right) \xrightarrow{d} \tau\mathcal{N},
$$

where $\tau^2 = (\rho + (1-\rho)/m)\gamma^2$ in the case $\rho > 0$. For $\rho = 0$ we have

$$
\sqrt{\frac{n}{m}}\left(\frac{\mathrm{Per}\,\mathbf{X}}{\binom{n}{m} m!\mu^m} - 1\right) \xrightarrow{d} \gamma\mathcal{N}.
$$

Acknowledgments

A significant part of this work was completed when the first author was visiting the Center for Mathematical Sciences at the University of Wisconsin-Madison. He would like to express his gratitude to the Center and its Acting Director Prof. Thomas G. Kurtz for their hospitality.

Appendix. WLLN for triangular arrays

Let us denote by $\{Y_{lk}\}$ for $l \geqslant 1$, $k = 1, \ldots, l$, an arbitrary triangular array of real rv's and consider the following conditions:

(A1) $\sup_l \frac{1}{l}\sum_{k=1}^{l} E|Y_{lk}| = \beta < \infty$.

(A2) $\frac{1}{l}\sum_{k=1}^{l} E|Y_{lk}| I\{|Y_{lk}| > c_l\} \to 0$ as $n \to \infty$, where the sequence c_l satisfies $c_l \to \infty$ and $c_l/l \to 0$.

Proposition A.1. *Let* $\{Y_{lk}\}$ *for* $l \geqslant 1$, $k = 1, \ldots, l$, *be a triangular array of rowwise independent*, *integrable rv's. If conditions* (A1) and (A2) *are satisfied then*

(i) $\frac{1}{l}\sum_{k=1}^{l}(Y_{lk} - EY_{lk}) \xrightarrow{P} 0$,

(ii) $\frac{1}{l^{\alpha}}\sum_{k=1}^{l}|Y_{lk}|^{\alpha} \xrightarrow{P} 0$ *for any* $\alpha > 1$.

Proof. For the proof of (i), let us consider the expression

$$\begin{aligned}\frac{1}{l}\sum_{k=1}^{l}(Y_{lk} - EY_{lk}) &= \frac{1}{l}\sum_{k=1}^{l}(Y_{lk}I\{|Y_{lk}| \leqslant c_l\} - EY_{lk}I\{|Y_{lk}| \leqslant c_l\}) \\ &\quad + \frac{1}{l}\sum_{k=1}^{l}(Y_{lk}I\{|Y_{lk}| > c_l\} - EY_{lk}I\{|Y_{lk}| > c_l\}) \\ &= (\mathrm{I}) + (\mathrm{II}). \end{aligned} \tag{A.1}$$

But in view of (A1) and (A2), expressions (I) and (II) both converge to zero in probability. Indeed, apropos (I), for any $\varepsilon > 0$,

$$\begin{aligned} &P\left(\left|\frac{1}{l}\sum_{k=1}^{l}(Y_{lk}I\{|Y_{lk}| \leqslant c_l\} - EY_{lk}I\{|Y_{lk}| \leqslant c_l\})\right| > \varepsilon\right) \\ &\quad \leqslant \frac{1}{\varepsilon^2 l^2}\sum_{k=1}^{l}EY_{lk}^2 I\{|Y_{lk}| \leqslant c_l\} \leqslant \frac{c_l}{\varepsilon^2 l}\beta \to 0 \end{aligned}$$

as $l \to \infty$ in view of (A1) and $c_l/l \to 0$.

Similarly, apropos (II), for any $\varepsilon > 0$ we have

$$\begin{aligned} &P\left(\left|\frac{1}{l}\sum_{k=1}^{l}(Y_{lk}I\{|Y_{lk}| > c_l\} - EY_{lk}I\{|Y_{lk}| > c_l\})\right| > \varepsilon\right) \\ &\quad \leqslant \frac{2}{l\varepsilon}\sum_{k=1}^{l}E|Y_{lk}|I\{|Y_{lk}| > c_l\} \to 0 \end{aligned}$$

as $l \to \infty$, in view of assumption (A2).

For the proof of (ii), let us consider a decomposition similar to (A.1)

$$\begin{aligned} &\frac{1}{l^{\alpha}}\sum_{k=1}^{l}|Y_{lk}|^{\alpha} \\ &\quad = \frac{1}{l^{\alpha}}\sum_{k=1}^{l}|Y_{lk}|^{\alpha}I\{|Y_{lk}| \leqslant c_l\} + \frac{1}{l^{\alpha}}\sum_{k=1}^{l}|Y_{lk}|^{\alpha}I\{|Y_{lk}| > c_l\} = (\mathrm{III}) + (\mathrm{IV}). \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Expression (III) converges to zero in probability in view of the inequality

$$P\left(\left|\frac{1}{l^{\alpha}}\sum_{k=1}^{l}|Y_{lk}|^{\alpha}I\{|Y_{lk}| \leqslant c_l\}\right| > \varepsilon\right) \leqslant \left(\frac{c_l}{l}\right)^{\alpha-1}\frac{\beta}{\varepsilon}$$

and expression (IV) converges to zero in probability in view of

$$P\left(\left|\frac{1}{l^\alpha}\sum_{k=1}^{l}|Y_{lk}|^\alpha I\{|Y_{lk}|>c_l\}\right|>\varepsilon\right)\leqslant P\left(\sum_{k=1}^{l}\frac{|Y_{lk}|}{l}I\{|Y_{lk}|>c_l\}>\varepsilon^{1/\alpha}\right)$$
$$\leqslant\frac{1}{\varepsilon^{1/\alpha}l}\sum_{k=1}^{l}E|Y_{lk}|I\{|Y_{lk}|>c_l\}$$

and condition (A2). □

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