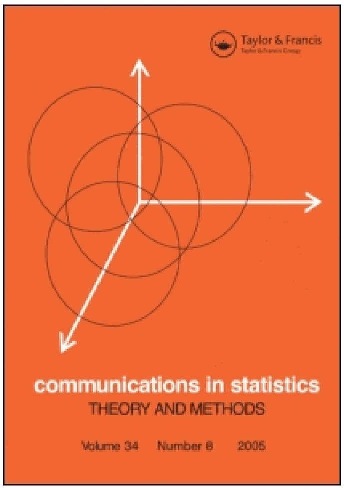


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# Martingales Defined by Reciprocals of Sums and Related Characterizations

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# Martingales Defined by Reciprocals of Sums and Related Characterizations

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## ABSTRACT

We prove that linearly transformed inverses of cumulative sums form backward martingales for gamma, inverse Gaussian, and Kendall and Borel–Tanner sequences of independent, identically distributed random variables. Conversely, a characterization of the family of these four distributions by linearity of regression of inverses of sums is obtained. The results in both directions are derived via the technique of variance functions of natural exponential families.

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## 1. INTRODUCTION

Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed (iid) random variables (rv's) and let $S_n = X_1 + \cdots + X_n$, $\mathscr{F}_n = \sigma(S_n, S_{n+1}, \ldots)$, $n = 1, 2, \ldots$. Consider a sequence of measurable functions $(f_n)_{n\geq 1}$ such that $E|f_n(S_n)| < \infty$ with the property that $\mathbf{F} = (f_n(S_n), \mathscr{F}_n)$ is a backward martingale. It is rather obvious that for $f_n(x) = \alpha_n x$, where $\alpha_n$ is a real number, $n = 1, 2, \ldots$, the sequence $\mathbf{F}$ is a backward martingale only if $E|X_1| < \infty$ and then necessarily, $\alpha_n = 1/n$, $n = 1, 2, \ldots$, and no additional assumptions on the distribution of $X_1$ is necessary. However, for other sequences of functions $(f_n)_{n\geq 1}$, only for special families of distributions $\mathbf{F}$ has the martingality property. For instance, if $f_n(x) = \alpha_n x^2$, then $\mathbf{F}$ is a backward martingale if and only if the observations come from the gamma distribution. The same holds true for $f_n(x) = \alpha_n x^r$, where $r$ is a positive number (for details see Hall and Simons, 1969). On the other hand, taking $f_n(x) = \alpha_n x^2 + \beta_n x + \gamma_n$ leads to the Morris (1982) class of six natural exponential families (nef's) with quadratic variance functions (normal, Poisson, binomial, negative binomial, gamma, and hyperbolic cosine); this path can be traced back to Laha and Lukacs (1960) quadratic regression characterizations. The case of $f(x) = \alpha_n x^3 + \beta_n x^2 + \gamma_n x + \delta$ may be resolved using the regression characterization of Fosam and Shanbhag (1997) which extends the Laha and Lukacs (1960) result to cubic regressions. Then the family of 12 distributions is characterized. This family was described earlier in Letac and Mora (1990) (abbreviated to LM in the sequel) as the family of all nef's with the cubic variance function. In that paper, a full description of possible cubic variance functions is given. Since the variance function uniquely determines the distribution, the LM classification is a useful tool for identifying distributions which will be used later on in the present article.

Here we are interested in the case of $f_n(x) = \alpha_n/x + \beta_n$. It will appear that except for the known case of the gamma distribution ($\beta_n \equiv 0$), the family also includes the inverse Gaussian law ($\alpha_n \equiv 2$), which, in view of the Seshadri (1983) observation, was expected, and, rather surprisingly, two other distributions are also included: The Kendall distribution introduced in the theory of dams in Kendall (1957), studied also in Khan and Jain (1978), and Jain and Khan (1979) as a kind of generalized gamma distribution. Then it was rediscovered (and named the Kendall–Ressel distribution) as one of the cubic variance nef's in LM. The Borel–Tanner distribution related to queuing theory, introduced by Borel (1942) and Tanner (1953), and was also, studied for instance, in Jain (1974) and identified by LM as another cubic variance nef (named there the Abel distribution). Here we chose to refer to the distributions as: Kendall rather than the Kendall–Ressel distribution and the Borel–Tanner rather than the Abel distribution, i.e., different than those used in LM, in an attempt to follow the tradition established in the statistical literature. This family of four distributions, being a subclass of nef's with the cubic variance functions, was characterized by the property that the reciprocal moments are affine functions of reciprocals of the mean in Letac and Seshadri (1989).

The simplest formulation of the characterization problem, in which we are also interested in here, lies in identifying all distributions for which

$$E\left(\frac{1}{X}\middle|X+Y\right)=\frac{a}{X+Y}+b \tag{1.1}$$

for some real numbers $a$ and $b$, where $X$ and $Y$ are iid positive non degenerate rv's. Observe that (1.1) equivalently can be written as

$$E\left(\frac{Y}{X}\middle|X+Y\right)=a-1+b(X+Y). \tag{1.2}$$

The case $b=0$ leads to the gamma distribution and was taken care of in Wesołowski (1990), while for $a=2$, we get the inverse Gaussian distribution according to the paper Seshadri (1983). It appears that for all other possible choices for $a$ and $b$ the condition (1.1) characterizes the Kendall or Borel–Tanner distributions. In a proof of this fact, see Sec. 2, in which we will use some facts from the theory of the variance functions of nef's (for its basics see, for instance, LM). Unexpectedly, the direct result, i.e., proving that (1.1) holds for iid rv's with the Kendall or Borel–Tanner distribution, appears to be somewhat more difficult that the converse problem. This is due to difficulties with finding analytic forms of some integrals related to the transformed Kendall density and sums related to the Borel–Tanner probability mass function (pmf).

It may be worth mentioning that condition (1.2) with $a=1$ has been recently considered in Wesołowski (2002) and earlier in Seshadri and Wesołowski (2001) in the context of a so-called Matsumoto–Yor property (see Matsumoto and Yor, 2001 or Letac and Wesołowski, 2000). It appears that for independent $X$ and $Y$ the property (1.2) together with another regression condition

$$E\left(\frac{X}{Y}\middle| X+Y\right) = \frac{c}{X+Y},$$

implies that the random variables $X$ and $Y$ have, respectively, generalized inverse Gaussian and gamma distributions.

The characterizations based on (1.1) are treated in Sec. 2. In Sec. 3, the results in both directions are extended to cumulative sums of iid rv's leading to backward martingale properties for the sequences of observations from the four distributions considered.

## 2. REGRESSION CHARACTERIZATIONS

Denote by $\gamma_{\lambda,p}$ the gamma distribution defined by the density:

$$\gamma_{\lambda,p}(dx) = \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x} I_{(0,\infty)}(dx),$$

where $\lambda$ and $p$ are positive constants. By $\mu_{\lambda,\sigma}$, denote the inverse Gaussian distribution with the density

$$\mu_{\lambda,\sigma}(dx) = e^{2\sqrt{\lambda\sigma}} \sqrt{\frac{\sigma}{2\pi}} x^{-3/2} e^{-\lambda x - \sigma/x} I_{(0,\infty)}(dx),$$

where $\lambda$ and $\sigma$ are positive constants. Define also the Kendall distribution $\kappa_{\lambda,\theta,r}$ by the density

$$\kappa_{\lambda,\theta,r}(dx) = \frac{\lambda \theta^{\lambda+rx} x^{\lambda+rx-1} e^{-\theta x}}{\Gamma(\lambda+rx+1)} I_{(0,\infty)}(dx),$$

which $\lambda$, $\theta$, and $r$ are positive numbers such that $\theta > r$. Finally, define the Borel–Tanner distribution $BT_p$ by the pmf

$$BT_p(k) = \frac{p(p+k)^{k-1}}{k!} e^{-(p+k)}, \quad k = 0, 1, \ldots$$

It appears that the above class of four distributions is completely characterized by the condition (1.1) with different possible choices for $a$ and $b$.

**Theorem 2.1.** *Let $X$ and $Y$ be iid non degenerate positive rvs, such that $E(X^{-1}) < \infty$. Then the condition* (1.1) *holds for some real numbers $a$ and $b$ iff one of the following cases occurs*:

(i) $a = 2$, $b > 0$, *and* $X \sim \mu_{\lambda,\sigma}$, *where* $\lambda > 0$ *and* $\sigma = \frac{1}{4b}$.
(ii) $a > 2$, $b = 0$, *and* $X \sim \gamma_{\lambda,p}$, *where* $p = \frac{a-1}{a-2} > 1$ *and* $\lambda > 0$.
(iii) $a > 2$, $b > 0$, *and* $X \sim \kappa_{\lambda,\theta,r}$, *where* $\lambda = \frac{a-1}{a-2} > 1$ *and* $r = \frac{2(a-1)b}{(a-2)^2}$.
(iv) $1 < a < 2$, $b > 0$, *and* $X = c(1 + Z/p)$, *where* $Z \sim BT_p$, $p = \frac{a-1}{2-a}$ *and* $c = \frac{2-a}{2b}$.

*Proof.* First assume that there exist probability distributions, i.e., also rv's $X$ and $Y$, such that (1.1) holds for some constants $a$ and $b$. Then, (1.1) is equivalent to

$$hh'' = (a-1)(h')^2 + 2bh'h'', \tag{2.1}$$

where $h(s) = E(X^{-1}\exp(sX))$, $s < s_0 \geq 0$, which implies that it is impossible to have $a < 1$ and $b \leq 0$ together. Now take derivatives of both sides of (2.1) to get

$$h'h'' + hh''' = 2(a-1)h'h'' + 2b(h'')^2 + 2bh'h'''.$$

Eliminating $h$ from the above equations, we get

$$(a-1)L^2L'' = (2a-3)LL'^2 + 2bL'^3,$$

where $L = h'$ is the Laplace transform of $X$. Hence,

$$(a-1)\left[\frac{L''}{L} - \left(\frac{L'}{L}\right)^2\right] = (a-2)\left(\frac{L'}{L}\right)^2 + 2b\left(\frac{L'}{L}\right)^3,$$

with $a = 1$ implies that the distribution of $X$ is degenerate. Thus, $a \neq 1$. Consider now the nef defined by the distribution of $X$. Then the above equation is equivalent to

$$(a-1)V(m) = m^2(a-2+2bm), \quad 0 \leq \alpha < m < \beta \leq \infty, \tag{2.2}$$

where $V$ is the variance function.

$$V(m) = \frac{a-2}{a-1} m^2 \left(1 + \frac{2b}{a-2} m\right), \quad \alpha < m < \beta. \tag{2.3}$$

Recall that earlier we observed that if $a < 1$ then $b \leq 0$ and $a = 1$ are impossible as well. Now by the uniqueness property of the variance function and the classification of nef's with cubic variance functions given in LM, it follows from (2.2) or (2.3) that $\alpha \geq 0$, $\beta = \infty$ and only the following cases are possible:

(i) $\alpha = 0$, $a = 2$, $b > 0$ and then the rv $X$ has an inverse Gaussian distribution.
(ii) $\alpha = 0$, $b = 0$, $(a-2)(a-1) > 0$ and then the rv $X$ has a gamma distribution.
(iii) $\alpha = 0$, $(a-1)(a-2) > 0$, $b(a-2) > 0$ and then the rv $X$ has a Kendall distribution.
(iv) $\alpha = c > 0$, $1 < a < 2$, $b > 0$ and then the rv $X$ has a linearly transformed Borel–Tanner distribution (see Prop. 2.4 (v) in LM).

Here, as emphasized in Sec. 1, due to historical reasons we use names for the last two distributions different from those used in LM.

Now, in order to find out the exact values of the constants $a$ and $b$ for each of the four distributions appearing in the formulation of the theorem, differentiate (2.2) with respect to $m$. Then

$$(a-1)V'(m) = 2(a-2)m + 6bm^2. \tag{2.4}$$

Combining (2.2) and (2.4), we find that for $m > 0$ (observe that $2m^2 + mV'(m) - 3V(m) > 0$),

$$a = \frac{2m^2 + mV'(m) - 3V(m)}{m^2 + mV'(m) - 3V(m)}$$

and

$$b = \frac{mV'(m) - 2V(m)}{2m[m^2 + mV'(m) - 3V(m)]}.$$

Recall, following Morris (1982), that for the gamma law $\gamma_{\lambda,p}$ the variance function has the form: $V(m) = m^2/p$. Then it follows that $a = \frac{2p-1}{p-1}$ and $b = 0$. Observe that $p > 1$ since it is assumed that the first inverse moment is finite and thus $a > 2$.

For the inverse Gaussian distribution $\mu_{\lambda,\sigma}$ we have $V(m) = m^3/\sigma$, which yields $a = 2$ and $b = 1/(4\sigma)$.

For the Kendall distribution $\kappa_{\lambda,\theta,r}$ we have $V(m) = \frac{m^2}{\lambda}(1 + \frac{r}{\lambda}m)$ and since $E(X^{-1})$ is finite, then $\lambda > 1$ (see Letac and Seshadri, 1989). Thus, $a = \frac{2\lambda-1}{\lambda-1} > 2$ and $b = \frac{r}{2\lambda(\lambda-1)}$.

For the random variable $Z$ with the Borel–Tanner distribution $BT_p$, we have $V_Z(\tilde{m}) = \tilde{m}\big(1 + \frac{\tilde{m}}{p}\big)$. If $X = c(1 + Z/p)$, $c > 0$, then $m = c\big(1 + \frac{\tilde{m}}{p}\big)$, where $m = E(X) > c$ and

$$V(m) = \left(\frac{c}{p}\right)^2 V_Z(\tilde{m}(m)) = \left(\frac{c}{p}\right)^2 \tilde{m}(m)\left(1 + \frac{\tilde{m}(m)}{p}\right)^2 = \frac{m^2}{c}\left(\frac{m}{c} - 1\right).$$

Thus $a = \frac{2p+1}{p+1}$ and $b = \frac{1}{2c(p+1)}$.

Now to complete the proof we have to show that there exist rv's $X$ and $Y$ for which (1.1) is satisfied. To this end let us define a function $v : (0, 1/(2b)) \to (0, \infty)$ by

$$v(x) = \frac{(a-2)x^2 + 2bx^3}{1 - 2bx}.$$

Observe that $v(x) = \sum_{k=2}^{\infty} c_k x^k$, where $c_2 = a - 1$ and $c_k = a(2b)^{k-2}$, $k = 3, 4, \ldots$. Consequently, for $a > 1$ the function $v$ is a variance function of some nef generated, say, by a positive measure $\nu$ (See Corollary 3.3 in LM). Hence, its cummulant function $k_\nu$ satisfies

$$k''_\nu = (a-2)k'^2_\nu + 2bk'^3_\nu + 2bk'_\nu k''_\nu.$$

Furthermore, its Laplace transform $L_\nu$ satisfies

$$L_\nu L''_\nu = (a-1)L'^2_\nu + 2bL_\nu L''_\nu.$$

Hence if we define a probabilistic measure $P_{\nu,\theta}$ by taking $P_{\nu,\theta}(dx) = e^{\theta x - k_\nu(\theta)}\nu(dx)$ for some $\theta \in \Theta$ and by $L$, we denote its Laplace transform, then since $L(s) = L_\nu(s+\theta)/L_\nu(\theta)$, it follows that the above equation also is satisfied for $L$. Now define a new probabilistic measure by taking $(x/c)P_{\nu,\theta}(dx) = \mu(dx)$, where $c$ is a normalizing constant. Then it follows that (1.1) holds for iid rv's $X$ and $Y$ with the common distribution $\mu$. $\square$

**Remark 2.1.** Another possibility in approaching the problem, which we solved above, may be based on the description of all nef's for which the expectation of reciprocal as the function of mean is affine in the reciprocal of the mean. It appears that the set of such nef's consists of the same four distributions we characterized in Theorem 1 (see Theorem 3.1 in Letac and Seshadri, 1989).

**Remark 2.2.** Observe that the direct computation of the conditional expectation of $X^{-1}$, given $X+Y$ for the Kendall distribution, is rather difficult. Due to the fact that for two independent Kendall random variables $U$ and $V$ with distributions $\kappa_{\lambda_1,\theta,r}$ and $\kappa_{\lambda_2,\theta,r}$, the distribution of the sum $U+V$ is also Kendall $\kappa_{\lambda_1+\lambda_2,\theta,r}$ (see, for instance, Jain and Khan, 1979) one needs to find out the value of the integral involving the conditional densities which has the form

$$\frac{\lambda}{2}\int_0^z \frac{x^{\lambda+rx-2}(z-x)^{\lambda+r(z-x)-1}}{z^{2\lambda+rz-1}} \frac{\Gamma(2\lambda+rz+1)}{\Gamma(\lambda+rx+1)\Gamma(\lambda+r(z-x)+1)}\,dx$$
$$=\frac{\lambda\Gamma(2\lambda+rz+1)}{2z}\int_0^1 \frac{t^{\lambda+rzt-2}(1-t)^{\lambda+rz(1-t)-1}}{\Gamma(\lambda+rzt+1)\Gamma(\lambda+rz(1-t)+1)}\,dt.$$

From (iii) of Theorem 2.1 it follows that the integral equals

$$\frac{1}{\lambda-1}\left(\frac{2\lambda-1}{z}+\frac{r}{2\lambda}\right).$$

**Remark 2.3.** Similarly, the direct computation of the conditional expectation of $X^{-1}$, given $X+Y$ for the Borel–Tanner distribution, is not simple. The property that for two Borel–Tanner independent random variables $U$ and $V$ with distributions $BT_{p_1}$ and $BT_{p_2}$, respectively, the distribution of the sum $U+V$ is Borel–Tanner $BT_{p_1+p_2}$ (see, for instance, Jain and Khan, 1979) is helpful and leads finally to the following sum

$$(2p+l)^{-(l-1)}\sum_{i=0}^{l}\binom{l}{i}(p+i)^{i-2}(p+l-i)^{l-i-1},$$

the computation of which seems to be rather involved. However, due to (iv) of Theorem 2.1 it follows that the above expression equals

$$\frac{1}{p+1}\left(\frac{2p+1}{2p+l}+\frac{1}{2p}\right).$$

## 3. MARTINGALE PROPERTIES

Consider now a sequence $(X_n)_{n\geq 1}$ of iid positive non degenerate rv's. Let $S_k = X_1 + \cdots + X_k$ for any $k = 1, 2, \ldots$. Assume that $E(X_1^{-1}) < \infty$. Then, obviously, $E(S_k^{-1}) < \infty$ for any $k = 1, 2, \ldots$.

In this section we are interested in the condition

$$E\left(\frac{1}{S_k}\middle| S_n\right) = \frac{a}{S_n} + b \tag{3.1}$$

for $1 \leq k \leq n$ and some constants $a$ and $b$ (possibly depending on $k$ and $n$). The situation is slightly different from the previous section since the random variables $S_k$ and $S_n - S_k$, though independent, are not identically distributed in general. However, we will proceed in a similar manner as in Sec. 2.

Here, the main issue is again the question if such sequences exist. If the existence is assumed, then, similarly as in the previous section, we have equivalence of (3.1) and

$$\begin{aligned} &E\left(\frac{1}{S_k}e^{tS_k}\right)E\Big((S_n - S_k)e^{t(S_n - S_k)}\Big) \\ &\quad = (a-1)E\big(e^{tS_n}\big) + bE\big(S_n e^{tS_n}\big), \quad t \leq 0. \end{aligned} \tag{3.2}$$

But (3.2) can be rewritten as

$$(n-k)h_k h_1'' = (a-1)(h_1')^{k+1} + nb(h_1')^k h_1'',$$

where $h_i(t) = E(S_i^{-1}\exp(tS_i))$ for $i = 1, k$. Since $h_k' = (h_1')^k$, repeating the argument from Sec. 2, we get

$$V(m) = \frac{m^2}{a-1}(ak - n + knbm), \quad 0 \leq \alpha < m < \beta \leq \infty,$$

as the variance function of the nef generated by the distribution of $X_1$.

Consequently, again using the LM classification of nef's with cubic variance functions, we conclude that only the four cases—gamma, inverse Gaussian, Kendall, and Borel–Tanner—are allowed. Differentiating the above formula for $V(m)$, we obtain the expressions for $a$ and $b$ (note that $km^2 + mV'(m) - 3V(m) > 0$):

$$a = \frac{nm^2 + mV'(m) - 3V(m)}{km^2 + mV'(m) - 3V(m)}, \quad b = \frac{(n-k)[mV'(m) - 2V(m)]}{knm[km^2 + mV'(m) - 3V(m)]}.$$

Hence we get:

(1) $X_1 \sim \gamma_{\lambda,p}$ $(p > 1)$ and then

$$a = \frac{np-1}{kp-1} > \frac{n}{k}, \quad b = 0.$$

(2) $X_1 \sim \mu_{\lambda,\sigma}$ and then

$$a = \frac{n}{k}, \quad b = \frac{n-k}{k^2 n 2\sigma} > 0.$$

(3) $X_1 \sim \kappa_{\lambda,\theta,r}$ $(\lambda > 1)$ and then

$$a = \frac{n\lambda - 1}{k\lambda - 1} > \frac{n}{k}, \quad b = \frac{(n-k)r}{kn\lambda(k\lambda - 1)} > 0.$$

(4) $X_1 \stackrel{d}{=} c(1 + Z/p)$, where $Z \sim BT_p$ and then

$$a = \frac{np+1}{kp+1} \in \left(1, \frac{n}{k}\right), \quad b = \frac{n-k}{knc(kp+1)} > 0.$$

Now we will resolve the question of existence of a sequence $(X_n)_{n\geq 1}$ satisfying (3.1). Observe that we cannot just apply the argument given in Sec. 2 since the rv's $S_k$ and $S_n - S_k$, in general, are not identically distributed. Regardless, we will try to modify the previously developed reasoning.

First consider a function $v$ on $(0, (n-k)/(nb))$ $(b > 0)$ defined by

$$v(x) = \frac{x^2(ak - n + nbx)}{n - k - nbx}.$$

Again using Corollary 3.3 from LM, we conclude that for $a > 1$ it is a variance function of a nef. Consequently, there exists a positive rv, say $V$, with infinitely divisible distribution, such that

$$(n-k)gg'' = k(a-1)(g')^2 + nbg'g'', \tag{3.3}$$

where $g(t) = E(V^{-1}\exp(tV))$. Thus $V \stackrel{d}{=} X_1 + \cdots + X_k = S_k$ for some positive iid rv's $X_1, \ldots, X_k$. Therefore, (3.3) can be written as

$$(n-k)h_k(h_1')^{k-1}h_1'' = (a-1)(h_1')^{2k} + knb(h_1')^{2k-1}h_1'',$$

which implies (3.2) and thus (3.1) is also satisfied.

Consequently, we are led to the martingale properties of four random sequences considered above (which, since any affine transformation of a martingale is a martingale again, are defined up to affinity).

**Theorem 3.1.** *For a sequence $(X_n)_{n\geq 1}$ of positive, non degenerate iid rv's define the sequence of tail σ-algebras by taking $\mathscr{F}_n = \sigma(S_n, S_{n+1}, \ldots)$. Then the sequence*

$$\left(\frac{\alpha_n}{S_n} - \beta_n, \mathscr{F}_n\right)_{n\geq 1}$$

*is a backward martingale only in the following four cases*:

(i) $X_1 \sim \gamma_{\lambda,p},\ \alpha_n = np - 1,\ \beta_n = 0,\ p > 1.$
(ii) $X_1 \sim \mu_{\lambda,\sigma},\ \alpha_n = n,\ \beta_n = \frac{1}{n2\sigma}.$
(iii) $X_1 \sim \kappa_{\lambda,\theta,r},\ \alpha_n = n\lambda - 1,\ \beta_n = \frac{r}{n\lambda},\ \lambda > 1.$
(iv) $X_1 \stackrel{d}{=} c(1 + Z/p)$, *where* $Z \sim BT_p,\ \alpha_n = np + 1,\ \beta_n = \frac{1}{cn},\ c > 0,\ p > 0.$

*Proof.* Observe that if $(X, Y, Z)$ is a random vector such that $(X, Y)$ and $Z$ are independent and $E|X| < \infty$, then $E(X|Y, Z) = E(X|Y)$ a.s. Hence, for any $1 \leq k \leq n$ and any measurable function $f$ such that $E|f(S_k)| < \infty$,

$$E(f(S_k)|\mathscr{F}_n) = E(f(S_k)|S_n, X_{n+1}, X_{n+2}, \ldots) = E(f(S_k)|S_n).$$

In view of (3.1) and 1–4 above, it proves the result. □

**Remark 3.1.** Note that in the two cases of the gamma and inverse Gaussian distributions the property (3.1) can be derived directly.

*The case of the gamma distribution.* Consider a rv $X$ with the gamma $\gamma_{\lambda,p}$ distribution, and assume that $p > 1$. Then

$$E\left(\frac{1}{X}e^{sX}\right) = \frac{\lambda^p}{(p-1)(\lambda - s)^{p-1}}, \quad s < \lambda.$$

Let $Y$ be independent of $X$ with the distribution $\gamma_{\lambda,q}$. Then it follows that

$$E\left(\frac{1}{X}e^{sX}\right)E\left(e^{sY}\right) = \frac{p+q-1}{p-1}E\left(\frac{1}{X+Y}e^{s(X+Y)}\right), \quad s < \lambda,$$

which is equivalent to

$$E\left(\frac{p-1}{X}\middle|X+Y\right)=\frac{p+q-1}{X+Y}\quad \text{a.s.}$$

For a sequence $(X_n)_{n\geq 1}$ of iid gamma $\gamma_{\lambda,p}$ $(p>1)$ rv's, define $S_k=X_1+\cdots+X_k$, $k=1,2,\ldots$. Then the above observations imply

$$E\left(\frac{kp-1}{S_k}\middle|S_n\right)=\frac{np-1}{S_n}$$

for any $1\leq k\leq n$.

*The case of the inverse Gaussian distribution.* Consider a rv $X$ with the inverse Gaussian $\mu_{\lambda,\sigma}$ distribution. Then

$$E\left(\frac{1}{X}e^{sX}\right)=\left(\sqrt{\frac{\lambda-s}{\sigma}}-\frac{1}{2\sigma}\right)e^{-2\sqrt{\sigma}(\sqrt{\lambda}-\sqrt{\lambda-s})},\quad s<\lambda.$$

Let $Y$ be independent of $X$ with the distribution $\mu_{\lambda,\tau}$. Then it follows that

$$\begin{aligned}E\left(\frac{1}{X}e^{sX}\right)E\left(e^{sY}\right)=&\frac{\sqrt{\sigma}+\sqrt{\tau}}{\sqrt{\sigma}}E\left(\frac{1}{X+Y}e^{s(X+Y)}\right)\\&+\frac{\sqrt{\tau}}{2\sigma(\sqrt{\sigma}+\sqrt{\tau})}E\left(e^{s(X+Y)}\right),\quad s<\lambda,\end{aligned}$$

which is equivalent to

$$E\left(\frac{\sqrt{\sigma}}{X}-\frac{1}{2\sqrt{\sigma}}\middle|X+Y\right)=\frac{\sqrt{\sigma}+\sqrt{\tau}}{X+Y}-\frac{1}{2(\sqrt{\sigma}+\sqrt{\tau})}\quad \text{a.s.}$$

For a sequence $(X_n)_{n\geq 1}$ of iid inverse Gaussian $\mu_{\lambda,\sigma}$ rv's, define $S_k=X_1+\cdots+X_k$, $k=1,2,\ldots$. Then the above observations imply

$$E\left(\frac{k}{S_k}-\frac{1}{2\sigma k}\middle|S_n\right)=\frac{n}{S_n}-\frac{1}{2\sigma n}$$

for any $1\leq k\leq n$.

**Remark 3.2.** To find the conditional expectation of $S_k^{-1}$ given $S_n$ for the Kendall distribution via the direct computation, one needs to find out the value of the integral involving the conditional densities of the form

$$\frac{k(n-k)\lambda}{n}\int_0^z \frac{x^{k\lambda+rx-2}(z-x)^{(n-k)\lambda+r(z-x)-1}\Gamma(n\lambda+rz+1)}{z^{n\lambda+rz-1}\Gamma(k\lambda+rx+1)\Gamma((n-k)\lambda+r(z-x)+1)}\,dx$$
$$=\frac{k(n-k)\lambda\Gamma(n\lambda+rz+1)}{nz}\int_0^1 \frac{t^{k\lambda+rzt-2}(1-t)^{(n-k)\lambda+rz(1-t)-1}}{\Gamma(\lambda+rzt+1)\Gamma(\lambda+rz(1-t)+1)}\,dt.$$

From the result of 3 in this section, it follows that this expression equals

$$\frac{1}{k\lambda-1}\left(\frac{n\lambda-1}{z}+\frac{(n-k)r}{kn\lambda}\right),$$

extending the formula given in Remark 2.2.

**Remark 3.3.** Similarly, for finding the conditional expectation of $S_k^{-1}$ given $S_n$ for the Borel–Tanner distribution directly, one has to compute the following sum

$$\frac{1}{(np+l)^{l-1}}\sum_{i=0}^{l}\binom{l}{i}\frac{(kp+i)^{i-1}}{p+i}[(n-k)p+l-i]^{l-i-1}.$$

From the result 4 in this section, we obtain the value of this expression as

$$\frac{1}{kp+1}\left(\frac{np+1}{np+l}+\frac{n-k}{knp}\right),$$

which is a straightforward extension of the formula obtained in Remark 2.3.

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