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Chapter 11

Bivariate Matsumoto–Yor Property and Related Characterizations

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ABSTRACT

In this paper, a bivariate version of the Matsumoto–Yor independence property for the generalized inverse Gaussian (GIG) and gamma distributions is considered. It appears that the property does not characterize the general families of bivariate gamma and GIG distributions but only special cases of random vectors with either independent or linearly dependent components.

KEYWORDS AND PHRASES: (Bivariate) gamma distribution, generalized inverse Gaussian distribution, Matsumoto–Yor property

11.1 INTRODUCTION

Considering functionals of the geometric Brownian motion, Matsumoto and Yor (2001) have recently observed that the map $\psi : (0,\infty)^2 \to (0,\infty)^2$, defined by $\psi(x,y) = ((x+y)^{-1}, x^{-1} - (x+y)^{-1})$, preserves a probability measure which is a product of the generalized inverse Gaussian (GIG) and the gamma distributions. Recall that the GIG distribution $\mu_{-p,a,b}$ is defined by

$$\mu_{-p,a,b}(dx) = K_1 x^{-p-1} \exp(-a^{-1}x - (bx)^{-1}) I_{(0,\infty)}(x)\, dx,$$

where $p \in \mathbf{R}$, $a, b \in (0,\infty)$, are the parameters. The gamma distribution $\gamma_{q,c}$ is defined by

$$\gamma_{q,c}(dy) = K_2 y^{q-1} \exp(-c^{-1}y) I_{(0,\infty)}(y)\, dy,$$

where $q, c \in (0,\infty)$ are parameters and K_1 and K_2 are normalizing constants. Matsumoto and Yor (2001) observed that if random variables X and Y are independent, X has the GIG distribution $\mu_{-p,a,a}$ $(p > 0)$, and Y has the gamma distribution $\gamma_{p,a}$, i.e., $(X,Y) \sim \mu_{-p,a,a} \otimes \gamma_{p,a}$, then the random vector

$$(U,V) = \psi(X,Y) = \left(\frac{1}{X+Y}, \frac{1}{X} - \frac{1}{X+Y}\right)$$

has the same distribution as (X,Y); hence, in particular, U and V are independent. As observed in Letac and Wesołowski (2000), the following extension of the Matsumoto–Yor property holds: if (X,Y) has the distribution $\mu_{-p,a,b} \otimes \gamma_{p,a}$, then (U,V) is distributed according to $\mu_{-p,b,a} \otimes \gamma_{p,b}$.

Matsumoto and Yor (2001) asked about a converse of their observation: Assume that X and Y are independent and that the random vector $(U,V) = \psi(X,Y)$ has independent components. Does (X,Y) have the distribution $\mu_{-p,a,b} \otimes \gamma_{p,a}$ (and consequently (U,V) is distributed according to $\mu_{-p,b,a} \otimes \gamma_{p,b}$)? This question has been answered in the affirmative by Letac and Wesołowski (2000) [a related problem involving constancy of regression of V or V^{-1} on U has been considered also in Seshadri and Wesołowski (2001) and solved finally in Wesołowski (2002)]. Also in that paper, the authors considered the Matsumoto–Yor property for distributions on the cone of positive definite symmetric matrices. The characterization given there was restricted to distributions having strictly positive, twice continuously differentiable densities. An extension assuming differentiable densities has been given in Wesołowski (2002). The problem for random matrices of different dimensions has been studied recently in Massam and Wesołowski (2004).

The Matsumoto–Yor property has never been treated up to now for random vectors. This paper is intended to partially fill this gap by considering the bivariate situation. It is interesting to note that the development of studies here is parallel to investigations concerning the Lukacs (1955) characterization of the gamma law: if X, Y are independent positive non-degenerate random variables and $X+Y$, $X/(X+Y)$ are also independent, then X, Y have gamma distributions. It was followed by the solution of the problem in the matrix variate case first—see Olkin and Rubin (1962), Casalis and Letac (1996), Letac and Massam (1998), and Bobecka and Wesołowski (2002). The case of random vectors was treated only recently, first, bivariate in Bobecka (2002), and then, n-variate in Bobecka and Wesołowski (2004).

11.2 CHARACTERIZATION

Below we present the characterization related to the Matsumoto–Yor property for bivariate random vectors. It appears that in this case the independence property (similarly as in the Lukacs characterization) imposes special structures of the bivariate gamma and GIG distributions. This is the main result of the paper presented in the theorem below.

THEOREM 11.2.1 *Let* $\bar{X} = (X_1, X_2)$ *and* $\bar{Y} = (Y_1, Y_2)$ *be independent random vectors with positive components. Assume that* $\bar{X}$ *or* $\bar{Y}$ *is not degenerate to the point. Let*

$$\bar{U} = (U_1, U_2) = \left(\frac{1}{X_1+Y_1}, \frac{1}{X_2+Y_2}\right)$$

and

$$\bar{V} = (V_1, V_2) = \left(\frac{1}{X_1} - \frac{1}{X_1+Y_1}, \frac{1}{X_2} - \frac{1}{X_2+Y_2}\right).$$

The random vectors $\bar{U}$ *and* $\bar{V}$ *are independent if and only if there exist positive constants* p_j, λ_j, κ_j, *such that* X_j *has a GIG distribution:* $\mu_{-p_j,\lambda_j,\kappa_j}$ *and* Y_j *has a gamma distribution:* γ_{p_j,λ_j}, $j = 1, 2$, *and either*

1. *the components of* $\bar{X}$ *and* $\bar{Y}$ *are independent*

or

2. *the components of* $\bar{X}$ *and* $\bar{Y}$ *are linearly dependent:* $X_1 = aX_2$, $Y_1 = aY_2$ *with* $a = \lambda_1/\lambda_2$, *and then* $p_1 = p_2$.

PROOF. Observe that if any one of $\bar{X}$ and $\bar{Y}$ is not degenerate to a point, then all four random vectors $\bar{X}$, $\bar{Y}$, $\bar{U}$, and $\bar{V}$ are not degenerate.

Necessity. The independence property and the identity

$$\frac{Y_j}{X_j} = \frac{V_j}{U_j}, \quad j = 1, 2,$$

imply

$$E\left(Y_1^{\alpha} Y_2^{\beta} e^{\sigma_1 Y_1 + \sigma_2 Y_2}\right) E\left(X_1^{-\alpha} X_2^{-\beta} A(\sigma_1, \sigma_2, \theta_1, \theta_2)\right)$$
$$= E\left(V_1^{\alpha} V_2^{\beta} e^{\theta_1 V_1 + \theta_2 V_2}\right) E\left(U_1^{-\alpha} U_2^{-\beta} B(\sigma_1, \sigma_2, \theta_1, \theta_2)\right), \tag{11.2.1}$$

where

$$A(\sigma_1, \sigma_2, \theta_1, \theta_2) = e^{\sigma_1 X_1 + \sigma_2 X_2 + \theta_1 X_1^{-1} + \theta_2 X_2^{-1}}$$

and

$$B(\sigma_1, \sigma_2, \theta_1, \theta_2) = e^{\sigma_1 U_1^{-1} + \sigma_2 U_2^{-1} + \theta_1 U_1 + \theta_2 U_2}$$

for any negative σ_1, σ_2, θ_1, θ_2 and fixed non-negative α and β.

Taking the logarithm of both sides of (11.2.1) and applying $\partial^2/\partial\sigma_1\partial\theta_1$, we obtain

$$\frac{E\left(X_1^{-\alpha+1} X_2^{-\beta} A(\sigma_1, \sigma_2, \theta_1, \theta_2)\right) E\left(X_1^{-\alpha-1} X_2^{-\beta} A(\sigma_1, \sigma_2, \theta_1, \theta_2)\right)}{\left[E\left(X_1^{-\alpha} X_2^{-\beta} A(\sigma_1, \sigma_2, \theta_1, \theta_2)\right)\right]^2}$$
$$= \frac{E\left(U_1^{-\alpha+1} U_2^{-\beta} B(\sigma_1, \sigma_2, \theta_1, \theta_2)\right) E\left(U_1^{-\alpha-1} U_2^{-\beta} B(\sigma_1, \sigma_2, \theta_1, \theta_2)\right)}{\left[E\left(U_1^{-\alpha} U_2^{-\beta} B(\sigma_1, \sigma_2, \theta_1, \theta_2)\right)\right]^2}. \tag{11.2.2}$$

Now applying (11.2.1) for α, $\alpha - 1$, and $\alpha + 1$ to (11.2.2), we arrive at

$$\frac{E\left(Y_1^{\alpha-1}Y_2^{\beta}e^{\sigma_1Y_1+\sigma_2Y_2}\right)E\left(Y_1^{\alpha+1}Y_2^{\beta}e^{\sigma_1Y_1+\sigma_2Y_2}\right)}{\left[E\left(Y_1^{\alpha}Y_2^{\beta}e^{\sigma_1Y_1+\sigma_2Y_2}\right)\right]^2} = \frac{E\left(V_1^{\alpha-1}V_2^{\beta}e^{\theta_1V_1+\theta_2V_2}\right)E\left(V_1^{\alpha+1}V_2^{\beta}e^{\theta_1V_1+\theta_2V_2}\right)}{\left[E\left(V_1^{\alpha}V_2^{\beta}e^{\theta_1V_1+\theta_2V_2}\right)\right]^2}. \tag{11.2.3}$$

Similarly, we obtain a dual relation

$$\frac{E\left(Y_1^{\alpha}Y_2^{\beta-1}e^{\sigma_1Y_1+\sigma_2Y_2}\right)E\left(Y_1^{\alpha}Y_2^{\beta+1}e^{\sigma_1Y_1+\sigma_2Y_2}\right)}{\left[E\left(Y_1^{\alpha}Y_2^{\beta}e^{\sigma_1Y_1+\sigma_2Y_2}\right)\right]^2} = \frac{E\left(V_1^{\alpha}V_2^{\beta-1}e^{\theta_1V_1+\theta_2V_2}\right)E\left(V_1^{\alpha}V_2^{\beta+1}e^{\theta_1V_1+\theta_2V_2}\right)}{\left[E\left(V_1^{\alpha}V_2^{\beta}e^{\theta_1V_1+\theta_2V_2}\right)\right]^2}. \tag{11.2.4}$$

Writing (11.2.3) for $\alpha = 1$, $\beta = 0$ and writing (11.2.4) for $\alpha = 0$, $\beta = 1$, we have

$$\frac{E\left(Y_j^2e^{\sigma_1Y_1+\sigma_2Y_2}\right)E\left(e^{\sigma_1Y_1+\sigma_2Y_2}\right)}{\left[E\left(Y_je^{\sigma_1Y_1+\sigma_2Y_2}\right)\right]^2} = \frac{E\left(V_j^2e^{\theta_1V_1+\theta_2V_2}\right)E(e^{\theta_1V_1+\theta_2V_2})}{\left[E\left(V_je^{\theta_1V_1+\theta_2V_2}\right)\right]^2} \tag{11.2.5}$$

for $j = 1, 2$. Then by the principle of separation of variables, (11.2.5) implies

$$\frac{\frac{\partial^2 f}{\partial\sigma_j^2}f}{\left(\frac{\partial f}{\partial\sigma_j}\right)^2} = c_j, \qquad \frac{\frac{\partial^2 g}{\partial\theta_j^2}g}{\left(\frac{\partial g}{\partial\theta_j}\right)^2} = c_j, \quad j = 1, 2, \tag{11.2.6}$$

where f and g are the Laplace transforms of $\bar{Y}$ and $\bar{V}$, respectively, and c_1, c_2 are some constants greater than one. Then as in Bobecka (2003), we conclude that only the following two cases are possible: either

1. $c_1 \neq c_2$ and then

$$f(\sigma_1, \sigma_2) = (1 - \lambda_1\sigma_1)^{-p_1}(1 - \lambda_2\sigma_2)^{-p_2},$$
$$(\sigma_1, \sigma_2) \in (-\infty, \lambda_1^{-1}) \times (-\infty, \lambda_2^{-1})$$

and

$$g(\theta_1, \theta_2) = (1 - \kappa_1\theta_1)^{-p_1}(1 - \kappa_2\sigma_2)^{-p_2},$$
$$(\theta_1, \theta_2) \in (-\infty, \kappa_1^{-1}) \times (-\infty, \kappa_2^{-1}),$$

where $p_j = 1/(c_j - 1) > 0$, and $\lambda_j > 0$, $\kappa_j > 0$, $j = 1, 2$, i.e., the random vectors $\bar{Y} = (Y_1, Y_2)$ and $\bar{V} = (V_1, V_2)$ have independent gamma components: $Y_j \sim \gamma_{p_j,\lambda_j^{-1}}$, $V_j \sim \gamma_{p_j,\kappa_j^{1}}$, $j = 1, 2$;

or

2. $c_1 = c_2 = c$ and then

$$f(\sigma_1, \sigma_2) = (1 - \lambda_1\sigma_1 - \lambda_2\sigma_2 + \lambda_3\sigma_1\sigma_2)^{-p},$$
$$\lambda_1\sigma_1 + \lambda_2\sigma_2 - \lambda_3\sigma_1\sigma_2 < 1, \qquad (11.2.7)$$

and

$$g(\theta_1, \theta_2) = (1 - \kappa_1\theta_1 - \kappa_2\theta_2 + \kappa_3\theta_1\theta_2)^{-p},$$
$$\kappa_1\theta_1 + \kappa_2\theta_2 - \kappa_3\theta_1\theta_2 < 1, \qquad (11.2.8)$$

where $p = 1/(c - 1) > 0$ and $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 \geq \lambda_3 \geq 0$, κ_1, $\kappa_2 > 0$, $\kappa_1\kappa_2 \geq \kappa_3 \geq 0$, i.e., the random vectors $\bar{Y}$, $\bar{V}$ have bivariate gamma distributions.

In the next step of the proof, it will be shown that in the above case 2 we have either $\lambda_3 = \lambda_1\lambda_2$ and $\kappa_3 = \kappa_1\kappa_2$, which implies that the components of $\bar{Y}$ and $\bar{V}$ are independent, or $\lambda_3 = 0$ and $\kappa_3 = 0$, which implies that the components of $\bar{Y}$ and $\bar{V}$ are linearly dependent gamma variables.

Again, we apply the principle of separation of variables to (11.2.3) with $\alpha = \beta = 1$ and $\sigma_2 = 0$, arriving at

$$E\left(Y_2 e^{\sigma_1 Y_1}\right) E\left(Y_1^2 Y_2 e^{\sigma_1 Y_1}\right) = d\left[E\left(Y_1 Y_2 e^{\sigma_1 Y_1}\right)\right]^2, \qquad (11.2.9)$$

where $d > 1$ is a constant. Now introduce a new random variable Z with the distribution defined by

$$P_Z(dy_1) = \frac{\int_0^\infty y_2 F(dy_1, dy_2)}{E(Y_2)},$$

where F is the df of $\bar{Y}$ and the integral in the numerator is with respect to y_2. Then after dividing both sides of (11.2.9) by $[E(Y_2)]^2$, we have

$$E\left(e^{\sigma_1 Z}\right) E\left(Z^2 e^{\sigma_1 Z}\right) = d\left[E\left(Z e^{\sigma_1 Z}\right)\right]^2,$$

which means that Z is a gamma random variable, $\gamma_{q,1/\alpha}$. Then in particular

$$E\left(e^{\sigma_1 Z}\right) = \frac{1}{(1-\alpha\sigma_1)^q}. \tag{11.2.10}$$

Now observe that

$$E\left(Y_2 e^{\sigma_1 Y_1}\right) = E\left(e^{\sigma_1 Z}\right) E\left(Y_2\right).$$

Using the fact that $\bar{Y}$ has the bivariate gamma distribution (with the Laplace transform (11.2.7)) and (11.2.10), we obtain the equation

$$(\lambda_2 - \lambda_3\sigma_1)(1-\alpha\sigma_1)^q = \lambda_2(1-\lambda_1\sigma_1)^{p+1} \tag{11.2.11}$$

for any $\sigma_1 < \lambda_1^{-1}$. Letting $\sigma_1 \uparrow \lambda_1^{-1}$, it follows that the right-hand side of (11.2.11) tends to zero. Consequently, either $\lambda_3 = \lambda_1\lambda_2$ or $\alpha = \lambda_1$. Thus, in the first case we have

$$(1-\alpha\sigma_1)^q = (1-\lambda_1\sigma_1)^p,$$

which implies $\alpha = \lambda_1$ and $q = p$. In the second case, it follows that

$$(\lambda_2 - \lambda_3\sigma_1) = \lambda_2(1-\lambda_1\sigma_1)^{p+1-q}$$

and, thus, either $\lambda_3 = 0$ and then $p+1 = q$ or $\lambda_3 \neq 0$ and then $q = p$, $\lambda_1\lambda_2 = \lambda_3$.

Summing up, only the following cases are possible: either $\lambda_3 = \lambda_1\lambda_2$ or $\lambda_3 = 0$. Similarly, we can show that either $\kappa_3 = \kappa_1\kappa_2$ or $\kappa_3 = 0$.

If $\lambda_3 = \lambda_1\lambda_2$ and $\kappa_3 = \kappa_1\kappa_2$, then $\bar{Y}$ and $\bar{V}$ have independent gamma components: $Y_j \sim \gamma_{p,\lambda_j^{-1}}$, $V_j \sim \gamma_{p,\kappa_j^{-1}}$, $j = 1, 2$.

If $\lambda_3 = 0$ and $\upsilon_3 = 0$, then the components of $\bar{Y}$ and $\bar{V}$ are linearly dependent, i.e., $Y_2 = aY_1$, $V_2 = bV_1$, where $Y_1 \sim \gamma_{p,\lambda_1^{-1}}$, $V_1 \sim \gamma_{p,\kappa_1^{-1}}$, $a = \lambda_2/\lambda_1$, $b = \kappa_2/\kappa_1$.

Observe that other cases are impossible. If $\lambda_3 = \lambda_1\lambda_2$ and $\kappa_3 = 0$, then (Y_1, Y_2) has a density and (V_1, V_2) doesn't have a density. However, if (Y_1, Y_2) has a density, then also $(X_1+Y_1, X_2+Y_2)=(U_1, U_2)$ has a density. Hence, $(U_1+V_1, U_2+V_2)=\left(\frac{1}{X_1}, \frac{1}{X_1}\right)$ has a density. Thus,

$$(V_1, V_2)=\left(\frac{1}{X_1}-\frac{1}{X_1+Y_1}, \frac{1}{X_2}-\frac{1}{X_2+Y_2}\right)$$

has also a density since it is a smooth function of the random vector $(\bar{X}, \bar{Y})$ with independent bivariate absolutely continuous components $\bar{X}$ and $\bar{Y}$. Consequently, $\kappa_3 \neq 0$. Similarly, the case $\lambda_3 = 0$ and $\kappa_3 = \kappa_1\kappa_2$ is impossible.

Summing up, we have the following two cases:

either

1. $\bar{Y}$ and $\bar{V}$ have independent gamma components: $Y_j \sim \gamma_{p_j,\lambda_j^{-1}}$, $V_j \sim \gamma_{p_j,\kappa_j^{-1}}$, $j = 1, 2$,

or

2. $\bar{Y}$ and $\bar{V}$ have linearly dependent gamma components: $Y_2 = aY_1$, $V_2 = bV_1$, where $Y_1 \sim \gamma_{p,\lambda_1^{-1}}$, $V_1 \sim \gamma_{p,\kappa_1^{-1}}$, $a = \lambda_2/\lambda_1$, $b = \kappa_2/\kappa_1$.

Case 1

In this case all the random vectors $\bar{X}$, $\bar{Y}$, $\bar{U}$, $\bar{V}$ have densities. Since $\bar{X}$, $\bar{Y}$ are independent and $\bar{U}$, $\bar{V}$ are independent, we have the following identity for the densities:

$$f_{\bar{U}}(u_1, u_2) f_{\bar{V}}(v_1, v_2) = \frac{f_{\bar{X}}\left(\frac{1}{u_1+v_1}, \frac{1}{u_2+v_2}\right) f_{\bar{Y}}\left(\frac{1}{u_1}-\frac{1}{u_1+v_1}, \frac{1}{u_2}-\frac{1}{u_2+v_2}\right)}{(u_1+v_1)^2 u_1^2 (u_2+v_2)^2 u_2^2}, \tag{11.2.12}$$

which holds a.e. with respect to the Lebesgue measure L_4 in $\mathbf{R}^4$ for $u_j, v_j \in (0, \infty)$, $j = 1, 2$. Using the fact that $\bar{Y}$ and $\bar{V}$ have independent gamma components, we obtain the following:

$$\begin{aligned} & f_{\bar{U}}(u_1, u_2) u_1^{p_1+1} u_2^{p_2+1} e^{\kappa_1^{-1}u_1} e^{\kappa_2^{-1}u_2} e^{\lambda_1^{-1}u_1^{-1}} e^{\lambda_2^{-1}u_2^{-1}} \\ &= c f_{\bar{X}}\left((u_1+v_1)^{-1}, (u_2+v_2)^{-1}\right)(u_1+v_1)^{-(p_1+1)}(u_2+v_2)^{-(p_2+1)} \\ &\times e^{\kappa_1^{-1}(u_1+v_1)} e^{\kappa_2^{-1}(u_2+v_2)} e^{\lambda_1^{-1}(u_1+v_1)^{-1}} e^{\lambda_2^{-1}(u_2+v_2)^{-1}}, \end{aligned} \tag{11.2.13}$$

for $u_j, v_j \in (0, \infty)$, $j = 1, 2$, L_4 a.e., where $c = const$.

Denoting $u_1 + v_1 = m_1$, $u_2 + v_2 = m_2$, the above equation can be written as

$$f_{\bar{U}}(u_1, u_2) = c(m_1, m_2) g_1(u_1) g_2(u_2), \tag{11.2.14}$$

where c is the right-hand side of (11.2.13) and

$$g_j(u_j) = u_j^{-p_j - 1} e^{-\kappa_j^{-1} u_j - \lambda_j^{-1} u_j^{-1}},$$

$j = 1, 2$. We can always choose m_1, m_2 such that (11.2.14) holds for $(u_1, u_2) \in (0, m_1) \times (0, m_2)$ L_2 a.e. Moreover, m_1 and m_2 can be chosen arbitrarily large. This implies that $\bar{U}$ has independent GIG components $U_j \sim \mu_{-p_j, \kappa_j, \lambda_j}$, $j = 1, 2$. Dually, by (11.2.13), it follows that $\bar{X}$ has also independent GIG components $X_j \sim \mu_{-p_j, \lambda_j, \kappa_j}$, $j = 1, 2$.

Case 2

Since $Y_2 = aY_1$, $V_2 = bV_1$ P-a.s. and $V_j = \frac{1}{X_j} - \frac{1}{X_j + Y_j}$, $j = 1, 2$, we obtain

$$\frac{bY_1}{X_1(X_1 + Y_1)} = \frac{aY_1}{X_2(X_2 + aY_1)} \quad P-\text{a.s.}$$

Since Y_1 is P-a.s. positive, we obtain

$$Y_1(X_1 - bX_2) = X_1^2 - \frac{b}{a} X_2^2 \quad P-\text{a.s.} \tag{11.2.15}$$

Assume now that $X_1 \neq bX_2$ on a set A of positive probability P. Then on A we have

$$Y_1 = \frac{X_1^2 - \frac{b}{a} X_2^2}{X_1 - bX_2},$$

which contradicts the independence of $\bar{X}$ and $\bar{Y}$. Thus, $X_1 = bX_2$ P-a.s. and by (11.2.15) $b = 1/a$. Thus, the components of $\bar{X}$ are linearly dependent: $X_2 = aX_1$. Since $U_j = \frac{1}{X_j + Y_j}$, $j = 1, 2$, we obtain immediately that the components of $\bar{U}$ are also linearly dependent: $U_2 = bU_1$.

Thus, the problem is reduced to the univariate case. Hence, by the result of Letac and Wesołowski (2000), we get that X_1 and U_1 have GIG distributions: $X_1 \sim \mu_{-p, \lambda_1, \kappa_1}$, $U_1 \sim \mu_{-p, \kappa_1, \lambda_1}$.

Sufficiency. Now we assume that the random vectors $\bar{X}$ and $\bar{Y}$ have the GIG and gamma distributions as given in the statement of the theorem. We will show that the random vectors $\bar{U}$ and $\bar{V}$ are independent.

First consider the case of independent components of $\bar{X}$ and $\bar{Y}$. Since, by the assumption of the theorem, $\bar{X}$ and $\bar{Y}$ are independent, it follows that the random vectors (X_1, Y_1) and (X_2, Y_2) are independent. It implies that the random vectors (U_1, V_1) and (U_2, V_2) are independent. However, by the univariate Matsumoto–Yor property (recall that the components of $\bar{X}$ are GIGs and the components of $\bar{Y}$ are gammas), it follows that U_1, V_1 are independent and U_2, V_2 are independent. Again, using the independence of (U_1, V_1) and (U_2, V_2), we conclude that $\bar{U} = (U_1, U_2)$ and $\bar{V} = (V_1, V_2)$ are independent.

Finally, consider the case of linearly dependent components of $\bar{U}$ and $\bar{V}$, i.e., $\bar{U} = (U_1, U_1/a)$ and $\bar{V} = (V_1, V_1/a)$, with U_1 being a GIG random variable and V_1 being a gamma random variable. Then, by the univariate Matsumoto–Yor property, it follows that U_1 and V_1 are independent. Consequently, $\bar{U}$ and $\bar{V}$ are also independent. ■

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