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Conditional moments of q-Meixner processes

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Abstract. We show that stochastic processes with linear conditional expectations and quadratic conditional variances are Markov, and their transition probabilities are related to a three-parameter family of orthogonal polynomials which generalize the Meixner polynomials. Special cases of these processes are known to arise from the non-commutative generalizations of the Lévy processes.

1. Introduction

1.1. Motivation

It has been known since the work of Biane [10] that every non-commutative process with free increments gives rise to a classical Markov process, whose transition probabilities "realize" the non-commutative free convolution of the corresponding measures. It is natural to ask how to recognize in classical probabilistic terms which Markov processes might arise from this construction. Unfortunately, the non-commutative freeness seems to be poorly reflected in the corresponding classical Markov process, which makes it hard to answer this question. A more general framework might be less constraining and easier to handle.

Non-commutative processes with free increments can be thought as a special case corresponding to the value $q = 0$ of the more general class of q-Lévy processes [3], [7]. Markov processes are known to arise in this more general setting in two important cases: Bożejko, Kümmerer, and Speicher, give explicit Markov transition probabilities for the q-Brownian motion, see [11, Theorem 1.10], and Anshelevich [6, Corollary A.1] proves the corresponding result for the q-Poisson process. Other q-Lévy processes are still not well understood, so it is not known whether Markov processes arise in the general case; for indications that Markov property may perhaps fail, see [4].

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This paper arose as an attempt to better understand the emergence of related Markov processes from probabilistic assumptions. We define our class of processes by assuming that the first two conditional moments are given respectively by the generic linear and quadratic expressions. Such assumptions are familiar from Lévy's characterization of the Wiener process as a martingale and a quadratic martingale with continuous trajectories. For more general processes the assumption of continuity of trajectories fails, so we replace it by conditioning with respect not only to the past, but also to the future. This approach originated with Plucińska [23] who proved that processes with linear conditional expectations and constant conditional variances are Gaussian. Subsequent papers covered discrete Gaussian sequences [16], L_2-differentiable processes [28], Poisson process [14], Gamma process [31]. Wesołowski [32] unified several partial results, identifying the general quadratic conditional variance problem which characterizes the five Lévy processes of interest in this note: Wiener, Poisson, Pascal, Gamma, and Meixner. Our main result, Theorem 3.5, extends [32, Theorem 2] to the more general quadratic conditional variances. Similar analysis of stationary sequences in [15] yields the classical versions of the non-commutative q-Gaussian processes of [11]. Further contributions to the stationary case can be found in [21].

Stochastic processes with linear conditional expectations and quadratic conditional variances turn out to depend on three numerical parameters $-\infty < \theta < \infty$, $\tau \geq 0$, and $-1 \leq q \leq 1$. They are Markov, and arise from the non-commutative constructions, at least for those values of the parameters when such constructions are known. To point out the connection with the orthogonal polynomials from which they are derived, we call them q-Meixner processes.

When $q = 1$, the q-Meixner processes have independent increments and we recover the five Lévy processes from [32, Theorem 2]. For other values of parameter q, we encounter several processes that arose in non-commutative probability. If $\tau = \theta = 0$, we get the classical version of the q-Brownian motion [11]. If $\tau = 0, \theta \neq 0$ the q-Meixner processes arise as the classical version from the q-Poisson process defined in [3, Def. 6.16]. When $q = 0$ the q-Meixner processes are related to the class of free Lévy processes considered by Anshelevich [5].

The reasons why these special cases of q-Meixner processes should arise from the Fock space constructions are not clear to us. It is not known whether the generic q-Meixner process arises as a classical version of a non-commutative process, but the situation must be more complex. The connection with the q-Levy processes on the q-Fock space as defined in [3] fails for the following reason. In Proposition 3.3 below we establish a polynomial martingale property (46) for all q-Meixner processes. But from Anshelevich [4, Appendix A.2] we know that a generic q-Levy process does not have martingale polynomials; the exceptions are $q = 0$, $q = 1$, the q-Poisson process, and the q-Brownian motion, and these are precisely the cases that we already mentioned above.

1.2. Assumptions

Throughout this paper $(X_t)_{t\geq 0}$ is a separable square-integrable stochastic process, normalized so that for all $t, s \geq 0$

$$E(X_t) = 0,\ E(X_t X_s) = \min\{t, s\}. \tag{1}$$

We are interested in the processes with linear conditional expectations and quadratic conditional variances. More specifically, we assume the following.

For all $0 \le s < t < u$,

$$E(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u}) = aX_s + bX_u, \tag{2}$$

where $a = a(s, t, u), b = b(s, t, u)$ are the deterministic functions of s, t, u, and $\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u}$ denotes the σ-field generated by $\{X_t : t \in [0, s] \cup [u, \infty)\}$.

For ease of reference, we list the following trivial consequences of (2). From the form of the covariance it follows that

$$a = \frac{u-t}{u-s},\ b = \frac{t-s}{u-s}. \tag{3}$$

Notice that from (2) we have

$$\begin{aligned} E(E(X_t|\mathcal{F}_s) - X_s)^2 &= E(E(E(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u})|\mathcal{F}_s) - X_s)^2 \\ &= b^2 E(E(X_u - X_s|\mathcal{F}_s))^2 \le (t-s)^2/(u-s). \end{aligned}$$

Passing to the limit as $u \to \infty$ we see that

$$E(X_t|\mathcal{F}_{\le s}) = X_s \tag{4}$$

for $0 \le s \le t$. Similarly, taking $s = 0$ in (2) we get

$$E(X_t|\mathcal{F}_{\ge u}) = \frac{t}{u} X_u. \tag{5}$$

Processes which satisfy condition (2) are sometimes called harnesses, see [20], [33]. We assume in addition that the conditional variance of X_t given $\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u}$ is given by a quadratic expression in X_s, X_u. Recall that the conditional variance of X with respect to a σ-field $\mathcal{F}$ is defined as

$$\mathrm{Var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - (E(X|\mathcal{F}))^2 .$$

For later calculations, it is convenient to express this assumption as follows.

For all $0 \le s < t < u$,

$$E(X_t^2|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u}) = AX_s^2 + BX_s X_u + CX_u^2 + D + \alpha X_s + \beta X_u, \tag{6}$$

where $A = A(s, t, u), B = B(s, t, u), C = C(s, t, u), D = (s, t, u)$, $\alpha = \alpha(s, t, u), \beta = \beta(s, t, u)$ are the deterministic functions of s, t, u.

Since $X_0 = 0$, the coefficients a, A, B, α are undefined at $s = 0$. In some formulas for definiteness we assign these values by continuity.

It turns out that under mild assumptions, the functions $A, B, C, D, \alpha, \beta$, are determined uniquely as explicit functions of s, t, u, up to some numerical constants. The next assumption specifies two of these constants by requesting that

$\mathrm{Var}(X_t|\mathcal{F}_{\le s}) = const$ for all $0 \le s \le t$. We use (4) to state this assumption in the following more explicit form.

$$E(X_t^2|\mathcal{F}_{\le s}) = X_s^2 + t - s. \tag{7}$$

Notice that equations (4) and (7) imply that $\{X_t : t \ge 0\}$ and $\{X_t^2 - t : t \ge 0\}$ are martingales with respect to the natural filtration $\mathcal{F}_{\le t}$; these two martingale conditions (and continuity of trajectories) are the usual assumptions in the Lévy theorem.

2. Conditional variances

It is interesting to note that under mild assumptions, assumption (6) can be written explicitly, up to some numerical constants. Two of these numerical constants appear already under one-sided conditioning.

Proposition 2.1. *Let $(X_t)_{t\ge 0}$ be a separable square integrable stochastic process which satisfies conditions (1), (2), and such that $1, X_t, X_t^2$ are linearly independent for all $t > 0$. If for every $0 < t < u$ the conditional expectation $E(X_t^2|\mathcal{F}_{\ge u})$ is a quadratic expression in variable X_u, then there are constants $\tau \in [0, \infty]$ and $\theta \in \mathbb{R}$ such that*

$$\mathrm{Var}(X_t|\mathcal{F}_{\ge u}) = \begin{cases} \frac{t(u-t)}{u+\tau}\left(\tau \frac{X_u^2}{u^2} + \theta \frac{X_u}{u} + 1\right) & \text{if } \tau < \infty, \\ t(u-t)\left(\frac{X_u^2}{u^2} + \theta \frac{X_u}{u}\right) & \text{if } \tau = \infty. \end{cases} \tag{8}$$

for all $0 \le t < u$.

Proof. By assumption, for any $0 < s < t$

$$E(X_s^2|\mathcal{F}_{\ge t}) = m(s,t)X_t^2 + n(s,t)X_t + o(s,t)\ , \tag{9}$$

where m, n, o are some functions.

On the other hand from (5) we get

$$E(X_sX_t|\mathcal{F}_{\ge u}) = E(E(X_s|\mathcal{F}_{\ge t})X_t|\mathcal{F}_{\ge u}) = \frac{s}{t}E(X_t^2|\mathcal{F}_{\ge u}),$$

and from (2) we get

$$\begin{aligned} E(X_sX_t|\mathcal{F}_{\ge u}) &= E(X_sE(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u})|\mathcal{F}_{\ge u}) \\ &= \frac{u-t}{u-s}E(X_s^2|\mathcal{F}_{\ge u}) + \frac{t-s}{u-s}X_uE(X_s|\mathcal{F}_{\ge u}) \\ &= \frac{u-t}{u-s}E(X_s^2|\mathcal{F}_{\ge u}) + \frac{(t-s)s}{(u-s)u}X_u^2\ . \end{aligned}$$

Combining the above two formulas we have

$$\frac{s}{t}E(X_t^2|\mathcal{F}_{\ge u}) = \frac{u-t}{u-s}E(X_s^2|\mathcal{F}_{\ge u}) + \frac{(t-s)s}{(u-s)u}X_u^2. \tag{10}$$

Now we substitute the conditional moments from (9) into (10), getting

$$\frac{s}{t}\left(m(t,u)X_u^2+n(t,u)X_u+o(t,u)\right)$$
$$=\frac{u-t}{u-s}\left(m(s,u)X_u^2+n(s,u)X_u+o(s,u)\right)+\frac{(t-s)s}{(u-s)u}X_u^2 .$$

Recall that $1, X_u, X_u^2$ are linearly independent. Comparing the coefficients of respective powers of X_u we obtain

$$\frac{s}{t}m(t,u)=\frac{u-t}{u-s}m(s,u)+\frac{(t-s)s}{(u-s)u} ,$$
$$\frac{s}{t}n(t,u)=\frac{u-t}{u-s}n(s,u) , \quad \frac{s}{t}o(t,u)=\frac{u-t}{u-s}o(s,u) .$$

The first equation leads to

$$\left(\frac{m(t,u)}{t}-\frac{1}{u}\right)\frac{1}{u-t}=\left(\frac{m(s,u)}{s}-\frac{1}{u}\right)\frac{1}{u-s},$$

and hence

$$m(t,u)=\frac{t}{u}+t(u-t)i(u)$$

for some function $i:\mathbb{R}\to\mathbb{R}$. The next two equations give

$$n(t,u)=t(u-t)j(u) \quad \text{and} \quad o(t,u)=t(u-t)k(u)$$

for some functions $j,k:\mathbb{R}\to\mathbb{R}$. Thus from (9) we get

$$E(X_t^2|\mathcal{F}_{\geq u})=\left(\frac{t}{u}+t(u-t)i(u)\right)X_u^2+t(u-t)j(u)X_u+t(u-t)k(u) .$$

Taking the expectations of both sides we get $t=t+tu(u-t)i(u)+t(u-t)k(u)$, so $k(u)=-ui(u)$. Finally we have

$$E(X_t^2|\mathcal{F}_{\geq u})=\frac{t}{u}X_u^2+t(u-t)\left[i(u)(X_u^2-u)+j(u)X_u\right] . \tag{11}$$

To identify the functions i and j we fix $s<t<u$ and insert (11) into the formula

$$E(X_s^2|\mathcal{F}_{\geq u})=E(E(X_s^2|\mathcal{F}_{\geq t})|\mathcal{F}_{\geq u}) .$$

This gives

$$\frac{s}{u}X_u^2+s(u-s)\left[i(u)(X_u^2-u)+j(u)X_u\right]$$
$$=E\left(\frac{s}{t}X_t^2+s(t-s)\left[i(t)(X_t^2-t)+j(t)X_t\right]\Big|\mathcal{F}_{\geq u}\right)$$
$$=\frac{s}{t}\left\{\frac{t}{u}X_u^2+t(u-t)\left[i(u)(X_u^2-u)+j(u)X_u\right]\right\}$$
$$+s(t-s)i(t)\left\{\frac{t}{u}X_u^2+t(u-t)\left[i(u)(X_u^2-u)+j(u)X_u\right]\right\}$$
$$+s(t-s)j(t)\frac{t}{u}X_u-st(t-s)i(t) .$$

Comparing the coefficients of respective powers of X_u we obtain

$$ui(u) = ti(t) + (u-t)ti(t)ui(u) \ , \tag{12}$$

$$uj(u) = tj(t) + (u-t)ti(t)uj(u) \ . \tag{13}$$

If i is non-zero for all $t > 0$ then (12) gives $\frac{1}{ti(t)} + t = \frac{1}{ui(u)} + u$. This means that $\frac{1}{ti(t)} + t = -\tau$ for some constant τ, and $\tau \geq 0$ since $1/i(t)$ cannot vanish for any $t > 0$. Hence

$$i(t) = -\frac{1}{t(t+\tau)} \ . \tag{14}$$

Using this in (13) we get $u(u+\tau)j(u) = t(t+\tau)j(t)$. Thus

$$j(t) = \frac{\theta}{t(t+\tau)}$$

for some real constant θ. We get

$$E(X_s^2|\mathcal{F}_{\geq t}) = \frac{s(s+\tau)}{t(t+\tau)}X_t^2 + \frac{s(t-s)}{t(t+\tau)}\theta X_t + \frac{s(t-s)}{t+\tau} \ . \tag{15}$$

Suppose now that $i(t) = 0$ for some $t > 0$. Then (12) implies that i is a zero function, corresponding to $\tau = \infty$ in (14). In this case (13) leads to $uj(u) = tj(t)$, which means that $j(t) = \theta/t$ for some real number θ. Thus in this case

$$E(X_s^2|\mathcal{F}_{\geq t}) = \frac{s}{t}X_t^2 + \frac{s(t-s)}{t}\theta X_t \ . \tag{16}$$

□

Notice that taking the expected value of both sides of (6), we get a trivial relation

$$t - As - Cu = Bs + D, \tag{17}$$

valid for all $0 \leq s < t < u$. We need additional relations between the coefficients in (6).

Lemma 2.2. *Let $(X_t)_{t\geq 0}$ be a separable square integrable stochastic process which satisfies conditions (1), (2), and such that $1, X_t, X_t^2$ are linearly independent for all $t > 0$. Suppose that condition (6) holds with $D(s,t,u) \neq 0$ for all $0 \leq s < t < u$. Then the conditional expectation $E(X_u^2|\mathcal{F}_{\leq t})$ is quadratic in X_t for any $0 \leq t < u$. Moreover,*

$$E(X_u^2 - u|\mathcal{F}_{\leq s}) = \left(1 + \frac{A+B+C-1}{b-C}\right)(X_s^2 - s) + \frac{\alpha+\beta}{b-C}X_s. \tag{18}$$

Proof. Equation (4) implies that $E(X_t^2|\mathcal{F}_{\le s}) = E(X_t X_u|\mathcal{F}_{\le s})$, so from (2) we get

$$E(X_t^2|\mathcal{F}_{\le s}) = aX_s^2 + bE(X_u^2|\mathcal{F}_{\le s}).$$

From (6) we get

$$E(X_t^2|\mathcal{F}_{\le s}) = AX_s^2 + BX_s^2 + CE(X_u^2|\mathcal{F}_{\le s}) + (\alpha+\beta)X_s + D.$$

Notice that this implies $C \neq b$. Indeed, if $C = b$ then subtracting the equations we get a quadratic equation for X_s. If this equation is non-trivial, then $1, X_s, X_s^2$ are linearly dependent. So the coefficients in the quadratic equation must all be zero; in particular, $D = 0$, contradicting the assumption.

Since $C \neq b$, we can solve the equations for $E(X_t^2|\mathcal{F}_{\le s})$ and $E(X_u^2|\mathcal{F}_{\le s})$. Using (17), we get (18) after a calculation. □

Lemma 2.3. *Let $(X_t)_{t\ge 0}$ be a separable square integrable stochastic process which satisfies conditions (1), (2), (7) and such that $1, X_t, X_t^2$ are linearly independent for all $t > 0$. Suppose that condition (6) holds with $D(s,t,u) \neq 0$ for all $0 \le s < t < u$. Then the conditional expectations $E(X_s^2|\mathcal{F}_{\ge t})$ are quadratic in X_t for any $0 \le s < t$. Moreover, there are constants $0 \le \tau < \infty$, $-\infty < \theta < \infty$ such that (8) holds true, and the parameters in (6), evaluated at $0 \le s < t < u$, satisfy the following equations.*

$$A + B + C = 1, \tag{19}$$

$$As^2 + Bsu + Cu^2 - t^2 = \tau D, \tag{20}$$

$$s\alpha + u\beta = \theta D, \tag{21}$$

$$\alpha + \beta = 0. \tag{22}$$

Proof. Comparing the coefficients in (7) and (18), we get (19), and (22).

Setting $s = 0$ in (6) we see that $E(X_t^2|\mathcal{F}_{\ge u})$ is quadratic in X_u. Thus Proposition 2.1 implies that (8) holds true. Notice that since $D(0,t,u) \neq 0$, we must have $\tau < \infty$, so (15) holds. We use the latter in

$$E(X_t^2|\mathcal{F}_{\ge u}) = AE(X_s^2|\mathcal{F}_{\ge u}) + \frac{s}{u}BX_u^2 + CX_u^2 + \Big(\frac{s}{u}\alpha + \beta\Big)X_u + D,$$

which follows from (6). We get (20) from the comparison of the quadratic terms, and (21) from the comparison of the linear terms. □

For future reference we state the following.

Remark 2.1. The system of equations (17), (19), (20), (21), (22) has the solution

$$\alpha = D\frac{-\theta}{u-s}, \tag{23}$$

$$\beta = D\frac{\theta}{u-s}, \tag{24}$$

$$A = \frac{ta}{s} - D\frac{u+\tau}{s(u-s)}, \tag{25}$$

$$B = D\frac{s+u+\tau}{s(u-s)} - \frac{u-s}{s}ab, \tag{26}$$

$$C = b - D\frac{1}{u-s}. \tag{27}$$

We need the following version of [32, Theorem 2].

Proposition 2.4. *Let $(X_t)_{t\geq 0}$ be a separable square integrable stochastic process which satisfies conditions (1), (2), (6), and (7). Suppose that the coefficient D in (6) satisfies $D(s,t,u) \neq 0$ for all $0 \leq s < t < u$, and that $1, X_t, X_t^2$ are linearly independent for all $t > 0$. Then $E(|X_t|^p) < \infty$ for all $p \geq 0$.*

Moreover, if (X_t) and (Y_t) satisfy these assumptions with the same coefficients in (6), then the joint moments of both processes are equal,

$$E(X_{t_1}^{n_1} X_{t_2}^{n_2} \dots X_{t_k}^{n_k}) = E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \dots Y_{t_k}^{n_k})$$

for all $t_1, t_2, \dots, t_k > 0, n_1, n_2, \dots, n_k \in \mathbb{N}, k \in \mathbb{N}$.

Proof. Fix $s < t$ and let $\{t_k : k \geq 0\}$ be an arbitrary infinite strictly increasing sequence which contains s and t as consecutive elements, say $s = t_N, t = t_{N+1}$ for some $N \in \mathbb{N}$.

We apply [32, Theorem 2] to the sequence $\xi_k = X_{t_k}$. Of course, $\sigma(\xi_1, \dots, \xi_k, \xi_{k+1}) \subset \mathcal{F}_{\leq t_{k-1}} \vee \mathcal{F}_{\geq t_{k+1}}$. Therefore, conditions (4), (2), (7), and (6) imply [32, (6), (7), (8), and (9)], respectively. Since $corr(\xi_{k-1}, \xi_k) = \sqrt{t_{k-1}/t_k} \neq 0, \pm 1$, the assumption [32, (10)] holds true, too. Finally, notice that Wesołowski's $\alpha_k = 1$, and his $\underline{a}_k = C(t_{k-1}, t_k, t_{k+1}) \neq a_k = b(t_{k-1}, t_k, t_{k+1})$ because from (27) we see that $D \neq 0$ if and only if $C \neq b$. Thus [32, (11)] hold true. From [32, Theorem 2] we see that $E(|X_t|^p) < \infty$ for all $p > 0$, and that for $n = 1, 2 \dots$, the conditional moment $E(X_t^n | X_{t_1}, \dots, X_{t_{N-1}}, X_s)$ is a unique polynomial of degree n in the variable X_s.

If two processes satisfy the assumptions, then the conditional moments of both processes can be expressed as polynomials with the same coefficients. This implies that all joint moments of the processes are equal. ◻

Next we give the general form of the conditional variance under the two-sided conditioning.

Proposition 2.5. *Let $(X_t)_{t\geq 0}$ be a separable square integrable stochastic process which satisfies conditions (1), (2), (6), and (7). Suppose that the coefficient D in (6) satisfies $D(s,t,u) \neq 0$ for all $0 \leq s < t < u$, and that $1, X_t, X_t^2$ are linearly independent for all $t > 0$. Then there are parameters $-\infty < \theta < \infty$, and $0 \leq \tau < \infty$ such that the first part of (8) holds true. In addition, there exists $-1 < q \leq 1$ such that*

$$\begin{aligned} &\mathrm{Var}(X_t | \mathcal{F}_{\leq s} \vee \mathcal{F}_{\geq u}) \\ &\quad = \frac{(u-t)(t-s)}{u + \tau - qs} \Bigg((1-q) \frac{(X_u - X_s)(sX_u - uX_s)}{(u-s)^2} \\ &\qquad\qquad + \tau \frac{(X_u - X_s)^2}{(u-s)^2} + \theta \frac{X_u - X_s}{u-s} + 1 \Bigg). \end{aligned} \tag{28}$$

Proof. By Proposition 2.4, all moments of X_t are finite. Fix $s < t$. Then from (5) and (7) we get $\frac{s}{t} E(X_t^3) = E(X_t^2 X_s) = EX_s^3$, so $E(X_t^3)/t$ does not depend on $t > 0$. On the other hand, from (15) we get

$$E(X_s^3) = E(X_s^2 X_t) = \frac{s(s+\tau)}{t(t+\tau)} E(X_t^3) + \frac{s(t-s)}{t+\tau}\theta.$$

Hence

$$EX_t^3 = t\theta. \tag{29}$$

Similarly, from (7) we get

$$E(X_t^2 X_s^2) = EX_s^4 + s(t-s),$$

and from (8) we get

$$E(X_t^2 X_s^2) = \frac{s(s+\tau)}{t(t+\tau)} EX_t^4 + \theta \frac{s(t-s)}{t(t+\tau)} E(X_t^3) + st\frac{t-s}{t+\tau}.$$

Using (29) we get after a calculation that $\frac{E(X_s^4)-s(s+\theta^2)}{s(s+\tau)}$ does not depend on s. Thus

$$E(X_t^4) = (1+q)t(t+\tau) + t(t+\theta^2) \tag{30}$$

for some constant $q \in \mathbb{R}$.

A calculation gives

$$E(X_t - X_s)^2 = t-s, \;\; E(X_t - X_s)^3 = \theta(t-s),$$

and

$$E(X_t - X_s)^4 = (t-s)\left(6s + \theta^2 - \tau + (2+q)(t+\tau-3s)\right).$$

Since the determinant

$$\begin{aligned}&\frac{1}{(t-s)^2}\det\begin{bmatrix} 1 & E(X_t - X_s) & E((X_t - X_s)^2) \\ E(X_t - X_s) & E((X_t - X_s)^2) & E((X_t - X_s)^3) \\ E((X_t - X_s)^2) & E((X_t - X_s)^3) & E((X_t - X_s)^4)\end{bmatrix}\\ &= q\,(t+\tau-3s) + s + t + \tau\end{aligned}$$

is non-negative, taking $s = t-1$ and $t \to \infty$, we get $q \le 1$. Since $1, X_t, X_t^2$ are linearly independent, the determinant evaluated at $s=0$ must be strictly positive, see [18, pg. 19]. This shows that $q > -1$.

It remains to determine the coefficients in (6). Fix $s < t < u$. Comparing the two representations of $E(X_t X_u^2 | \mathcal{F}_{\le s})$ as

$$E(E(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u})X_u^2|\mathcal{F}_{\le s}) = E(X_t E(X_u^2|\mathcal{F}_{\le t})|\mathcal{F}_{\le s}),$$

and the similar two expressions for $E(X_t^2 X_u|\mathcal{F}_{\le s})$, we get two different expressions for $E(X_t^3|\mathcal{F}_{\le s})$. Equating them, we get

$$\begin{aligned}aX_s^3 + bE(X_u^3|\mathcal{F}_{\le s}) = AX_s^3 + BX_s^3 + BX_s(u-s) + CE(X_u^3|\mathcal{F}_{\le s})\\ +DX_s + (\alpha+\beta)X_s^2 + \beta(u-s).\end{aligned}$$

We can solve this equation for $E(X_u^3|\mathcal{F}_{\le s})$, as (27) implies that $C \neq b$. Using (24) and (19), the answer simplifies to

$$E(X_u^3|\mathcal{F}_{\le s}) = X_s^3 + \frac{B(u-s)+D}{b-C} X_s + (u-s)\frac{\beta}{b-C}.$$

From this we get

$$\frac{s}{u} E(X_u^4) = E(X_s X_u^3) = E(X_s^4) + \frac{B(u-s)+D}{D}(u-s)s.$$

Substituting (30) we deduce the following equation

$$\frac{(u-s)B}{D} = 1+q. \tag{31}$$

Solving together equations (19), (20), (21), (22), and (31) for $A, B, C, D, \alpha, \beta$ we obtain (28). □

Remark 2.2. Solving together equations (19), (20), (21), (22), and (31) for $A, B, C, D, \alpha, \beta$ we get

$$A = \frac{u-t}{u-s} \times \frac{u+\tau-qt}{u+\tau-qs}, \tag{32}$$

$$B = (1+q)\frac{t-s}{u-s} \times \frac{u-t}{u+\tau-qs}, \tag{33}$$

$$C = \frac{t-s}{u-s} \times \frac{t+\tau-qs}{u+\tau-qs}, \tag{34}$$

$$D = \frac{(u-t)(t-s)}{u+\tau-qs}, \tag{35}$$

$$\alpha = -\theta \frac{(u-t)(t-s)}{(u-s)(u+\tau-qs)}, \tag{36}$$

$$\beta = \theta \frac{(u-t)(t-s)}{(u-s)(u+\tau-qs)}. \tag{37}$$

Remark 2.3. From the formula for $E(X_t - X_s)^4$ we see that except for the case $q = 1$, the increments of the process X_t are not stationary. For $\tau = 0$, the increments of the corresponding non-commutative processes are stationary, but this property is not inherited by the classical version.

3. *q*-Meixner Markov processes

We use the standard notation

$$[n]_q = 1 + q + \cdots + q^{n-1},$$
$$[n]_q! = [1]_q [2]_q \ldots [n]_q,$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!},$$

with the usual conventions $[0]_q = 0, [0]_q! = 1$. For fixed real parameters x, s, t, q, θ, τ, define the polynomials Q_n in variable y by the three step recurrence

$$y Q_n(y|x) = Q_{n+1}(y|x) + (\theta[n]_q + xq^n) Q_n(y|x) \\ +(t - sq^{n-1} + \tau[n-1]_q)[n]_q Q_{n-1}(y|x), \tag{38}$$

where $n \geq 1$, and $Q_{-1}(y|x) = 0$, $Q_0(y|x) = 1$, so $Q_1(y|x) = y - x$. It is well known that such polynomials are orthogonal with respect to a probability measure if the last coefficient of the three step recurrence is positive, see [18, Theorem I.4.4]. Therefore, (38) defines a probability measure whenever $x, \theta \in \mathbb{R}, 0 < s < t, \tau \geq 0, -1 \leq q \leq 1$. Moreover, in this case

$$\sum_n \frac{1}{\sqrt{(t - sq^{n-1} + \tau[n-1]_q)[n]_q}} = \infty, \tag{39}$$

so from Carleman's criterion (see [27, page 59]), this measure is unique. We denote this unique probability measure by $\mu_{x,s,t}(dy)$.

Of course, $\mu_{x,s,t}(dy) = \mu_{x,s,t,q,\theta,\tau}(dy)$ depends on all the parameters of the recurrence (38). It is worth noting explicitly that if $q = -1$ then $[2]_q = 0$, so $\mu_{x,s,t}(dy)$ is supported on two points only. In general, more explicit expressions for $\mu_{x,s,t}(dy)$ can perhaps be derived from [9, Theorem 2.5] by taking their parameters $b = c = 0$, $ad = -(s(1-q)+\tau)/(t(1-q)+\tau)$, $a + d = ((1-q)x - 1)/\sqrt{t + \tau/(1-q)}$.

If we need to indicate the dependence of the polynomials in (38) on the additional parameters in the recurrence (38), we write $Q_n(y|x, s, t)$.

We will need two algebraic identities; the first one resembles [2, (2.3)] but is in fact different; the second one is a slight generalization of [17, Theorem 1].

Lemma 3.1. *For every $x, y, z \in \mathbb{R}$, $n \in \mathbb{N}$, and $0 \leq s \leq t \leq u$ we have*

$$Q_n(z|x, s, u) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q Q_{n-k}(y|x, s, t) Q_k(z|y, t, u). \tag{40}$$

Furthermore,

$$Q_n(z|y, t, u) \\ = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q Q_{n-k}(0|y, t, 0) \left(Q_k(z|0, 0, u) - Q_k(y|0, 0, t)\right). \tag{41}$$

Proof. Consider first the case $|q| < 1$. It is easy to check by q-differentiation with respect to ζ that the generating function

$$\phi(\zeta, y, x, s, t) = \sum_{n=0}^{\infty} \frac{\zeta^n}{[n]_q!} Q_n(y|x, s, t)$$

of the polynomials Q_n is given by

$$\phi(\zeta, y, x, s, t) = \prod_{k=0}^{\infty} \frac{1 + \theta\zeta q^k - (1-q)x\zeta q^k + ((1-q)s + \tau)\zeta^2 q^{2k}}{1 + \theta\zeta q^k - (1-q)y\zeta q^k + ((1-q)t + \tau)\zeta^2 q^{2k}}.$$

For details, see [2]. Notice that for $|q| < 1$, the series defining $\phi(\zeta, y, x, s, t)$ converges for all $|\zeta|$ small enough. Indeed, from (38) we get by induction $|Q_{n+1}| \le C^n$ with $C = \max\{1, (|x| + |y| + |\theta| + \tau + t + s)/(1 - |q|)^2\}$.

Therefore,

$$\phi(\zeta, z, x, s, u) = \phi(\zeta, y, x, s, t)\phi(\zeta, z, y, t, u), \tag{42}$$

which implies (40) for all $n \ge 0$ and $|q| < 1$. Since (40) is an identity between the polynomial expressions in variables z, y, q, it must hold for all q.

Since $1/\phi(\zeta, y, x, s, t) = \phi(\zeta, x, y, t, s)$, from (42) we get

$$\begin{aligned}\phi(\zeta, z, y, t, u) &= \frac{\phi(\zeta, z, x, s, u)}{\phi(\zeta, y, x, s, t)} \\ &= 1 + \frac{1}{\phi(\zeta, y, x, s, t)} \left(\phi(\zeta, z, x, s, u) - \phi(\zeta, y, x, s, t)\right) \\ &= 1 + \phi(\zeta, x, y, t, s) \left(\phi(\zeta, z, x, s, u) - \phi(\zeta, y, x, s, t)\right).\end{aligned}$$

Evaluating this at $s = 0, x = 0$ we get

$$\phi(\zeta, z, y, t, u) = 1 + \phi(\zeta, 0, y, t, 0) \left(\phi(\zeta, z, 0, 0, u) - \phi(\zeta, y, 0, 0, t)\right).$$

This shows that (41) holds for all $n \ge 1$ and $|q| < 1$. Since (41) is an identity between the polynomial expressions in variables z, y, q, it must hold for all q. □

We now verify that $\mu_{x,s,t}(dy)$ are the transition probabilities of a Markov process.

Proposition 3.2. *If* $0 \le s < t < u$, *then*

$$\mu_{x,s,u}(\cdot) = \int \mu_{y,t,u}(\cdot)\mu_{x,s,t}(dy).$$

Proof. Let $\nu(dz) = \int \mu_{x,s,t}(dy)\mu_{y,t,u}(dz)$. To show that $\nu(dz) = \mu_{x,s,u}(dz)$, we verify that $Q_n(z|x, s, u)$ are orthogonal with respect to $\nu(dz)$. Since $Q_n(z|x, s, u)$ satisfy the three-step recurrence (38), we need only to show that for $n \ge 1$ these polynomials integrate to zero. Since $\int Q_k(z|y, t, u)\mu_{y,t,u}(dz) = 0$ for $k \ge 1$, by (40) we have

$$\begin{aligned}&\int Q_n(z|x, s, u)\nu(dz) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \int \left(\int Q_k(z|y, t, u)\mu_{y,t,u}(dz)\right) Q_{n-k}(y|x, s, t)\mu_{x,s,t}(dy) \\ &= \int Q_n(y|x, s, t)\mu_{x,s,t}(dy) = 0,\end{aligned}$$

as $n \ge 1$. □

Let (X_t) be a Markov process with the transition probabilities defined for $0 \leq s < t$ by

$$P_{s,t}(x, dy) = \mu_{x,s,t}(dy), \tag{43}$$

where $\mu_{x,s,t}(dy)$ is the distribution orthogonalizing the polynomials (38), $X_0 = 0$. Since the distribution of X_t is $\mu_{0,0,t}(dx)$, the monic polynomials $p_n(x, t)$ orthogonal with respect to the distribution of X_t are $p_n(x, t) = Q_n(x|0, 0, t)$. These polynomials satisfy a somewhat simpler three-step recurrence

$$\begin{aligned} xp_n(x,t) = {} & p_{n+1}(x,t) + \theta[n]_q p_n(x,t) \\ & +(t+\tau[n-1]_q)[n]_q p_{n-1}(x,t), \ n \geq 1. \end{aligned} \tag{44}$$

Identity (41) can be re-written as

$$Q_n(y|x, s, t) = \sum_{k=1}^{n} B_{n-k}(x) \left(p_k(y, t) - p_k(x, s)\right), \tag{45}$$

where $B_k(x)$ are polynomials in variable x such that $B_0 = 1$.

If $-1 \leq q < 1$, then the coefficients of the recurrence (44) are uniformly bounded. Therefore, the distribution of X_t has bounded support, see [30, Theorem 69.1]. If $q = 1$, these are classical Meixner polynomials (see [18, Ch. VI.3] or [26, Sections 4.2 and 4.3]), and their distributions have analytic characteristic functions. This implies that polynomials are dense in $L_2(X_s, X_u)$, see [19, Theorem 3.1.18].

We use these observations to extend [26, (4.4)] to some non-Lévy processes.

Proposition 3.3. *If (X_t) is the Markov process with transition probabilities (43) and $X_0 = 0$, then for $t > s$ and $n \geq 0$ we have*

$$E(p_n(X_t, t)|\mathcal{F}_{\leq s}) = p_n(X_s, s). \tag{46}$$

Proof. Notice that for $n \geq 1$ we have $E(Q_n(X_t|X_s, s, t)|X_s) = 0$, as $Q_n(y|x, s, t)$ is orthogonal to $Q_0 = 1$ under the conditional probability (43). We use this to prove (46) by induction.

Since $p_0 = 1$, (46) holds true for $n = 0$. Suppose (46) holds true for all $0 \leq n \leq N$. From (45) and the induction assumption it follows that

$$0 = E(Q_{N+1}(X_t|X_s, s, t)|X_s) = B_0(X_s) \left(E(p_{N+1}(X_t, t)|X_s) - p_{N+1}(X_s, s)\right).$$

Since $B_0 = 1$, this proves that $E(p_{N+1}(X_t, t)|X_s) = p_{N+1}(X_s, s)$, which by the Markov property implies (46) for $n = N + 1$. $\square$

Proposition 3.4. *If $-1 \leq q \leq 1$ and (X_t) is the Markov process with transition probabilities (43) and $X_0 = 0$, then (1), (2), (7), and (28) hold true.*

Proof. Let $p_n(x, t)$ be the monic polynomials which are orthogonal with respect to the distribution of X_t. For the first part of the proof we will write their three step recurrence (44) as

$$xp_n(x, t) = p_{n+1}(x, t) + a_n(t)p_n(x, t) + b_n(t)p_{n-1}(x, t), \tag{47}$$

where the coefficients are

$$a_n(t) = \theta[n]_q,\ b_n(t) = (t + \tau[n-1]_q)[n]_q. \tag{48}$$

We will also use the notation

$$a_n(t) = \alpha_n + t\beta_n,\ b_n(t) = \gamma_n + t\delta_n. \tag{49}$$

Recall that

$$E(p_{n+1}^2(X_t, t)) = b_{n+1}(t)E(p_n^2(X_t, t)), \tag{50}$$

see [18, page 19].

We first verify (7). Since $p_1(x, t) = x$, $p_2(x, t) = x^2 - \theta x - t$, from (46) we get $E(X_t^2|X_s) = E(p_2(X_t, t)|X_s) + \theta E(p_1(X_t, t)|X_s) + t = p_2(X_s, s) + \theta p_1(X_s, s) + t = X_s^2 + t - s$.

Condition (1) holds true as $E(X_t) = E(p_1(X_t, t)p_0(X_t, t)) = 0$, and for $s < t$ we have $E(X_s X_t) = E(X_s p_1(X_s, s)) = E(p_2(X_s, s) + \theta p_1(X_s, s) + s) = s$.

To verify (2), we use the fact that polynomials are dense in $L_2(X_s, X_u)$. Thus by the Markov property to prove (2) we only need to verify that

$$\begin{aligned} &E\left(p_n(X_s, s)X_t p_m(X_u, u)\right) \\ &\quad = aE\left(X_s p_n(X_s, s)p_m(X_u, u)\right) + bE\left(p_n(X_s, s)X_u p_m(X_u, u)\right) \end{aligned} \tag{51}$$

for all $m, n \in \mathbb{N}$ and $0 < s < t$. To prove this, we invoke Proposition 3.3. By (46)

$$E(p_n(X_s, s)X_t p_m(X_u, u)) = E(p_n(X_s, s)X_t p_m(X_t, t)).$$

Then by (47) and again using (46) we get that the left hand side of (51) is

$$\begin{aligned} &E(p_n(X_s, s)p_{m+1}(X_s, s)) + a_m(t)E(p_n(X_s, s)p_m(X_s, s)) \\ &\quad + b_m(t)E(p_n(X_s, s)p_{m-1}(X_s, s)). \end{aligned}$$

Thus the left hand side of the equation is zero, except when $n = m + 1$, $n = m$, or $n = m - 1$.

Similar argument applies to the right hand side of (51). Thus, writing Ep_m^2 for $E(p_m^2(X_s, s))$, equation (51) takes the form $0 = 0$, except for the following three cases.

(i) Case $n = m + 1$. Then the equation reads

$$Ep_{m+1}^2 = a(s, t, u)b_{m+1}(s)Ep_m^2 + b(s, t, u)Ep_{m+1}^2.$$

By (50) this holds true as $a + b = 1$, see (3).

(ii) Case $n = m$. Then the equation reads

$$a_m(t)Ep_m^2 = a(s, t, u)a_m(s)Ep_m^2 + b(s, t, u)a_m(u)Ep_m^2.$$

By (3), this equation holds true for any three step recurrence (47) with the coefficients $a_n(t)$ that are linear in variable t.

(iii) Case $n = m - 1$. In this case, (51) reads

$$b_m(t)Ep_{m-1}^2 = a(s,t,u)Ep_m^2 + b(s,t,u)b_m(u)Ep_{m-1}^2.$$

By (50) this is equivalent to $b_m(t) = a(s,t,u)b_m(s) + b(s,t,u)b_m(u)Ep_{m-1}^2$, which by (3) holds true for any three step recurrence (47) with the coefficients $b_n(t)$ that are linear in variable t.

The proof of (28) follows the same plan. We verify that (6) holds true with the parameters given by formulas (32), (33), (34), (35), (36), (37). (In fact, our proof indicates also how these formulas and the recurrence (44) were initially derived.) To do so, from the three step recurrence (47) we derive

$$\begin{aligned} x^2 p_{n-1}(x) = {} & p_{n+1}(x) + (a_n + a_{n-1})p_n(x) + (a_{n-1}^2 + b_n + b_{n-1})p_{n-1}(x) \\ & + b_{n-1}(a_{n-1} + a_{n-2})p_{n-2}(x) + b_{n-1}b_{n-2}p_{n-3}(x) \end{aligned} \tag{52}$$

for $n \geq 2$. (Recall that we use the convention $p_{-1}(x) = 0$.)

We need to prove that for any $n, m \in \mathbb{N}$ and $0 < s < t$

$$\begin{aligned} & E\left(p_n(X_s,s)X_t^2 p_m(X_u,u)\right) \\ & \quad = AE\left(X_s^2 p_n(X_s,s)p_m(X_u,u)\right) + BE\left(X_s p_n(X_s,s)X_u p_m(X_u,u)\right) \\ & \qquad + CE\left(p_n(X_s,s)X_u^2 p_m(X_u,u)\right) + \alpha E\left(X_s p_n(X_s,s)p_m(X_u,u)\right) \\ & \qquad + \beta E\left(p_n(X_s,s)X_u p_m(X_u,u)\right) + DE\left(p_n(X_s,s)p_m(X_u,u)\right). \end{aligned} \tag{53}$$

For the remainder of the proof, all the polynomials are evaluated at (X_s, s). Using (52), (47) and (46), we get

$$\begin{aligned} & Ep_np_{m+2} + (a_{m+1}(t) + a_m(t))Ep_np_{m+1} + (a_m^2(t) + b_{m+1}(t) + b_m(t))Ep_np_m \\ & \quad + b_m(t)(a_m(t) + a_{m-1}(t))Ep_np_{m-1} + b_m(t)b_{m-1}(t)Ep_np_{m-2} \\ & = A(Ep_{n+2}p_m + (a_{n+1}(s) + a_n(s))Ep_{n+1}p_m \\ & \quad + (a_n^2(s) + b_{n+1}(s) + b_n(s))Ep_np_m + b_n(s)(a_n(s) \\ & \quad + a_{n-1}(s))Ep_{n-1}p_m + b_n(s)b_{n-1}(s)Ep_{n-2}p_m) \\ & \quad + BE\left((p_{n+1} + a_n(s)p_n + b_n(s)p_{n-1})(p_{m+1} + a_m(u)p_m + b_m(u)p_{m-1})\right) \\ & \quad + C(Ep_np_{m+2} + (a_{m+1}(u) + a_m(u))Ep_np_{m+1} + (a_m^2(u) + b_{m+1}(u) \\ & \quad + b_m(u))Ep_np_m + b_m(u)(a_m(u) + a_{m-1}(u))Ep_np_{m-1} \\ & \quad + b_m(u)b_{m-1}(u)Ep_np_{m-2}) + \alpha(Ep_{n+1}p_m \\ & \quad + a_n(s)Ep_np_m + b_n(s)Ep_{n-1}p_m) + \beta(Ep_np_{m+1} \\ & \quad + a_m(u)Ep_np_m + b_m(u)Ep_np_{m-1}) + DEp_np_m. \end{aligned}$$

Thus the equation (53) takes the form $0 = 0$, except for the following five cases:

(i) Case $n = m + 2$. In this case, equation (53) reads

$$Ep_{m+2}^2 = Ab_{m+2}(s)b_{m+1}(s)Ep_m^2 + Bb_{m+2}(s)Ep_{m+1}^2 + CEp_{m+2}^2.$$

By (50), this is equivalent to (19), which holds true by our choice of A, B, C.

(ii) Case $n = m + 1$. In this case, equation (53) reads

$$\begin{aligned}&(a_{m+1}(t) + a_m(t))\, Ep_{m+1}^2\\ &\quad = Ab_{m+1}(s)(a_{m+1}(s) + a_m(s))Ep_m^2\\ &\qquad + B(a_{m+1}(s)Ep_{m+1}^2 + b_{m+1}(s)a_m(u)Ep_m^2)\\ &\qquad + C(a_{m+1}(u) + a_m(u))Ep_{m+1}^2 + \alpha b_{m+1}(s)Ep_m^2 + \beta Ep_{m+1}^2.\end{aligned}$$

By (49) and (50), this reduces to equation $(\beta_n + \beta_{n-1}) = \frac{(u-s)B}{D}\beta_{n-1}$, which holds true since $\beta_n = 0$, see (48).

(iii) Case $n = m$. In this case, equation (53) reads

$$\begin{aligned}&\left(a_m^2(t) + b_{m+1}(t) + b_m(t)\right) Ep_m^2\\ &\quad = A\left(a_m^2(s) + b_{m+1}(s) + b_m(s)\right) Ep_m^2\\ &\qquad + B\left(Ep_{m+1}^2 + a_m(s)a_m(u)Ep_m^2 + b_m(s)b_m(u)Ep_{m-1}^2\right)\\ &\qquad + C\left(a_m^2(u) + b_{m+1}(u) + b_m(u)\right) Ep_m^2\\ &\qquad + \alpha a_m(s)Ep_m^2 + \beta a_m(u)Ep_m^2 + DEp_m^2.\end{aligned}$$

After a calculation, this reduces to equation $\delta_n + \delta_{n-1} = \delta_{n-1}\frac{(u-s)B}{D} + 1$. The latter holds true by (48) and (31).

(iv) Case $n = m - 1$. In this case, equation (53) reads

$$\begin{aligned}&b_m(t)(a_m(t) + a_{m-1}(t))Ep_{m-1}^2\\ &\quad = Ab_m(s)(a_m(s) + a_{m-1}(s))Ep_{m-1}^2\\ &\qquad + B(a_m(u)Ep_m^2 + a_{m-1}(s)b_m(u)Ep_{m-1}^2)\\ &\qquad + Cb_m(u)(a_m(u) + a_{m-1}(u))Ep_{m-1}^2 + \alpha Ep_m^2 + \beta b_m(u)Ep_{m-1}^2.\end{aligned}$$

After a calculation, this reduces to equation

$$(\alpha_{n-1} + \alpha_{n-2})\delta_{n-1} = (1+q)\delta_{n-1}\alpha_{n-2} + \delta_{n-1}\frac{s\alpha + u\beta}{D(s,t,u)}.$$

The latter holds true for all $n \geq 2$ by (48) and (21).

(v) Case $n = m - 2$. In this case, equation (53) reads

$$b_m(t)b_{m-1}(t)Ep_{m-2}^2 = AEp_m^2 + Bb_m(u)Ep_{m-1}^2 + Cb_m(u)b_{m-1}(u)Ep_{m-2}^2.$$

After a calculation, this reduces to equation

$$\delta_{n-1}\gamma_{n-2} + \delta_{n-2}\gamma_{n-1} = (1+q)\delta_{n-1}\gamma_{n-2} + \delta_{n-1}\delta_{n-2}\frac{As^2 + Bsu + Cu^2 - t^2}{D}.$$

Using relation (20), this gives

$$\delta_{n-2}\gamma_{n-1} = \tau\delta_{n-1}\delta_{n-2} + q\delta_{n-1}\gamma_{n-2}.$$

The latter is satisfied with the initial condition $\gamma_1 = 0$ whenever

$$\gamma_n = \tau[n-1]_q\delta_n. \qquad \square$$

From Proposition 2.5 we see that the conditional variance of a stochastic process (X_t) that satisfies (1), (2), (6) with $D \neq 0$, (7), and which has at least 3-point support is given by (58) with parameters $-\infty < \theta < \infty$, $-1 < q \leq 1$, $\tau \geq 0$.

Let (Y_t) be the Markov process with the transition probabilities (43) and the same parameters. By Proposition 3.4, this process satisfies (1), (2), (7), and (28).

Since processes (X_t) and (Y_t) satisfy (1), (2), (7), and (28) with the same parameters q, θ, τ, and the distribution of (Y_t) is determined uniquely by moments, therefore by Proposition 2.4 the processes have the same finite dimensional distributions. This establishes our main result.

Theorem 3.5. *Let $(X_t)_{t\geq 0}$ be a separable square integrable stochastic process which satisfies conditions (1), (2), (6), and (7). Suppose that the coefficient D in (6) satisfies $D(s,t,u) \neq 0$ for all $0 \leq s < t < u$, and that $1, X_t, X_t^2$ are linearly independent for all $t > 0$. Then there are parameters $-1 < q \leq 1$, $\theta \in \mathbb{R}$, and $\tau \geq 0$ such that (X_t) is a Markov process, with the transition probabilities (43), $X_0 = 0$.*

Conversely, for any $-1 < q \leq 1, \tau \geq 0, \theta \in \mathbb{R}$, the Markov process with transition probabilities (43) satisfies (1), (2), (6), and (7).

Remark 3.1. If $1, X_u, X_u^2$ are linearly dependent, then the coefficients in (6) are not unique; in particular, one can modify $\beta(s,t,u)$ and $C(s,t,u)$ to get $D(s,t,u) = 0$ for all $s < t < u$, and the assumption $D \neq 0$ makes little sense. However, this can sometimes be circumvented, see Theorem 4.1.

Remark 3.2. For $q = 1$, expression (28) depends on the increments of (X_t) only, i.e., it takes the form analyzed in [32, Theorem 1], see also Theorem 4.2. It is tempting to use this case as a model and define the q-generalizations of the five types of Lévy processes determined in [32]:

(i) q-Wiener processes: $\tau = 0, \theta = 0$.
(ii) q-Poisson type processes: $\tau = 0, \theta \neq 0$.
(iii) q-Pascal type processes: $\tau > 0, \theta^2 > 4\tau$.
(iv) q-Gamma type processes: $\tau > 0, \theta^2 = 4\tau$.
(v) q-Meixner type processes: $\theta^2 < 4\tau$.

Some of these generalizations have already been studied in the non-commutative probability; for the q-Brownian motion see [11], for the q-Poisson process see [6], [22], [24], and the references therein. Anshelevich [5, Remark 6] states a recurrence which is equivalent to (38) for $s = 0, x = 0$; the latter, written as (44), plays the role in our proof of Theorem 3.5.

However, it is also possible that for $|q| < 1$ the differences between these processes are less pronounced; when $q = 0$, the transition probabilities in Theorem 4.3 share the continuous component and its discrete components also admit a common interpretation, dispensing with the "cases". The case of $q = 0$ is especially interesting, as it corresponds to certain free Lévy processes. As we already pointed out in the introduction, all free Lévy non-commutative processes have classical Markov versions by [10, Theorem 3.1].

4. Some special cases and examples

As we already mentioned in the introduction, some of the examples we encounter are classical versions of the non-commutative processes that already have been studied. It might be useful to clarify terminology. A non-commutative (real) process $(\mathbf{X}_t)_{t\in[0,\infty)}$ is a family of elements of a unital $*$-algebra $\mathcal{A}$ equipped with a state (i.e., normalized positive linear functional) $\Phi : \mathcal{A} \to \mathbb{C}$ such that $\mathbf{X}_t^* = \mathbf{X}_t$. A classical version of a non-commutative process $(\mathbf{X}_t)$ is a stochastic process (X_t) such that for every finite choice $0 \le t_1 \le t_2 \le \cdots \le t_k$ the corresponding moments match:

$$\Phi(\mathbf{X}_{t_1} \dots \mathbf{X}_{t_k}) = E(X_{t_1} \dots X_{t_k}). \tag{54}$$

If $\sum a^n \Phi(\mathbf{X}_t^{2n})/2n! < \infty$ for some $a > 0$, i.e., X_t has finite exponential moments, this condition determines uniquely the finite-dimensional distributions of (X_t). Of course, the left hand side of (54) depends on the order of $\{t_j\}$, which cannot be permuted.

4.1. q-Brownian process

For $-1 \le q \le 1$, the classical version of the q-Brownian motion, see [11, Definition 3.5 and Theorem 4.6], is a Markov process with the transition probabilities $P_{s,t}(x, dy)$ for $0 < s < t$ given by

$$\begin{cases} \frac{1}{2}\left(1+\sqrt{s/t}\right)\delta_{x\sqrt{t/s}}(dy) + \frac{1}{2}\left(1-\sqrt{s/t}\right)\delta_{-x\sqrt{t/s}}(dy) & \text{if } q=-1, \\ \frac{\sqrt{1-q}}{2\pi\sqrt{4t-(1-q)y^2}} & \\ \quad\times \prod_{k=0}^{\infty} \frac{(t-sq^k)\left(1-q^{k+1}\right)\left(t(1+q^k)^2-(1-q)y^2q^k\right)}{(t-sq^{2k})^2-(1-q)q^k(t+sq^{2k})xy+(1-q)(sy^2+tx^2)q^{2k}}\,dy & \text{if } -1<q<1, \\ \frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(y-x)^2}{2(t-s)}\right)dy & \text{if } q=1. \end{cases} \tag{55}$$

The support consists of two-point $\pm\frac{\sqrt{t}}{\sqrt{s}}x$ when $q = -1$, and is bounded $|y| < 2\sqrt{t}/\sqrt{1-q}$ when $-1 < q < 1$.

The univariate distribution of X_t, $t > 0$ is given by the transitions $P_{0,t}(0, dy)$ from $X_0 = 0$, which are given by

$$\begin{cases} \frac{1}{2}\delta_{\sqrt{t}}(dy) + \frac{1}{2}\delta_{-\sqrt{t}}(dy) & \text{if } q=-1, \\ \frac{\sqrt{1-q}}{2\pi\sqrt{4t-(1-q)y^2}}\prod_{k=0}^{\infty}\left((1+q^k)^2-(1-q)\frac{y^2}{t}q^k\right) & \\ \quad\times\prod_{k=0}^{\infty}(1-q^{k+1}) & \text{if } -1<q<1, \\ \frac{1}{\sqrt{2\pi t}}\exp(-\frac{y^2}{2t})\,dy & \text{if } q=1. \end{cases} \tag{56}$$

The following shows that the q-Brownian motion is characterized by the assumption that conditional variances are quadratic, coupled with the additional assumption that for $t < u$ the conditional variances $\mathrm{Var}(X_t|\mathcal{F}_{\geq u})$ are non-random.

Theorem 4.1. *Suppose that $(X_t)_{t\geq 0}$ is a square-integrable separable process such that (1), (2), (6), (7) hold true, and in addition*

$$\mathrm{Var}(X_t|\mathcal{F}_{\geq u}) = \frac{t}{u}(u-t), \tag{57}$$

for all $t < u$. Then there exists $q \in [-1, 1]$ such that

$$\mathrm{Var}(X_t|\mathcal{F}_{\leq s} \vee \mathcal{F}_{\geq u}) = \frac{(t-s)(u-t)}{u-qs}\left(\frac{(1-q)}{(u-s)^2}(X_u - X_s)(sX_u - uX_s) + 1\right). \tag{58}$$

Moreover, then (X_t) is Markov with transition probabilities (55) and (56). Conversely, a Markov process, $X_0 = 0$, with the transition probabilities given by (55) satisfies conditions (2), (6), (7), and (57).

Proof. Formulas (7) and (8) hold true with $\tau = \theta = 0$ by assumption. The proof of (30) relies only on these two formulas. Therefore, $E(X_t^4) = (2+q)t^2$ for some $-1 \leq q \leq 1$. In particular, $q = -1$ iff $(E(X_t^2))^2 = E(X_t)^4$, i.e., $X_t = \pm\sqrt{t}$ with equal probabilities. We need to consider separately cases $q = -1$ and $q > -1$.

If $q = -1$, the joint moments are uniquely determined from (4). Namely, if $s < t$ and m is odd then $E(X_t^m|\mathcal{F}_{\leq s}) = t^{(m-1)/2}E(X_t|\mathcal{F}_{\leq s}) = t^{(m-1)/2}X_s$. This determines all mixed moments uniquely: if $n_1, \dots, n_k$ are even numbers, $m_1, m_2, \dots, m_\ell$ are odd numbers, $s_1 < s_2 < \dots < s_\ell$, and ℓ is even then we have

$$E\left(X_{t_1}^{n_1}X_{t_2}^{n_2}\dots X_{t_k}^{n_k}X_{s_1}^{m_1}X_{s_2}^{m_2}\dots X_{s_\ell}^{m_\ell}\right) = \prod_{j=1}^{k} t_j^{n_j/2}\prod_{j=1}^{\ell/2}\left(s_{2j-1}^{(m_{2j-1}+1)/2}s_{2j}^{(m_{2j}-1)/2}\right).$$

If ℓ is odd, then $E\left(X_{t_1}^{n_1}X_{t_2}^{n_2}\dots X_{t_k}^{n_k}X_{s_1}^{m_1}X_{s_2}^{m_2}\dots X_{s_\ell}^{m_\ell}\right) = 0$. Since the same holds true for the two-valued Markov chain, and its conditional variance can be written as (58), this ends the proof when $q = -1$.

If $-1 < q \leq 1$, then $1, X_t, X_t^2$ are linearly independent for all $t > 0$. To apply Theorem 3.5 we need to verify that $D(s,t,u) \neq 0$ for all $s < t < u$. Suppose $D(s,t,u) = 0$ for some $0 \leq s < t < u$. Inspecting the proof of Lemma 2.3 we see that equations (7), (16) (which hold true by assumption) and linear independence imply (23),(24), (25), (26), and (27) with $D = 0$.

We now use these values and the value $E(X_sX_t^2X_u)$ to derive a contradiction. Notice that (4) and (5) imply that $E(X_sX_t^2X_u) = s/tE(X_t^4) = (2+q)st$. On the other hand, since $E(X_t^3) = 0$ and $D = 0$, from (6) we get

$$E(X_sX_t^2X_u) = AE(X_s^4) + BE(X_s^2X_u^2) + \frac{s}{u}CE(X_u^4).$$

Since $E(X_s^4) = (2+q)s^2$, and A, B, C are given explicitly, a calculation shows that this equation holds true only if $(u-t)(t-s) = 0$. Thus $D(s,t,u) \neq 0$ for all $0 \leq s < t < u$.

This shows that the assumptions of Theorem 3.5 are satisfied. Theorem 3.5 shows that X_t is Markov with uniquely determined transition probabilities. Formulas (55) and (56) give the distribution which orthogonalizes the corresponding Al-Salam–Chihara polynomials, see [8]. □

4.2. Lévy processes with quadratic conditional variance

A special choice of the coefficients in (6) casts the conditional variance as a quadratic function of the increments of the process,

$$\operatorname{Var}(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u}) = C_2(X_u - X_s)^2 + C_1(X_u - X_s) + C_0, \tag{59}$$

where $C_0 = C_0(s,t,u)$, $C_1 = C_1(s,t,u)$, $C_2 = C_2(s,t,u)$ are deterministic functions of $s < t < u$.

As an application of Theorem 3.5, we give the following version of [32, Theorem 1].

Theorem 4.2 (Wesolowski). *Let $(X_t)_{t\ge 0}$ be a square integrable separable stochastic process such that the conditions (1), (2), and (59) hold true, and $C_2 \ne ab$. If for every $t > 0$ the distribution of X_t has at least 3 point support, then there are numbers $\theta \in \mathbb{R}$, $\tau \ge 0$ such that the conditional variance (59) is given by*

$$\operatorname{Var}(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u}) = \frac{(u-t)(t-s)}{u-s+\tau}\left(\tau\frac{(X_u-X_s)^2}{(u-s)^2} + \theta\frac{X_u-X_s}{u-s} + 1\right). \tag{60}$$

Moreover, one of the following holds:

(i) $\tau = 0, \theta = 0$, *and (X_t) is the Wiener processes,*

$$E(\exp(iuX_t)) = \exp(-tu^2/2).$$

(ii) $\tau = 0, \theta \ne 0$, *and (X_t) is a Poisson type processes,*

$$E(\exp(iuX_t)) = \exp\left(\frac{t}{\theta^2}e^{iu\theta} - i\frac{ut}{\theta}\right).$$

(iii) $\tau > 0$ *and* $\theta^2 > 4\tau$, *and (X_t) is a Pascal (negative-binomial) type process,*

$$E(\exp(iuX_t)) = (p\exp(iu\delta_1) + (1-p)\exp(iu\delta_2))^{-t/\tau},$$

where $\delta_1 < \delta_2$ are the roots of $1 - \theta x + \tau x^2 = 0$ and $p = 1 - \delta_1/\delta_2$.

(iv) $\tau > 0$ *and* $\theta^2 = 4\tau$, *and (X_t) is a Gamma type process,*

$$E(\exp(iuX_t)) = \exp\left(-2iut/\theta^2\right)\left(1 - i\frac{u\theta}{2}\right)^{-4t/\theta^2}.$$

(v) $\theta^2 < 4\tau$, *and (X_t) is a Meixner (hyperbolic-secant) type process,*

$$E(\exp(iuX_t)) = \exp\left(-i\frac{u\theta t}{2\tau}\right)\left(\cosh(\frac{\sqrt{4\tau-\theta^2}u}{2}) + i\frac{\theta}{\sqrt{4\tau-\theta^2}}\sinh(\frac{\sqrt{4\tau-\theta^2}u}{2})\right)^{-t/\tau}.$$

Proof. We verify that the assumptions of Theorem 3.5 are satisfied.

Assumption (59) implies that (6) holds true with parameters $A = C_2 + a^2$, $B = 2ab - 2C_2$, $C = C_2 + b^2$, and $\alpha + \beta = 0$. Therefore, $A + B + C = 1$, which together with (17) implies (27). Since $C_2 \neq ab$ is the same as $C \neq b$, the latter implies that $D \neq 0$. Thus we can use Lemma 2.2. From (18) we get (7). Theorem 3.5 implies that (X_t) is a Markov process with the transition probabilities which are identified uniquely from their orthogonal polynomials, see [18, Ch VI.3]; see also [26, Sections 4.2 and 4.3]. In particular, (X_t) has independent and homogeneous increments, with the distribution of $X_{t+s} - X_s \cong X_t$ as listed in the theorem.

From separability, the usual properties of the trajectories of the Wiener and Poisson processes follow. □

4.3. Free Lévy processes with quadratic conditional variance

A special choice of the coefficients in (6) leads to the following conditional variance

$$\begin{aligned}&\mathrm{Var}(X_t|\mathcal{F}_{\le s} \vee \mathcal{F}_{\ge u})\\ &= ab\left(\frac{(X_u - X_s)(sX_u - uX_s)}{u+\tau} + \tau\frac{(X_u - X_s)^2}{(u-s)^2} + \theta\frac{X_u - X_s}{u-s} + 1\right), \quad (61)\end{aligned}$$

where a, b are the coefficients from (2). This formula seems hard to separate by natural assumptions from the general expression (28), but the fact that $q = 0$ leads to considerable computational simplifications. Theorem 3.5 in this setting takes the following form, with explicit foarmulas for the transition probabilities.

Theorem 4.3. *Let $(X_t)_{t\ge 0}$ be a square integrable separable stochastic process such that the conditions (1), (2), and (61) hold true. If for every $t > 0$ the distribution of X_t has at least 3 point support, then (X_t) is a Markov process with the transition probabilities $P_{s,t}(x, dy)$ given for $0 \le s < t$ by the Stieltjes-Cauchy transform*

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{z-y} P_{s,t}(x, dy)\\ &= \frac{1}{2}\frac{(t+s+2\tau)(z-x) + (t-s)\theta - (t-s)\sqrt{(z-\theta)^2 - 4(t+\tau)}}{\tau(z-x)^2 + \theta(t-s)(z-x) + tx^2 + sz^2 - (s+t)xz + (t-s)^2}. \quad (62)\end{aligned}$$

The absolutely continuous part of $P_{s,t}(x, dy)$ is given by the density

$$\frac{1}{2\pi}\frac{(t-s)\sqrt{4(t+\tau) - (y-\theta)^2}}{\tau(y-x)^2 + \theta(t-s)(y-x) + tx^2 + sy^2 - (s+t)xy + (t-s)^2},$$

supported on $(y-\theta)^2 < 4(t+\tau)$; the singular part is zero, and the discrete part is zero except for the following cases.

(i) *$\tau = 0, \theta \neq 0$. Then the discrete part of $P_{s,t}(x, dy)$ is non-zero only for $x = -s/\theta, 0 < s < t < \theta^2$ and is then*

$$\frac{1 - t/\theta^2}{1 - s/\theta^2}\delta_{-t/\theta}.$$

(ii) $\tau > 0$ *and* $\theta^2 > 4\tau$. *Then the discrete part of* $P_{s,t}(x, dy)$ *is non-zero only if* $x = y_*(s)$ *and is then*

$$\frac{\left(1 - \frac{t}{2\tau}\frac{|\theta|-\sqrt{\theta^2-4\tau}}{\sqrt{\theta^2-4\tau}}\right)^+}{1 - \frac{s}{2\tau}\frac{|\theta|-\sqrt{\theta^2-4\tau}}{\sqrt{\theta^2-4\tau}}}\delta_{y_*(t)},$$

where

$$y_*(t) = \begin{cases} -t\frac{\theta-\sqrt{\theta^2-4\tau}}{2\tau} & \text{if } \theta > 0 \\ -t\frac{\theta+\sqrt{\theta^2-4\tau}}{2\tau} & \text{if } \theta < 0 \end{cases}.$$

Proof. From (61) it follows that $D = ab \neq 0$ and $A + B + C = 1$. Since $1, X_t, X_t^2$ are linearly independent by assumption, from (18) we deduce (7). Thus by Theorem 3.5, (X_t) is a Markov process with the transition probabilities defined by (38). It remains to find the Cauchy-Stieltjes transform of the distribution.

It is well known that the Cauchy-Stieltjes transform

$$G_{x,s,t}(z) = \int_{\mathbb{R}} \frac{1}{z-y} P_{s,t}(x, dy)$$

is given by the continued fraction expansion associated with the orthogonal polynomials, [18, page 85]. The initial polynomials are

$$Q_0(y) = 1,\ Q_1(y) = y - x,\ Q_2(y) = y^2 - (x+\theta)y + \theta x - (t-s).$$

For $n \geq 2$, we have

$$yQ_n(y) = Q_{n+1}(y) + \theta Q_n(y) + (t+\tau)Q_{n-1}(y),$$

so for $n \geq 2$ this is a constant-coefficients recurrence. Thus the corresponding continued fraction is

$$G_{x,s,t}(z) = \cfrac{1}{z - x - \cfrac{t-s}{z - \theta - \cfrac{t+\tau}{z - \theta - \cfrac{t+\tau}{\ddots}}}}.$$

This gives

$$G_{x,s,t}(z) = \cfrac{1}{z - x - \cfrac{t-s}{\phi(z)}},$$

where

$$\phi(z) = \frac{z - \theta + \sqrt{(z-\theta)^2 - 4(t+\tau)}}{2}$$

solves the quadratic equation

$$\phi(z) = z - \theta - \frac{t+\tau}{\phi(z)}.$$

The branch of the root should be taken so that the imaginary parts satisfy $\Im(z)\Im(G_{x,s,t}(z)) \leq 0$. This branch should be taken as the regular branch when $\theta > x$ (with the cut from $-\infty$ to 0), and as the negative of the regular branch when $\theta < x$.

To get the explicit transition probabilities, we use the Stieltjes inversion formula: $P_{s,t}(x, dy)$ is the weak limit $\lim_{\varepsilon\to 0^+} -\frac{1}{\pi}\Im G(y+i\varepsilon)dy$, see [1, page 125], [18, (4.9)], [30, (65.4)].

For computational purposes, the following form is more convenient

$$G_{x,s,t}(z) = \frac{1}{2}\frac{(t+s+2\tau)(z-x)+(t-s)\theta-(t-s)\sqrt{(z-\theta)^2-4(t+\tau)}}{\tau(z-x)^2+\theta(t-s)(z-x)+tx^2+sz^2-(s+t)xz+(t-s)^2}.$$

The calculations are cumbersome but routine, and an equivalent calculation has been done by several authors, see [25, Theorem 2.1], [5, Theorem 4]. To get the answer given above, one relies on Markov property to determine the values of x which can be reached from 0 at time s. ◻

Remark 4.1. The transition probabilities from Theorem 4.3 can be cast into the form resembling Theorem 4.2. Since the continuous part varies smoothly as we vary the parameters, the main distinctions between the "five" processes are in the presence of the discrete component. Accordingly, we have the following cases:

(i) $\tau = 0, \theta = 0$, and (X_t) is the free Brownian motion with the law of X_t given by
$$\frac{1}{2\pi t}\sqrt{4t-x^2}\, 1_{x^2\leq 4t}\, dx,$$
see [10, Section 5.3].

(ii) $\tau = 0, \theta \neq 0$, and (X_t) is a free Poisson type processes with the law of X_t given by
$$\left(1-t/\theta^2\right)^+ \delta_{-t/\theta}(dx) + \frac{1}{2\pi}\frac{1}{\theta x+t}\sqrt{4t-(x-\theta)^2}\, 1_{(x-\theta)^2\leq 4t}\, dx,$$
compare [29, Section 2.7].

(iii) $\tau > 0$ and $\theta^2 > 4\tau$, and (X_t) is a free Pascal (Negative binomial) process with the law of X_t given by
$$p_*(t)\delta_{x_*} + \frac{1}{2\pi}\frac{t}{\tau x^2+t\theta x+t^2}\sqrt{4(t+\tau)-(x-\theta)^2}\, 1_{(x-\theta)^2\leq 4(t+\tau)}\, dx,$$
where
$$p_*(t) = \left(1-\frac{t}{2\tau}\frac{|\theta|-\sqrt{\theta^2-4\tau}}{\sqrt{\theta^2-4\tau}}\right)^+,$$
and
$$x_*(t) = \begin{cases} t(\sqrt{\theta^2-4\tau}-\theta)/(2\tau) & \text{if } \theta > 0, \\ \\ -t(\sqrt{\theta^2-4\tau}+\theta)/(2\tau) & \text{if } \theta < 0, \end{cases}$$
compare [5, Theorem 4].

(iv) $\tau > 0$ and $\theta^2 = 4\tau$ and (X_t) is a free Gamma type process with the law of X_t given by

$$\frac{1}{2\pi}\frac{4t}{(x\theta+2t)^2}\sqrt{4t+\theta^2-(x-\theta)^2}\,1_{(x-\theta)^2\leq 4t+\theta^2}\,dx,$$

compare [5, Theorem 4].

(v) $\theta^2 < 4\tau$, and (X_t) is a free Meixner (hyperbolic-secant) type process with the law of X_t given by

$$\frac{1}{2\pi}\frac{t}{\tau x^2+t\theta x+t^2}\sqrt{4(t+\tau)-(x-\theta)^2}\,1_{(x-\theta)^2\leq 4t}\,dx,$$

compare the "Continuous Binomial process" in [5, Theorem 4].

We remark that these measures occurred in the literature. The free Brownian and free Poisson processes have been studied in considerable detail, see [29] and the references therein. Symmetric free Meixner distribution appears in [13, Theorem 3], and in [12]. According to [25, Theorem 3.2(2)], these laws are infinitely divisible with respect to the free convolution, with explicit Lévy representations. All five distributions occur in Anshelevich [5, Theorem 4]; Anshelevich also points out that the correspondence between the classical and free Levy processes based on the values of parameters θ, τ does not match the Bercovici-Pata bijection.

4.4. Binomial Example

The coefficients in (2) and (6) alone do not determine the distribution of a process, and (2) and (6) may be satisfied by processes with univariate distributions different than those listed in Theorem 3.5.

Proposition 4.4. *Let $p : [0,\infty) \to [0,\infty)$ be such that $\int_0^\infty p(x)\,dx < 1$. Fix $m \in \mathbb{N}$ and let $\pi(s,t) = \int_s^t p(x)\,dx$. The Markov process $(Y_s)_{s\geq 0}$ with $Y_0 = 0$ and the transition probabilities*

$$P(Y_t = j|Y_s = i) = \frac{(m-i)!}{(j-i)!(m-j)!}\frac{(\pi(s,t))^{j-i}\,(1-\pi(0,t))^{m-j}}{(1-\pi(0,s))^{m-i}}\ ,$$

for $0 \leq i \leq j \leq m$ and any $0 \leq s < t$, satisfies (2) and (6) with the coefficients that do not depend on the parameter $m \in \mathbb{N}$. Namely,

$$E(Y_t|\mathcal{F}_{\leq s} \vee \mathcal{F}_{\geq u}) = \frac{\pi(t,u)}{\pi(s,u)}Y_s + \frac{\pi(s,t)}{\pi(s,u)}Y_u \tag{63}$$

and

$$\mathrm{Var}(Y_t|\mathcal{F}_{\leq s} \vee \mathcal{F}_{\geq u}) = \frac{\pi(s,t)\pi(t,u)}{(\pi(s,u))^2}(Y_u - Y_s). \tag{64}$$

Proof. We first show that the transition probabilities are consistent. For any $0 \le s < t < u$ and integers $i, n \ge 0$, $i + n \le m$

$$\begin{aligned}
&P(Y_u = i+n | Y_s = i) \\
&= \sum_{j=0}^{n} P(Y_u = i+n | Y_t = i+j) P(Y_t = i+j | Y_s = i) \\
&= \sum_{j=0}^{n} \frac{(m-i)!\,(\pi(t,u))^{n-j}\,(1-\pi(0,u))^{m-i-n}\,(\pi(s,t))^{j}}{j!(n-j)!(m-i-n)!\,(1-\pi(0,s))^{m-i}} \\
&= \frac{(m-i)!\,(1-\pi(0,u))^{m-i-n}}{n!(m-i-n)!\,(1-\pi(0,s))^{m-i}} \sum_{j=0}^{n} \binom{n}{j} (\pi(s,t))^{j}\,(\pi(t,u))^{n-j} \\
&= \frac{(m-i)!}{n!(m-i-n)!} \frac{(1-\pi(0,u))^{m-i-n}}{(1-\pi(0,s))^{m-i}} (\pi(s,t)+\pi(t,u))^{n}\,. \\
&= \frac{(m-i)!}{n!(m-i-n)!} \frac{(1-\pi(0,u))^{m-i-n}}{(1-\pi(0,s))^{m-i}} (\pi(s,u))^{n}\,.
\end{aligned}$$

Then the joint distribution of (Y_s, Y_t, Y_u) is given by

$$\begin{aligned}
&P(Y_u = i+j+k,\ Y_t = i+j,\ Y_s = i) \\
&= P(Y_u = i+j+k | Y_t = i+j) P(Y_t = i+j | Y_s = i) P(Y_s = i | Y_0 = 0) \\
&= \binom{m-i-j}{k} \frac{(\pi(t,u))^{k}\,(1-\pi(0,u))^{m-i-j-k}}{(1-\pi(0,t))^{m-i-j}} \\
&\quad \times \binom{m-i}{j} \frac{(\pi(s,t))^{j}\,(1-\pi(0,t))^{m-i-j}}{(1-\pi(0,s))^{m-i}} \\
&\quad \times \binom{m}{i} (\pi(0,s))^{i}\,(1-\pi(0,s))^{m-i} \\
&= \frac{m!}{i!j!k!(m-i-j-k)!} (\pi(0,s))^{i}\,(\pi(s,t))^{j}\,(\pi(t,u))^{k} \\
&\quad \times (1-\pi(0,u))^{m-i-j-k}\,.
\end{aligned}$$

From this, it is easy to see that conditionally on Y_s, Y_u, the increment $Y_t - Y_s$ has the binomial distribution with $Y_u - Y_s$ trials and the probability of success $\pi(s,t)/\pi(s,u)$, i.e.,

$$P(Y_t = k+i | Y_s = i, Y_u = i+n) = \binom{n}{k} \left(\frac{\pi(s,t)}{\pi(s,u)}\right)^{k} \left(\frac{\pi(t,u)}{\pi(s,u)}\right)^{n-k}.$$

Therefore

$$E(Y_t | Y_s, Y_u) = Y_s + \frac{\pi(s,t)}{\pi(s,u)} (Y_u - Y_s),$$

and (63) follows from the Markov property. Similarly, (64) is a consequence of Markov property and the formula for the variance of the binomial distribution. □

Remark 4.2. For $s \leq t$ the conditional distribution of $Y_t - Y_s$ given Y_s is binomial $b(m - Y_s, \pi(s,t)/(1-\pi(0,s)))$, which gives

$$Cov(Y_s, Y_t) = m\pi(0,s)\,(1-\pi(0,t))\,.$$

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