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Approximation theorems for random permanents and associated stochastic processes

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Abstract. The limit theorems for certain stochastic processes generated by permanents of random matrices of independent columns with exchangeable components are established. The results are based on the martingale decomposition of a random permanent function similar to the one known for U -statistics and on relating the components of this decomposition to some multiple stochastic integrals.

1. Introduction

Denote by $\mathbb{A} = [a_{ij}]$ an $m \times n$ real matrix with $m \leq n$. Then a permanent of the matrix \mathbb{A} is defined by

$$\text{Per } \mathbb{A} = \sum_{(i_1, \dots, i_m): \{i_1, \dots, i_m\} \subset \{1, \dots, n\}} a_{1i_1} \cdots a_{mi_m}.$$

The permanent function has a long history, having been first introduced by Cauchy in 1812 in his celebrated memoir on determinants. More recently, several problems in statistical mechanics, quantum field theory and chemistry, as well as enumeration problems in combinatorics and linear algebra have been reduced to the computation of a permanent. Unfortunately, the fastest known algorithm for computing a permanent of $n \times n$ matrix runs, as shown by Ryser [16], in $O(n2^n)$ time. Moreover, strong evidence for the apparent complexity of the problem was provided by Valiant [20] who showed that evaluating a permanent is $\#P$ -complete, even when restricted to 0–1 matrix. In view of these results, the approximation theory for a permanent function in case of a large dimension of the matrix \mathbb{A} has become a very active area of research over past twenty years (for most recent results for both random and non-random settings cf. e.g., Pate [12], Rempala and

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Wesołowski [15], Forbert and Marx [5], Coelho, and Duffner [4] and references therein).

Let us note that if the entries of \mathbb{A} satisfy $a_{ij} = a_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, then $\text{Per } \mathbb{A}$ reduces, in essence, to an elementary symmetric polynomial of degree m in variables a_1, \dots, a_n . In probability theory a *random permanent* function is, therefore, of interest since it can be viewed as a natural generalization of a concept of a *random elementary symmetric polynomial*, i.e., a U -statistic based on the product kernel which asymptotic behavior has been intensely studied (cf. e.g., Székely [18], Móri and Székely [11], van Es and Helmers [21] or Székely and Szeldi [17]).

In the present work we shall be concerned with the setting where $\mathbb{A} = \mathbb{X}^{(n)} = [X_{ij}]$ is an $m \times n$ ($m \leq n$) real random matrix of square integrable components and such that its columns are build from the first m terms of independent identically distributed (iid) sequences $(X_{i,1})_{i \geq 1}, (X_{i,2})_{i \geq 1}, \dots, (X_{i,n})_{i \geq 1}$ of exchangeable random variables (r.v.'s). Note that under these assumptions all entries of the matrix $\mathbb{X}^{(n)}$ are identically distributed, albeit not necessarily independent. For $i, k = 1, \dots, m$ and $j = 1, \dots, n$ we denote $\mu = E X_{ij}, \sigma^2 = \text{Var } X_{ij}$ and $\rho = \text{Corr}(X_{kj}, X_{ij})$. Observe that, via De-Finetti's theorem, necessarily $\rho \geq 0$. Assuming in the sequel that $\mu \neq 0$ we denote additionally by $\gamma = \sigma/\mu$ the variation coefficient. Finally, we let \mathcal{N} denote a standard normal rv and \xrightarrow{d} stand for convergence in distribution. In what follows $m = m_n \leq n$ is always a non-decreasing sequence. In this notation we derive herein the following.

Theorem 1 (Approximation theorem for random permanents).

(i) Suppose that $\rho = 0$.

If $m/n \rightarrow \lambda > 0$ as $n \rightarrow \infty$ then

$$\frac{1}{\binom{n}{m} m! \mu^m} \text{Per } (\mathbb{X}^{(n)}) \xrightarrow{d} \exp(\sqrt{\lambda} \gamma \mathcal{N} - \lambda \gamma^2 / 2).$$

If $m/n \rightarrow \lambda = 0$ and $m = m_n \rightarrow \infty$ as $n \rightarrow \infty$ then

$$\sqrt{\frac{n}{m}} \left(\frac{\text{Per } (\mathbb{X}^{(n)})}{\binom{n}{m} m! \mu^m} - 1 \right) \xrightarrow{d} \gamma \mathcal{N}.$$

(ii) Suppose that $\rho > 0$.

If $m^2/n \rightarrow \lambda > 0$ as $n \rightarrow \infty$ then

$$\frac{1}{\binom{n}{m} m! \mu^m} \text{Per } (\mathbb{X}^{(n)}) \xrightarrow{d} \exp(\sqrt{\lambda \rho} \gamma \mathcal{N} - \lambda \rho \gamma^2 / 2).$$

If $m^2/n \rightarrow \lambda = 0$ and $m = m_n \rightarrow \infty$ as $n \rightarrow \infty$ then

$$\frac{\sqrt{n}}{m} \left(\frac{\text{Per } (\mathbb{X}^{(n)})}{\binom{n}{m} m! \mu^m} - 1 \right) \xrightarrow{d} \sqrt{\rho} \gamma \mathcal{N}.$$

This general result extends and gives the common ground to some special cases of the above reported at various times in the literature. See, for instance, van Es and Helmers [21], Korolyuk and Borovskikh [8], Janson [7], and Rempała and Wesolowski [13]. As a motivation for the potential applicability of the result, consider the following simple example.

Example 1. Counting matchings in a bipartite random graph. Let $G_{m,n,p} = (V_1, V_2; E)$ be a bipartite random graph with $V_1 = \{r_1, r_2, \dots, r_m\}$, $V_2 = \{c_1, c_2, \dots, c_n\}$, ($m \leq n$) and $E \subset V_1 \times V_2$. Assume that the edges occur independently with a fixed probability $0 < p < 1$. In this setting, the reduced adjacency matrix of $G_{m,n,p}$ is a random $m \times n$ matrix $X = [X_{i,j}]$ of independent Bernoulli $B(p)$ random variables. If $m = n$ it is well known (cf. e.g., [2] chapter 7) that the number of *perfect matchings* in $G_{n,n,p}$, equals $\text{Per}(X)$. Extending the concept of a perfect matching to the case when $m < n$, we shall say that a matching is *fully saturating* if it saturates the set V_1 . Denoting the number of fully saturating matchings by $\mathcal{H}(G_{m,n,p})$ we have again $\mathcal{H}(G_{m,n,p}) = \text{Per}(X)$. Observe that according to our earlier notation we have $\mu = p$, $\gamma = \sqrt{(1-p)/p}$. Note also that in this particular case the distribution of $\mathcal{H}(G_{m,n,p})$ is the same as that of a random variable $m! \binom{T_{mn}}{m}$, where T_{mn} is a binomial $b(mn, p)$. From Theorem 1 it follows now that if $n, m \rightarrow \infty$ and $m/n \rightarrow \lambda > 0$ then

$$\frac{\mathcal{H}(G, m, n, p)}{\binom{n}{m} m! p^m} \xrightarrow{d} \exp(\sqrt{\lambda} \gamma \mathcal{N} - \lambda \gamma^2 / 2).$$

On the other hand, if $n - m \rightarrow \infty$ and $m/n \rightarrow 0$ then,

$$\sqrt{\frac{mp}{n(1-p)}} \left(\frac{\mathcal{H}(G, m, n, p)}{\binom{n}{m} m! p^m} - 1 \right) \xrightarrow{d} \mathcal{N}.$$

In the case $m = n$ the first of the above relations was noted, in a slightly different form, in [7].

In the current paper we show that the assertions of Theorem 1 are consequences of the corresponding functional limit theorems based on some general results from the stochastic integration theory. For a different approach see [15]. In order to consider a functional version of Theorem 1 we extend the definition of a permanent function to a stochastic process (in the sequel referred to as “permanent stochastic process” or PSP) in a manner similar to that of extending the concept of a “ U -statistic” to a “ U -process” (see [10]). That such a construction is possible follows from the orthogonal decomposition of a random permanent introduced in Rempała and Wesolowski [14], similar to the famous Hoeffding decomposition for U -statistics (cf, e.g., [10]) and, in fact, being in some sense its natural extension. In order to obtain the final result we first derive a general limit theorem for the elementary symmetric polynomial process (ESPP) based on a triangular array of rowwise independent, zero mean, square integrable random variables (rv’s). Our theorem in the special case of the independent, identically distributed (iid) sequence of random variables reduces to the one obtained in Móricz and Székely [11] and Kurtz and Protter [9]. For a concise proof, we adapt to our setting the idea of Kurtz and Protter

taken from [9] (referred to as KP in the sequel) of representing ESPP as a multiple stochastic integral.

The paper is organized as follows. In the reminder of this section we define PSP via orthogonal decomposition of a permanent. In Section 2 we recall some basic facts from the stochastic integration theory and then derive the invariance principle for an ESPP via stochastic integrals convergence theorem of KP. The result is presented as Theorem 2. In Section 3 we relate the results of Section 2 to the asymptotic behavior of the component processes of PSP (Proposition 1). Once this result is established the results for PSP's follow via the truncation-type argument along with the basic properties of the Prohorov distance and the martingale properties of the component processes. These are the main results of the paper and are presented in the last two theorems of Section 3. In particular, from the theorems of Section 3 the approximation theorems of Rempała and Wesolowski quoted above follow immediately.

To define the appropriate random functions related to permanents, let us first note the following useful representation of a random permanent. Namely, it has been proved in [14] that

$$\frac{\text{Per } \mathbb{X}^{(n)}}{\binom{n}{m} m! \mu^m} = 1 + \sum_{c=1}^m \binom{m}{c} U_c^{(m,n)}, \tag{1}$$

where

$$U_c^{(m,n)} = \binom{n}{c}^{-1} \binom{m}{c}^{-1} c!^{-1} \sum_{1 \leq i_1 < \dots < i_c \leq m} \sum_{1 \leq j_1 < \dots < j_c \leq n} \text{Per} [\tilde{X}_{i_u j_v}]_{\substack{u=1, \dots, c \\ v=1, \dots, c}}$$

for $\tilde{X}_{ij} = X_{ij}/\mu - 1, i = 1, \dots, m, j = 1, \dots, n$. Moreover, the $U_c^{(m,n)}$'s are orthogonal, i.e.,

$$\text{Cov} \left(U_{c_1}^{(m,n)}, U_{c_2}^{(m,n)} \right) = 0 \quad \text{for } c_1 \neq c_2. \tag{2}$$

as well as

$$\text{Var } U_c^{(m,n)} = \binom{n}{c}^{-1} \binom{m}{c}^{-1} \gamma^{2c} \sum_{r=0}^c \binom{m-r}{c-r} \frac{\rho^{c-r} (1-\rho)^r}{r!}. \tag{3}$$

Extending the definition of orthogonal components $U_c^{(m,n)}$ ($c=1,2, \dots$) to random functions is straightforward. For any real number x let $[x]$ denote the largest integer less or equal to x . Then for any $c=1,2, \dots$ and for any $t \in [0, 1]$ such that $c \leq [nt]$ let us define

$$U_c^{(m,n)}(t) = \frac{1}{\binom{n}{c} \binom{m}{c} c!} \sum_{1 \leq i_1 < \dots < i_c \leq m} \sum_{1 \leq j_1 < \dots < j_c \leq [nt]} \text{Per} [\tilde{X}_{i_u j_v}]_{\substack{u=1, \dots, c \\ v=1, \dots, c}}$$

and put $U_c^{(m,n)}(t) = 0$ if $c > [nt]$.

Now, the decomposition (1) suggests that on the interval $[0, 1]$ we may define the stochastic process associated with $\text{Per}(\mathbb{X}^{(n)})$, say $\mathcal{PSP}\mathbb{X}^{(n)}(\cdot)$, by the relation

$$\mathcal{PSP}\mathbb{X}^{(n)}(t) = \binom{n}{m} m! \mu^m \left[1 + \sum_{c=1}^m \binom{m}{c} U_c^{(m,n)}(t) \right] \quad t \in [0, 1].$$

It follows from (1) that the trajectory of $\mathcal{PSP}\mathbb{X}^{(n)}(\cdot)$ coincides with a permanent function for $t = 1$, i.e.,

$$\mathcal{PSP}\mathbb{X}^{(n)}(1) = \text{Per}(\mathbb{X}^{(n)}). \tag{4}$$

2. Weak Convergence of the ESPP

In this section we derive a general result on weak convergence of ESPP based on any rowwise independent (i.e., independent across any row) triangular array of zero mean, square integrable rv's. The result can be viewed as the extension of Donsker's central limit theorem. Recall that in accordance with our initial convention here and elsewhere the symbol \xrightarrow{d} denotes weak convergence.

Theorem 2. *Let $Y_{l,k}$, $k = 1, \dots, l$, $l = 1, 2, \dots$, be a double array of square integrable, zero-mean, rowwise independent rv's. Assume that*

$$\lim_{l \rightarrow \infty} \frac{1}{l} \sum_{k=1}^l \text{Var} Y_{l,k} = \sigma^2 > 0, \tag{5}$$

$$\forall \varepsilon > 0 \quad \lim_{l \rightarrow \infty} \frac{1}{l} \sum_{k=1}^l E Y_{l,k}^2 I\{|Y_{l,k}| > \varepsilon\sqrt{l}\} = 0. \tag{6}$$

Then, for any positive integer c ,

$$\left[\left(\frac{S_{[lt]}^{(l)}(1)}{\sqrt{l}} \right)_{t \in [0,1]}, \dots, \left(\frac{S_{[lt]}^{(l)}(c)}{(\sqrt{l})^c} \right)_{t \in [0,1]} \right]^T \xrightarrow{d} \left[\left(\frac{\sigma t^{1/2}}{1!} H_1(B_t/\sqrt{t}) \right)_{t \in [0,1]}, \dots, \left(\frac{\sigma^c t^{c/2}}{c!} H_c(B_t/\sqrt{t}) \right)_{t \in [0,1]} \right]^T \tag{7}$$

in the Skorokhod space $D_{\mathbf{R}^c}([0, 1])$, where

$$S_{[lt]}^{(l)}(j) = \sum_{1 \leq k_1 < \dots < k_j \leq [lt]} Y_{l,k_1} \dots Y_{l,k_j},$$

is an elementary symmetric polynomial of degree j (where a sum over empty set of indices is taken as zero), H_j is the j -th Hermite polynomial with leading coefficient equal to one, $j = 1, 2, \dots$, and $\mathbf{B} = (B_t)_{t \in [0,1]}$ is the standard Brownian motion.

Remark 1. The asymptotic distribution of $l^{-c/2} S_{[l]}^{(l)}(c)$ for fixed c and iid rv's appears in Móri and Székely [11]. The relation (7), again for iid rv's only, appears in KP.

Remark 2. The above theorem can be also viewed as a functional version of the result on elementary symmetric polynomial statistics in [15].

Several different ways of proving the statement (7) are possible. In particular, the approach utilizing the Newton identities along with some properties of elementary symmetric polynomials and the Marcinkiewicz-Zygmund-type law of large numbers could be adapted to the present setting. However, a somewhat simpler proof is also possible, based on a quite general result on the convergence of stochastic integrals, due to KP (see also Jakubowski, Memin and Pagés [6]). We have chosen the latter approach below since the application of the stochastic integrals theory allows us to present a more concise argument, avoiding many technical details that otherwise would need to be addressed.

In order to proceed we shall recall first some basic facts and definitions. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}$ be a filtration in a probability space (Ω, \mathcal{F}, P) . If $Y = (Y_t)_{t \in [0,1]}$ is a right continuous \mathbb{F} martingale then its quadratic variation process $([Y]_t)_{t \in [0,1]}$ is defined for any $t \in [0, 1]$ by

$$\sum_{k=1}^n (Y_{u_{k+1}^{(n)}} - Y_{u_k^{(n)}})^2 \xrightarrow{P} [Y]_t,$$

where $(u_k^{(n)})$ is a sequence of (non-random) partitions of $[0, t]$ such that $\max_k (u_{k+1}^{(n)} - u_k^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$. Denote by $D_{\mathbf{R}^k}$ the Skorokhod space of cadlag functions on $[0, 1]$ with values in \mathbf{R}^k . Let Y be an \mathbb{F} martingale with trajectories in $D_{\mathbf{R}}$ and let $X = (X_t)_{t \in [0,1]}$ be an \mathbb{F} adapted process with trajectories in $D_{\mathbf{R}^k}$. Then, following KP we define the stochastic integral as a limit in probability of the Riemann-Stieltjes sums

$$\int_0^t X_s dY_s = \lim \sum X_{t_i^{(n)}} (Y_{t_{i+1}^{(n)}} - Y_{t_i^{(n)}}),$$

for (non-random) partitions $(t_i^{(n)})$ of $[0, t]$ such that $\max(t_{i+1}^{(n)} - t_i^{(n)})$ converges to zero. The integral exists if the respective limit in probability of the Riemann-Stieltjes sums exists. We will use the notation $\int X dY$ for the process $(\int_0^t X_s dY_s)_{t \in [0,1]}$.

The version of the convergence result for stochastic integrals which suits our purposes can be formulated in the following way (cf. Theorem 2.2 in [9]).

Theorem 3 (KP). *Assume that $(X, Y), (X_1, Y_1), (X_2, Y_2), \dots$ are pairs of stochastic processes such that trajectories of the their first components are in $D_{\mathbf{R}^k}$ and trajectories of the second components are in $D_{\mathbf{R}}$. Let Y_n be a martingale with respect to a filtration $\mathbb{F}_n, n = 1, 2, \dots$, and $\sup_{n \geq 1} E([Y_n]_t) < \infty$ for any $t \in [0, 1]$, where $([Y_n]_t)_{t \in [0,1]}$ is the quadratic variation process for the square integrable martingale Y_n . Assume also that the process X_n is \mathbb{F}_n adapted, $n = 1, 2, \dots$. If (X_n, Y_n) converges weakly to (X, Y) in $D_{\mathbf{R}^{k+1}}$ then $(X_n, Y_n, \int X_n dY_n)$ converges weakly to $(X, Y, \int X dY)$ in $D_{\mathbf{R}^{2k+1}}$.*

We shall use the above result to give a simple proof of our Theorem 2. The idea of representing elementary symmetric polynomials via stochastic integrals comes from Example 3.1 of KP.

Proof. (of Theorem 2) Define for any $k = 1, 2, \dots, c$

$$Y_n^{(k)}(t) = \frac{S_{[nt]}^{(n)}(k)}{(\sqrt{n})^k} \quad t \in [0, 1].$$

Observe that

$$Y_n^{(k)} = \int Y_n^{(k-1)} dY_n^{(1)}.$$

Also define

$$Y^{(k)}(t) = \left(\frac{\sigma^k t^{k/2}}{k!} H_k(B_t/\sqrt{t}) \right) \quad t \in [0, 1].$$

Note that a simple application of Ito's formula gives

$$Y^{(k)} = \int Y^{(k-1)} dY^{(1)}.$$

Let $Z_n^{(k)} = [Y_n^{(1)}, \dots, Y_n^{(k)}]$, $n = 1, 2, \dots$ and let $Z^{(k)} = [Y^{(1)}, \dots, Y^{(k)}]$, $k = 1, 2, \dots, c$.

To prove that

$$Z_n^{(k)} \xrightarrow{d} Z^{(k)},$$

in the Skorokhod space $D_{\mathbf{R}^k}$, $k = 1, \dots, c$, we use induction with respect to k .

The result for $k = 1$ follows in view of the conditions (5) and (6) which imply, by the Donsker theorem for triangular arrays (see, e.g. Billingsley [1], page 194), that $Z_n^{(1)} = Y_n^{(1)} \xrightarrow{d} \mathbf{B} = Y^{(1)}$ in $D_{\mathbf{R}}$. Observe also that the process $Y_n^{(1)}$ is a martingale since it is based on the summation of independent rv's. Moreover, since the process $Y_n^{(1)}$ just cumulates jumps while remaining constant for given n in-between jumps, then its quadratic variation is

$$[Y_n^{(1)}]_t = n^{-1} \sum_{k=1}^{[nt]} Y_{n,k}^2.$$

Consequently,

$$\sup_n E([Y_n^{(1)}]_t) = \sup_n \frac{[nt]\sigma^2}{n} \leq t\sigma^2.$$

Let us also note that all the processes $Y_n^{(k)}$, $k = 1, \dots, c$, are adapted to the same filtration, generated by $Y_n^{(1)}$, $n = 1, 2, \dots$, and similarly, the processes $Y^{(k)}$, $k = 1, \dots, c$, are adapted to the same filtration, generated by the Wiener process \mathbf{B} .

Assume now that $Z_n^{(k-1)} \xrightarrow{d} Z^{(k-1)}$ in $D_{\mathbf{R}^{k-1}}$. Then, obviously, $(Z_n^{(k-1)}, Y_n^{(1)}) \xrightarrow{d} (Z^{(k-1)}, Y^{(1)})$ in $D_{\mathbf{R}^k}$ as $Y_n^{(1)}$ and $Y^{(1)}$ are components of $Z_n^{(k-1)}$ and $Z^{(k-1)}$, respectively. Consequently, by the KP theorem it follows that

$$(Z_n^{(k-1)}, Y_n^{(1)}, \int Z_n^{(k-1)} dY_n^{(1)}) \xrightarrow{d} (Z^{(k-1)}, Y^{(1)}, \int Z^{(k-1)} dY^{(1)}).$$

But the above convergence yields immediately the final result since $(Y_n^{(1)}, \int Z_n^{(k-1)} dY_n^{(1)}) = Z_n^{(k)}$ and $(Y^{(1)}, \int Z^{(k-1)} dY^{(1)}) = Z^{(k)}$. \square

3. Convergence of the component processes

Recall that by definition

$$\frac{\mathcal{PSP} \mathbb{X}^{(n)}(t)}{\binom{n}{m} m! \mu^m} = \left[1 + \sum_{c=1}^m \binom{m}{c} U_c^{(m,n)}(t) \right] \quad t \in [0, 1].$$

We shall first establish the limiting result for $U_c^{(m,n)}(\cdot)$ ($c = 1, 2, \dots$).

To this end, let us consider for any $c = 1, 2, \dots$ a process $(W_c^{(n)}(t))$ defined by the following rescaling of $U_c^{(m,n)}(t)$.

$$W_c^{(n)}(t) = \binom{n}{c} \binom{m}{c} c! U_c^{(m,n)}(t) = \sum_{1 \leq i_1 < \dots < i_c \leq m} \sum_{1 \leq j_1 < \dots < j_c \leq [nt]} \mathcal{PSP}[\tilde{X}_{i_u j_v}](t)_{\substack{u=1, \dots, c \\ v=1, \dots, c}}$$

for $t \in [c/n, 1]$ and $W_c^{(n)}(t) = 0$ for $t \in [0, c/n)$. The reminder of this section is devoted to the proof of the following

Proposition 1. *Let c be an arbitrary positive integer. Assume that $m = m_n \rightarrow \infty$ as $n \rightarrow \infty$.*

If $\rho = 0$ then

$$\begin{aligned} & \left[\frac{W_1^{(n)}(t)}{\sqrt{mn}}, \dots, \frac{W_c^{(n)}(t)}{(\sqrt{mn})^c} \right]_{t \in [0,1]}^T \\ & \xrightarrow{d} \left[\frac{\gamma \sqrt{t} H_1\left(\frac{B_t}{\sqrt{t}}\right)}{1!}, \dots, \frac{(\gamma \sqrt{t})^c H_c\left(\frac{B_t}{\sqrt{t}}\right)}{c!} \right]_{t \in [0,1]}^T \quad \text{in } D_{\mathbf{R}^c}. \end{aligned}$$

If $\rho > 0$ then

$$\begin{aligned} & \left[\frac{W_1^{(n)}(t)}{m \sqrt{n}}, \dots, \frac{W_c^{(n)}(t)}{(m \sqrt{n})^c} \right]_{t \in [0,1]}^T \\ & \xrightarrow{d} \left[\frac{\gamma \sqrt{\rho t} H_1\left(\frac{B_t}{\sqrt{t}}\right)}{1!}, \dots, \frac{(\gamma \sqrt{\rho t})^c H_c\left(\frac{B_t}{\sqrt{t}}\right)}{c!} \right]_{t \in [0,1]}^T \quad \text{in } D_{\mathbf{R}^c}. \end{aligned}$$

Proof. Consider first the case $\rho = 0$. For an arbitrary fixed positive integer c define a process $V_c^{(n)}$ by

$$V_c^{(n)}(t) = n^{-c/2} \sum_{1 \leq j_1 < \dots < j_c \leq [nt]} Y_{n,j_1} \dots Y_{n,j_c}$$

for $t \in [c/n, 1]$, where

$$Y_{n,j} = m^{-1/2} \sum_{i=1}^m \tilde{X}_{i,j}, \quad j = 1, \dots, n,$$

and $V_c^{(n)}(t) = 0$ for $t \in [0, c/n)$.

Observe that by Theorem 2 the convergence

$$[V_1^{(n)}(t), \dots, V_c^{(n)}(t)]_{t \in [0,1]} \xrightarrow{d} \left[\frac{\gamma \sqrt{t} H_1\left(\frac{B_t}{\sqrt{t}}\right)}{1!}, \dots, \frac{(\gamma \sqrt{t})^c H_c\left(\frac{B_t}{\sqrt{t}}\right)}{c!} \right]_{t \in [0,1]}^T, \quad (8)$$

in $D_{\mathbf{R}^c}$ follows as soon as we verify (5) and the Lindeberg condition (6), which in our current setting takes the form

$$E(Y_{n,1}^2 I(|Y_{n,1}| > \sqrt{n\varepsilon})) \rightarrow 0$$

for any $\varepsilon > 0$ as $n \rightarrow \infty$. To prove that the condition is satisfied observe first that

$$E(Y_{n,1}^2 I(|Y_{n,1}| > \sqrt{n\varepsilon})) \leq \sup_{k \geq 1} E(Y_{k,1}^2 I(Y_{k,1}^2 > n\varepsilon^2)).$$

Consequently, it suffices to show that the sequence of rv's $(Y_{k,1}^2)_{k \geq 1}$ is uniformly integrable. To this end, let us note the following.

- (i) By a central limit theorem for exchangeable sequences - see, e.g., Thm 2.1.2 p.30 in Taylor et al. [19] - it follows that $Y_{k,1}^2$ converges in distribution to $E(\tilde{X}_{1,1}^2 | \mathcal{F}) \mathcal{N}^2$, where \mathcal{F} is the σ -algebra of permutable events for the exchangeable sequence $(\tilde{X}_{i,1})_{i \geq 1}$, and \mathcal{N} is a standard normal rv independent of \mathcal{F} , and
- (ii) $E(E(\tilde{X}_{1,1}^2 | \mathcal{F}) \mathcal{N}^2) = E(\tilde{X}_{1,1}^2) = \gamma^2$ which, on the other hand, equals $E(Y_{1,k}^2)$ for any $k \geq 1$.

Hence, we conclude that the positive sequence of random variables $(Y_{k,1}^2)_{k \geq 1}$ is uniformly integrable since it converges in distribution and the corresponding sequence of expectations also converges (all being equal) to the suitable limit. This entails (5) and (6) and thus, via Theorem 2, also (8).

Observe that for any $k = 1, \dots, c$

$$\frac{W_k^{(n)}(t)}{(\sqrt{mn})^k} = V_k^{(n)}(t) - \frac{R_k^{(n)}(t)}{(\sqrt{mn})^k}, \quad (9)$$

where $(R_k^{(n)}(t))$ is a process such that for any $t \in [k/n, 1]$ it is a sum of different products $\tilde{X}_{i_1, j_1} \dots \tilde{X}_{i_k, j_k}$ such that $1 \leq j_1 < \dots < j_k \leq [nt]$, $(i_1, \dots, i_k) \in$

$\{1, \dots, m\}^k$ and at least one of the indices i_1, \dots, i_k in the sequence (i_1, \dots, i_k) repeats. For $t \in [0, k/n]$ we define $R_k^{(n)}(t)$ to be equal to zero.

Note that $(R_k^{(n)}(t))_{t \in [0,1]}$ is a martingale for any fixed n , in view of the assumed independence of columns of $\mathbb{X}^{(n)}$. Consequently, by the maximal inequality,

$$P \left(\sup_{t \in [0,1]} \frac{|R_k^{(n)}(t)|}{(\sqrt{mn})^k} > \varepsilon \right) \leq \frac{E (R_k^{(n)}(1))^2}{\varepsilon^2 (mn)^k} = \frac{Var (R_k^{(n)}(1))}{\varepsilon^2 (mn)^k}.$$

Using the fact that $R_k^{(n)}(1)$ is a sum of orthogonal products (observe that the covariance of any two of such different products equals zero since the columns are independent and elements in each column have zero correlation) and that the variance of any of such single product equals γ^{2k} while the number of products in $R_k^{(n)}(1)$ equals $\binom{n}{k}(m^k - \binom{m}{k}k!)$ we obtain

$$Var \left(\frac{R_k^{(n)}(1)}{(\sqrt{nm})^k} \right) = \gamma^{2k} \frac{\binom{n}{k}(m^k - \binom{m}{k}k!)}{n^k m^k} \leq \frac{\gamma^{2k}}{k!} \frac{m^k - \binom{m}{k}k!}{m^k} = O(1/m) \rightarrow 0,$$

as $n \rightarrow \infty$ (since the numerator is of order $O(m^{k-1})$).

Consequently, the c -variate process

$$\left[\frac{R_1^{(n)}(t)}{\sqrt{mn}}, \dots, \frac{R_c^{(n)}(t)}{(\sqrt{mn})^c} \right]_{t \in [0,1]}^T \xrightarrow{P} 0$$

in $D_{\mathbb{R}^c}$ and thus the first result follows immediately by (8) and (9).

Next, let us consider the case $\rho > 0$. As above, let us define for any $c = 1, 2, \dots$ the process $(V_c^{(n)}(t))_{t \in [0,1]}$ by

$$V_c^{(n)}(t) = n^{-c/2} \sum_{1 \leq j_1 < \dots < j_c \leq [nt]} Y_{n,j_1} \dots Y_{n,j_c},$$

where

$$Y_{n,j} = \frac{1}{m} \sum_{i=1}^m \tilde{X}_{ij},$$

$j = 1, 2, \dots, n$.

Similarly to the previous case, in order to apply the result of Theorem 2, we only need to show that the Lindeberg condition (6) and the condition (5) are satisfied. As above, we shall argue that the sequence $(Y_{k,1}^2)_{k \geq 1}$ is uniformly integrable. To this end, we observe first that by a version of the law of large numbers for exchangeable sequences (see, e.g., [3], Ch. 7, p.223) it follows that

$$Y_{k,1}^2 \xrightarrow{d} E^2(\tilde{X}_{1,1} | \mathcal{F}),$$

where \mathcal{F} is the σ -algebra of permutable events. Further, by the de Finetti theorem, we have

$$\begin{aligned} E(E^2(\tilde{X}_{1,1}|\mathcal{F})) &= E(E(\tilde{X}_{1,1}|\mathcal{F})E(\tilde{X}_{2,1}|\mathcal{F})) \\ &= E(E(\tilde{X}_{1,2}\tilde{X}_{2,1}|\mathcal{F})) = E(\tilde{X}_{1,1}\tilde{X}_{1,2}) = \rho\gamma^2. \end{aligned}$$

We also note that

$$E(Y_{k,1}^2) = \frac{1}{m^2}[m\gamma^2 + m(m-1)\rho\gamma^2] \rightarrow \rho\gamma^2, \quad m = m_k \rightarrow \infty.$$

Thus, the sequence $(Y_{k,1}^2)_{k \geq 1}$ is uniformly integrable and the conditions (5) and (6) of Theorem 2 are satisfied. This allows us to conclude that in the sense of the appropriate Skorokhod space

$$\begin{aligned} &[V_1^{(n)}(t), \dots, V_c^{(n)}(t)]_{t \in [0,1]}^T \\ &\xrightarrow{d} \left[\frac{(\sqrt{\rho t} \gamma)^1}{1!} H_1(B_t/\sqrt{t}), \dots, \frac{(\sqrt{\rho t} \gamma)^c}{c!} H_c(B_t/\sqrt{t}) \right]_{t \in [0,1]}^T. \end{aligned} \quad (10)$$

Similarly to the previous case of uncorrelated components, for any $k = 1, \dots, c$, we have

$$\frac{W_k^{(n)}(t)}{m^k(\sqrt{n})^k} = V_k^{(n)}(t) - \frac{R_k^{(n)}(t)}{m^k(\sqrt{n})^k}, \quad (11)$$

however, note that this time some of the elements of $R_k^{(n)}(1)$ are correlated - this is true for the pairs of products originating from exactly the same columns; if at least one column in the pair of products is different then their correlation is zero. Consequently,

$$\begin{aligned} \text{Var} \left(\frac{R_k^{(n)}(1)}{m^k \sqrt{n}^k} \right) &= \frac{1}{m^{2k} n^k} \sum_{1 \leq j_1 < \dots < j_k \leq n} \text{Var} (R_{j_1, \dots, j_k}(n)) \\ &= \frac{\binom{n}{k}}{m^{2k} n^k} \text{Var} (R_{1, \dots, k}(n)), \end{aligned}$$

where $R_{j_1, \dots, j_k}(n)$ denotes sum of respective products arising from the columns j_1, \dots, j_k . Since

$$|\text{Cov}(\tilde{X}_{i_1, j_1} \dots \tilde{X}_{i_k, j_k}, \tilde{X}_{l_1, j_1} \dots \tilde{X}_{l_k, j_k})| \leq \gamma^{2k}$$

for any choices of rows (i_1, \dots, i_k) and (l_1, \dots, l_k) , we conclude that

$$\text{Var} (R_{1, \dots, k}(n)) < \left(m^k - \binom{m}{k} k! \right)^2 \gamma^{2k}.$$

Hence, it follows that

$$\text{Var} \left(\frac{R_k^{(n)}(1)}{m^k(\sqrt{n})^k} \right) < \frac{\binom{n}{k} (m^k - \binom{m}{k} k!)^2 \gamma^{2k}}{m^{2k} n^k} < \frac{\gamma^{2k}}{k!} \frac{(m^k - \binom{m}{k} k!)^2}{m^{2k}} \rightarrow 0$$

as $n \rightarrow \infty$. As before the process $(R_k^{(n)}(t))_{t \in [0,1]}$ is a martingale for fixed n and thus once again the maximal inequality yields

$$\left[\frac{R_1^{(n)}(t)}{m\sqrt{n}}, \dots, \frac{R_c^{(n)}(t)}{(m\sqrt{n})^c} \right]_{t \in [0,1]}^T \xrightarrow{P} 0 \quad \text{in } D_{\mathbf{R}^c}.$$

Now, the second assertion of the theorem follows from (10) and (11) and the proof of Proposition 1 is complete. \square

4. Main results

Having established the result of Proposition 1 in the previous section, we are finally in position to state our two main theorems which are functional versions of the limiting result of Theorem 1. In particular, due to (4) they entail the latest.

To begin, we consider the case when the entries of matrix \mathbb{X} are uncorrelated.

Theorem 4. *Assume that $\rho = 0$ and let $(B_t)_{t \in [0,1]}$ denote the standard Brownian motion.*

If $m/n \rightarrow \lambda > 0$ as $n \rightarrow \infty$ then

$$\frac{(\mathcal{PSP} \mathbf{X}^{(n)}(t))_{t \in [0,1]}}{\binom{n}{m} m! \mu^m} \xrightarrow{d} \left(\exp \left(\sqrt{\lambda} \gamma B_t - \frac{\lambda t \gamma^2}{2} \right) \right)_{t \in [0,1]} \quad \text{in } D_{\mathbf{R}}. \quad (12)$$

If $m/n \rightarrow 0$ and $m = m_n \rightarrow \infty$ as $n \rightarrow \infty$ then

$$\sqrt{\frac{n}{m}} \left(\frac{(\mathcal{PSP} \mathbf{X}^{(n)}(t))_{t \in [0,1]}}{\binom{n}{m} m! \mu^m} - 1 \right) \xrightarrow{d} \gamma (B_t)_{t \in [0,1]} \quad \text{in } D_{\mathbf{R}}. \quad (13)$$

Proof. Consider first the case $\lambda > 0$. For any n and any N such that $N < m_n$ define a process $\mathbf{S}_N^{(n)} = (S_N^{(n)}(t))_{t \in [0,1]}$ by

$$S_N^{(n)}(t) = 1 + \sum_{c=1}^N \binom{m}{c} U_c^{(m,n)}(t) = 1 + \sum_{c=1}^N \frac{(\sqrt{mn})^c}{\binom{n}{c} c!} \frac{W_c^{(n)}(t)}{(\sqrt{mn})^c}.$$

Observe that by the first assertion of Proposition 1 it follows that

$$(S_N^{(n)}(t))_{t \in [0,1]} \xrightarrow{d} \mathbf{G}_N = \left(\sum_{c=0}^N \frac{(\lambda t \gamma^2)^{c/2}}{c!} H_c(B_t/\sqrt{t}) \right)_{t \in [0,1]} \quad (14)$$

in $D_{\mathbf{R}}$ as $n \rightarrow \infty$, since for any $c = 1, 2, \dots$

$$\frac{(\sqrt{mn})^c}{\binom{n}{c} c!} \rightarrow \lambda^{c/2}.$$

Define also a process $\mathbf{T}_N^{(n)} = (T_N^{(n)}(t))_{t \in [0,1]}$ by

$$T_N^{(n)}(t) = \sum_{c=N+1}^{m_n} \binom{m}{c} U_c^{(m,n)}(t).$$

And let $\mathbf{G}_\infty = (G_\infty(t))_{t \in [0,1]}$ be defined by

$$(G_\infty(t))_{t \in [0,1]} = \left(\sum_{c=0}^{\infty} \frac{(\lambda t \gamma^2)^{c/2}}{c!} H_c(B_t/\sqrt{t}) \right)_{t \in [0,1]} = \exp \left(\sqrt{\lambda} \gamma B_t - \frac{\lambda t \gamma^2}{2} \right)_{t \in [0,1]} .$$

Let ϱ denote the Prokhorov distance between the random elements in $D_{\mathbf{R}}$. Then

$$\varrho(\mathbf{Z}^{(n)}, \mathbf{G}_\infty) \leq \varrho(\mathbf{Z}^{(n)}, \mathbf{S}_N^{(n)}) + \varrho(\mathbf{S}_N^{(n)}, \mathbf{G}_N) + \varrho(\mathbf{G}_N, \mathbf{G}_\infty), \tag{15}$$

where $\mathbf{Z}^{(n)} = \mathbf{S}_N^{(n)} + \mathbf{T}_N^{(n)}$. Let us note that for any $t \in [0, 1]$

$$\mathcal{PSP} \mathbf{X}^{(n)}(t) = \binom{n}{m} m! \mu^m Z^{(n)}(t)$$

and that (14) implies $\varrho(\mathbf{S}_N^{(n)}, \mathbf{G}_N) \rightarrow 0$ as $n \rightarrow \infty$ for any fixed N , as well as, that we have $\varrho(\mathbf{G}_N, \mathbf{G}_\infty) \rightarrow 0$ as $N \rightarrow \infty$. Therefore, in order to argue that $\varrho(\mathbf{Z}^{(n)}, \mathbf{G}_\infty) \rightarrow 0$ as $n \rightarrow \infty$ we only need to show that $\varrho(\mathbf{Z}^{(n)}, \mathbf{S}_N^{(n)})$ tends to zero uniformly in n as $N \rightarrow \infty$.

To this end observe that by the definition of the Prohorov distance

$$\varrho(\mathbf{Z}^{(n)}, \mathbf{S}_N^{(n)}) \leq \inf\{\varepsilon > 0 : P(\sup_{t \in [0,1]} |T_N^{(n)}(t)| > \varepsilon) \leq \varepsilon\}.$$

On the other hand, as before, it is easy to see that due to the assumed independence of columns of $\mathbb{X}^{(n)}$ the process $\mathbf{T}_N^{(n)}$ is a martingale for fixed n, N and thus, via the maximal inequality and the relations (2) and (3) for $\rho = 0$, we have for any $\varepsilon > 0$

$$\begin{aligned} P(\sup_{t \in [0,1]} |T_N^{(n)}(t)| > \varepsilon) &\leq \varepsilon^{-2} \text{Var } T_N^{(n)}(1) = \varepsilon^{-2} \sum_{c=N+1}^m \frac{\binom{m}{c} \gamma^2}{\binom{n}{c} c!} \\ &\leq \varepsilon^{-2} \sum_{c=N+1}^{\infty} \frac{\gamma^{2c}}{c!} = \varepsilon^{-2} \alpha_N \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$, which entails

$$\varrho(\mathbf{Z}^{(n)}, \mathbf{S}_N^{(n)}) \leq \sqrt[3]{\alpha_N} \rightarrow 0,$$

uniformly in n , as $N \rightarrow \infty$ and hence, by (15) also $\varrho(\mathbf{Z}^{(n)}, \mathbf{G}_\infty) \rightarrow 0$ as $n \rightarrow \infty$.

Thus, we conclude that $\mathbf{Z}^{(n)}$ converges weakly to \mathbf{G}_∞ in $D_{\mathbf{R}}$.

For the proof in the case $\lambda = 0$, let us observe that

$$\sqrt{\frac{n}{m}} \sum_{c=1}^m \binom{m}{c} U_c^{(m,n)}(t) = \frac{W_1^{(n)}(t)}{\sqrt{mn}} + R^{(m,n)}(t), \tag{16}$$

where

$$R^{(m,n)}(t) = \sqrt{\frac{n}{m}} \sum_{c=2}^m \binom{m}{c} U_c^{(m,n)}(t).$$

Observe that due to the independence of columns of the matrix \mathbb{X} the process $(R^{(m,n)}(t))$ is a martingale for fixed values of m, n . Consequently, by the maximal inequality we get

$$\begin{aligned} (\sup_{t \in [0,1]} |R^{(m,n)}(t)| > \varepsilon) &\leq \varepsilon^{-2} \text{Var } R^{(m,n)}(1) = \varepsilon^{-2} \frac{n}{m} \sum_{c=2}^m \frac{\binom{m}{c} \gamma^{2c}}{\binom{n}{c} c!} \\ &\leq \frac{m}{n} \exp(\gamma^2) \rightarrow 0 \end{aligned} \tag{17}$$

and conclude that $(R^{(m,n)}(t))$ converges to zero in probability in $D_{\mathbf{R}}$. Hence, the final result follows now directly by Proposition 1. \square

Remark 3. A quick inspection of the proof of the second part of the theorem reveals that a slight modification of the argument shows the validity of the second part of the hypothesis also for m remaining fixed. Indeed, note that since (17) holds for fixed m thus in view of (16) it is enough to argue that the conclusion of Proposition 1 holds for $W_1^{(n)}(t)/\sqrt{mn}$. This is immediate, however, upon noticing that in (9) we have $R_1^{(n)} = 0$ which entails

$$W_1^{(n)}(t)/\sqrt{mn} = V_1^{(n)}(t),$$

and hence Theorem 4 holds true for fixed m as well. In particular, let us note that for $m = 1$ we obtain simply Donsker’s theorem.

The result of Theorem 4 is complemented with the corresponding one in the case when $\rho > 0$.

Theorem 5. *Assume that $\rho > 0$ and, as before, let $(B_t)_{t \in [0,1]}$ denote the standard Brownian motion.*

If $m^2/n \rightarrow \lambda > 0$ as $n \rightarrow \infty$ then

$$\frac{(\mathcal{PSP} \mathbb{X}^{(n)}(t))_{t \in [0,1]}}{\binom{n}{m} m! \mu^m} \xrightarrow{d} \exp\left(\sqrt{\lambda \rho} \gamma B_t - \frac{\lambda \rho t \gamma^2}{2}\right)_{t \in [0,1]} \quad \text{in } D_{\mathbf{R}}.$$

If $m^2/n \rightarrow 0$ and $m = m_n \rightarrow \infty$ as $n \rightarrow \infty$ then

$$\frac{\sqrt{n}}{m} \left(\frac{(\mathcal{PSP} \mathbf{X}^{(n)}(t))_{t \in [0,1]}}{\binom{n}{m} m! \mu^m} - 1 \right) \xrightarrow{d} \sqrt{\rho} \gamma (B_t)_{t \in [0,1]} \quad \text{in } D_{\mathbf{R}}.$$

Proof. The proof of the result parallels, to large extent, that of Theorem 4. As before, consider first the case $\lambda > 0$. For any n and N such that $N < m_n$ define a process $\mathbf{S}_N^{(n)} = (S_N^{(n)}(t))_{t \in [0,1]}$ by

$$S_N^{(n)}(t) = 1 + \sum_{c=1}^N \binom{m}{n} U_c^{(m,n)}(t) = 1 + \sum_{c=1}^N \frac{(\sqrt{m} n)^c}{\binom{n}{c} c!} \frac{W_c^{(n)}(t)}{(\sqrt{m} n)^c}.$$

Observe that by the second assertion of Proposition 1 it follows that

$$(S_N^{(n)}(t))_{t \in [0,1]} \xrightarrow{d} \mathbf{G}_N = \left(\sum_{c=0}^N \frac{(\lambda \rho t \gamma^2)^{c/2}}{c!} H_c(B_t / \sqrt{t}) \right)_{t \in [0,1]}$$

in $D_{\mathbf{R}}$ as $n \rightarrow \infty$, since for any $c = 1, 2, \dots$

$$\frac{(\sqrt{n} m)^c}{\binom{n}{c} c!} \rightarrow \lambda^{c/2} .$$

Define also a process $\mathbf{T}_N^{(n)} = (T_N^{(n)}(t))_{t \in [0,1]}$ by

$$T_N^{(n)}(t) = \sum_{c=N+1}^{m_n} \binom{m}{c} U_c^{(m,n)}(t) .$$

And let $\mathbf{G}_\infty = (G_\infty(t))_{t \in [0,1]}$ be defined by

$$\begin{aligned} (G_\infty(t))_{t \in [0,1]} &= \left(\sum_{c=0}^\infty \frac{(\lambda \rho t \gamma^2)^{c/2}}{c!} H_c(B_t / t) \right)_{t \in [0,1]} \\ &= \exp \left(\sqrt{\lambda \rho} \gamma B_t - \frac{\lambda \rho t \gamma^2}{2} \right)_{t \in [0,1]} . \end{aligned}$$

As before, if ϱ denotes the Prokhorov distance between the random elements in $D_{\mathbf{R}}$, then

$$\varrho(\mathbf{Z}^{(n)}, \mathbf{G}_\infty) \leq \varrho(\mathbf{Z}^{(n)}, \mathbf{S}_N^{(n)}) + \varrho(\mathbf{S}_N^{(n)}, \mathbf{G}_N) + \varrho(\mathbf{G}_N, \mathbf{G}_\infty) .$$

where $\mathbf{Z}^{(n)} = \mathbf{S}_N^{(n)} + \mathbf{T}_N^{(n)}$. In order to complete the argument along the lines of the first part of the proof of Theorem 4 we only need to argue that for any $\varepsilon > 0$

$$P \left(\sup_{t \in [0,1]} |T_N^{(n)}(t)| > \varepsilon \right) \rightarrow 0 ,$$

uniformly in n as $N \rightarrow \infty$. To this end note that again by the martingale property and the maximal inequality as well as by the relations (2) and (3)

$$\begin{aligned} P \left(\sup_{t \in [0,1]} |T_N^{(n)}(t)| > \varepsilon \right) &\leq \varepsilon^{-2} \text{Var } T_N^{(n)}(1) \\ &= \varepsilon^{-2} \frac{n}{m^2} \sum_{c=N+1}^m \frac{\binom{m}{c} \gamma^{2c}}{\binom{n}{c}} \sum_{r=0}^c \frac{1}{r!} \binom{m-r}{c-r} (1-\rho)^r \rho^{c-r} \\ &\leq \varepsilon^{-2} \exp(1) \sum_{c=N+1}^m \left(\frac{m^2}{n} \right)^c \frac{\gamma^{2c}}{c!} , \end{aligned}$$

in view of the inequalities $\binom{m-r}{c-r} \leq \binom{m}{c}$ and $c! \frac{\binom{m}{c}^2}{\binom{n}{c}} \leq \left(\frac{m^2}{n} \right)^c$ for $0 \leq r \leq c \leq m$. Therefore,

$$\begin{aligned}
 P\left(\sup_{t \in [0,1]} |T_N^{(n)}(t)| > \varepsilon\right) &\leq \varepsilon^{-2} \exp(1) \sum_{c=N+1}^m \left(\frac{m^2}{n}\right)^c \frac{\gamma^{2c}}{c!} \\
 &\leq \varepsilon^{-2} \exp(1) \sum_{c=N+1}^{\infty} \frac{(2\lambda \gamma^2)^c}{c!} = \varepsilon^{-2} \alpha_N \rightarrow 0
 \end{aligned}$$

as $N \rightarrow \infty$, and the first part of the theorem follows.

The proof in the case $\lambda = 0$, follows similarly to the second part of the proof of Theorem 4 with obvious modifications. Consider

$$\frac{\sqrt{n}}{m} \sum_{c=1}^m \binom{m}{c} U_c^{(m,n)}(t) = \frac{W_1^{(n)}(t)}{m \sqrt{n}} + R^{(m,n)}(t),$$

where

$$R^{(m,n)}(t) = \frac{\sqrt{n}}{m} \sum_{c=2}^m \binom{m}{c} U_c^{(m,n)}(t).$$

The process $(R^{(m,n)}(t))$ is a martingale for fixed values of m, n and thus

$$P\left(\sup_{t \in [0,1]} |R^{(m,n)}(t)| > \varepsilon\right) \leq \varepsilon^{-2} \text{Var } R^{(m,n)}(1) \leq \varepsilon^{-2} \frac{m^2}{n} \exp(1 + \gamma^2) \rightarrow 0,$$

follows, in view of

$$\begin{aligned}
 \text{Var } R_{m,n}(1) &= \frac{n}{m^2} \sum_{c=2}^m \frac{\binom{m}{c} \gamma^{2c}}{\binom{n}{c}} \sum_{r=0}^c \frac{1}{r!} \binom{m-r}{c-r} (1-\rho)^r \rho^{c-r} \\
 &\leq \exp(1) \frac{n}{m^2} \sum_{c=2}^m \left(\frac{m^2}{n}\right)^c \frac{\gamma^{2c}}{c!} \leq \exp(1) \frac{m^2}{n} \sum_{c=2}^m \left(\frac{m^2}{n}\right)^{c-2} \frac{\gamma^{2c}}{c!} \\
 &\leq \exp(1) \frac{m^2}{n} \sum_{c=2}^m \frac{\gamma^{2c}}{c!} \leq \frac{m^2}{n} \exp(1 + \gamma^2)
 \end{aligned}$$

if only n is large enough to have $m^2/n < 1$. The result now follows as in the proof of Theorem 4. □

Remark 4. Let us again note that the above result remains valid for m being a fixed constant (see the remark after the proof of Theorem 4).

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