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Asymptotics for products of independent sums with an application to Wishart determinants

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Abstract

We derive a lognormal limit theorem for products of independent sums of positive random variables or, in general, products of non-degenerate independent U-statistics. An application of the result gives a limit theorem for the determinant of a Wishart matrix.

MSC: primary 60F05; 62G32

Keywords: Central limit theorem; Lognormal distribution; Products of sums of iid rv's; Wishart determinant; U-statistics

1. Introduction

The asymptotic behavior of a product of partial sums of a sequence of independent and identically distributed (iid) positive random variables has been recently studied in several papers (see, e.g., Qi, 2003 for a brief review). In particular, it was shown in Rempała and Wesołowski

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(2002) that if (X_n) is a sequence of iid positive square integrable random variables with $E(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$ then setting $S_k = \sum_{i=1}^{k} X_i$ we have as $n \to \infty$

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{d} \mathrm{e}^{\sqrt{2}\mathcal{N}},$$

where $\xrightarrow{d}$ stands for convergence in distribution and $\mathcal{N}$ is a standard normal random variable. This result was recently extended in Qi (2003) and Lu and Qi (2004) to a general limit theorem covering the case when the underling distribution is in the domain of attraction of a stable law with index from the interval [1,2].

The purpose of the current note is to obtain a limit theorem for $\prod_{k=1}^{n} S_k$ in case when the partial sums S_k are mutually independent and have square integrable components. This particular setup seems to be of interest as it pertains to a limit theorem for random determinants of Wishart matrices. Our main result is provided in Theorem 1 of Section 2 below. In Section 3 we discuss its extensions to non-iid case as well as to a case of so-called non-degenerate U-statistics. Section 4 contains an application of our result to Wishart determinants. For the convenience of the readership we have also provided in Section 4 a brief outline of some basic facts on Wishart matrices. We note that the result obtained therein overlaps with that of Girko (1990, 1997) but we believe that our method of the proof is in general much simpler and, in particular, requires virtually no background in random matrices theory.

2. Main result

Our main result of this note is the following limit theorem.

Theorem 1. *Let $(X_{k,i})_{i=1,\dots,k}$; $k = 1, 2, \dots$ be a triangular array of iid positive square integrable rv's with finite absolute moment of order $p > 2$. Denote $\mu = E(X_1) > 0$, $\gamma = \sigma/\mu$, where $\sigma^2 = \mathrm{Var}(X_1)$, and $S_k = X_{k,1} + \cdots + X_{k,k}$, $k = 1, 2, \dots$. Then as $n \to \infty$*

$$\left(n^{\frac{\gamma^2}{2}} \frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{\log(n)}}} \xrightarrow{d} \mathrm{e}^{\mathcal{N}},$$

where $\mathcal{N}$ is a standard normal rv.

Before proving the above result we will establish a version of the classical central limit theorem, essentially, for scaled iid rv's. To this end we will use the clt for triangular arrays, so the basic step in the proof will rely on verifying the Lindeberg condition. First we recall an elementary fact about the moments of sums of iid variables (e.g., Lee, 1990, p. 22). In the sequel, for notational convenience, we set $C_k = S_k/(\mu k)$.

Lemma 1 (*Burkholder Inequality*). *Let $p \geqslant 2$. Under the assumptions of Theorem 1 there exists a universal constant D_p (i.e., depending on p but not on k) such that for $k \geqslant 1$*

$$E|(C_k - 1)|^p \leqslant \frac{D_p}{k^{p/2}}.$$

Lemma 2. *Under the assumptions of Theorem 1, as* $n \to \infty$

$$\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{n}(C_k - 1)\xrightarrow{d} \mathcal{N}.$$

Proof. Since

$$\operatorname{Var}\left(\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right)\right) = \gamma^2\sum_{k=1}^{n}\frac{1}{k}\to\infty \quad \text{as } n\to\infty$$

and by Lemma 1 also

$$\limsup_{n}\sum_{k=1}^{n}E\left|\frac{S_k}{\mu k}-1\right| \leqslant D_p\sum_{k=1}^{\infty}\frac{1}{k^{p/2}}<\infty.$$

Thus

$$\frac{\left(\sum_{k=1}^{n}E\left|\frac{S_k}{\mu k}-1\right|^p\right)^{2/p}}{\operatorname{Var}\left(\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right)\right)}\to 0 \quad \text{as } n\to\infty$$

so the Lyapounov and hence the Lindeberg condition is satisfied. □

Lemma 3. *Under the assumptions of Theorem 1, as* $n \to \infty$

(i) $\frac{1}{\sqrt{\log(n)}}\sum_{k=1}^{n}[(C_k-1)^2-\frac{\gamma^2}{k}]\xrightarrow{P} 0,$
(ii) $\frac{1}{\sqrt{\log(n)}}\sum_{k=1}^{n}|C_k-1|^3\xrightarrow{P} 0.$

Proof. For the proof of (i) denote $Z_k = k(C_k-1)^2-\gamma^2$ and note that $EZ_k = 0$ and by our assumptions and Lemma 1 there exists $0<\alpha<1$ such that $\sup_k E|Z_k|^{1+\alpha}<\infty$. Let a_n be any numeric sequence such that $a_n\to\infty$ but $a_n^{1-\alpha}/\log(n)\to 0$ (e.g., $a_n=\log(n)$ will do). Define $Z_k' = Z_k\, I(|Z_k|/k\leqslant a_n)$ and note that for some universal constant D_α

$$P\left(\sum_{k=1}^{n}Z_k/k\neq\sum_{k=1}^{n}Z_k'/k\right)\leqslant\sum_{k=1}^{n}P(|Z_k|>k\,a_n)\leqslant D_\alpha/a_n^{1+\alpha}\to 0 \quad \text{as } n\to\infty. \tag{1}$$

We now show that for the weighted sum of Z_k''s the weak law of large numbers holds. Indeed, for any $\varepsilon>0$

$$P\left(\left|\sum_{k=1}^{n}(Z_k'/k-EZ_k'/k)\right|>\varepsilon\sqrt{\log(n)}\right)\leqslant\frac{1}{\varepsilon^2\log(n)}\sum_{k=1}^{n}E(Z_k'/k-EZ_k'/k)^2$$
$$\leqslant\frac{1}{\varepsilon^2\log(n)}\sum_{k=1}^{n}E|Z_k'/k|^2\leqslant\frac{a_n^{1-\alpha}}{\varepsilon^2\log(n)}\sum_{k=1}^{n}E|Z_k/k|^{1+\alpha}\to 0 \quad \text{as } n\to\infty. \tag{2}$$

Finally, note that since $EZ_k = 0$ then

$$\left|\sum_{k=1}^{n} EZ'_k/k\right| = \left|\sum_{k=1}^{n} \frac{1}{k}\, EZ_k\, I(|Z_k/k| > a_n)\right| \leqslant \frac{1}{a_n^{\alpha}} \sum_{k=1}^{n} E|Z_k/k|^{1+\alpha} \to 0 \quad \text{as } n \to \infty. \tag{3}$$

The relations (1)–(3) imply (i). In order to show the second assertion we proceed similarly denoting this time $W_k = k(C_k - 1)^2$ and $W'_k = W_k\, I(|W_k| \leqslant k\, b_n)$ where b_n is any numeric sequence satisfying $b_n \to \infty$ but $b_n^{1-2\alpha}/\log(n) \to 0$. Note that as before, $\sup_k E|W_k|^{1+\alpha} < \infty$. In this notation, the relation (ii) follows for $|W'_k/k|^{3/2}$ in view of the Markov inequality since

$$\frac{1}{\sqrt{\log(n)}} \sum_{k=1}^{n} E|W'_k/k|^{3/2} \leqslant \frac{b_n^{1/2-\alpha}}{\sqrt{\log(n)}} \sum_{k=1}^{n} E|W_k/k|^{1+\alpha} \to 0 \quad \text{as } n \to \infty.$$

The fact that this also implies (ii) for the C_k's is immediate since

$$P\left(\sum_{k=1}^{n} |W_k/k|^{3/2} \neq \sum_{k=1}^{n} |W'_k/k|^{3/2}\right) \leqslant \sum_{k=1}^{n} P(|W_k| > k\, b_n) \leqslant \sum_{k=1}^{n} E|W_k/(k\, b_n)|^{1+\alpha}$$
$$\leqslant D_\alpha/b_n^{1+\alpha} \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Finally, we are in position to prove our main result.

Proof of Theorem 1. We first note that $C_k = S_k/(\mu k)$ converges almost surely to one. Indeed, for any $\delta > 0$ we have

$$P\left(\sup_{k \geqslant r} |C_k - 1| > \delta\right) \leqslant \sum_{k=r}^{\infty} P(|C_k - 1| > \delta) \leqslant \frac{1}{\delta} \sum_{k=r}^{\infty} E|C_k - 1|^p \to 0 \quad \text{as } r \to \infty.$$

Consequently, there exist two sequences $(\delta_m) \downarrow 0$ $(\delta_1 = 1/2)$ and $(R_m) \uparrow \infty$ such that

$$P\left(\sup_{k \geqslant R_m} |C_k - 1| > \delta_m\right) < \delta_m.$$

Take now any real x and any m. Then

$$P\left(\frac{1}{\gamma\sqrt{\log(n)}} \sum_{k=1}^{n} \left(\log(C_k) + \frac{\gamma^2}{2k}\right) \leqslant x\right) = P\left(\frac{1}{\gamma\sqrt{\log(n)}} \sum_{k=1}^{n} \left(\log(C_k) + \frac{\gamma^2}{2k}\right)\right.$$
$$\left.\leqslant x,\ \sup_{k > R_m} |C_k - 1| > \delta_m\right) + P\left(\frac{1}{\gamma\sqrt{\log(n)}} \sum_{k=1}^{n} \left(\log(C_k) + \frac{\gamma^2}{2k}\right)\right.$$
$$\left.\leqslant x,\ \sup_{k > R_m} |C_k - 1| \leqslant \delta_m\right) = A_{m,n} + B_{m,n}$$

and $A_{m,n} < \delta_m$.

To compute $B_{m,n}$ we will expand the logarithm: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$, where $\theta \in (0,1)$ depends on $x \in (-1,1)$. Thus,

$$
\begin{aligned}
B_{m,n} &= P\Bigg(\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{R_m}\left(\log(C_k)+\frac{\gamma^2}{2k}\right)+\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}\left(\log(1+(C_k-1))+\frac{\gamma^2}{2k}\right)\\
&\qquad \leqslant x,\ \sup_{k>R_m}|C_k-1|\leqslant\delta_m\Bigg)\\
&= P\Bigg(\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{R_m}\left(\log(C_k)+\frac{\gamma^2}{2k}\right)+\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}(C_k-1)\\
&\qquad -\frac{1}{2\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}\left[(C_k-1)^2-\frac{\gamma^2}{k}\right]\\
&\qquad +\frac{1}{3\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}\frac{(C_k-1)^3}{(1+\theta_k(C_k-1))^3}\leqslant x,\ \sup_{k>R_m}|C_k-1|\leqslant\delta_m\Bigg)\\
&= P\Bigg(\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{R_m}\left(\log(C_k)+\frac{\gamma^2}{2k}\right)+\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}(C_k-1)\\
&\qquad -\frac{1}{2\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}\left[(C_k-1)^2-\frac{\gamma^2}{k}\right]+\frac{1}{3\gamma\sqrt{\log(n)}}\\
&\qquad \times\left[\sum_{k=R_m+1}^{n}\frac{(C_k-1)^3}{(1+\theta_k(C_k-1))^3}\right]I\left(\sup_{k>R_m}|C_k-1|\leqslant\delta_m\right)\leqslant x\Bigg)\\
&\qquad - P\Bigg(\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{R_m}\left(\log(C_k)+\frac{\gamma^2}{2k}\right)+\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}(C_k-1)\\
&\qquad -\frac{1}{2\gamma\sqrt{\log(n)}}\sum_{k=R_m+1}^{n}\left[(C_k-1)^2-\frac{\gamma^2}{k}\right]\leqslant x,\ \sup_{k>R_m}|C_k-1|>\delta_m\Bigg)=D_{m,n}+F_{m,n},
\end{aligned}
$$

where θ_k, $k=1,\ldots,n$ are $(0,1)$-valued rv's and $F_{m,n}<\delta_m$.

Rewrite now $D_{m,n}$ as

$$
\begin{aligned}
D_{m,n} &= P\Bigg(\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{R_m}\left(\log(C_k)-C_k+1+\frac{(C_k-1)^2}{2}-\frac{\gamma^2}{2k}\right)+\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{n}(C_k-1)\\
&\qquad -\frac{1}{2\gamma\sqrt{\log(n)}}\sum_{k=1}^{n}\left[(C_k-1)^2-\frac{\gamma^2}{k}\right]+\frac{1}{3\gamma\sqrt{\log(n)}}\left[\sum_{k=R_m+1}^{n}\frac{(C_k-1)^3}{(1+\theta_k(C_k-1))^3}\right]I\\
&\qquad \times\left(\sup_{k>R_m}|C_k-1|<\delta_m\right)\leqslant x\Bigg).
\end{aligned}
$$

Observe that for any fixed m

$$\frac{1}{\gamma\sqrt{2n}}\sum_{k=1}^{R_m}\left(\log(C_k)-C_k+1+\frac{(C_k-1)^2}{2}-\frac{\gamma^2}{2k}\right)\xrightarrow{P}0 \quad \text{as } n\to\infty \tag{4}$$

(as a matter of fact, this sequence converges to zero a.s.).

Invoking Lemma 3(i) we see

$$P\left(\frac{1}{2\gamma\sqrt{\log(n)}}\sum_{k=1}^{n}\left[(C_k-1)^2-\frac{\gamma^2}{k}\right]>\varepsilon\right)\to 0.$$

Note that for $|x|<1/2$ and any $\theta\in(0,1)$ it follows that $|x|^3/|1+\theta x|^3\leqslant 8|x|^3$. Thus for any m

$$\begin{aligned}&\frac{1/3}{\gamma\sqrt{\log(n)}}\left[\sum_{k=R_m+1}^{n}\frac{|C_k-1|^3}{|1+\theta_k(C_k-1)|^3}\right]I\left(\sup_{k>R_m}|C_k-1|<\delta_m\right)\\ &\quad\leqslant\frac{8/3}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{n}|C_k-1|^3\xrightarrow{P}0,\end{aligned} \tag{5}$$

as $n\to\infty$ by Lemma 3(ii).

Since, on the other hand, by Lemma 1 it follows that

$$\frac{1}{\gamma\sqrt{\log(n)}}\sum_{k=1}^{n}(C_k-1)\xrightarrow{d}\mathcal{N}$$

as $n\to\infty$ then by (4) and (5) we conclude that for any fixed m

$$D_{m,n}\to\Phi(x),$$

where Φ is the standard normal distribution function.

Finally, observe that

$$\begin{aligned}P\left(\log\left(n^{\frac{\gamma^2}{2}}\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{\log(n)}}}\leqslant x\right)&=P\left(\frac{1}{\gamma\sqrt{\log(n)}}\left(\sum_{k=1}^{n}\log(C_k)+\frac{\gamma^2}{2}\log(n)\right)\leqslant x\right)\\&=A_{m,n}+D_{m,n}+F_{m,n},\end{aligned}$$

which implies the assertion of Theorem 1, since $A_{m,n}+F_{m,n}<2\delta_m\to 0$ as $m\to\infty$, uniformly in n and

$$\frac{\log(n)-\sum_{k=1}^{n}\frac{1}{k}}{\sqrt{\log(n)}}\to 0 \quad \text{as } n\to\infty. \qquad \square$$

Remark 1. It is perhaps worth to notice that as soon as we have $S_k/k\to\mu$ a.s. (in particular, under the assumptions of Theorem 1) then by the property of the geometric mean it follows directly that as $n \to \infty$

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!}\right)^{1/n} \to \mu \quad \text{a.s.}$$

3. Extensions

The following extension of Theorem 1 covering the non-all-iid setting is rather straightforward.

Theorem 2. *Let $(X_{k,i})_{i=1,\dots,k}$; $k=1,2,\dots$ be a triangular array of independent and rowwise identically distributed, positive rv's with finite absolute moment of order $p>2$. Denote, as before, $S_k = X_{k,1} + \cdots + X_{k,k}$, with $\mu_k = E(X_{k,1})>0$, $\sigma_k^2 = \mathrm{Var}(X_{k,1})$, and $\gamma_k = \sigma_k/\mu_k$. Let $c_n^2 = \sum_{k=1}^{n} \gamma_k^2/k$. If*

(i) $c_n \to \infty$ *as* $n \to \infty$,
(ii) $\sum_{k=1}^{\infty} E|\frac{S_k - k\mu_k}{k\mu_k}|^p < \infty$,

then as $n \to \infty$

$$\left(\mathrm{e}^{\frac{c_n^2}{2}} \frac{\prod_{k=1}^{n} S_k/\mu_k}{n!}\right)^{1/c_n} \xrightarrow{d} \mathrm{e}^{\mathcal{N}},$$

where $\mathcal{N}$ is a standard normal rv.

Another direction leading to an extension of the result of Theorem 1 is via a notion of a U-statistic introduced by Hoeffding (1948). Let U-statistic U_n be defined as

$$U_n = \binom{n}{m}^{-1} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} h(X_{i_1}, \dots, X_{i_m}), \tag{6}$$

where h is a symmetric real function of m arguments, the X_i's are iid rv's, and the summation is carried over all possible choices of m distinct indices out of the set $\{1, 2, \dots, n\}$. Let us note that if $m=1$ and $h(x)=x$ then the above definition gives simply S_n/n. If we assume that $Eh(X_1, \dots, X_m)^2 < \infty$ and define $h_1(x) = Eh(x, X_2, \dots, X_m)$ as well as

$$\hat{U}_n = \left[\frac{m}{n}\sum_{i=1}^{n}(h_1(X_i) - Eh) + Eh\right],$$

then we may write

$$U_n = \hat{U}_n + R_n, \tag{7}$$

where

$$R_n = \binom{n}{m}^{-1} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} H(X_{i_1}, \dots, X_{i_m}),$$

and

$$H(x_1,\dots,x_m)=h(x_1,\dots,x_m)-\sum_{i=1}^{m}(h_1(x_i)-Eh)-Eh.$$

It is well known (cf. e.g., Lee, 1990, Chapter 1) that,

$$\mathrm{Cov}(\hat{U}_n,R_n)=0 \tag{8}$$

and

$$\mathrm{Var}\,R_n=\mathrm{O}(n^{-2}). \tag{9}$$

The result of Theorem 1 can be extended to U-statistics as follows.

Theorem 3. *Let $(X_{k,i})_{i=1,\dots,k}$; $k=m,m+1,\dots$ be a triangular array of iid random variables. Let U_k be a statistic given by (6) based on $X_{k,1},\dots,X_{k,k}$. Assume $E|h|^p<\infty$ for some $p>2$ and $P(h(X_1,\dots,X_m)>0)=1$, as well as $\sigma^2=m^2\,\mathrm{Var}(h_1(X_1))\neq 0$. Denote $\mu=Eh>0$ and let $\gamma=\sigma/\mu>0$ be the coefficient of variation. Then, as $n\to\infty$*

$$\left(n^{\frac{\gamma^2}{2}}\frac{\prod_{k=1}^{n}U_k}{\mu^n}\right)^{\frac{1}{\gamma\sqrt{\log(n)}}}\xrightarrow{d}\mathrm{e}^{\mathcal{N}},$$

where $\mathcal{N}$ is a standard normal rv.

Proof. Set now $C_k=U_k/\mu$ and let γ be defined as above. Retaining the notation of Section 2 with these modifications we find that the result of Lemma 1 still holds true (Lee, 1990, p. 21). Regarding the extension of the conclusion of Lemma 2 set $z_n=\gamma(\log(n))^{-1/2}$ and note that by (7)

$$z_n\sum_{k=m}^{n}(C_k-1)=z_n\sum_{k=m}^{n}\left(\frac{\hat{U}_k}{\mu}-1+\frac{R_n}{\mu}\right)=z_n\sum_{k=m}^{n}\left(\frac{\hat{U}_k}{\mu}-1\right)+z_n\sum_{k=m}^{n}\frac{R_n}{\mu}.$$

The first sum in the latest expression above is asymptotically standard normal in view of the result of Lemma 2 of Section 2 and the second one vanishes asymptotically in probability since by (9)

$$z_n^2\,\mathrm{Var}\left(\sum_{k=1}^{n}\frac{R_n}{\mu}\right)=z_n^2\sum_{k=1}^{n}\left(\frac{\mathrm{Var}\,R_n}{\mu^2}\right)\to 0\quad\text{as } n\to\infty.$$

Hence, the result of Lemma 2 remains valid for U-statistics and a similar argument can be invoked to argue that the results of Lemma 3 are true for U-statistics as well. Finally, in view of the SLLN for U-statistics (see, e.g., Lee, 1990, p. 122) which implies that under our assumptions $C_k\to 1$ as $k\to\infty$, we may virtually repeat the expansion argument used in the proof of Theorem 1 to obtain the required assertion of Theorem 3. □

4. Asymptotics for Wishart determinants

Asymptotic distribution of random determinants for squares of matrices of iid Gaussian-like entries is derived in Girko (1990, Theorem 6.3.1). Unfortunately, the proof is rather complicated and difficult to follow. In this section we are concerned with non-iid Gaussian entries, more precisely with a sequence of Wishart matrices $W_n(n,\Sigma_n)$, $n=1,2,\dots$. It appears that in this case which on one side is more general than Girko's since the entries of the matrix which is squared are non-iid, on the other hand is more restrictive since the entries are Gaussian, the limiting law of determinants (properly normalized) is lognormal, as expected. The main advantage of our approach is the fact that the proof we offer herein is an immediate consequence of our asymptotic result for the product of independent sums as given in Theorem 1.

We first recall some basic facts about the classical Wishart distribution. For further references, see, e.g., Anderson (1984).

Let $Y_1, \dots, Y_n$ be iid d-dimensional Gaussian zero-mean random vectors with a positive definite covariance matrix Σ. The $d \times d$ dimensional random matrix $\mathbf{A} = \sum_{i=1}^n Y_i Y_i^{\mathrm{T}}$ is said to have the classical Wishart distribution $W_d(n, \Sigma)$. If $n \geqslant d$ then the distribution of $\mathbf{A}$ is concentrated on the open cone of $d \times d$ positive definite symmetric matrices $\mathscr{V}_d^+$ and its density with respect to the appropriate Lebesgue measure is

$$f(x) = \frac{\det(x)^{\frac{n-d-1}{2}} \exp[-\frac{1}{2}(\sigma^{-1}, x)]}{2^{\frac{nd}{2}} \det(\Sigma)^{\frac{n}{2}} \Gamma_d(\frac{n}{2})}, \quad x \in \mathscr{V}_d^+.$$

If $n < d$ then the Wishart measures is singular with respect to the Lebesgue measure and is concentrated on the boundary of the cone $\mathscr{V}_d^+$.

Let $\mathbf{A} \sim W_d(n, \Sigma)$ be decomposed into blocks according to the dimensions p and q, $p + q = d$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_2 \end{pmatrix},$$

such that $\mathbf{A}_1$ is a $p \times p$, $\mathbf{A}_{12} = \mathbf{A}_{21}^{\mathrm{T}}$ is a $p \times q$ and $\mathbf{A}_2$ is a $q \times q$ matrix. Similarly we can decompose

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_2 \end{pmatrix}.$$

It is well known that:

1. $\mathbf{A}_1 \sim W_p(n, \Sigma_1)$,
2. $\mathbf{A}_{2\cdot 1} = \mathbf{A}_2 - \mathbf{A}_{21}\mathbf{A}_1^{-1}\mathbf{A}_{12} \sim W_q(n - p, \Sigma_{2\cdot 1})$ with $\Sigma_{2\cdot 1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{12}$,
3. the pair $(\mathbf{A}_1, \mathbf{A}_{12})$ and $\mathbf{A}_{2\cdot 1}$ are independent,
4. $\det(\mathbf{A}) = \det(\mathbf{A}_1)\det(\mathbf{A}_{2\cdot 1})$.

Now we decompose $\mathbf{A} = [a_{ij}]$ step by step. First $\mathbf{A} = \mathbf{A}_{(1\dots d)}$ into blocks: $\mathbf{A}_{(1\dots d-1)}$ of dimensions $(d-1) \times (d-1)$ and $\mathbf{A}_d = a_{dd}$ of dimensions 1×1, then $\mathbf{A}_{(1\dots d-1)}$ into blocks $\mathbf{A}_{(1\dots d-2)}$ of dimensions $(d-2) \times (d-2)$ and $\mathbf{A}_{d-1} = a_{d-1,d-1}$ of dimensions 1×1, ending up with $\mathbf{A}_{(12)}$ decomposed into $\mathbf{A}_1 = a_{11}$ and $\mathbf{A}_2 = a_{22}$ both of dimensions 1×1. By properties (1–4) above we have the following multiplicative representation for the determinant of $\mathbf{A}$

$$\det(\mathbf{A}) = \mathbf{A}_{d\cdot(1\dots d-1)}\mathbf{A}_{d-1\cdot(1\dots d-2)} \cdot \dots \cdot \mathbf{A}_{2\cdot 1}\mathbf{A}_1,$$

where the factors are independent gamma variables:

$$Y_{n+1-k} = \mathbf{A}_{k\cdot 1\dots k-1} \sim G\left(\frac{n+1-k}{2}, \frac{1}{2\Sigma_{k\cdot(1\dots k-1)}}\right), \quad k = 1, \dots, d$$

understanding that $Y_1 = \mathbf{A}_1$. Here the gamma distribution $G(a,p)$ is defined through its density of the form $f(x) \propto x^{p-1}e^{-ax}I_{(0,\infty)}(x)$ for $a,p>0$. Thus for a triangular array of iid χ^2 variables with one degree of freedom X_{kj}, $j=1,\ldots,k$, $k=1,2,\ldots,d$, we have

$$Y_k \stackrel{d}{=} c_{kn}\sum_{l=1}^{k} X_{kl},$$

where $c_{kn} = \Sigma_{n+1-k\cdot(1\ldots n-k)}$, $k = n-d+1,\ldots,n$.

Taking all what was said above into account and using our Theorem 1 we can obtain the following asymptotic result for determinants of Wishart matrices.

Theorem 4. *Let* $\mathbf{A}_n \sim W_n(n, \Sigma_n)$, $n = 1,2,\ldots$. *Then*

$$\left(\frac{\det(\mathbf{A}_n)}{\det(\Sigma_n)(n-1)!}\right)^{\frac{1}{\sqrt{2\log(n)}}} \stackrel{d}{\to} e^{\mathcal{N}}. \tag{10}$$

Proof. Note that putting $d = n$ in the considerations prior to the formulation of Theorem 4, we have

$$\frac{\det(\mathbf{A}_n)}{\prod_{k=1}^{n} c_{kn}} \stackrel{d}{=} \prod_{k=1}^{n}\left(\sum_{l=1}^{n} X_{kl}\right),$$

where (X_{kl}) are iid $\chi^2(1)$ random variables. Thus in the notation of Theorem 1 we have $\mu = 1$ and $\gamma^2 = 2$. Moreover $\prod_{k=1}^{n} c_{kn} = \det(\Sigma_n)$. Now the result follows directly from Theorem 1. □

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