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The classical bi-Poisson process: An invertible quadratic harness ☆

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Abstract

We give an elementary construction of a time-invertible Markov process which is discrete except at one instance. The process is one of the quadratic harnesses studied in Bryc and Wesołowski [2005. Conditional moments of q-Meixner processes. Probab. Theory Related Fields 131, 415–441 ⟨arxiv.org/abs/math.PR/0403016⟩], Bryc et al. [2005b. Quadratic harnesses, q-commutations, and orthogonal martingale polynomials. Trans. Amer. Math. Soc. ⟨arxiv.org/abs/math.PR/0504194⟩, to appear], and Bryc et al. [2005a. The bi-Poisson process: a quadratic harness ⟨arxiv.org/abs/math.PR/0510208⟩]. It can be constructed from a pair of independent Poisson processes with the same gamma-distributed intensity.

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1. Introduction

The aim of this paper is to provide an elementary construction of a special case of Markov processes which in Bryc et al. (2005b) we call the bi-Poisson processes. Our interest in this construction is two-fold: it provides an example of a time-invertible process without smooth transition probabilities, and it is a non-trivial quadratic harness that can be analyzed in detail.

According to Watanabe (1974), a stochastic process $(X_t)_{t>0}$ has the time-inversion property, if it has the same finite-dimensional distributions as the process $(tX_{1/t})_{t>0}$. In papers Gallardo and Yor (2005) and Lawi (2005) the authors give criteria for the time invertibility of Markov processes with transition probabilities that have smooth densities with respect to the Lebesgue measure. It is of interest to give examples of time-invertible Markov processes without smooth transition probabilities. By Proposition 4.2 our construction gives an example of a time-invertible Markov process for which all transitions except between times $s<1$ and $t=1$ are

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discrete. Our previous example (Bryc and Wesołowski, 2004, Corollary 3.4) was much less elementary and while some of its transition probabilities did not possess a density, they always had an absolutely continuous and smooth component.

The general class of bi-Poisson processes was introduced in Bryc et al. (2005b, Example 4.8, Proposition 4.13). A bi-Poisson process with parameters (η, θ, q) is a square-integrable Markov process (X_t) which is uniquely determined by the following three properties:

$$\mathbb{E}(X_t) = 0, \quad \mathbb{E}(X_t X_s) = s, \tag{1}$$

$$\mathbb{E}(X_t|\mathscr{F}_{s,u}) = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u, \tag{2}$$

$$\mathrm{Var}(X_t|\mathscr{F}_{s,u}) = \frac{(u-t)(t-s)}{u-qs}\left(1+\eta\,\frac{uX_s - sX_u}{u-s} + \theta\,\frac{X_u - X_s}{u-s} - (1-q)\frac{(uX_s - sX_u)(X_u - X_s)}{(u-s)^2}\right), \tag{3}$$

for all $0 \leqslant s < t < u$, where $\mathscr{F}_{s,u} = \sigma\{X_t : 0 \leqslant t \leqslant s \text{ or } t \geqslant u\}$.

Processes with property (2) are sometimes called harnesses, see Mansuy and Yor (2005). Condition (3) is a special case of the quadratic harness property introduced in Bryc et al. (2005b). The adjective "classical" refers to the value of parameter $q = 1$, compare Bryc et al. (2005b, Section 4.2) and Bryc and Wesołowski (2005, Section 4.2). Thus, this paper gives an elementary construction of the bi-Poisson process with parameters $(\eta, \theta, 1)$. Since throughout most of the paper the value of the third parameter is fixed at $q = 1$, we will refer to such processes as bi-Poisson processes with parameters (η, θ).

Our previous constructions in Bryc et al. (2005a) and Bryc and Wesołowski (2004, 2005) relied heavily on cumbersome identities between certain multi-parameter families of orthogonal polynomials, and identified the transition probabilities in implicit form only. From the elementary construction in this paper we can see the structure of the process in more detail. In Proposition 4.1, we show that our process can be constructed by pasting together two conditionally independent Poisson processes which are coupled together by sharing a random gamma-distributed intensity. The pasting of these processes is accomplished by appropriate affine transformations and deterministic changes of time.

The plan of the paper is as follows. In Section 2 we give the construction of the process. In Section 3 we verify that the construction indeed gives a bi-Poisson process. In Section 4 we deduce some additional properties, including time invertibility.

2. Construction

It is known, see Bryc et al. (2005b, Proposition 4.13), that the parameters (η, θ) of a bi-Poisson process (X_t) with $q = 1$ satisfy inequality $\eta\theta \geqslant 0$. In the degenerate case $\theta\eta = 0$ it is known that the bi-Poisson process (X_t) is either $X_t = B_t$, where (B_t) is the standard Brownian motion for $\eta = \theta = 0$, or $X_t = \theta N_{t/\theta^2} - t/\theta$, where (N_t) is the standard Poisson process when $\eta = 0$, $\theta \neq 0$, see Wesołowski (1993, Theorem 1). Passing to the time inverse $(tX_{1/t})$, we see that $X_t = \eta t N_{1/(t\eta^2)} - 1/\eta$ in the remaining degenerate case $\theta = 0$, $\eta \neq 0$.

We will therefore concentrate on the case $\theta\eta > 0$. Passing to $(-X_t)$ preserves (3) replacing parameters (η, θ) by $(-\eta, -\theta)$, so we may assume $\eta, \theta > 0$. Replacing the process (X_t) by the process $(\sqrt{\eta/\theta} X_{t\theta/\eta})$, we get the bi-Poisson process with parameters $(\sqrt{\eta\theta}, \sqrt{\eta\theta})$. Thus, without loss of generality we may assume that $\eta = \theta > 0$.

The moment of time $t = 1$ is preserved by the time inversion and plays a special role in the construction. The bi-Poisson process traverses a family of deterministic lines, with jumps in the upward direction when $t < 1$ and in the downward direction when $t > 1$, see Fig. 1.

The process is determined by specifying the (random) integer that describes the line being followed at time t. The integers that describe the upward jumps form a linear pure birth process with immigration with the time transformed to run on the interval $[0, 1)$. At time $t = 1$ instead of being infinite, the process takes the continuous spectrum of real values. For $t > 1$, the downward jumps form a linear pure death process which "returns from ∞" by a Poisson entrance law, again with the time transformed to run on the interval $(1, \infty)$. The deterministic time transformations are logarithmic and introduce a rather simple non-homogeneity into

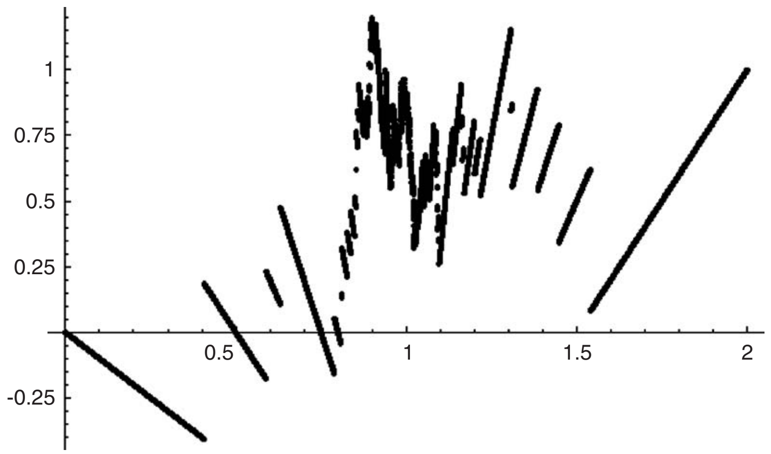

Fig. 1. Simulated sample trajectory of the bi-Poisson process. The process follows the segments $y = \theta(1-t)j - t/\theta$, $0<t<1$, $j = 0, 1, 2, \ldots$, except for the upward jumps, and then follows the half-lines $y = \theta(t-1)j - 1/\theta$, $t>1$, $j = \ldots, 1, 0$, with the downward jumps.

the birth rates and the death rates of the process. However, they force infinite number of jumps before and after $t = 1$.

For a more formal description of (X_t), we set

$$X_t = \begin{cases} \theta(1-t)Z_t - \dfrac{t}{\theta}, & 0 \leqslant t < 1, \\ \theta Z_1 - \dfrac{1}{\theta}, & t = 1, \\ \theta(t-1)Z_t - \dfrac{1}{\theta}, & t > 1, \end{cases} \tag{4}$$

where the random variables Z_t are $\{0, 1, 2, \ldots\}$-valued for $t \neq 1$. We will construct the appropriate process $(Z_t)_{t \geqslant 0}$ in three steps: we first define $(Z_t)_{0 \leqslant t < 1}$ as a pure birth process, then we extend it to $t = 1$ by passing to the limit, and finally we extend the process to $t > 1$ as a pure death process with a Z_1-dependent Poisson entrance law.

2.1. The pure birth phase

As $(Z_t)_{0 \leqslant t < 1}$ we take the non-homogeneous linear pure birth process with immigration with the birth rate

$$\lambda_n(t) = \frac{n + 1/\theta^2}{1-t}.$$

The properties of such a process are well-known. For small enough $|z|$, Parzen (1962, Exercise 5.1) gives the following generating function of the transition probabilities for the more general non-homogeneous linear pure birth process with the birth rate $\lambda_n(t) = v(t) + n\lambda(t)$.

$$\sum_{k=0}^{\infty} z^k p_{j,j+k}(s,t) = z^{v(s)/\lambda(s) - v(t)/\lambda(t)} \left(\frac{p(s,t)}{1 - z(1-p(s,t))} \right)^{j + v(s)/\lambda(s)},$$

where $p(s,t) = \mathrm{e}^{-\int_s^t \lambda(u)\,\mathrm{d}u}$. In our setting, $p(s,t) = (1-t)/(1-s)$ and with $z = \mathrm{e}^{u(1-t)}$ we get

$$\mathbb{E}(\mathrm{e}^{u(1-t)(Z_t - Z_s)} | \mathscr{F}_{\leqslant s}) = \left(\frac{1-t}{1-s-(t-s)\mathrm{e}^{u(1-t)}} \right)^{Z_s + 1/\theta^2}, \tag{5}$$

Table 1
Laws that appear as transition probabilities

Name	Parameters	Distribution	$E(e^{uZ})$
Poisson	$\lambda > 0$	$e^{-\lambda}\lambda^k/k!, k = 0, 1, 2, \ldots$	$\exp(\lambda(e^u - 1))$
Gamma	$p > 0,\ \sigma > 0$	$f(x) = \frac{1}{\sigma^p \Gamma(p)} x^{p-1} e^{-x/\sigma},\ x > 0$	$(1 - \sigma u)^{-p}$
Negative binomial	$r > 0,\ 0 < p < 1$	$\frac{\Gamma(k+r)}{\Gamma(r)k!} p^r (1-p)^k, k = 0, 1, \ldots$	$\frac{p^r}{(1-(1-p)e^u)^r}$
Binomial	$n \geqslant 0,\ 0 \leqslant p \leqslant 1$	$\binom{n}{k} p^k (1-p)^{n-k},\ k = 0, 1, \ldots, n.$	$(1 - p + pe^u)^n$

where $\mathscr{F}_{\leqslant s} = \sigma\{X_r : r \leqslant s\} = \sigma\{Z_r : r \leqslant s\}$. Thus, the conditional distribution $\mathscr{L}(Z_t - Z_s | Z_s)$ is negative binomial with parameters $r = Z_s + 1/\theta^2$, $p = (1-t)/(1-s)$. (Table 1 lists the parameterizations of the distributions we use in this note.)

Differentiating (5) at $u = 0$ and using (4) we verify that $(X_t, \mathscr{F}_{\leqslant t})_{0 \leqslant t < 1}$ is a martingale. Setting $s = 0$, from (5) we get $\mathbb{E}(X_t) = 0$ and differentiating (5) again, after a calculation, we get $\mathbb{E}(X_t^2) = t$.

Matringale property implies that $X_1 := \lim_{t \to 1} X_t$ exists almost surely; this defines $Z_1 := \lim_{t \to 1}(1-t)Z_t$. Taking the limit in (5) we see that

$$\mathbb{E}(e^{uZ_1} | Z_s) = (1 - u(1-s))^{-Z_s - 1/\theta^2}, \tag{6}$$

thus $\mathscr{L}(Z_1 | Z_s)$ is gamma with shape parameter $p = Z_s + 1/\theta^2$ and scale parameter $\sigma = 1 - s$, see Table 1. In particular, Z_1 is gamma with $r = 1/\theta^2$, $\sigma = 1$, and the support of X_1 is $[-1/\theta, \infty)$.

2.2. The pure death phase

We now extend (Z_t) from $0 \leqslant t \leqslant 1$ to $t > 1$ by specifying $(Z_t)_{t>1}$ as a pure death process with the death rate

$$\mu_n(t) = \frac{n}{t-1},$$

and the Z_1-dependent entrance law $\mathscr{L}(Z_t | Z_1)$ which we take as the Poisson law with parameter $\lambda = Z_1/(t-1)$. Thus

$$\mathbb{E}(e^{uZ_t} | Z_1) = \exp\left(Z_1 \frac{e^u - 1}{t-1} \right). \tag{7}$$

A well-known property of the linear pure death process is that for $1 < s < t$ the transition probabilities $\mathscr{L}(Z_t | Z_s)$ are binomial with parameters $n = Z_s$, $p = (s-1)/(t-1)$, so

$$\mathbb{E}(e^{uZ_t} | Z_s) = \left(\frac{t - s + (s-1)e^u}{t-1} \right)^{Z_s}. \tag{8}$$

The Poisson distribution is indeed the entrance law: given $1 < s < t$ we have

$$\Pr(Z_t = i | Z_1) = \sum_{n=i}^{\infty} \Pr(Z_t = i | Z_s = n) \Pr(Z_s = n | Z_1).$$

Indeed, the right-hand side is

$$\frac{Z_1^i}{i!(t-1)^i} e^{-Z_1/(s-1)} \sum_{n=i}^{\infty} \frac{Z_1^{n-i}(t-s)^{n-i}}{(n-i)!(s-1)^{n-i}(t-1)^{n-i}} = \frac{Z_1^i}{i!(t-1)^i} e^{-Z_1/(t-1)}.$$

We now verify that the two pieces of the process fit together into a well-defined Markov process $(Z_t)_{t\geqslant 0}$. For $0\leqslant s<1<t$, by conditioning on Z_1 we get

$$\begin{aligned}\Pr(Z_t=j|Z_s) &= \frac{1}{j!(t-1)^j}\mathbb{E}(Z_1^j \mathrm{e}^{-Z_1/(t-1)}|Z_s)\\ &= \int_0^\infty \frac{x^{Z_s+j+1/\theta^2-1}}{j!(t-1)^j(1-s)^{Z_s+1/\theta^2}\Gamma(Z_s+1/\theta^2)}\mathrm{e}^{-x/(1-s)}\,\mathrm{d}x\\ &= \frac{\Gamma(Z_s+j+1/\theta^2)}{j!\Gamma(Z_s+1/\theta^2)}\left(\frac{t-1}{t-s}\right)^{Z_s+1/\theta^2}\left(\frac{1-s}{t-s}\right)^j.\end{aligned}$$

Thus, $\mathscr{L}(Z_t|Z_s)$ is negative binomial with $r=Z_s+1/\theta^2$ and $p=(t-1)/(t-s)$. In particular, Z_t is negative binomial with $r=1/\theta^2$ and $p=1-1/t$. An elementary calculation shows that X_t defined by (4) has mean zero and variance t.

A straightforward calculation leads now to the verification of the Chapman–Kolmogorov equations in the remaining two cases:

(i) If $0<s_1<s_2<1<t$ and $i,k\geqslant 0$ then

$$\Pr(Z_t=k|Z_{s_1}=i)=\sum_{j=i}^{\infty}\Pr(Z_t=k|Z_{s_2}=j)\Pr(Z_{s_2}=j|Z_{s_1}=i).$$

To verify this, we rewrite the right-hand side as

$$\begin{aligned}&\frac{1}{k!\Gamma(i+1/\theta^2)}\left(\frac{t-1}{1-s_1}\right)^{i+1/\theta^2}\sum_{j=i}^{\infty}\frac{\Gamma(j+k+1/\theta^2)(1-s_2)^{i+k+1/\theta^2}(t-1)^{j-i}(s_2-s_1)^{j-i}}{(j-i)!(1-s_1)^{j-i}(t-s_2)^{j-i}(t-s_2)^{i+k+1/\theta^2}}\\ &\quad=\frac{\Gamma(i+k+1/\theta^2)}{k!\Gamma(i+1/\theta^2)}\left(\frac{t-1}{1-s_1}\right)^{i+1/\theta^2}\frac{(1-s_2)^{i+k+1/\theta^2}}{(t-s_2)^{i+k+1/\theta^2}}\left(1-\frac{(t-1)(s_2-s_1)}{(1-s_1)(t-s_2)}\right)^{-(i+k+1/\theta^2)}\\ &\quad=\frac{\Gamma(i+k+1/\theta^2)}{k!\Gamma(i+1/\theta^2)}\left(\frac{t-1}{t-s_1}\right)^{i+1/\theta^2}\left(\frac{1-s_1}{t-s_1}\right)^{k}.\end{aligned}$$

(ii) If $0<s<1<t_1<t_2$ and $i,k\geqslant 0$ then

$$\Pr(Z_{t_2}=k|Z_s=i)=\sum_{n=k}^{\infty}\Pr(Z_{t_2}=k|Z_{t_1}=n)\Pr(Z_{t_1}=n|Z_s=i).$$

We verify this by writing the right-hand side as

$$\begin{aligned}&\frac{(1-s)^k(t_1-1)^{i+k+1/\theta^2}}{k!\Gamma(i+1/\theta^2)(t_1-s)^{i+k+1/\theta^2}(t_2-1)^k}\sum_{n=k}^{\infty}\frac{\Gamma(n+i+1/\theta^2)}{(n-k)!}\left(\frac{(t_2-t_1)(1-s)}{(t_2-1)(t_1-s)}\right)^{n-k}\\ &\quad=\frac{\Gamma(k+i+1/\theta^2)}{k!\Gamma(i+1/\theta^2)}\left(\frac{t_1-1}{t_2-s}\right)^{i+1/\theta^2}\left(\frac{1-s}{t_2-s}\right)^{k}.\end{aligned}$$

Thus $(Z_t)_{t\geqslant 0}$ is a well-defined Markov process which determines Markov process $(X_t)_{t\geqslant 0}$ through the one-to-one transformation (4).

3. Conditional moments

We now verify that $(X_t)_{t\geqslant 0}$ is a quadratic harness.

Theorem 3.1. *For $\theta>0$, let (Z_t) be the Markov process defined in the previous section. Let (X_t) be defined by* (4). *Then (X_t) is the bi-Poisson process with parameters (θ,θ), i.e. it has covariance* (1), *conditional moments* (2) *and* (3) *with $\eta=\theta$ and $q=1$.*

Proof. In Section 2 we already verified that $\mathbb{E}(X_t) = 0$, $\mathbb{E}X_t^2 = t$. Since $\mathscr{L}(Z_t|Z_s)$ is binomial for $1 \leqslant s < t$, we have $\mathbb{E}(Z_t|Z_s) = ((s-1)/(t-1))Z_s$. Combining this with the already established martingale property for $t<1$, we see that $(X_t, \mathscr{F}_{\leqslant t})_{t \geqslant 0}$ is a martingale. From the martingale property we get (1).

To compute the conditional moments, we calculate explicitly the conditional distribution of $\mathscr{L}(Z_t|Z_s, Z_u)$. These are routine calculations, so we just state the final answers, and omit most of the calculations of the corresponding moments.

(i) If $0<s<t<u<1$ then $\mathscr{L}(Z_t - Z_s|Z_s, Z_u)$ is binomial with parameters $n = Z_u - Z_s$ and $p = (1-u)(t-s)/((1-t)(u-s))$. Therefore,

$$\mathbb{E}(Z_t|\mathscr{F}_{s,u}) = Z_s + \frac{(1-u)(t-s)}{(1-t)(u-s)}(Z_u - Z_s)$$
$$= \frac{(u-t)(1-s)}{(1-t)(u-s)}Z_s + \frac{(1-u)(t-s)}{(1-t)(u-s)}Z_u.$$

Using (4) we get

$$\mathbb{E}(X_t|\mathscr{F}_{s,u}) = -t/\theta + \frac{u-t}{u-s}(X_s + s/\theta) + \frac{t-s}{u-s}(X_u + u/\theta),$$

which gives (2). Similarly,

$$\mathrm{Var}(Z_t|\mathscr{F}_{s,u}) = \frac{(1-u)(t-s)(u-t)(1-s)}{(1-t)^2(u-s)^2}(Z_u - Z_s)$$

which gives

$$\mathrm{Var}(X_t|\mathscr{F}_{s,u}) = \frac{(t-s)(u-t)}{(u-s)^2}(\theta(1-s)(X_u + u/\theta) - \theta(1-u)(X_s + s/\theta)).$$

A further calculation gives (3).

(ii) If $0<s<t<1<u$ then $\mathscr{L}(Z_t - Z_s|Z_s, Z_u)$ is negative binomial with parameters $r = Z_s + Z_u + 1/\theta^2$ and $p = (1-t)(u-s)/((1-s)(u-t))$. Therefore,

$$\mathbb{E}(Z_t|\mathscr{F}_{s,u}) = Z_s + r(1-p)/p$$
$$= \frac{(1-s)(u-t)}{(1-t)(u-s)}Z_s + \frac{(u-1)(t-s)}{(1-t)(u-s)}Z_u + \frac{(u-1)(t-s)}{\theta^2(1-t)(u-s)},$$

which leads to (2) and

$$\mathrm{Var}(Z_t|\mathscr{F}_{s,u}) = \frac{r(1-p)}{p^2}$$
$$= \frac{(u-1)(1-s)(t-s)(u-t)}{(1-t)^2(u-s)^2}(Z_s + Z_u + 1/\theta^2),$$

which after a calculation leads to (3).

(iii) If $0<s<1<t<u$ then $\mathscr{L}(Z_t - Z_u|Z_s, Z_u)$ is negative binomial with parameters $r = Z_s + Z_u + 1/\theta^2$ and $p = (t-1)(u-s)/((t-s)(u-1))$. We omit the details of a calculation which verifies (2) and (3).

(iv) If $1<s<t<u$ then $\mathscr{L}(Z_t - Z_u|Z_s, Z_u)$ is binomial with $n = Z_s - Z_u$ and $p = (s-1)(u-t)/((t-1)(u-s))$. We omit calculations which verify (2) and (3).

The conditional moments for the remaining choices of $s<t<u$ follow by continuity. □

4. Additional properties

Proposition 4.1 (*Poisson representation*). *Let* (N_t^λ) *and* (M_t^λ) *be two independent Poisson processes with intensity* $\lambda>0$. *If* (X_t) *is a bi-Poisson process with positive parameters* (η,θ) *then*

$$\mathscr{L}\left(\left(t\left(h(t)X_{\theta/(\eta h(t))}+\frac{1}{\eta}\right)\right)_{t>0},\left(t\left(X_{\theta h(t)/\eta}+\frac{1}{\eta}\right)\right)_{t>0}\middle|X_{\theta/\eta}=\lambda-\frac{1}{\eta}\right)=\mathscr{L}((N_t^\lambda)_{t>0},(M_t^\lambda)_{t>0}),$$

where

$$h(t)=\frac{1+\theta t}{\theta t}.$$

Proof. Without loss of generality we assume $\eta=\theta$. By (4) and the Markov property, it suffices to prove that

$$\mathscr{L}((Z_{1/h(t)})_{t>0}|Z_1=\lambda)=\mathscr{L}((N_t^\lambda)_{t>0}) \tag{9}$$

and

$$\mathscr{L}((Z_{h(t)})_{t>0}|Z_1=\lambda)=\mathscr{L}((M_t^\lambda)_{t>0}). \tag{10}$$

Both equalities follow now from elementary calculations of finite-dimensional distributions using the conditional distributions identified in Section 2.

To prove (10), take $t_n<t_{n-1}<\cdots<t_1$ so that $1<h(t_1)<h(t_2)<\cdots<h(t_n)$. Then for $k_1\geqslant k_2\geqslant\cdots\geqslant k_n$ denoting $Y_j=Z_{h(t_j)}$ we have

$$\begin{aligned}&\Pr(Y_1=k_1,Y_2=k_2,\ldots,Y_n=k_n|Z_1=\lambda)\\&\quad=\Pr(Y_n=k_n|Y_{n-1}=k_{n-1})\ldots\Pr(Y_2=k_2|Y_1=k_1)\Pr(Y_1=k_1|Z_1=\lambda)\\&\quad=\frac{\lambda^{k_1}\exp(-\lambda t_1)}{k_n!(k_{n-1}-k_n)!\ldots(k_1-k_2)!}t_n^{k_n}(t_{n-1}-t_n)^{k_{n-1}-k_n}\ldots(t_1-t_2)^{k_1-k_2},\end{aligned}$$

which proves (10).

The proof of (9) is similar after using the generalized Bayes formula. Let $t_1<t_2<\cdots<t_n$ so that $0<1/h(t_1)<1/h(t_2)<\cdots<1/h(t_n)<1$. Then for $k_1\leqslant k_2\leqslant\cdots\leqslant k_n$, denoting $Y_j=Z_{1/h(t_j)}$ we have

$$\begin{aligned}&\Pr(Y_1=k_1,Y_2=k_2,\ldots,Y_n=k_n|Z_1=\lambda)\\&\quad=\frac{f_{Z_1|Y_n=k_n}(\lambda)}{f_{Z_1}(\lambda)}\Pr(Y_n=k_n|Y_{n-1}=k_{n-1})\ldots\Pr(Y_2=k_2|Y_1=k_1)\Pr(Y_1=k_1),\end{aligned}$$

where $f_{Z_1|Y_n=k_n}$ is the conditional density of Z_1 given $Y_n=k_n$, and f_{Z_1} is the density of Z_1, which are both gamma, see the last paragraph of Section 2.1. Elementary calculations now prove (9). □

Proposition 4.2 (*Time inversion*). *If* (X_t) *is a bi-Poisson process with parameters* (θ,θ,q), *then* $(tX_{1/t})_{t>0}$ *has the same distribution as* $(X_t)_{t>0}$. (*Compare* Gallardo and Yor, 2005; Lawi, 2005.)

Proof. This follows from the fact that process $(tX_{1/t})$ satisfies (1), (2) and (3), with the same parameters as process (X_t). By Bryc et al. (2005b, Theorem 4.5, Proposition 4.13) the finite-dimensional distributions of such a process are determined uniquely. For $q=1$, the conclusion can also be derived directly from (4) using the fact that Markov process $(Z_t)_{t>0}$ has the same transition probabilities as $(Z_{1/t})_{t>0}$. □

Proposition 4.3 (*Distribution of upward jumps*). *For a bi-Poisson process* (X_t) *with parameters* (θ,θ), *where* $\theta>0$, *define*

$$\Gamma_i=\sup\{s\in[0,1):\ Z_s=i\},\quad i=0,1,\ldots,$$

i.e. Γ_i *is the time of the* $(i+1)$*th jump of the process* (X_t) *from the line* $y=\theta(1-t)i-t/\theta$, $0\leqslant t<1$, *for* $i=0,1,\ldots$, *see Fig.* 1. *Then the joint density of the random vector* $(\Gamma_0,\Gamma_1,\ldots,\Gamma_k)$ is

$$f_{(\Gamma_0,\Gamma_1,\ldots,\Gamma_k)}(s_0,s_1,\ldots,s_k)=\frac{\Gamma(1/\theta^2+k+1)(1-s_k)^{1/\theta^2+k-1}}{\Gamma(1/\theta^2)(1-s_0)^2(1-s_1)^2\ldots(1-s_{k-1})^2} \tag{11}$$

for $0\leqslant s_0<s_1<\cdots<s_k<1$ (*and* 0 *otherwise*).

Proof. Let $(M_t)_{t\geqslant 0}$ be a homogeneous pure birth process with birth rates $\lambda_n=n+1/\theta^2$, $n=0,1,\ldots$. It is well known that the sojourn times τ_j of (M_t) in state j are exponential with parameter $j+1/\theta^2$, $j=0,1,\ldots$. For $0\leqslant t<1$ we have $Z_t=M_{-\ln(1-t)}$, so $-\log(1-\Gamma_k)=\sum_{j=0}^k\tau_j$. Therefore, $\tau_k=\ln(1-\Gamma_{k-1})-\ln(1-\Gamma_k)$ (here, we set $\Gamma_{-1}=0$). Since the Jacobian of the transformation $s_j\mapsto\ln(1-s_{j-1})-\ln(1-s_j)$, $j=0,\ldots,k$, is $J(s_0,\ldots,s_k)=\prod_{j=0}^k(1-s_j)^{-1}$, and $\tau_0,\tau_1,\ldots,\tau_k$ are independent, the joint density of $(\Gamma_0,\Gamma_1,\ldots,\Gamma_k)$ is

$$J(s_0,\ldots,s_k)\prod_{j=0}^k\left((j+1/\theta^2)\exp\left(-(j+1/\theta^2)\ln\frac{1-s_{j-1}}{1-s_j}\right)\right)$$

which simplifies to (11).

Alternatively, we can use the fact that

$$\begin{aligned}R&=P(s_0<\Gamma_0<s_1<\Gamma_1<s_2<\cdots<s_{k-1}<\Gamma_{k-1}<s_k<\Gamma_k)\\&=P(Z_{s_0}=0,Z_{s_1}=1,\ldots,Z_{s_k}=k).\end{aligned}$$

The formula

$$f_{(\Gamma_0,\Gamma_1,\ldots,\Gamma_k)}(s_0,s_1,\ldots,s_k)=(-1)^k\frac{\partial^{k+1}R}{\partial s_k\partial s_{k-1}\ldots\partial s_1\partial s_0}$$

yields (11) after a calculation. □

Proposition 4.4 (*Distribution of downward jumps*). *Consider*

$$\Delta_i=\inf\{t>1:\ Z_t=i\},\quad i=0,1,\ldots,$$

i.e. Δ_i *is the time of entrance of the process* (X_t) *onto the line* $y=\theta(t-1)i-1/\theta$, $t>1$, *for* $i=0,1,\ldots$. *Then the joint density of the random vector* $(\Delta_0,\Delta_1\ldots,\Delta_k)$ *is*

$$f_{(\Delta_0,\Delta_1\ldots,\Delta_k)}(t_k,\ldots,t_1,t_0)=\frac{\Gamma\left(1/\theta^2+k+1\right)(t_k-1)^{1/\theta^2+k-1}}{\Gamma\left(1/\theta^2\right)t_k^{1/\theta^2+k+1}(t_0-1)^2(t_1-1)^2\ldots(t_{k-1}-1)^2}$$

for $1<t_k<\cdots<t_1<t_0$ (*and* 0 *otherwise*).

Proof. From time invertibility of the process, $(\Delta_0,\Delta_1\ldots,\Delta_k)$ has the same distribution as $(1/\Gamma_0,1/\Gamma_1,\ldots,1/\Gamma_k)$. Thus,

$$f_{(\Delta_0,\Delta_1\ldots,\Delta_k)}(t_k,\ldots,t_1,t_0)=\frac{1}{t_0^2t_1^2\ldots t_k^2}f_{(\Gamma_0,\Gamma_1,\ldots,\Gamma_k)}(1/t_0,1/t_1,\ldots,1/t_k),$$

which simplifies to the expression above. □

Proposition 4.5 (*Time to reach lower boundary*). *The time* Δ_0 *in which a bi-Poisson process* (X_t) *with parameters* (θ,θ) *with* $\theta>0$ *reaches the horizontal line* $-1/\theta$ (*on which it then stays forever*) *is finite but has infinite expectation.*

Proof. The distribution of $\Delta_0=\inf\{t>1:X_t=-1/\theta\}$ is a special case of the distribution of jumps, but it is just as easy to give an independent derivation. Since Z_t is negative binomial, for $t>1$ we have $\Pr(\Delta_0>t)=1-\Pr(Z_t=0)=1-(1-1/t)^{1/\theta^2}$.

From the inequalities $(1-x)^p \leqslant 1-px$ when $0<p<1$, $x \geqslant 0$ and $(1-x)^p \leqslant (1-x)$ when $p>1$, $0 \leqslant x \leqslant 1$ we get

$$\mathbb{E}(\Delta_0) = \int_1^\infty \left(1 - \left(1 - \frac{1}{t}\right)^{1/\theta^2}\right) \mathrm{d}t \geqslant \min\{1, 1/\theta^2\} \int_1^\infty \frac{\mathrm{d}t}{t} = \infty. \qquad \square$$

Proposition 4.6 (*Poisson limit*). *For $\eta\theta > 0$ let $(X_t^{(\eta,\theta)})$ be a bi-Poisson process with parameters (η, θ), and let (N_t) be the Poisson process with parameter $\lambda = 1$. As $\eta \to 0$ the process $((1/\theta)X_{t\theta^2}^{(\eta,\theta)})$ converges in $D[0,\infty)$ to the Poisson-type process $(N_t - t)_{t \geqslant 0}$.*

Proof. Calculating the conditional variances one can check that $((1/\theta)X_{t\theta^2}^{(\eta,\theta)})$ is a bi-Poisson process with parameters $(\eta\theta, 1)$.

Consider now the bi-Poisson process $(X_t) = (X_t^{(\varepsilon)})$ with parameters $(\varepsilon, \varepsilon)$ for $\varepsilon = \sqrt{\eta\theta}$. Then by the previous argument, process $Y^\varepsilon = ((1/\varepsilon)X_{t\varepsilon^2}^{(\varepsilon)})_{t \geqslant 0}$ has the same distribution as process $((1/\theta)X_{t\theta^2}^{(\eta,\theta)})$. Therefore, it suffices to show that as $\varepsilon \to 0$, the process Y^ε converges in $D[0,\infty)$ to the Poisson-type process $(N_t - t)_{t \geqslant 0}$.

We first verify the convergence of finite-dimensional distributions. For $0 \leqslant t < 1/\varepsilon^2$, the appropriate version of (4) is

$$\frac{1}{\varepsilon} X_{t\varepsilon^2}^{(\varepsilon)} = (1 - t\varepsilon^2) Z_{t\varepsilon^2}^{(\varepsilon)} - t, \tag{12}$$

where $Z_{t\varepsilon^2}^{(\varepsilon)}$ is a (non-homogeneous) pure birth process on $0 \leqslant t < 1/\varepsilon^2$. We will verify that the finite-dimensional distributions of $(Z_{t\varepsilon^2}^{(\varepsilon)})$ converge to the finite-dimensional distributions of (N_t).

Fix arbitrary $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and $u_1, u_2, \ldots, u_n \leqslant 0$. It suffices to show that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\exp\left(\sum_{j=1}^n u_j Z_{t_j\varepsilon^2}^{(\varepsilon)}\right)\right) = \prod_{j=1}^n \exp(-(t_j - t_{j-1})(\mathrm{e}^{\sum_{i=j}^n u_i} - 1)). \tag{13}$$

We rely on the following observation, which can be regarded as special case of Slutsky's theorem: if $u_\alpha \to u$ and $(W_1^{(\alpha)}, W_2^{(\alpha)}, \ldots, W_n^{(\alpha)})$ converges weakly to a random vector $(W_1, W_2, \ldots, W_n)$ as $\alpha \to 0$ and appropriate exponential moments exist, then

$$\lim_{\alpha \to 0} \mathbb{E}\left(\exp\left(\sum_{j=1}^n u_j W_j^{(\alpha)} + u_\alpha W_n^{(\alpha)}\right)\right) = \mathbb{E}\left(\exp\left(\sum_{j=1}^n u_j W_j + u W_n\right)\right). \tag{14}$$

From (5) we have

$$\mathbb{E}\left(\exp\left(\sum_{j=1}^n u_j Z_{t_j\varepsilon^2}^{(\varepsilon)}\right)\right) = A_\varepsilon \mathbb{E}\left(\exp\left(\sum_{j=1}^{n-1} u_j Z_{t_j\varepsilon^2}^{(\varepsilon)} + (u_n + u_\varepsilon) Z_{t_{n-1}\varepsilon^2}^{(\varepsilon)}\right)\right),$$

where

$$\mathrm{e}^{u_\varepsilon} = \frac{1 - \varepsilon^2 t_n}{1 - \varepsilon^2 t_{n-1} - \varepsilon^2 (t_n - t_{n-1}) \mathrm{e}^{u_n}} \to 1$$

and

$$A_\varepsilon = \mathrm{e}^{u_\varepsilon/\varepsilon^2} \to \mathrm{e}^{-(t_n - t_{n-1})(\mathrm{e}^{u_n} - 1)}.$$

This proves (13) for $n = 1$, and shows that (13) holding for $n - 1$ implies (13) for n, ending the proof by induction.

Therefore, the increments of $(Z_{t\varepsilon^2}^{(\varepsilon)})$ are asymptotically independent Poisson distributed with parameter $\lambda = t - s$, and the finite-dimensional distributions of $(Z_{t\varepsilon^2}^{(\varepsilon)})$ converge to the corresponding distributions of (N_t).

To prove convergence in $D[0,\infty)$, we can restrict our attention to $\varepsilon \to 0$ over a countable set. Then tightness of Y^ε in $D[0,\infty)$ follows from Lindvall (1973, Corollary, p. 120), which says that for processes that have jumps located at absolutely continuous moments of time tightness in $D[0,\infty)$ is equivalent to tightness in $D[0,T]$ for all deterministic $T>0$. Namely, for a fixed $T>0$ and $1/\varepsilon^2>T$, our process is a simple linear transformation (12) of the pure-jump process $Z^{(\varepsilon)}_{t\varepsilon^2}$ to which we can apply Gut and Janson (2001, Theorem 2.1, Remark 4.2). Indeed, their condition (4.3) holds as Y^ε is a martingale and $E(Y^\varepsilon(t))^2=t$. Their condition (4.4) holds as for fixed $a_1\leqslant s\leqslant a_2$ and all ε small enough from (5) we get

$$\Pr(Z^{(\varepsilon)}_{t\varepsilon^2}\text{ has at least two jumps in }[s,s+\delta))\leqslant \Pr(Z^{(\varepsilon)}_{(s+\delta)\varepsilon^2}-Z^{(\varepsilon)}_{s\varepsilon^2}\geqslant 2)\leqslant \delta^2 C(a_1,a_2)\mathbb{E}(1+\varepsilon^2 Z^{(\varepsilon)}_{a_2\varepsilon^2})^2.$$

(Here we used the fact that if X has $nb(r,p)$ law then $\Pr(X>1)=1-p^r-r(1-p)p^r=(1-p)^2\sum_{k=1}^r kp^{k-1}\leqslant (1-p)^2 r(r+1)/2$.) Thus,

$$\delta^{-1}\limsup_{\varepsilon\to 0}\sup_{a_1\leqslant s\leqslant a_2}\Pr(\text{at least two }Z^{(\varepsilon)}_{t\varepsilon^2}\text{ jumps in }[s,s+\delta))\leqslant \delta C(a_1,a_2)\to 0\text{ as }\delta\to 0.$$

Finally, their condition (2.5) holds as $Z^{(\varepsilon)}_0=0$ and for fixed $\eta>0$ and all $a>0$ small enough

$$\limsup_{\varepsilon\to 0}\Pr\left(\sup_{s\leqslant a}|Z^{(\varepsilon)}_{a\varepsilon^2}|>\eta\right)\leqslant \limsup_{\varepsilon\to 0}\Pr(Z^{(\varepsilon)}_{a\varepsilon^2}\geqslant 1)=a\mathrm{e}^{-a}\to 0\quad\text{as } a\to 0.$$

Proposition 4.7 (*Brownian limit*). *Let $(X_t)_{t\geqslant 0}=(X^{(\theta)}_t)_{t\geqslant 0}$ be a bi-Poisson process with parameters (θ,θ). Then as $\theta\to 0^+$ the process $(X^{(\theta)}_t)_{t\geqslant 0}$ converges in $D[0,\infty)$ to the standard Brownian motion $(B_t)_{t\geqslant 0}$.*

Proof. The convergence of finite-dimensional distributions follows from uniqueness of the quadratic harnesses (see Wesołowski, 1993 for the special case we need here) and the fact that the limiting process must satisfy (3) by the uniform integrability of $\{(X^{(\theta)}_t)^2:\theta\leqslant 1\}$; in fact, $\mathbb{E}(X^4_t)=2t^2+(t+3t^2+t^3)\theta^2$.

Tightness in $D[0,\infty)$ now follows from Aldous (1989, Proposition 1.2), as $(X^{(\theta)}_t:0\leqslant t<\infty)$ is a martingale for each θ, and the limiting process is continuous.

Here we give a simple direct argument for the convergence of finite-dimensional distributions. It is enough to prove that for all $0=t_0<t_1<t_2<\cdots<t_n$ and all $u_1,u_2,\ldots,u_n$ close enough to zero

$$\lim_{\theta\to 0}\mathbb{E}\left(\exp\left(\sum_{j=1}^n u_jX^{(\theta)}_{t_j}\right)\right)=\exp\frac{1}{2}\left(\sum_{k=1}^n(t_k-t_{k-1})\left(\sum_{j=k}^n u_j\right)^2\right).\tag{15}$$

We proceed by induction, suppressing θ in $X^{(\theta)}_t$ to shorten the expressions. Without loss of generality we may assume $t_1<1$ (set $u_1=0$, if necessary). To verify (15) for $n=1$ and $t_1<1$ we use (4) and (5) with $s=0$, which gives

$$\mathbb{E}(\mathrm{e}^{u_1X_{t_1}})=\left(\frac{1-t_1}{\mathrm{e}^{u_1t_1\theta}-t_1\mathrm{e}^{u_1\theta}}\right)^{1/\theta^2}\to\exp\left(\frac{1}{2}u_1^2t_1\right).$$

Suppose (15) holds for $0=t_0<t_1<t_2<\cdots<t_{n-1}<1$ and let $t_n\in(t_{n-1},1)$. Then again using (4) and (5) we get

$$\mathbb{E}\left(\exp\left(\sum_{j=1}^n u_jX_{t_j}\right)\right)=A_\theta\mathbb{E}\left(\exp\left(\sum_{j=1}^{n-1}u_jX_{t_j}+u_\theta X_{t_{n-1}}\right)\right),\tag{16}$$

where

$$A_\theta=\left(\frac{1-t_n}{1-t_{n-1}-(t_n-t_{n-1})\mathrm{e}^{\theta u_n(1-t_n)}}\right)^{1/(\theta^2(1-t_{n-1}))}\exp\left(\frac{-u_n(t_n-t_{n-1})}{\theta(1-t_{n-1})}\right),$$

$$\mathrm{e}^{u_\theta}=\left(\frac{1-t_n}{1-t_{n-1}-(t_n-t_{n-1})\mathrm{e}^{\theta u_n(1-t_n)}}\right)^{1/(\theta(1-t_{n-1}))}\exp\left(u_n\frac{1-t_n}{1-t_{n-1}}\right).$$

Since

$$A_\theta=\left(\frac{1-t_n}{(1-t_{n-1})\mathrm{e}^{\theta u_n(t_n-t_{n-1})}-(t_n-t_{n-1})\mathrm{e}^{\theta u_n(1-t_{n-1})}}\right)^{1/(\theta^2(1-t_{n-1}))},$$

by Taylor expansion $\lim_{\theta\to 0}A_\theta=\mathrm{e}^{(1/2)u_n^2(t_n-t_{n-1})}$. Similarly, $\lim_{\theta\to 0}u_\theta=u_n$, which upon using (14) and induction assumption proves (15).

Now we know that (15) holds for all $n\geqslant 1$ and $0=t_0<t_1<t_2<\cdots<t_{n-1}<1$, and we take $t_n=1$. Using (4) and (6) we see that (16) holds with

$$e^{u_\theta}=\left(\frac{1}{1-(1-t_{n-1})u_n\theta}\right)^{1/(\theta(1-t_{n-1}))}\to e^{u_n}$$

and

$$A_\theta=e^{(u_\theta-u_n)/\theta}\to e^{(1/2)u_n^2(1-t_{n-1})}.$$

By (14) this proves (15).

Now we know that (15) holds for all $0=t_0<t_1<t_2<\cdots<t_{n-2}<t_{n-1}=1$, $n\geqslant 1$ and we take $t_n>1$. By (4) and (7) we get (16) with

$$e^{u_\theta}=\exp\left(\frac{e^{\theta u_n(t_n-1)}-1}{\theta(t_n-1)}\right)\to e^{u_n}$$

and

$$A_\theta=e^{(u_\theta-u_n)/\theta}\to e^{(1/2)u_n^2(t_n-1)}.$$

Again using (14) we get (15).

Finally, we assume that (15) holds for all $0=t_0<t_1<t_2<\cdots<t_{n-1}$, $n\geqslant 1$ with $t_{n-1}>1$ and we take $t_n>t_{n-1}$. Using (4) and (6) we see that (16) holds with

$$e^{u_\theta}=\left(\frac{t_n-t_{n-1}+(t_{n-1}-1)e^{\theta u_n t_n-1}}{t_n-1}\right)^{1/(\theta(t_{n-1}-1))}\to e^{u_n}$$

and

$$A_\theta=e^{(u_\theta-u_n)/\theta}\to e^{(1/2)u_n^2(t_n-t_{n-1})}.$$

By (14) this proves (15). □

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