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Multivariate Matsumoto–Yor property is rather restrictive

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Abstract

Matsumoto and Yor [2001. An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. Part II: the role of the GIG laws. Nagoya Math. J. 162, 65–86] discovered an interesting invariance property of a product of the generalized inverse Gaussian (GIG) and the gamma distributions. For univariate random variables or symmetric positive definite random matrices it is a characteristic property for this pair of distributions. It appears that for random vectors the Matsumoto–Yor property characterizes only very special families of multivariate GIG and gamma distributions: components of the respective random vectors are grouped into independent subvectors, each subvector having linearly dependent components. This complements the version of the multivariate Matsumoto–Yor property on trees and related characterization obtained in Massam and Wesołowski [2004. The Matsumoto–Yor property on trees. Bernoulli 10, 685–700].

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1. Introduction

Consider two random variables: X having the generalized inverse Gaussian (GIG) distribution $\mu_{-p,a,a}$ and Y having the gamma distribution $\gamma_{p,a}$. Recall, that the GIG distribution $\mu_{-p,a,b}$, where $p \in R,\ a, b > 0$ are the parameters, is defined by

$$\mu_{-p,a,b}(\mathrm{d}x) = K_1 x^{-p-1} \exp(-a^{-1}x - (bx)^{-1}) I_{(0,\infty)}(x)\,\mathrm{d}x$$

and the gamma distribution $\gamma_{q,c}$, where $q, c > 0$ are parameters, is defined by

$$\gamma_{q,c}(\mathrm{d}y) = K_2 y^{q-1} \exp(-c^{-1}y) I_{(0,\infty)}(y)\,\mathrm{d}y$$

(K_1, K_2 are normalizing constants). Matsumoto and Yor (2001) observed that if random variables X and Y are independent then $U = (X+Y)^{-1}$ and $V = X^{-1} - (X+Y)^{-1}$ are also independent and have the same distributions as X and Y,

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respectively. The following extension of the Matsumoto–Yor (MY in the sequel) property is immediate: if (X, Y) has the distribution $\mu_{-p,a,b} \otimes \gamma_{p,a}$ ($\otimes$ denotes the product measure) then (U, V) is distributed according to $\mu_{-p,b,a} \otimes \gamma_{p,b}$. Its interpretation in terms of Brownian motion and related stochastic processes was given in Matsumoto and Yor (2003). Letac and Wesołowski (2000) obtained its converse with the proof based on an application of the Laplace transform technique (see Wesołowski, 2002b for an alternative approach based on densities). They dealt additionally with a matrix variate version of the MY property deriving a related characterization under the assumption that densities of random matrices X and Y exist and are strictly positive real functions of the class C_2. Wesołowski (2002a) extended this result by weakening the smoothness assumption, imposed on densities, to differentiability. Regression-type characterizations obtained in that paper have been recently refined in Chou and Huang (2004). The MY property for random matrices of different dimensions and related characterization has been studied in Massam and Wesołowski (2006), where a connection with the conditional structure of Wishart matrices was strongly emphasized.

The bivariate version of the MY property and a respective characterization has been considered recently in Bobecka and Wesołowski (2005). It has been proved there that in the bivariate setting, the property does not characterize general families of bivariate GIG and gamma distributions, but is more restrictive: it implies that respective random vectors have independent or linearly dependent components.

Here we are concerned with the MY property for random vectors of higher dimensions. The bivariate case is only a starting point of our induction argument. It appears that random vectors with the MY property have to have components grouped into independent subvectors, each vector having linearly dependent GIG or gamma components, respectively. So, similarly as in two dimensions only very special types of multivariate GIG and gamma distributions are involved. In the proof we borrow a lot from Bobecka and Wesołowski (2004), which will be referred to as BW in the sequel.

It has to be emphasized that another version of the multivariate MY property, defined in the language of directed trees, has been studied recently in Massam and Wesołowski (2004). These authors showed that the MY-like independence properties (defined through taking different roots in the given undirected tree) characterize multivariate distributions $W(q, K_G, a)$ with densities

$$f(k) \propto |\mathbf{k}|^{q-1} \mathrm{e}^{-(a,k)}, \quad k \in M(G, K_G),$$

where $G = (V, E)$ is an undirected tree with p vertices (V is the set of vertices and E the set of vertices),

$$K_G = \{k_{i,j} = k_{j,i} \neq 0, (i, j) \in E, k_{i,j} = k_{j,i} = 0, (i, j) \notin E\},$$

$$M(G, K_G) = \{k = (k_1, \ldots, k_p) \in \mathbf{R}^p : \mathbf{k} = [k_{i,j}] \in \Omega_p^+, k_{i,i} = k_i, k_{i,j} \in K_G, i \neq j\},$$

where Ω_p^+ is the cone of $p \times p$ positive definite symmetric matrices. Such distributions, due to the shape of their density function, can be regarded as versions of a multivariate gamma law. However, their univariate marginals are not of the gamma type, moreover those attached to the leaves of the tree are independent GIGs.

Let us note that the development of studies related to the MY property is somehow parallel to investigations concerning the Lukacs characterization of the gamma law. The Lukacs theorem (1955) for the univariate case was followed by its matrix variate analogue—see Olkin and Rubin (1962), Casalis and Letac (1996), Letac and Massam (1998), and Bobecka and Wesołowski (2002). The case of random vectors, first studied in the bivariate case of constant regressions by Wang (1981), only recently has been considerably expanded—see Bobecka (2002), Bobecka and Wesołowski (2003) and BW.

2. Characterization

In the sequel, we will use the following definitions:

We will say that a positive random vector $\bar{Y} = (Y_1, \ldots, Y_n)$ has a distribution $MG^*(\bar{A}, \bar{p}, \bar{\lambda})$, where $\bar{A} = (A_1, \ldots, A_r)$ is such that $\bigcup_{i=1}^r A_i = \{1, \ldots, n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\bar{p} = (p_1, \ldots, p_r)$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $p_i, \lambda_j > 0$, if a Laplace transform of $\bar{Y}$ is of the form

$$L_{\bar{Y}}(\sigma_1, \ldots, \sigma_n) = \prod_{i=1}^{r} \left(1 - \sum_{j \in A_i} \lambda_j \sigma_j \right)^{-p_i},$$

i.e. Y_j has gamma distribution: γ_{p_j,λ_j}, $j=1,\ldots,n$, and the components of $\bar{Y}$ are grouped into independent subvectors: $\bar{Z}_1=(Y_l)_{l\in A_1},\ldots,\bar{Z}_r=(Y_l)_{l\in A_r}$, each subvector having linearly dependent components, i.e. $\forall i\ \exists k\in A_i$ such that $Y_j=(\lambda_j/\lambda_k)\,Y_k\ \forall j\in A_i$ (note that in the case $r=n$ the components of $\bar{Y}$ are independent).

We will say that a positive random vector $\bar{X}=(X_1,\ldots,X_n)$ has a distribution $MGIG^*(\bar{A},\bar{p},\bar{\lambda},\bar{\kappa})$, where $\bar{A}=(A_1,\ldots,A_r)$ is such that $\bigcup_{i=1}^r A_i=\{1,\ldots,n\}$, $A_i\cap A_j=\emptyset$, $i\neq j$ and $\bar{p}=(p_1,\ldots,p_r)$, $\bar{\lambda}=(\lambda_1,\ldots,\lambda_n)$, $\bar{\kappa}=(\kappa_1,\ldots,\kappa_n)$, where $p_i\in\mathbf{R}$, $\lambda_j,\kappa_j>0$, if X_j has GIG distribution: $\mu_{-p_j,\lambda_j,\kappa_j}$, $j=1,\ldots,n$, and the components of $\bar{X}$ are grouped into independent subvectors: $\bar{Z}_1=(X_l)_{l\in A_1},\ldots,\bar{Z}_r=(X_l)_{l\in A_r}$, each subvector having linearly dependent components, i.e. $\forall i\ \exists k\in A_i$ such that $X_j=(\lambda_j/\lambda_k)X_k\ \forall j\in A_i$ (note that in the case $r=n$ the components of $\bar{X}$ are independent).

Now we are ready to state the main result of this paper which characterizes the MG^* and $MGIG^*$ distributions through the MY property. In the proof we combine and develop techniques used in the univariate (Letac and Wesołowski, 2000) and the bivariate (Bobecka and Wesołowski, 2005) characterizations related to the MY property as well as those used in the multivariate Lukacs characterization (BW).

Theorem 1. *Let $\bar{X}=(X_1,\ldots,X_n)$ and $\bar{Y}=(Y_1,\ldots,Y_n)$ be independent non-degenerate n-variate positive random vectors. Let*

$$\bar{U}=(U_1,\ldots,U_n)=\left(\frac{1}{X_1+Y_1},\ldots,\frac{1}{X_n+Y_n}\right)$$

and

$$\bar{V}=(V_1,\ldots,V_n)=\left(\frac{1}{X_1}-\frac{1}{X_1+Y_1},\ldots,\frac{1}{X_n}-\frac{1}{X_n+Y_n}\right).$$

The random vectors $\bar{U}$ and $\bar{V}$ are independent iff the random vectors $\bar{X}$ and $\bar{Y}$ have distributions $MGIG^*(\bar{A},\bar{p},\bar{\lambda},\bar{\kappa})$ and $MG^*(\bar{A},\bar{p},\bar{\lambda})$, respectively, for some $\bar{A},\bar{p},\bar{\lambda},\bar{\kappa}$.

Proof. First, we will show that

Claim 1. *$\exists\bar{A}=(A_1,\ldots,A_r)$ such that $\bigcup_{i=1}^r A_i=\{1,\ldots,n\}$, $A_i\cap A_j=\emptyset$, $i\neq j$, and $\bar{Y}\sim MG^*(\bar{A},\bar{p},\bar{\lambda})$, $\bar{V}\sim MG^*(\bar{A},\bar{p},\bar{\kappa})$.*

The proof of this claim is by induction on n. For $n=2$ theorem holds (see Bobecka and Wesołowski, 2005) and thus the claim is true. Assume that it is true for $n-1\geqslant 2$. We will show that it is true for n.

Observe that if any one of $\bar{X}$ and $\bar{Y}$ is not degenerate at a point then all four random vectors $\bar{X},\bar{Y},\bar{U}$ and $\bar{V}$ are not degenerate. The independence property and the identity

$$\frac{Y_j}{X_j}=\frac{V_j}{U_j},\quad j=1,\ldots,n,$$

imply

$$E\left(\prod_{j=1}^n Y_j^{\alpha_j}\exp\left(\sum_{j=1}^n\sigma_jY_j\right)\right)E\left(\prod_{j=1}^n X_j^{-\alpha_j}\exp\left(\sum_{j=1}^n(\sigma_jX_j+\theta_jX_j^{-1})\right)\right)$$
$$=E\left(\prod_{j=1}^n V_j^{\alpha_j}\exp\left(\sum_{j=1}^n\theta_jV_j\right)\right)E\left(\prod_{j=1}^n U_j^{-\alpha_j}\exp\left(\sum_{j=1}^n(\sigma_jU_j^{-1}+\theta_jU_j)\right)\right),\tag{1}$$

for any negative σ_j,θ_j and fixed non-negative α_j, $j=1,\ldots,n$.

Taking logarithm of both sides of (1) and applying $\partial^2/\partial\sigma_j\partial\theta_j$ we obtain

$$\frac{E(X_j^{-\alpha_j+1}\prod_{i\neq j,i=1}^n X_i^{-\alpha_i}\exp(\sum_{j=1}^n(\sigma_j X_j+\theta_j X_j^{-1})))E(X_j^{-\alpha_j-1}\prod_{i\neq j,i=1}^n X_i^{-\alpha_i}\exp(\sum_{j=1}^n(\sigma_j X_j+\theta_j X_j^{-1})))}{[E(X_j^{-\alpha_j}\prod_{i\neq j,i=1}^n X_i^{-\alpha_i}\exp(\sum_{j=1}^n(\sigma_j X_j+\theta_j X_j^{-1})))]^2}$$
$$=\frac{E(U_j^{-\alpha_j+1}\prod_{i\neq j,i=1}^n U_i^{-\alpha_i}\exp(\sum_{j=1}^n(\sigma_j U_j^{-1}+\theta_j U_j)))E(U_j^{-\alpha_j-1}\prod_{i\neq j,i=1}^n U_i^{-\alpha_i}\exp(\sum_{j=1}^n(\sigma_j U_j^{-1}+\theta_j U_j)))}{[E(U_j^{-\alpha_j}\prod_{i\neq j,i=1}^n U_i^{-\alpha_i}\exp(\sum_{j=1}^n(\sigma_j U_j^{-1}+\theta_j U_j)))]^2}, \tag{2}$$

$j=1,\ldots,n$.

Now applying (1) for α_j, α_j-1 and α_j+1 to (2) we arrive at

$$\frac{E(Y_j^{\alpha_j-1}\prod_{i\neq j,i=1}^n Y_i^{\alpha_i}\exp(\sum_{j=1}^n\sigma_j Y_j))E(Y_j^{\alpha_j+1}\prod_{i\neq j,i=1}^n Y_i^{\alpha_i}\exp(\sum_{j=1}^n\sigma_j Y_j))}{[E(Y_j^{\alpha_j}\prod_{i\neq j,i=1}^n Y_i^{\alpha_i}\exp(\sum_{j=1}^n\sigma_j Y_j))]^2}$$
$$=\frac{E(V_j^{\alpha_j-1}\prod_{i\neq j,i=1}^n V_i^{\alpha_i}\exp(\sum_{j=1}^n\theta_j V_j))E(V_j^{\alpha_j+1}\prod_{i\neq j,i=1}^n V_i^{\alpha_i}\exp(\sum_{j=1}^n\theta_j V_j))}{[E(V_j^{\alpha_j}\prod_{i\neq j,i=1}^n V_i^{\alpha_i}\exp(\sum_{j=1}^n\theta_j V_j))]^2}, \tag{3}$$

$j=1,\ldots,n$.

Inserting $\alpha_j=1$ and $\alpha_i=0$ for $i\neq j$ into (3) we obtain

$$\frac{E(Y_j^2\exp(\sum_{j=1}^n\sigma_j Y_j))E(\exp(\sum_{j=1}^n\sigma_j Y_j))}{[E(Y_j\exp(\sum_{j=1}^n\sigma_j Y_j))]^2}=\frac{E(V_j^2\exp(\sum_{j=1}^n\theta_j V_j))E(\exp(\sum_{j=1}^n\theta_j V_j))}{[E(V_j\exp(\sum_{j=1}^n\theta_j V_j))]^2}, \tag{4}$$

for $j=1,\ldots,n$. Then by the principle of separation of variables (4) implies

$$\frac{\dfrac{\partial^2 f}{\partial\sigma_j^2}f}{\left(\dfrac{\partial f}{\partial\sigma_j}\right)^2}=c_j,\quad \frac{\dfrac{\partial^2 g}{\partial\theta_j^2}g}{\left(\dfrac{\partial g}{\partial\theta_j}\right)^2}=c_j,\quad j=1,\ldots,n, \tag{5}$$

where f and g are the Laplace transforms of $\bar{Y}$ and $\bar{V}$, respectively, and c_j, $j=1,\ldots,n$, are some constants greater than one.

Observe that there are two possible cases:

Case I: For any $\{i_1,\ldots,i_{n-1}\}\subset\{1,\ldots,n\}$ the components of $(Y_{i_1},\ldots,Y_{i_{n-1}})$ are independent. Then it follows that the components of the vector $(V_{i_1},\ldots,V_{i_{n-1}})$ are independent.

Case II: There exists $\{j_1,\ldots,j_{n-1}\}\subset\{1,\ldots,n\}$ such that the components of the vector $(Y_{j_1},\ldots,Y_{j_{n-1}})$ are not independent. Then the same holds for the vector $(V_{j_1},\ldots,V_{j_{n-1}})$.

Proof of Claim 1 in Case I. As in Theorem 2 in BW, using the induction assumption, we conclude that only the following two situations are possible:

either

1. $\exists i, j$ such that $c_i\neq c_j$ and then (see Case I.1, p. 153 in BW)

$$f(\bar{\sigma})=\prod_{j=1}^n(1-\lambda_j\sigma_j)^{-p_j},\quad \sigma_j<\lambda_j^{-1},\ j=1,\ldots,n$$

and

$$g(\bar{\theta})=\prod_{j=1}^n(1-\kappa_j\theta_j)^{-p_j},\quad \theta_j<\kappa_j^{-1},\ j=1,\ldots,n,$$

where $p_j = 1/(c_j - 1) > 0$ and $\lambda_j > 0, \kappa_j > 0$, $j = 1, \dots, n$, i.e. the random vectors $\bar{Y} = (Y_1, \dots, Y_n)$ and $\bar{V} = (V_1, \dots, V_n)$ have independent gamma components: $Y_j \sim \gamma_{p_j, \lambda_j^{-1}}$, $V_j \sim \gamma_{p_j, \kappa_j^{-1}}$, $j = 1, \dots, n$;

or

2. $c_j = c \; \forall j = 1, \dots, n$, and then (see Case I.2, p. 154 in BW)

$$f(\bar{\sigma}) = \left[\prod_{j=1}^{n}(1 - \lambda_j \sigma_j) + M \prod_{j=1}^{n} \sigma_j\right]^{-p}, \quad \sigma_j < \lambda_j^{-1}, \quad j = 1, \dots, n \tag{6}$$

and

$$g(\bar{\theta}) = \left[\prod_{j=1}^{n}(1 - \kappa_j \theta_j) + N \prod_{j=1}^{n} \theta_j\right]^{-p}, \quad \theta_j < \kappa_j^{-1}, \quad j = 1, \dots, n, \tag{7}$$

where $p = 1/(c-1) > 0$ and $\lambda_j > 0$, $\kappa_j > 0$, $j = 1, \dots, n$.

In the next step of the proof it will be shown that we have $M = 0$ and $N = 0$ which implies that the components of $\bar{Y}$ and $\bar{V}$ are independent: $Y_j \sim \gamma_{p_j, \lambda_j^{-1}}$, $V_j \sim \gamma_{p_j, \kappa_j^{-1}}$, $j = 1, \dots, n$.

Again, we apply the principle of separation of variables to (3) with $\alpha_j = \alpha_i = 1$, $j, i = 1, \dots, n$ arriving at

$$E\left(\prod_{i \neq j, i=1}^{n} Y_i \exp\left(\sum_{j=1}^{n} \sigma_j Y_j\right)\right) E\left(Y_j^2 \prod_{i \neq j, i=1}^{n} Y_i \exp\left(\sum_{j=1}^{n} \sigma_j Y_j\right)\right)$$
$$= d\left[E\left(Y_j \prod_{i \neq j, i=1}^{n} Y_i \exp\left(\sum_{j=1}^{n} \sigma_j Y_j\right)\right)\right]^2, \tag{8}$$

where $d > 1$ is a constant, $j = 1, \dots, n$. Now we fix j and σ_i for $i \neq j$, $i = 1, \dots, n$, and introduce a new random variable $Z = Z_{\sigma_i, i \neq j, i=1,\dots,n}$ with the distribution defined by

$$P_Z(\mathrm{d}y_j) = \frac{\int_0^\infty \cdots \int_0^\infty \prod_{i \neq j, i=1}^{n} y_i \mathrm{e}^{\sum_{i \neq j, i=1}^{n} \sigma_i y_i} F(\mathrm{d}y_1, \dots, \mathrm{d}y_n)}{E(\prod_{i \neq j, i=1}^{n} Y_i \mathrm{e}^{\sum_{i \neq j, i=1}^{n} \sigma_i Y_i})},$$

where F is the df of $\bar{Y}$ and the integral in the numerator is with respect to y_i, $i \neq j$, $i = 1, \dots, n$. Then after dividing both sides of (8) by $[E(\prod_{i \neq j, i=1}^{n} Y_i \mathrm{e}^{\sum_{i \neq j, i=1}^{n} \sigma_i Y_i})]^2$ we have

$$E(\mathrm{e}^{\sigma_j Z}) E(Z^2 \mathrm{e}^{\sigma_j Z}) = d[E(Z \mathrm{e}^{\sigma_j Z})]^2,$$

which means that Z is a gamma random variable, $\gamma_{q(\sigma_i, i \neq j, i=1,\dots,n), 1/a(\sigma_i, i \neq j, i=1,\dots,n)}$, where q and a are some positive functions depending on σ_i, $i \neq j$, $i = 1, \dots, n$. The functions depend on j also, but we will keep the letters q and a with no subscripts since we always deal with only one j at a time. Then in particular

$$E(\mathrm{e}^{\sigma_j Z}) = \frac{1}{(1 - a(\sigma_i, i \neq j, i = 1, \dots, n)\sigma_j)^{q(\sigma_i, i \neq j, i=1,\dots,n)}}. \tag{9}$$

Now observe that

$$E\left(\prod_{i \neq j, i=1}^{n} Y_i \exp\left(\sum_{j=1}^{n} \sigma_j Y_j\right)\right) = E(\mathrm{e}^{\sigma_j Z}) E\left(\prod_{i \neq j, i=1}^{n} Y_i \exp\left(\sum_{i \neq j, i=1}^{n} \sigma_i Y_i\right)\right),$$

$j = 1, \dots, n$.

Let $j=n$. Inserting $\sigma_j=0$ for $j=1,\ldots,n-2$ into the above equation we obtain

$$E\left(\prod_{i=1}^{n-1} Y_i \exp(\sigma_{n-1}Y_{n-1}+\sigma_n Y_n)\right)=E(\mathrm{e}^{\sigma_n Z})E\left(\prod_{i=1}^{n-1} Y_i \exp(\sigma_{n-1}Y_{n-1})\right).$$

Differentiating the Laplace transform (6) of $\bar{Y}$ and using (9) we obtain the following equation:

$$\frac{(1-\lambda_{n-1}\sigma_{n-1})[p^{n-1}(1-\lambda_n\sigma_n)\prod_{j=1}^{n-1}\lambda_j-pM\sigma_n]-p(p+1)M\sigma_{n-1}\sigma_n\lambda_{n-1}}{(1-\lambda_{n-1}\sigma_{n-1})^{p+2}(1-\lambda_n\sigma_n)^{p+1}}$$

$$=\frac{1}{(1-\tilde{a}(\sigma_{n-1})\sigma_n)^{\tilde{q}(\sigma_{n-1})}}\,\frac{p^{n-1}\prod_{j=1}^{n-1}\lambda_j}{(1-\lambda_{n-1}\sigma_{n-1})^{p+1}},$$

which can be rewritten in the form

$$\frac{(1-\lambda_{n-1}\sigma_{n-1})(1-\lambda_n\sigma_n)-\dfrac{M}{p^{n-2}\prod_{j=1}^{n-1}\lambda_j}(1+p\lambda_{n-1}\sigma_{n-1})\sigma_n}{(1-\lambda_{n-1}\sigma_{n-1})(1-\lambda_n\sigma_n)^{p+1}}=\frac{1}{(1-\tilde{a}(\sigma_{n-1})\sigma_n)^{\tilde{q}(\sigma_{n-1})}}, \tag{10}$$

for any $\sigma_{n-1}<\lambda_{n-1}^{-1}$, $\sigma_n<\lambda_n^{-1}$, where $\tilde{a}(\sigma_{n-1})=a(0,\ldots,0,\sigma_{n-1})$, $\tilde{q}(\sigma_{n-1})=q(0,\ldots,0,\sigma_{n-1})$. Letting $\sigma_n\nearrow\lambda_n^{-1}$ we get that $\tilde{a}(\sigma_{n-1})=\lambda_n$ for any $\sigma_{n-1}<\lambda_{n-1}^{-1}$ and (10) takes the form

$$\frac{(1-\lambda_{n-1}\sigma_{n-1})(1-\lambda_n\sigma_n)-\dfrac{M}{p^{n-2}\prod_{j=1}^{n-1}\lambda_j}(1+p\lambda_{n-1}\sigma_{n-1})\sigma_n}{1-\lambda_{n-1}\sigma_{n-1}}=(1-\lambda_n\sigma_n)^{p-\tilde{q}(\sigma_{n-1})+1}, \tag{11}$$

for any $\sigma_{n-1}<\lambda_{n-1}^{-1}$, $\sigma_n<\lambda_n^{-1}$. Inserting $\sigma_{n-1}=0$ in (11) we get

$$1-\left(\lambda_n+\frac{M}{p^{n-2}\prod_{j=1}^{n-1}\lambda_j}\right)\sigma_n=(1-\lambda_n\sigma_n)^{p-q+1} \tag{12}$$

for any $\sigma_n<\lambda_n^{-1}$, where $q=\tilde{q}(0)=q(0,\ldots,0)$. Thus, it follows that either $\lambda_n+M/(p^{n-2}\prod_{j=1}^{n-1}\lambda_j)\neq 0$ which implies $q=p$ and thus $M=0$ or $\lambda_n+M/(p^{n-2}\prod_{j=1}^{n-1}\lambda_j)=0$, that is $M=-p^{n-2}\prod_{j=1}^{n}\lambda_j$, and then $q=p+1$.

Now observe that the case $M=-p^{n-2}\prod_{j=1}^{n}\lambda_j$ is impossible: putting back $M=-p^{n-2}\prod_{j=1}^{n}\lambda_j$ into (11) we obtain

$$\frac{(1-\lambda_{n-1}\sigma_{n-1})(1-\lambda_n\sigma_n)+(1+p\lambda_{n-1}\sigma_{n-1})\lambda_n\sigma_n}{1-\lambda_{n-1}\sigma_{n-1}}=(1-\lambda_n\sigma_n)^{p-\tilde{q}(\sigma_{n-1})+1},$$

which is equivalent to

$$1+\frac{(p+1)\lambda_{n-1}\sigma_{n-1}}{1-\lambda_{n-1}\sigma_{n-1}}\,\lambda_n\sigma_n=(1-\lambda_n\sigma_n)^{p-\tilde{q}(\sigma_{n-1})+1}, \tag{13}$$

for any $\sigma_{n-1}<\lambda_{n-1}^{-1}$, $\sigma_n<\lambda_n^{-1}$. For any, fixed σ_{n-1}, the left-hand side of (13) is a linear function of σ_n. Thus, $\tilde{q}(\sigma_{n-1})=p$ for any $\sigma_{n-1}<\lambda_{n-1}^{-1}$, $\sigma_{n-1}\neq 0$. Putting it back into (13) we get

$$(p-1)\lambda_{n-1}\sigma_{n-1}+1=0,$$

for any $\sigma_{n-1}<\lambda_{n-1}^{-1}$, $\sigma_{n-1}\neq 0$. This is a contradiction.

Similarly, we can show that $N=0$. □

Proof of Claim 1 in Case II. We proceed as in Theorem 2 in BW (Case II, p. 158).

If there exists $\{j_1,\ldots,j_{n-1}\}\subset\{1,\ldots,n\}$ such that the components of the vector $(Y_{j_1},\ldots,Y_{j_{n-1}})$ are not independent, we can assume, without loss of generality, that this is the vector $(Y_1,\ldots,Y_{n-1})$. Then, by the induction assumption, $\exists \tilde{A}_1, \ldots, \tilde{A}_r : \bigcup_{i=1}^r \tilde{A}_i = \{1, \ldots, n-1\}$, $\tilde{A}_i \cap \tilde{A}_j = \emptyset, i \neq j$ and $\exists i\ \sharp \tilde{A}_i > 1$ such that the Laplace transform of $(Y_1, \ldots, Y_{n-1})$ is of the form

$$f(\sigma_1, \ldots, \sigma_{n-1}) = \prod_{k=1}^r \left(1 - \sum_{j \in \tilde{A}_k} \lambda_j \sigma_j\right)^{-p_k}.$$

Now take $Y_{i_1} \in \tilde{A}_1, \ldots, Y_{i_r} \in \tilde{A}_r$ (note that we have $r \leqslant n-2$) and consider the vector $(Y_{i_1}, \ldots, Y_{i_r}, Y_n)$. Since the dimension of $(Y_{i_1}, \ldots, Y_{i_r}, Y_n)$ is not greater than $n-1$, its Laplace transform has the desired form by the induction assumption, that is either Y_n is independent of $Y_{i_1}, \ldots, Y_{i_r}$ or there exists k such that $Y_n = aY_{i_k}$, where $a = \lambda_n / \lambda_{i_k}$, $k = 1, \ldots, r$.

Similarly, the Laplace transform of $(V_1, \ldots, V_{n-1})$ is of the form

$$g(\theta_1, \ldots, \theta_{n-1}) = \prod_{k=1}^r \left(1 - \sum_{j \in \tilde{A}_k} \kappa_j \theta_j\right)^{-p_k}.$$

Now take $V_{i_1} \in \tilde{A}_1, \ldots, V_{i_s} \in \tilde{A}_r$ and consider the vector $(V_{i_1}, \ldots, V_{i_r}, V_n)$. Since the dimension of $(V_{i_1}, \ldots, V_{i_r}, V_n)$ is not greater than $n-1$, its Laplace transform has the desired form by the induction assumption, that is either V_n is independent of $V_{i_1}, \ldots, V_{i_r}$ or there exists l such that $V_n = bV_{i_l}$, where $b = \kappa_n / \kappa_{i_l}$, $l = 1, \ldots, r$.

Hence, in this case, $\bar{Y} \sim MG^*(\bar{A}, \bar{p}, \bar{\lambda})$, $\bar{V} \sim MG^*(\bar{B}, \bar{p}, \bar{\kappa})$.

Moreover, observe that it has to be $\bar{A} = \bar{B}$.

We have: $\bar{A} = (A_1, \ldots, A_r)$, $\bar{B} = (B_1, \ldots, B_r)$ such that $\bigcup_{i=1}^r A_i = \{1, \ldots, n\}$, $A_i \cap A_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^r B_r = \{1, \ldots, n\}$, $B_i \cap B_j = \emptyset$, $i \neq j$. Suppose that $\bar{A} \neq \bar{B}$. Take $j_1 \in A_1, \ldots, j_r \in A_r$. Since $\bar{A} \neq \bar{B}$ there exists $j_k$such that $j_k \in A_k$ and $j_k \notin B_k$, that is $j_k \in B_l$, $l \neq k$, $k = 1, 2, \ldots, r$. We can assume, without loss of generality, that $j_k = j_1$ and $l = r$. Then $(Y_{j_1}, \ldots, Y_{j_r})$ has a density and$(V_{j_1}, \ldots, V_{j_r})$ does not have a density ($V_{j_1}, V_{j_r} \in B_r$ which means that V_{j_1}, V_{j_r} are linearly dependent). But if $(Y_{j_1}, \ldots, Y_{j_r})$ has a density then also $(X_{j_1} + Y_{j_1}, \ldots, X_{j_r} + Y_{j_r}) = (1/U_{j_1}, \ldots, 1/U_{j_r})$ has a density. Hence $(U_{j_1} + V_{j_1}, \ldots, U_{j_r} + V_{j_r}) = (1/X_{j_1}, \ldots, 1/X_{j_r})$ has a density. Thus, $(V_{j_1}, \ldots, V_{j_r}) = (1/X_{j_1} - 1/(X_{j_1} + Y_{j_1}), \ldots, 1/X_{j_r} - 1/(X_{j_r} + Y_{j_r}))$ has also a density since it is a smooth function of the r-variate random vectors $(X_{j_1}, \ldots, X_{j_r})$ and $(Y_{j_1}, \ldots, Y_{j_r})$ with independent absolutely continuous components. This is a contradiction. □

Summing up, we have the following two cases:

either

Case I: $\bar{Y}$ and $\bar{V}$ have independent gamma components: $Y_j \sim \gamma_{p_j, \lambda_j^{-1}}$, $V_j \sim \gamma_{p_j, \kappa_j^{-1}}$, $j = 1, \ldots, n$

or

Case II: $\bar{Y} \sim MG^*(\bar{A}, \bar{p}, \bar{\lambda})$ and $\bar{V} \sim MG^*(\bar{A}, \bar{p}, \bar{\kappa})$ that is $\bar{Y}$ and $\bar{V}$ have gamma components: $Y_j \sim \gamma_{q_j, \lambda_j^{-1}}$, $V_j \sim \gamma_{q_j, \kappa_j^{-1}}$, $j = 1, \ldots, n$ ($\forall j \in A_i\ q_j = p_i$, $i = 1, \ldots, r$) and the components of $\bar{Y}$ and $\bar{V}$ are grouped into independent subvectors: $\bar{Z}_1 = (Y_l)_{l \in A_1}, \ldots, \bar{Z}_r = (Y_l)_{l \in A_r}$, $\bar{W}_1 = (V_l)_{l \in A_1}, \ldots, \bar{W}_r = (V_l)_{l \in A_r}$ each subvector having linearly dependent components: $\forall i\ \exists k \in A_i$ such that $Y_j = (\lambda_j / \lambda_k) Y_k$, $V_j = (\kappa_j / \kappa_k) V_k$ (P-a.s.) $\forall j \in A_i$ (note that if $r = n$ then $\bar{Y}$ and $\bar{V}$ are as in Case I and if $r = 1$ we have a univariate case).

Now, to finish the proof, we will show the following.

Claim 2. *$\bar{X} \sim MGIG^*(\bar{A}, \bar{p}, \bar{\lambda}, \bar{\kappa})$ and $\bar{U} \sim MGIG^*(\bar{A}, \bar{p}, \bar{\kappa}, \bar{\lambda})$.*

Proof of Claim 2 in Case I. In this case all the random vectors $\bar{X}, \bar{Y}, \bar{U}, \bar{V}$ have densities. Since $\bar{X}, \bar{Y}$ are independent and $\bar{U}, \bar{V}$ are independent we have the following identity for the densities:

$$f_{\bar{U}}(u_1, ..u_n) f_{\bar{V}}(v_1, \ldots, v_n) = \frac{f_{\bar{X}}\left(\frac{1}{u_1 + v_1}, \ldots, \frac{1}{u_n + v_n}\right) f_{\bar{Y}}\left(\frac{1}{u_1} - \frac{1}{u_1 + v_1}, \ldots, \frac{1}{u_n} - \frac{1}{u_n + v_n}\right)}{\prod_{j=1}^n (u_j + v_j)^2 u_j^2},$$

which holds a.e. with respect to the Lebesgue measure L_{2n} in $\mathbf{R}^{2n}$ for $u_j, v_j \in (0, \infty)$, $j = 1, \ldots, n$. Using the fact that $\bar{Y}$ and $\bar{V}$ have independent gamma components we obtain the following:

$$f_{\bar{U}}(u_1, \ldots, u_n) \prod_{j=1}^{n} u_j^{p_j+1} e^{\kappa_j^{-1} u_j} e^{\lambda_j^{-1} u_j^{-1}}$$

$$= d f_{\bar{X}}((u_1+v_1)^{-1}, \ldots, (u_n+v_n)^{-1}) \prod_{j=1}^{n} (u_j+v_j)^{-(p_j+1)} e^{\kappa_j^{-1}(u_j+v_j)} e^{\lambda_j^{-1}(u_j+v_j)^{-1}} \tag{14}$$

for $u_j, v_j \in (0, \infty)$, $j = 1, \ldots, n$, L_{2n} a.e., where $d = \text{const}$.

Denoting $u_j + v_j = m_j$, $j = 1, \ldots, n$, the above equation can be written as

$$f_{\bar{U}}(u_1, \ldots, u_n) = c(m_1, \ldots, m_n) \prod_{j=1}^{n} g_j(u_j), \tag{15}$$

where $c\,(m_1, \ldots, m_n)$ is the right-hand side of (14) and

$$g_j(u_j) = u_j^{-p_j-1} e^{-\kappa_j^{-1} u_j - \lambda_j^{-1} u_j^{-1}},$$

$j = 1, \ldots, n$. We can choose $m_1, \ldots, m_n$ sufficiently large such that (15) holds for $(u_1, \ldots, u_n) \in (0, m_1) \times \cdots \times (0, m_n)$ L_n a.e. This implies that $\bar{U}$ has independent GIG components $U_j \sim \mu_{-p_j, \kappa_j, \lambda_j}$, $j = 1, \ldots, n$. Dually, by (14), it follows that $\bar{X}$ has also independent GIG components $X_j \sim \mu_{-p_j, \lambda_j, \kappa_j}$, $j = 1, \ldots, n$. □

Proof of Claim 2 in Case II. Since $\bar{Y} \sim MG^*(\bar{A}, \bar{p}, \bar{\lambda})$ and $\bar{V} \sim MG^*(\bar{A}, \bar{p}, \bar{\kappa})$ we have: $\forall i\ \exists k \in A_i$ such that $Y_j = (\lambda_j/\lambda_k) Y_k$, $V_j = (\kappa_j/\kappa_k) V_k$ (P-a.s.) $\forall j \in A_i$. Moreover, $V_j = 1/X_j - 1/(X_j + Y_j)$, $j = 1, \ldots, n$ which implies

$$\frac{\frac{\kappa_j}{\kappa_k} Y_k}{X_k(X_k + Y_k)} = \frac{\frac{\lambda_j}{\lambda_k} Y_k}{X_j\left(X_j + \frac{\lambda_j}{\lambda_k} Y_k\right)} \quad (P\text{-a.s.})\ \forall j \in A_i.$$

Since Y_k is P-a.s. positive we obtain

$$Y_k\left(X_k - \frac{\kappa_j}{\kappa_k} X_j\right) = X_k^2 - \frac{\kappa_j \lambda_k}{\kappa_k \lambda_j} X_j^2 \quad (P\text{-a.s.})\ \forall j \in A_i. \tag{16}$$

Assume now that $X_k \neq (\kappa_j/\kappa_k) X_j$ on a set G of positive probability P. Then on G we have

$$Y_k = \frac{X_k^2 - \frac{\kappa_j \lambda_k}{\kappa_k \lambda_j} X_j^2}{X_k - \frac{\kappa_j}{\kappa_k} X_j} \quad \forall j \in A_i,$$

which contradicts the independence of $\bar{X}$ and $\bar{Y}$. Thus, $X_k = (\kappa_j/\kappa_k) X_j$ P-a.s. $\forall j \in A_i$ and by (16) $\kappa_j/\kappa_k = \lambda_k/\lambda_j$. Thus, $\forall i\ \exists k \in A_i$ such that $X_j = (\lambda_j/\lambda_k) X_k$ (P-a.s.) $\forall j \in A_i$. Since $U_j = 1/(X_j + Y_j)$, $j = 1, \ldots, n$, we obtain immediately that $\forall i\ \exists k \in A_i$ such that $U_j = (\kappa_j/\kappa_k) U_k$ (P-a.s.) $\forall j \in A_i$.

Now take $j_1 \in A_1, \ldots, j_r \in A_r$. Since $(Y_{j_1}, \ldots, Y_{j_r})$ and $(V_{j_1}, \ldots, V_{j_r})$ have independent gamma components we get (as in case 1 above) that $(X_{j_1}, \ldots, X_{j_r})$ and $(U_{j_1}, \ldots, U_{j_r})$ have independent GIG components.

It means that the components of $\bar{X}$ and $\bar{U}$ are grouped into independent subvectors, each subvector having linearly dependent components.

Thus, $\bar{X} \sim MGIG^*(\bar{A}, \bar{p}, \bar{\lambda}, \bar{\kappa})$ and $\bar{U} \sim MGIG^*(\bar{A}, \bar{p}, \bar{\kappa}, \bar{\lambda})$. □

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