Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 10, No. 2 (2007) 277–291

BI-POISSON PROCESS

WŁODZIMIERZ BRYC
*Department of Mathematics, University of Cincinnati,
P. O. Box 210025, Cincinnati, OH 45221–0025, USA*
wlodzimierz.bryc@uc.edu

JACEK WESOŁOWSKI
*Faculty of Mathematics and Information Science, Warsaw University of Technology,
pl. Politechniki 1, 00-661 Warszawa, Poland*
wesolo@mini.pw.edu.pl

Received 6 September 2006
Communicated by M. Bożejko

Bi-Poisson processes are defined as a two-parameter family of processes with linear regressions and linear conditional variances. We give conditions for the existence and uniqueness of such processes. We point out that one-dimensional distributions of the bi-Poisson process are closed under a generalized free convolution introduced by Bożejko and Speicher.[7]

Keywords: Linear conditional expectations; quadratic conditional variances; hypergeometric orthogonal polynomials; quadratic harnesses; generalized free convolution.

AMS Subject Classification: 60J25, 46L54

1. Introduction

The family of stochastic processes with uncorrelated increments, linear regressions, and quadratic conditional variances contains a number of processes of importance. It includes the Wiener, Poisson, Gamma and Pascal processes; for details and additional references, see Theorem 1 of Ref. 24. It includes their free counterparts which are Markov processes whose bivariate distributions match the bivariate distributions of free Brownian motion, free Poisson process; for details and additional references, see Ref. 3, Theorem 4.3 of Ref. 15 and Proposition 3.4 of Ref. 4. It includes the classical version of the q-Brownian motion, see Ref. 5, and the classical version of the q-Poisson process, see Ref. 2.

All of the above-mentioned examples share the property that the conditional variance with respect to the past sigma field is constant. In this paper we expand this class of processes by constructing the first example of a Markov process with linear regressions and quadratic conditional variances which has non-constant conditional variances with respect to both past and future sigma fields. In a subsequent paper,[12] we construct additional examples.

Throughout this paper $(X_t)_{t\geq 0}$ is a square integrable stochastic process such that for all t, $s \geq 0$

$$E(X_t) = 0\,, \qquad E(X_t X_s) = \min\{t, s\}\,. \tag{1}$$

Consider the σ-fields $\mathcal{G}_{s,u} = \sigma\{X_t : t \in [0,s] \cup [u,\infty)\}$, $\mathcal{F}_s = \sigma\{X_t : t \in [0,s]\}$, $\mathcal{G}_u = \sigma\{X_t : t \in [u,\infty)\}$. We assume that the process has linear regressions,

Assumption 1.1. For all $0 \leq s < t < u$,

$$E(X_t|\mathcal{G}_{s,u}) = \mathbf{a}X_s + \mathbf{b}X_u\,, \tag{2}$$

where

$$\mathbf{a} = \mathbf{a}_{t,s,u} = \frac{u-t}{u-s}\,, \qquad \mathbf{b} = \mathbf{b}_{t,s,u} = \frac{t-s}{u-s} \tag{3}$$

are the deterministic functions of $0 \leq s < t < u$.

We also assume that the process has quadratic conditional variances,

$$E(X_t^2|\mathcal{G}_{s,u}) = \mathbf{A}X_s^2 + \mathbf{B}X_sX_u + \mathbf{C}X_u^2 + \mathbf{D} + \alpha X_s + \beta X_u\,, \tag{4}$$

where $\mathbf{A} = \mathbf{A}_{t,s,u}$, $\mathbf{B} = \mathbf{B}_{t,s,u}$, $\mathbf{C} = \mathbf{C}_{t,s,u}$, $\mathbf{D} = \mathbf{D}_{t,s,u}$, $\alpha = \alpha_{t,s,u}$, $\beta = \beta_{t,s,u}$ are the deterministic functions of $0 < s < t < u$. Generically, see Theorem 2.2 of Ref. 13, conditions (1), (2) and (4) imply that there are five real parameters q, η, θ, σ, τ such that

$$\mathbf{A}_{t,s,u} = \frac{(u-t)(u(1+\sigma t)+\tau-qt)}{(u-s)(u(1+\sigma s)+\tau-qs)}\,, \tag{5}$$

$$\mathbf{B}_{t,s,u} = \frac{(u-t)(t-s)(1+q)}{(u-s)(u(1+\sigma s)+\tau-qs)}\,, \tag{6}$$

$$\mathbf{C}_{t,s,u} = \frac{(t-s)(t(1+\sigma s)+\tau-qs)}{(u-s)(u(1+\sigma s)+\tau-qs)}\,, \tag{7}$$

$$\mathbf{D}_{t,s,u} = \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-qs}\,, \tag{8}$$

$$\alpha_{t,s,u} = \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-qs} \times \frac{u\eta-\theta}{u-s}\,, \tag{9}$$

$$\beta_{t,s,u} = \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-qs} \times \frac{\theta-s\eta}{u-s}\,. \tag{10}$$

Equivalently,

$$\begin{aligned}\mathrm{Var}(X_t|\mathcal{G}_{s,u}) &\\ &= \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-qs}\left(1+\sigma\frac{(uX_s-sX_u)^2}{(u-s)^2}+\eta\frac{uX_s-sX_u}{u-s}\right.\\ &\quad\left.+\tau\frac{(X_u-X_s)^2}{(u-s)^2}+\theta\frac{X_u-X_s}{u-s}+(1-q)\frac{(X_u-X_s)(sX_u-uX_s)}{(u-s)^2}\right), \qquad (11)\end{aligned}$$

compare Proposition 2.5 of Ref. 15. (Recall that the conditional variance of X with respect to a σ-field $\mathcal{F}$ is defined as $\mathrm{Var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - (E(X|\mathcal{F}))^2$.) In Ref. 15 we prove that the solution of Eqs. (1), (2) and (11) exists and is unique when $-1 < q \leq 1$, and $\sigma = \eta = 0$; it is then given by the Markov process which we called q-Meixner process. (The case $q = 1$ yields Lévy processes, and was studied earlier by several authors, see Ref. 24, and the references therein.) Due to the invariance of this problem under the time inversion that maps (X_t) to the process $(tX_{1/t})$, processes that satisfy (11) with $-1 < q \leq 1$, $\tau = \theta = 0$ are also Markov, and can be expressed in terms of the q-Meixner processes as $(tX_{1/t})$.

In this paper we consider the next simplest case, which one may call the free bi-Poisson processes. The adjective "free" refers to our choice of $q = 0$ in (11). The name "bi-Poisson" was originally motivated by the fact that under our choice of $\sigma = \tau = 0$ in (11), one has linear conditional variances under each uni-directional conditioning; the q-Poisson processes, and in particular, the classical Poisson process and the free Poisson process, have linear conditional variances when conditioned with respect to the future, and constant conditional variances when conditioned with respect to the past. After the first version of this paper was written, we noticed in Ref. 16 that when $q = 1$ the respective bi-Poisson process can indeed be constructed from two Poisson processes and suitable changes of time.

The role of these simplifying assumptions seems technical: $\sigma = \tau = 0$ allows us to prove that all moments are finite, see Lemma 3.2; the assumption that $q = 0$ allows us to guess useful algebraic identities between the orthogonal polynomials in Proposition 2.2. These considerations lead to the following.

Assumption 1.2. For all $0 \leq s < t < u$,

$$\begin{aligned}\mathrm{Var}(X_t|\mathcal{G}_{s,u}) &\\ &= \frac{(u-t)(t-s)}{u}\left(1+\eta\frac{uX_s-sX_u}{u-s}+\theta\frac{X_u-X_s}{u-s}+\frac{(X_u-X_s)(sX_u-uX_s)}{(u-s)^2}\right). \qquad (12)\end{aligned}$$

In Sec. 2 we construct the Markov process with covariances (1), linear regressions (2), and conditional variances (12) for a large set of real parameters η, θ. Our construction relies on guessing appropriate families of orthogonal polynomials and on information about their connection coefficients. In Sec. 3 we show that the solution is unique. In Sec. 4 we point out an intriguing connection with free probability:

we show that the one-dimensional distributions of the bi-Poisson process are closed under the c-free convolution.

2. Existence

When $\eta = 0$, expression (12) coincides with (28) of Ref. 15 for $\tau = q = 0$, so the corresponding Markov process exists and is determined uniquely, see Theorem 3.5 of Ref. 15. Since the transformation $X_t \mapsto tX_{1/t}$ switches the roles of η, θ, it follows that the corresponding Markov process also exists when $\theta = 0$. We may therefore restrict our attention to the case $\eta\theta \neq 0$. The construction of the processes is based on the idea already exploited in Ref. 15; namely, we construct the transition probabilities of the suitable Markov process, by defining the corresponding orthogonal polynomials. The construction relies on auxiliary identities between the orthogonal polynomials, which are used to verify the martingale polynomial property (26).

2.1. *One dimensional distributions*

We begin by carefully examining the "candidate" for the one dimensional distribution of X_t. For $t > 0$, let $p_0(x;t) = 1$, and consider the following monic polynomials $\{p_n(x;t) : n \geq 1\}$ in variable x.

$$xp_0(x;t) = p_1(x;t)\,, \tag{13}$$

$$xp_1(x;t) = p_2(x;t) + (t\eta + \theta)p_1(x;t) + tp_0(x;t)\,, \tag{14}$$

$$xp_n(x;t) = p_{n+1}(x) + (t\eta + \theta)p_n(x;t) + t(1 + \eta\theta)p_{n-1}(x;t)\,, \qquad n \geq 2\,. \tag{15}$$

From the general theory of orthogonal polynomials it follows that if $1 + \eta\theta \geq 0$, then $p_n(x;t)$ are orthogonal with respect to the unique probability measure π_t, see Ref. 17. We will need the following.

Lemma 2.1. *If $\eta\theta \neq 0$, then*

$$\pi_t(\{x : 1 + \eta x < 0\}) = 0\,. \tag{16}$$

Proof. If $1 + \eta\theta = 0$, then the recurrence is degenerate and the distribution is supported at the zeros of polynomial $p_2(x) = x^2 - (t\eta + \theta)x - t = (x + t/\theta)(x - \theta)x$; this follows from the fact that all higher order polynomials are the multiples of p_2. The support $\mathrm{supp}(\pi_t) = \{-t/\theta, -1/\eta\}$ is disjoint with the open set $\{x : 1 + \eta x < 0\}$, ending the proof in this case.

If $1 + \eta\theta > 0$, then (15) is a constant coefficient recurrence which has been analyzed by several authors, see Ref. 22 and the references therein. The Cauchy transform

$$G(z) = \int \frac{1}{z - x}\pi_t(dx)$$

is given by the corresponding continued fraction,

$$G(z) = \cfrac{1}{z - \cfrac{t}{z-(t\eta+\theta) - \cfrac{t(1+\eta\theta)}{z-(t\eta+\theta)-\cfrac{t(1+\eta\theta)}{\ddots}}}}$$

which after some calculation gives

$$G(z) = \frac{z(1+2\eta\theta)+t\eta+\theta-\sqrt{(z-(t\eta+\theta))^2-4t(1+\eta\theta)}}{2(1+z\eta)(t+z\theta)}\,. \tag{17}$$

The Stieltjes inversion formula gives the distribution π_t as the limit in distribution as $\varepsilon \to 0^+$ of the absolutely continuous measures $-\frac{1}{\pi}\Im G(x+i\varepsilon)dx$. This gives

$$\begin{aligned}\pi_t(dx) = {} & \frac{\sqrt{4t(1+\eta\theta)-(x-t\eta-\theta)^2}}{2\pi(x\eta+1)(x\theta+t)} 1_{(x-t\eta-\theta)^2<4t(1+\eta\theta)} \\ & + p(t)\delta_{-t/\theta} + q(t)\delta_{-1/\eta}\,,\end{aligned} \tag{18}$$

where δ_x denotes the point mass at x. The weights at the discrete point masses are

$$p(t) = \frac{-((1+\eta\theta)t-\theta^2)/\theta+\varepsilon|(1+\eta\theta)t-\theta^2|/|\theta|}{2(\theta-\eta t)}$$

and

$$q(t) = \frac{\eta(t-(1+\eta\theta)/\eta^2)+\varepsilon|\eta||t-(1+\eta\theta)/\eta^2|}{2(\eta t-\theta)}\,,$$

where the sign $\varepsilon = \varepsilon(t,\eta,\theta) = \pm 1$ is selected simultaneously for both expressions by the appropriate choice of the branch of the square root. We found that a practical way to choose the sign is to select $\varepsilon = \pm 1$ so that both expressions give a number in the interval $[0, 1]$; in our setting this determines ε uniquely for every choice of the parameters. Denote by U the open set $\{x : 1 + x\eta < 0\}$. It is easy to check that the support of the absolutely continuous part of π_t does not intersect U. Thus the only possibility for the set U to carry positive π_t-probability is when $-t/\theta \in U$. This is possible only if $\eta\theta > 0$ and t is large enough. The Stieltjes inversion formula gives the weight of the point mass at $-t/\theta$ as

$$p(t) = \frac{(\theta^2-(1+\eta\theta)t)_+}{\theta^2 - t\eta\theta}\,.$$

This shows that the point $-t/\theta$ carries positive probability $p(t)$ only for $t < \frac{\theta^2}{1+\eta\theta}$. On the other hand, $-t/\theta \in U$ only for $t > \theta/\eta$. Since trivially $\theta/\eta \geq \frac{\theta^2}{1+\eta\theta}$, this shows that $\pi_t(U) = 0$. $\square$

2.2. Transition probabilities

Fix $0 < s < t$, and let $x \in \mathbb{R}$ be such that $1 + x\eta \geq 0$. Consider monic polynomials in variable y defined by the three-step recurrence

$$Q_0(y; x, t, s) = 1\,,$$

$$Q_1(y; x, t, s) = y - x\,,$$

$$yQ_1(y; x, t, s) = Q_2(y; x, t, s) + ((t-s)\eta + \theta)Q_1(y; x, t, s) + (t-s)(1 + x\eta)Q_0(y; x, t, s)\,,$$

and for $n \geq 2$ by the constant coefficients recurrence

$$\begin{aligned} yQ_n(y; x, t, s) &= Q_{n+1}(y; x, t, s) \\ &\quad + (t\eta + \theta)Q_n(y; x, t, s) + t(1 + \eta\theta)Q_{n-1}(y; x, t, s)\,. \end{aligned} \tag{19}$$

Notice that recurrence (15) is a special case of (19) corresponding to $x = s = 0$. We want these polynomials to be orthogonal with respect to the conditional distribution $\mathcal{L}(X_t|X_s)$.

To this end, we define $P_{s,t}(x, dy)$ as the (unique) probability measure which makes the polynomials $\{Q_n(y; x, t, s) : n \in \mathbb{N}\}$ orthogonal. This is possible whenever $1 + \eta\theta \geq 0$ and $1 + x\eta \geq 0$; the latter holds true for all x from the support of the probability measure $\pi_s(dy) = P_{0,s}(0, dy)$, see Lemma 2.1. Since the coefficients of the three-step recurrence (19) are bounded, it is well known that measures $P_{s,t}(x, dy)$ have bounded support.

The next step is to prove that $P_{s,t}(x, dy)$ form a consistent family of measures, so that they indeed define the transition probabilities of the Markov chain which starts at the origin. To this end, we need the following algebraic relations between the polynomials. These relations are a more complicated version of Theorem 1 of Ref. 14 and Lemma 3.1 of Ref. 15.

Proposition 2.2. *For $n \geq 0$*

$$Q_n(z; x, u, s) = Q_n(y; x, t, s) + \sum_{k=0}^{n-1} B_k(y; x, t, s)Q_{n-k}(z; y, u, t)\,, \tag{20}$$

where $B_0 = 1$ and

$$\begin{aligned} &B_1(y; x, t, s) = Q_1(y; x, t, s) - (t - s)\eta B_0\,, \\ &B_k(y; x, t, s) = Q_k(y; x, t, s) - t\eta B_{k-1}(y; x, t, s)\,, \qquad k = 2, 3, \ldots\,. \end{aligned} \tag{21}$$

In addition, for $n \geq 1$

$$Q_n(y; x, t, s) = \sum_{k=0}^{n} \tilde{B}_{n-k}(x; s)(p_k(y; t) - p_k(x; s))\,, \tag{22}$$

where $\tilde{B}_k(x; s) = B_k(0; x, 0, s)$ are linear (affine) functions in variable x.

Proof. Let

$$\phi(\zeta; y, x, t, s) = \sum_{n=0}^{\infty} \zeta^n Q_n(y; x, t, s)$$

be the generating function of Q_n. Since

$$\phi(\zeta; y, x, t, s) = 1 + \zeta \sum_{n=0}^{\infty} \zeta^n Q_{n+1}(y; x, t, s)\,,$$

a calculation based on recurrence (19) shows that

$$\phi(\zeta; y, x, t, s) = \frac{1 + \zeta(t\eta + \theta - x) + \zeta^2(s + sy\eta - tx\eta + t\eta\theta)}{1 + \zeta(t\eta + \theta - y) + \zeta^2 t(1 + \eta\theta)}\,.$$

From (21) we get a similar expression for the generating function of B_n. Namely,

$$\psi(\zeta; y, x, t, s) = \sum_{n=0}^{\infty} \zeta^n B_n(y; x, t, s) = \frac{\phi(\zeta; y, x, t, s) + \eta s\zeta}{1 + \eta t\zeta}\,.$$

This gives

$$\psi(\zeta; y, x, t, s) = \frac{1 + \zeta(s\eta + \theta - x) + s(1 + \eta\theta)\zeta^2}{1 + \zeta(t\eta + \theta - y) + t(1 + \eta\theta)\zeta^2}\,.$$

It is now easy to verify that the two generating functions satisfy the identity:

$$\phi(\zeta; z, x, u, s) - \phi(\zeta; y, x, t, s) = \psi(\zeta; y, x, t, s)(\phi(\zeta; z, y, u, t) - 1)\,, \tag{23}$$

which implies (20). Since $\psi(\zeta, y, x, t, s)\psi(\zeta, x, y, s, t) = 1$ from (23) we get

$$\phi(\zeta; z, y, u, t) = 1 + \psi(\zeta; x, y, s, t)(\phi(\zeta; z, x, u, s) - \phi(\zeta; y, x, t, s))\,.$$

Since $p_n(x, t) = Q_n(x; 0, t, 0)$ setting $x = 0$, $s = 0$ proves (22). □

We now follow the argument from Proposition 3.2 of Ref. 15 and verify that probability measures $P_{s,t}(x, dy)$ are the transition probabilities of a Markov process. Let $U = \{x : 1 + \eta x \geq 0\}$.

Proposition 2.3. *If* $0 \leq s < t < u$, $1 + \eta\theta \geq 0$, *and* $x \in \operatorname{supp}(\pi_s)$ *then*

$$P_{s,u}(x, \cdot) = \int_U P_{t,u}(y, \cdot) P_{s,t}(x, dy)\,. \tag{24}$$

Proof. Since (19) is a constant-coefficient recurrence, the explicit expression for $P_{s,u}(x, \cdot)$ is known; in particular $U \ni x \mapsto P_{s,u}(x, U)$ is a continuous function. We first prove that (24) holds true for $s = 0$, $x = 0$. Let $\nu(A) = \int_U P_{t,u}(y, A)\pi_t(dy)$. To show that $\nu(dz) = \pi_u(dz)$, we verify that the polynomials $Q_n(z; 0, u, 0) = p_n(z; u)$ are orthogonal with respect to $\nu(dz)$. This is established by applying (20) and Lemma 2.1 as follows. By (20) we have

$$\begin{aligned}\int_{\mathbb{R}} p_n(z; u)\nu(dz) &= \iint_{U \times \mathbb{R}} p_n(z; u) P_{t,u}(y, dz)\pi_t(dy) \\ &= \iint_{U \times \mathbb{R}} p_n(y; t) P_{t,u}(y, dz)\pi_t(dy)\end{aligned}$$

$$+ \iint_{U\times\mathbb{R}} \sum_{k=0}^{n-1} B_k(y;0,t,0)Q_{n-k}(z;y,u,t)P_{t,u}(y,dz)\pi_t(dy)$$

$$= \int_U p_n(y;t)\pi_t(dy)$$

$$+ \int_U \sum_{k=0}^{n-1} B_k(y;0,t,0)\left(\int_{\mathbb{R}} Q_{n-k}(z;y,u,t)P_{t,u}(y,dz)\right)\pi_t(dy)\,.$$

From $p_0 = 1$ we see that $\int_U p_n(y;t)\pi_t(dy) = \int_{\mathbb{R}} p_n(y;t)\pi_t(dy) = 0$ for $n \geq 1$; similarly $\int_{\mathbb{R}} Q_j(z;y,u,t)P_{t,u}(y,dz) = 0$ for $j \geq 1$. Therefore, $\int_{\mathbb{R}} p_n(z;u)\nu(dz) = 0$ for all $n \geq 1$. Since polynomials $\{p_n\}$ satisfy the three-step recurrence (15), this shows that ν is their orthogonality measure. By uniquencess of the moment problem, $\nu = \pi_u$.

From the fact that (24) holds for $s = 0$, $x = 0$, we deduce that

$$P_{s,t}(x,U) = 1 \text{ for all } x \in \operatorname{supp}(\pi_s)\,. \tag{25}$$

To see this, we use the already established part of (24) as follows. Lemma 2.1 implies that

$$1 = \pi_t(U) = \int_{\mathbb{R}} P_{s,t}(x,U)\pi_s(dx)\,,$$

so $P_{s,t}(x,U) = 1$ on the set of x of π_s-probability one. However, since $P_{s,t}(x,U)$ is a continuous function of x, the conclusion follows for all $x \in \operatorname{supp}(\pi_s)$.

To end the proof of (24), left $\nu(A) = \int_U P_{t,u}(y,A)P_{s,t}(x,dy)$. We will show that $\nu(dz) = P_{s,u}(x,dz)$ by verifying that the polynomials $Q_n(z;x,u,s)$ are orthogonal with respect to $\nu(dz)$. Polynomials $\{Q_n\}$ satisfy the three-step recurrence (19); it suffices therefore to show that for $n \geq 1$ these polynomials integrate to zero. Since $\int_{\mathbb{R}} Q_k(z;y,u,t)P_{t,u}(y,dz) = 0$ for $k \geq 1$, by (20) we have

$$\int_{\mathbb{R}} Q_n(z;x,u,s)\nu(dz)$$

$$= \int_U Q_n(y;x,t,s)P_{s,t}(x,dy)$$

$$+ \sum_{k=0}^{n-1} \int_U B_k(y;x,t,s)\left(\int_{\mathbb{R}} Q_{n-k}(z;y,u,t)P_{t,u}(y,dz)\right)P_{s,t}(x,dy)$$

$$= \int_U Q_n(y;x,t,s)P_{s,t}(x,dy)\,.$$

By (25), this equals to $\int_{\mathbb{R}} Q_n(y;x,t,s)P_{s,t}(x,dy) = 0$, ending the proof. □

For $1 + \eta\theta \geq 0$, let (X_t) be the Markov process with the transition probabilities $P_{s,t}(x,dy)$, $X_0 = 0$. The following martingale polynomial property holds, compare Sec. 2.5 of Ref. 1 and Proposition 3.3 of Ref. 15.

Lemma 2.4. *For $t > s$, $n \in \mathbb{N}$ we have*

$$E(p_n(X_t;t)|\mathcal{F}_s) = p_n(X_s;s)\,. \tag{26}$$

Proof. By definition, for $n \geq 1$ we have $E(Q_n(X_t;X_s,t,s)|X_s) = 0$. Since $p_1(x;t) = x$, and $Q_1(y;x,t,s) = y - x$, by the Markov property (26) holds true for $n = 1$. Suppose that (26) holds true for all $n \leq N$. Then (22) implies

$$0 = E(Q_{N+1}(X_t;X_s,t,s)|X_s) = \tilde{B}_0(X_s;s)(E(p_{N+1}(X_t;t)|X_s) - p_{N+1}(X_s;s))\,.$$

Since $\tilde{B}_0 = 1$, this proves that $E(p_{N+1}(X_t;t)|X_s) = p_{N+1}(X_s;s)$, which by the Markov property implies (26) for $N+1$. $\square$

Theorem 2.5. *Suppose $1 + \eta\theta \geq 0$ and (X_t) is the Markov process with $X_0 = 0$ and with the transition probabilities $P_{s,t}(x,dy)$ defined via* (19)*. Then* (1), (2) *and* (12) *hold true.*

Proof. Condition (1) holds true. Indeed, $E(X_t) = \int p_1(x;t)p_0(x;t)\pi_t(dx) = 0$. Let $s < t$. From (26) we get $E(X_sX_t) = E(X_sE(p_1(X_t;t)|\mathcal{F}_s)) = \int p_1^2(x;s)\pi_s(dx) = \int(p_2(x;s)+(s\eta+\theta)p_1(x;s)+s)\pi_s(dx) = s$. Since random variables X_t are bounded, polynomials are dense in $L_2(X_s,X_u)$. Thus by the Markov property to prove (2) we only need to verify that

$$\begin{aligned} &E(p_n(X_s;s)X_tp_m(X_u;u)) \\ &\quad = \mathbf{a}_{t,s,u}E(X_sp_n(X_s;s)p_m(X_u;u)) + \mathbf{b}_{t,s,u}E(p_n(X_s;s)X_up_m(X_u;u)) \end{aligned} \tag{27}$$

for all m, $n \in \mathbb{N}$ and $0 < s < t$. For the proof of (12), we need to verify that for any n, $m \geq 1$ and $0 < s < t$

$$\begin{aligned} &E(p_n(X_s;s)X_t^2p_m(X_u;u)) \\ &\quad = \mathbf{A}E(X_s^2p_n(X_s;s)p_m(X_u;u)) + \mathbf{B}E(X_sp_n(X_s;s)X_up_m(X_u;u)) \\ &\qquad + \mathbf{C}E(p_n(X_s;s)X_u^2p_m(X_u;u)) + \alpha E(X_sp_n(X_s;s)p_m(X_u;u)) \\ &\qquad + \beta E(p_n(X_s;s)X_up_m(X_u,u)) + \mathbf{D}E(p_n(X_s;s)p_m(X_u;u))\,, \end{aligned} \tag{28}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, α, β are given by Eqs. (5)–(10) with $\sigma = \tau = q = 0$; recall that (15) implies $E(p_1^2(X_s;s)) = s$, and for $n \geq 1$

$$E(p_{n+1}^2(X_s;s)) = s(1+\eta\theta)E(p_n^2(X_s;s))\,, \tag{29}$$

see p. 19 of Ref. 17. An efficient way to verify (27) and (28) is to use generating functions. For $s \leq u$, let

$$\phi_0(z_1,z_2,s) = \sum_{m,n=0}^{\infty} z_1^n z_2^m E(p_n(X_s;s)p_m(X_u;u))\,.$$

From (26) it follows that $\phi_0(z_1, z_2, s)$ does not depend on u, and from (29) it follows that

$$\phi_0(z_1, z_2, s) = \frac{1 - z_1 z_2 \eta\theta s}{1 - z_1 z_2 s(1 + \eta\theta)}\,.$$

Consider now the generating function

$$\phi_1(z_1, z_2, s, t) = \sum_{m,n=0}^{\infty} z_1^n z_2^m E(p_n(X_s; s) X_t p_m(X_u; u))\,.$$

From (26) and (15) we get

$$\begin{aligned}\phi_1(z_1, z_2, s, t) &= \sum_{n=0}^{\infty} z_1^n E\left(p_n(X_s; s)\left(X_t + z_2 X_t p_1(X_t; t) + \sum_{m=2}^{\infty} z_2^m X_t p_m(X_t; t)\right)\right) \\ &= \sum_{n=0}^{\infty} z_1^n E(p_n(p_1 + z_2(p_2 + (t\eta + \theta)p_1 + tp_0))) \\ &\quad + \sum_{n=0}^{\infty} z_1^n E\left(p_n\left(\sum_{m=2}^{\infty} z_2^m (p_{m+1} + (t\eta + \theta)p_m + t(1 + \eta\theta)p_{m-1})\right)\right)\,.\end{aligned}$$

Thus

$$\phi_1(z_1, z_2, s, t) = \left(\frac{1}{z_2} + t\eta + \theta\right)(\phi_0(z_1, z_2, s) - 1) + z_2 t(1 + \eta\theta)\phi_0(z_1, z_2, s) - \eta\theta t z_2\,,$$

which gives

$$\phi_1(z_1, z_2, s, t) = \frac{s z_1 + t z_2 + s z_1 z_2(t\eta + \theta)}{1 - s z_1 z_2(1 + \eta\theta)}\,.$$

A calculation verifies that

$$\phi_1(z_1, z_2, s, t) = \mathbf{a}_{t,s,u}\phi_1(z_1, z_2, s, s) + \mathbf{b}_{t,s,u}\phi_1(z_1, z_2, s, u)\,,$$

(see (3)); thus (27) follows. Finally, for $s \le t_1 \le t_2 \le u$ consider the generating function

$$\phi_2(z_1, z_2, s, t_1, t_2) = \sum_{m,n=0}^{\infty} z_1^n z_2^m E(p_n(X_s; s) X_{t_1} X_{t_2} p_m(X_u; u))\,.$$

Another calculation based on (26) and (15) gives

$$\begin{aligned}&\phi_2(z_1, z_2, s, t_1, t_2) \\ &\qquad = \left(\frac{1}{z_2} + t_2\eta + \theta\right)(\phi_1(z_1, z_2, s, t_1) - \phi_1(z_1, 0, s, t_1)) \\ &\qquad\quad + z_2 t_2(1 + \eta\theta)\phi_1(z_1, z_2, s, t_1) - z_1 z_2 s t_2 \eta\theta\,.\end{aligned}$$

A computer-assisted calculation now verifies that

$$\begin{aligned}
&\phi_2(z_1, z_2, s, t, t) \\
&\quad = \mathbf{A}\phi_2(z_1, z_2, s, s, s) + \mathbf{B}\phi_2(z_1, z_2, s, s, u) + \mathbf{C}\phi_2(z_1, z_2, s, u, u) + \mathbf{D}\phi_0(z_1, z_2, s) \\
&\qquad + \alpha\phi_1(z_1, z_2, s, s) + \beta\phi_1(z_1, z_2, s, u)\,,
\end{aligned}$$

which proves (28). □

3. Uniqueness

We first state the main result of this section.

Theorem 3.1. *Suppose $(X_t)_{t\geq 0}$ is a centered square-integrable stochastic process with covariance* (1). *If (X_t) satisfies* (2) *and* (12) *with $1 + \eta\theta \geq 0$, then (X_t) is the Markov process, as defined in Theorem* 2.5.

The proof of Theorem 3.1 is based on the method of moments, and we begin by verifying that the moments exist.

Lemma 3.2. *Under the assumptions of Theorem* 3.1 *$E(|X_t|^p) < \infty$ for all $p > 0$.*

Proof. This result follows from Corollary 4 of Ref. 10. To use this result, fix $t_1 < t_2$ and let $\xi_1 = t_1^{-1/2} X_{t_1}$, $\xi_2 = t_2^{-1/2} X_{t_2}$. Then their correlation $\rho = E(\xi_1\xi_2) = \sqrt{t_1/t_2} \in (0, 1)$. It remains to notice that $E(\xi_i|\xi_j) = \rho\xi_j$ and the variances $\mathrm{Var}(\xi_i|\xi_j) = 1 - \rho^2 + a_j\xi_j$ for some $a_j \in \mathbb{R}$; these relations follow from taking the limits $s \to 0$ or $u \to \infty$ in (2) and (12). Thus by Corollary 4 of Ref. 10, $E(|\xi_1|^p) < \infty$ for all $p > 0$. □

The next result is closely related to Proposition 3.1 of Ref. 11 and Theorem 2 of Ref. 24.

Lemma 3.3. *Suppose X_t has covariance* (1), *and satisfies conditions* (2) *and* (12). *Then for every $k \geq 0$ and $0 \leq s < t$ there exists a monic polynomial $p_k(x)$ of degree k with coefficients determined uniquely from s, t and the parameters in* (12) *such that $E(X_t^k|\mathcal{F}_s) = p_k(X_s)$.*

Proof. By Lemma 3.2, $E(|X_t^n|) < \infty$ for all n. We proceed by induction on the degree k of the monomial. Clearly, $E(X_t^k|\mathcal{F}_s)$ is a unique monic polynomial of degree k when $k = 0, 1$. Suppose that the conclusion holds true for all $s < t$ and all $k \leq n$ for some integer $n \geq 1$. Multiplying (2) by X_u^n and applying to both sides conditional expectation $E(\cdot|\mathcal{F}_s)$, we get

$$E(X_t E(X_u^n|\mathcal{F}_t)|\mathcal{F}_s) = \mathbf{a}X_s E(X_u^n|\mathcal{F}_s) + \mathbf{b}E(X_u^{n+1}|\mathcal{F}_s)\,.$$

Using the induction assumption, we can write this equation as

$$E(X_t^{n+1}|\mathcal{F}_s) = \mathbf{a}X_s^{n+1} + \mathbf{b}E(X_u^{n+1}|\mathcal{F}_s) + f_n(X_s)\,, \tag{30}$$

where f_n is a unique polynomial of degree at most n. Multiplying (4) by X_u^{n-1} and applying $E(\cdot|\mathcal{F}_s)$ to both sides, we get

$$E(X_t^2 E(X_u^{n-1}|\mathcal{F}_t)|\mathcal{F}_s) = \mathbf{A}X_s^2 E(X_u^{n-1}|\mathcal{F}_s) + \mathbf{B}X_s E(X_u^n|\mathcal{F}_s) + \mathbf{C}E(X_u^{n+1}|\mathcal{F}_s) + \cdots .$$

Using the induction assumption, we can write this equation as

$$E(X_t^{n+1}|\mathcal{F}_s) = (\mathbf{A}+\mathbf{B})X_s^{n+1} + \mathbf{C}E(X_u^{n+1}|\mathcal{F}_s) + g_n(X_s)\,, \tag{31}$$

where g_n is a unique polynomial of degree at most n. Since $\mathbf{b} - \mathbf{C} \neq 0$, subtracting (30) from (31) we get

$$E(X_u^{n+1}|\mathcal{F}_s) = \frac{\mathbf{a}-\mathbf{A}-\mathbf{B}}{\mathbf{C}-\mathbf{b}} X_s^{n+1} + h_n(X_s)\,,$$

where h_n is a (unique) polynomial of degree at most n. From (5)–(7) we get

$$\frac{\mathbf{a}-\mathbf{A}-\mathbf{B}}{\mathbf{C}-\mathbf{b}} = \frac{1+\sigma u}{1+\sigma s} = 1\,,$$

as $\sigma = 0$. Thus $E(X_t^{n+1}|\mathcal{F}_s) = X_s^{n+1} + h_n(X_s)$ is a monic polynomial of degree $n+1$ in variable X_s with uniquely determined coefficients. □

Proof of Theorem 3.1. Denote by (Y_t) the Markov process from Theorem 2.5. We will verify by the method of moments that (X_t) and (Y_t) have the same finite dimensional distributions.

Process (X_t) satisfies the assumptions of Lemma 3.3; by Theorem 2.5, process (Y_t) also satisfies the assumptions of Lemma 3.3. Therefore, for $n \geq 0$

$$E(Y_t^n|\mathcal{F}_s) = Y_s^n + h_{n-1}(Y_s)\,, \tag{32}$$

$$E(X_t^n|\mathcal{F}_s) = X_s^n + h_{n-1}(X_s) \tag{33}$$

with the same polynomial h_{n-1}. From this, we use induction to deduce that all mixed moments are equal. Taking $s = 0$, from (32) and (33) we see that $E(X_t^n) = E(Y_t^n)$ for all $n \in \mathbb{N}$, $t > 0$. Suppose that for some $k \geq 1$ and all $0 < t_1 < t_2 < \cdots < t_k$, all $n_1, \ldots, n_k \in \mathbb{N}$ we have

$$E(X_{t_1}^{n_1} X_{t_2}^{n_2} \cdots X_{t_k}^{n_k}) = E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \cdots Y_{t_k}^{n_k})\,.$$

Then from (32) and (33), and the induction assumption for any $t > t_k$ and $n \in \mathbb{N}$ we get

$$\begin{aligned} E(X_{t_1}^{n_1} X_{t_2}^{n_2} \cdots X_{t_k}^{n_k} X_t^n) &= E(X_{t_1}^{n_1} X_{t_2}^{n_2} \cdots X_{t_k}^{n_k} E(X_t^n|\mathcal{F}_{t_k})) \\ &= E(X_{t_1}^{n_1} X_{t_2}^{n_2} \cdots X_{t_{k-1}}^{n_{k-1}} X_{t_k}^{n_k} (X_{t_k}^n + h_{n-1}(X_{t_k}))) \\ &= E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \cdots Y_{t_{k-1}}^{n_{k-1}} Y_{t_k}^{n_k} (Y_{t_k}^n + h_{n-1}(Y_{t_k}))) \\ &= E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \cdots Y_{t_k}^{n_k} E(Y_t^n|\mathcal{F}_{t_k})) = E(Y_{t_1}^{n_1} Y_{t_2}^{n_2} \cdots Y_{t_k}^{n_k} Y_t^n)\,. \end{aligned}$$

Since $t > t_k$ and $n \in \mathbb{N}$ are arbitrary and the random variables are bounded, this shows that all mixed moments of the $(k+1)$-dimensional distributions match. Thus (X_t) is a version of the Markov process (Y_t). □

Corollary 3.4. *Suppose (X_t) is a Markov process from Theorem* 2.5 *with parameters $\eta = \theta$. Then the process $(tX_{1/t})_{t>0}$ has the same finite dimensional distributions as process $(X_t)_{t>0}$.*

Proof. It is well known that (1), and hence (2), are preserved by the transformation $(X_t) \mapsto (tX_{1/t})$. A calculation shows that if $\eta = \theta$, then the conditional variance (12) is also preserved by this transformation. Thus by Theorem 3.1, both processes have the same distribution. □

We remark that Corollary 3.4 gives an example of a Markov process with time-inversion property which is not covered by the criterion in Ref. 18. Other lesser known examples of Markov processes with time-inversion property are classical versions of the q-Brownian motions for $-1 \le q \le 1$.

4. Relations to Noncommutative Probability

The c-convolution $\bigstar_c$ was introduced in Ref. 7 and studied in Refs. 6, 8, 9, 20 and 21. It is a binary operation on the pairs of probability measures (μ, ν). For our purposes the most convenient definition is the analytic approach from Theorem 5.2 of Ref. 6. According to this result, the c-convolution (μ, ν) of pairs of probability measures (μ_1, ν_1) and (μ_2, ν_2) can be defined using the Cauchy transforms

$$G_j(z) = \int \frac{1}{z-x}\mu_j(dx)\,, \qquad g_j(z) = \int \frac{1}{z-x}\nu_j(dx)\,, \qquad j = 1, 2\,.$$

Let $k_j(z)$ be the inverse function of $g_j(z)$ in a neighborhood of ∞, and define

$$r_j(z) = k_j(z) - 1/z\,. \tag{34}$$

On the second component the c-convolution acts as the free convolution, $\nu = \nu_1 \boxplus \nu_2$. Recall that the free convolution ν of measures ν_1, ν_2 is the unique probability measure with the Cauchy transform $g(z)$ which solves the equation:

$$g(z) = \frac{1}{z - r_1(g(z)) - r_2(g(z))}\,,$$

see Ref. 23. To define the action of the c-convolution on the first component, let

$$R_j(z) = k_j(z) - 1/G_j(k_j(z))\,.$$

The first component of the c-convolution is defined as the unique probability measure μ with the Cauchy transform

$$G(z) = \frac{1}{z - R_1(g(z)) - R_2(g(z))}\,.$$

Functions r, R define the c-free cumulants with interesting combinational interpretation, see Ref. 7. We write

$$(\mu, \nu) = (\mu_1, \nu_1)\bigstar_c(\mu_2, \nu_2)\,.$$

Let (X_t) be the Markov process from Theorem 2.5 with parameters $1+\eta\theta \geq 0$. As previously, denote by π_t the distribution of X_t.

Proposition 4.1. *For every $t \geq 0$, there exists a probability measure ν_t such that the pairs (π_t, ν_t) form a semigroup with respect to the c-convolution,*

$$(\pi_{t+s}, \nu_{t+s}) = (\pi_t, \nu_t) \bigstar_c (\pi_s, \nu_s)\,.$$

Proof. Fix η, θ. Let (N_t) be the Markov process from Theorem 2.5 taken with parameters $\eta = 0$, and the value of θ as fixed above. Then (N_t) is obtained from the free Poisson process with parameter $\lambda = 1/\theta^2$ by centering and multiplication by θ. Let ν_t be the distribution of the random variable $N_{t(1+\eta\theta)} + t\eta$. It is easy to verify that (34) gives $r_t(z) = t\frac{z+\eta}{1-z\theta}$; clearly, $r_{t+s} = r_t + r_s$ and measures ν_t form a semigroup with respect to the free convolution. The Cauchy transform $G_t(z)$ of π_t is given by Eq. (17). Simplifying the expression $R_t(z) = k_t(z) - 1/G_t(k_t(z))$, where $k_t(z) = r_t(z) + 1/z$ we get

$$R_t(z) = \frac{tz}{1 - z\theta}\,.$$

Since $R_{t+s}(z) = R_t(z) + R_s(z)$, this verifies the c-convolution property for π_t. □

Remark 4.2. Measures π_t for $\theta = 1$ occur also in the Poisson Limit Theorem for c-convolutions; up to centering and reparametrization, the Cauchy transform derived on p. 380 of Ref. 6 is equivalent to (17).

Remark 4.3. Krystek and Yoshida[21] generalize the t-transformation of Ref. 9 to a two-parameter operation and define the corresponding **t**-deformed free convolution that acts on single probability measures rather than on pairs of measures. A version of Proposition 4.1 that uses this **t**-deformed free convolution holds true, at least when $\theta = 1$. For further related generalizations see Ref. 19.

Remark 4.4. When $\eta = \theta = \rho$, the law π_l of X_1 is the "free game law"; to see this, compare (17) with $t = 1$, to formula (2) in Ref. 4 with $a = 2\rho$, $b = \rho^2$. This is a "free analogue" of the classical case[16] with $q = 1$, where X_1 has the centered gamma law.

Acknowledgments

We would like to thank M. Bożejko and A. Krystek for information about the generalized convolutions and related Fock space constructions. Research partially supported by NSF grants #INT-0332062, #DMS-0504198, by the C.P. Taft Memorial Fund, and University of Cincinnati's Summer Faculty Research Fellowship Program.

References

1. M. Anshelevich, Free martingale polynomials, *J. Funct. Anal.* **201** (2003) 228–261.
2. M. Anshelevich, q-Lévy processes, *J. Reine Angew. Math.* **576** (2004) 181–207.
3. P. Biane, Processes with free increments, *Math. Z.* **227** (1998) 143–174.
4. M. Bożejko and W. Bryc, On a class of free Lévy laws related to a regression problem, *J. Funct. Anal.* **236** (2006) 59–77.
5. M. Bożejko, B. Kümmerer and R. Speicher, q-Gaussian processes: noncommutative and classical aspects, *Comm. Math. Phys.* **185** (1997) 129–154.
6. M. Bożejko, M. Leinert and R. Speicher, Convolution and limit theorems for conditionally free random variables, *Pac. J. Math.* **175** (1996) 357–388.
7. M. Bożejko and R. Speicher, ψ-independent and symmetrized white noises, in *Quantum Probability & Related Topics*, QP–PQ, Vol. VI (World Scientific, 1991), pp. 219–236.
8. M. Bożejko and J. Wysoczański, New examples of convolutions and non-commutative central limit theorems, in *Quantum Probability* (Gdańsk, 1997), Banach Center Publ., Vol. 43 (Polish Acad. Sci., 1998), pp. 95–103.
9. M. Bożejko and J. Wysoczański, Remarks on t-transformations of measures and convolutions, *Ann. Inst. H. Poincaré Probab. Statist.* **37** (2001) 737–761.
10. W. Bryc, Some remarks on random vectors with nice enough behavior of conditional moments, *Bull. Polish Acad. Sci.* **33** (1985) 677–683.
11. W. Bryc, A characterization of the Poisson process by conditional moments, *Stoch.* **20** (1987) 17–26.
12. W. Bryc, W. Matysiak and J. Wesołowski, The bi-Poisson process: a quadratic harness, org/abs/math.PR/0510208, *Ann. Probab.*, to appear.
13. W. Bryc, W. Matysiak and J. Wesołowski, Quadratic harnesses, q-commutations, and orthogonal martingale polynomials, *Trans. Amer. Math. Soc.* (2007), to appear, org/abs/math.PR/0504194.
14. W. Bryc, P. Szabłowski and W. Matysiak, Probabilistic aspects of Al-Salam–Chihara polynomials, *Proc. Amer. Math. Soc.* **133** (2005) 1127–1134.
15. W. Bryc and J. Wesołowski, Conditional moments of q-Meixner processes, *Probab. Th. Rel. Fields* **131** (2005) 415–441.
16. W. Bryc and J. Wesołowski, Classical bi-Poisson process: an invertible quadratic harness, *Statist. Probab. Lett.* **76** (2006) 1664–1674.
17. T. S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, 1978).
18. L. Gallardo and M. Yor, Some new examples of Markov processes which enjoy the time-inversion property, *Probab. Th. Rel. Fields* **132** (2005) 150–162.
19. A. Krystek and Ł. Wojakowski, Associative convolutions arising from conditionally free convolution, *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **8** (2005) 515–545.
20. A. Krystek and H. Yoshida, The combinatorics of the r-free convolution, *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **6** (2003) 619–627.
21. A. Krystek and H. Yoshida, Generalized t-transformations of probability measures and deformed convolutions, *Probab. Math. Statist.* **24** (2004) 97–119.
22. N. Saitoh and H. Yoshida, The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory, *Probab. Math. Statist.* (*Wrocław*) **21** (2001) 159–170.
23. D. Voiculescu, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66** (1986) 323–346.
24. J. Wesołowski, Stochastic processes with linear conditional expectation and quadratic conditional variance, *Probab. Math. Statist.* (*Wrocław*) **14** (1993) 33–44.