

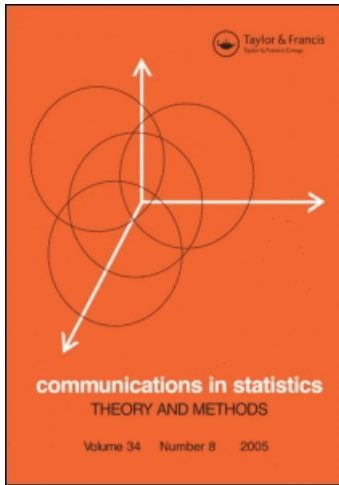
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Ordered Data and Applications

How Many Observations Fall in a Neighborhood of an Order Statistic?

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Asymptotic behavior of the number of independent identically distributed observations in a left or right neighborhood of k_n th order statistic from the sample of size n , for $k_n/n \rightarrow \alpha \in [0, 1]$, is studied. It appears that the limiting laws are of the Poisson type.

Keywords Limit theorems; Near order statistic observations; Order statistics; Poisson distribution; Swept discrete distributions.

Mathematics Subject Classification Primary 60G70; Secondary 62G30.

1. Introduction

Let (X_n) be a sequence of independent identically distributed (iid) random variables (rv's) of the continuous type. For any $n \geq 1$ by $X_{1:n} \leq \dots \leq X_{n:n}$, we denote the order statistics based on the sample (X_1, \dots, X_n) . The two basic objects which are studied in this article are

$$K_-(n, k, a) = \#\{j \in \{1, \dots, n\} : X_j \in (X_{k:n} - a, X_{k:n})\} \quad (1)$$

and

$$K_+(n, k, a) = \#\{j \in \{1, \dots, n\} : X_j \in (X_{k:n}, X_{k:n} + a)\}, \quad (2)$$

where $a > 0$ and $k \in \{1, \dots, n\}$. They are, respectively, numbers of observations falling in the open left or right a -vicinity of the k th order statistic.

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Different rv's closely related to particular forms of the two rv's defined above have been objects of quite intensive studies in the past ten years. The works of Eisenberg et al. (1993), Brands et al. (1994), Baryshnikov et al. (1994), Qi (1997), and Qi and Wilms (1997) have all focused on the problem of "a tie for the first place" for integer valued rv's. They were interested in asymptotics of $K(n) = \sum_{i=1}^n I\{X_i = X_{n:n}\}$, which might be considered as a discrete analog of $K_- + 1$ or $K_+ + 1$ for $k = n$.

Pakes and Steutel (1997) were the first who considered the case of continuous rv's. They described the limit behavior of $K_-(n, n, a) + 1$ based on the assumptions regarding the asymptotics of tails of the parent distribution function (df) F of the sequence (X_n) . They also studied the situation when $a = a_n$ depends on n for special choices of the sequence a_n in different domains of attraction of extreme distributions. The limit distributions they obtained included geometric and mixed Poisson laws. The analogs of those results for $K_-(n, n - k + 1, a_n)$ were given in Pakes and Li (1998), where again as limit distributions mixed Poisson laws showed up. Pakes (2000) studied the asymptotics $K_-(n, n - k + 1, a) + 1$ (a being fixed) for, so-called, thin-tailed distributions. He showed, for instance, that under suitable conditions, $K_-(n, n - k + 1, a)$ converges in distribution to a negative binomial law. Binomial limits for $K_+(n, n - k + 1, a)$ have been obtained recently in Balakrishnan and Stepanov (2005). Other results, related to those, quickly described above, can be also found in Khmaladze et al. (1997), Li and Pakes (1998), Li (1999), Hu and Su (2003), and Balakrishnan and Stepanov (2004).

The main novelty of this article is that while studying the limiting distributions of K_- and K_+ , we allow not only $a = a_n$, but also $k = k_n$ to vary in such a way that $k_n/n \rightarrow \alpha \in [0, 1]$. Sections 4, 5, and 6 are, respectively, devoted to studying the three cases $\alpha = 0$, $\alpha \in (0, 1)$ and $\alpha = 1$, while basic distribution properties are considered in Sec. 2 and a warming-up exponential example is worked out in Sec. 3.

2. Basic Distributional Properties

Directly from the definition (1) it follows that for any $j = 0, 1, \dots, k - 1$,

$$P(K_-(n, k, a) = j) = \frac{n!}{(k-j-1)!j!(n-k)!} \int_{\mathbf{R}} F^{k-j-1}(x-a)[F(x) - F(x-a)]^j \bar{F}^{n-k}(x) dF(x) \quad (3)$$

$$= \binom{k-1}{j} \int_{\mathbf{R}} \left[\frac{F(x-a)}{F(x)} \right]^{k-j-1} \left[1 - \frac{F(x-a)}{F(x)} \right]^j dF_{k:n}(x), \quad (4)$$

where $\bar{F} = 1 - F$ and $F_{k:n}$ is a df of $X_{k:n}$; see for instance Arnold et al. (1992) or David and Nagaraja (2003).

Similarly, the definition (2) results in

$$P(K_+(n, k, a) = j) = \frac{n!}{(k-1)!j!(n-k-j)!} \int_{\mathbf{R}} F^{k-1}(x)[F(x+a) - F(x)]^j \bar{F}^{n-k-j}(x+a) dF(x) \quad (5)$$

$$= \binom{n-k}{j} \int_{\mathbf{R}} \left[1 - \frac{\bar{F}(x+a)}{\bar{F}(x)} \right]^j \left[\frac{\bar{F}(x+a)}{\bar{F}(x)} \right]^{n-k-j} dF_{k:n}(x), \quad (6)$$

which is valid for any $j = 0, 1, \dots, n - k$.

Also for $j = 2, \dots, k - 1$

$$P(K_-(n, k, a) \leq j - 1) = P(X_{k:n} - X_{k-j:n} \geq a) \tag{7}$$

and for $j = 2, \dots, n - k$

$$P(K_+(n, k, a) \leq j - 1) = P(X_{k+j:n} - X_{k:n} \geq a). \tag{8}$$

Observe that (7) and (8) relate distributions of K_- and K_+ to distributions of spacings based on order statistics, and at least potentially may be used to deduce asymptotic properties of spacings from limiting distributions of K_- and K_+ . Also, they can be used to obtain distributional relations between K_+ and K_- , as briefly indicated in Pakes and Steutel (1997). Let (Y_n) be an iid sequence of rv's such that $Y_1 \stackrel{d}{=} -X_1$. Then $(Y_{k:n}, k = 1, \dots, n) \stackrel{d}{=} (-X_{n-k+1:n}, k = 1, \dots, n)$. Consequently, using first (7) and then (8), we get

$$K_-^{(X)}(n, k, a) \stackrel{d}{=} K_+^{(Y)}(n, n - k + 1, a) \tag{9}$$

or, equivalently,

$$K_+^{(X)}(n, k, a) \stackrel{d}{=} K_-^{(Y)}(n, n - k + 1, a), \tag{10}$$

where the superscript (X) or (Y) refers to the sequence of rv's for which K_- or K_+ is defined.

Thus, studying the asymptotics of $K_+(n, k_n, a_n)$ for $k_n/n \rightarrow \alpha$ is parallel to considering $K_-(n, k_n, a_n)$ with $k_n/n \rightarrow 1 - \alpha$ at least for $\alpha \in (0, 1)$. If $\alpha \in \{0, 1\}$ then one has to take into account if $l_F = \inf\{x \in \mathbf{R} : F(x) > 0\}$ and $r_F = \sup\{x \in \mathbf{R} : F(x) < 1\}$ are finite or not.

3. Exponential Observations

Our guiding example will be the case of standard (mean equal one) exponential iid rv's (X_n) . Apparently, this is the situation in which all the computations are rather straightforward and explicit. Therefore it gives a reasonable insight at what might be expected in more general cases.

For instance, using (3), one directly gets

$$P(K_-(n, k, a) = j) = \binom{n - k + j}{j} (1 - e^{-a})^j e^{-(n-k+1)a}, \quad j = 0, 1, \dots, k - 2,$$

and

$$P(K_-(n, k, a) = k - 1) = 1 - \sum_{j=0}^{k-2} P(K_-(n, k, a) = j).$$

Observe that the probability mass function (pmf) at $j = 0, 1, \dots, k - 2$, is negative binomial $nb(n - k + 1, e^{-a})$ and the rest of the mass of the $nb(n - k + 1, e^{-a})$ distribution is assigned to a single value $k - 1$. Such a distribution will

be called negative binomial swept at $k - 1$ and denoted by $nb_{k-1}(n - k + 1, e^{-a})$. The term “swept” follows Kingamn (1966), where it was used in the context of the Poisson distribution. Note that if \mathcal{L} is the distribution of a rv X then the rv $\min\{X, k\}$ has the distribution \mathcal{L} swept at k .

Note that as $n \rightarrow \infty$,

$$K_-(n, k, a) \xrightarrow{P} k - 1$$

and

$$P(K_-(n, k_n, a) = j) \rightarrow 0$$

for any j where $n - k_n \rightarrow \infty, k_n \rightarrow \infty$. Also, we have

$$nb_{n-k-1}(k + 1, e^{-a}) \sim K_-(n, n - k, a) \xrightarrow{d} N \sim nb(k + 1, e^{-a}).$$

Observe that though $K_-(n, k_n, a)$ is not a negative binomial rv it can be represented as $K_n(n, k_n, a) = T_n I(T_n < k_n - 1) + (k_n - 1) I(T_n \geq k_n - 1)$, where $T_n \sim nb(n - k_n + 1, e^{-a})$. For $k_n/n \rightarrow g \in (1 - e^{-a}, 1]$ via the normal approximation, it can be seen that $I(T_n < k_n - 1) \xrightarrow{P} 1$ and $(k_n - 1) I(T_n \geq k_n - 1) \xrightarrow{P} 0$. Further, by the clt for the negative binomial distribution, which is the law of a sum of iid geometric rv’s, in this case one gets

$$\frac{K_-(n, k_n, a) - (n - k_n)(e^a - 1)}{\sqrt{(n - k_n)e^a(e^a - 1)}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ denotes the standard normal distribution.

Now, for a given $\lambda > 0$, let us define $a = a_n = -\log(1 - \frac{\lambda}{n}) > 0$ for sufficiently large n . Then the formula for the pmf, via the Stirling approximation: $n! \approx (\frac{n}{e})^n \sqrt{2\pi n}$ (where for two sequences (b_n) and (d_n) of real numbers $d_n \approx b_n$ means $\frac{d_n}{b_n} \rightarrow 1$), yields

$$P(K_-(n, k, a_n) = j) \approx \frac{(n - k + j)^j}{j!} \left(\frac{\lambda}{n}\right)^j \left(1 - \frac{\lambda}{n}\right)^{n-k+1} \rightarrow e^{-\lambda} \frac{\lambda^j}{j!}, \quad j = 0, 1, \dots, k - 2,$$

and consequently,

$$K_-(n, k, a_n) \xrightarrow{d} \Lambda_{k-1} \sim \mathcal{P}_{k-1}(\lambda); ,$$

where $\mathcal{P}_{k-1}(\lambda)$ denotes the Poisson distribution with the mean λ swept at $k - 1$, i.e.,

$$P(\Lambda_{k-1} = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \quad j = 0, 1, \dots, k - 2,$$

and

$$P(\Lambda_{k-1} = k - 1) = 1 - \sum_{j=0}^{k-2} P(\Lambda_{k-1} = j).$$

Also, with the same sequence (a_n) as defined above and $k_n \rightarrow \infty, k_n/n \rightarrow 0$, we get for any $j = 0, 1, \dots$, and sufficiently large n that

$$P(K_-(n, k_n, a_n) = j) \approx \frac{(n - k_n + j)^j}{j!} \left(\frac{\lambda}{n}\right)^j \left(1 - \frac{\lambda}{n}\right)^{n-k_n+1} \rightarrow e^{-\lambda} \frac{\lambda^j}{j!},$$

and thus $K_-(n, k_n, a_n) \xrightarrow{d} \Lambda \sim \mathcal{P}(\lambda)$.

Let $k_n/n \rightarrow \alpha \in [0, 1)$. For a given $\lambda > 0$ define $a_n = \log(1 + \frac{\lambda}{(1-\alpha)n}) > 0$. Then, similarly as earlier for any $j = 0, 1, \dots$, we get

$$P(K_-(n, k_n, a_n) = j) \approx \frac{(n - k_n + j)^j}{j!} \frac{(\frac{\lambda}{(1-\alpha)n})^j}{(1 + \frac{\lambda}{(1-\alpha)n})^{n-k_n+j+1}} \rightarrow e^{-\lambda} \frac{\lambda^j}{j!},$$

which means that also in this case the limiting distribution for $K_-(n, k_n, a_n)$ is Poisson $\mathcal{P}(\lambda)$.

Similarly as for K_- , but using (5), we derive the exact distribution of $K_+(n, k, a)$ as

$$P(K_+(n, k, a) = j) = \binom{n-k}{j} (1 - e^{-a})^j e^{-(n-k-j)a}, \quad j = 0, 1, \dots, n - k,$$

i.e., $K_+(n, k, a)$ is a binomial, $b(n - k, 1 - e^{-a})$, rv.

So as $n - k_n \rightarrow \infty$ it follows that

$$(K_+(n, k_n, a) = j) \rightarrow 0$$

for any fixed j , which means that $K_+(n, k_n, a) \xrightarrow{P} \infty$ (compare Proposition 4.1 and 5.1 below). On the other hand,

$$K_+(n, n - k, a) \sim b(k, 1 - e^{-a})$$

does not depend on n . By the normal approximation of the binomial distribution we get immediately that

$$\frac{e^a K_+(n, k, a) - n(e^a - 1)}{\sqrt{n(e^a - 1)}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

Now we take $\lambda > 0$ and define $a = a_n = -\log(1 - \frac{\lambda}{(1-\alpha)n}) > 0$ for n sufficiently large. Then using the pmf one obtains the convergence in law to the Poisson distribution:

$$K_+(n, k_n, a_n) \xrightarrow{d} \Lambda \sim \mathcal{P}(\lambda),$$

for any sequence (k_n) such that $k_n/n \rightarrow \alpha \in [0, 1)$ (for instance, $k_n = k, n = 1, 2, \dots$, is admissible). The same limit is obtained for $k_n/n \rightarrow \alpha \in (0, 1], n - k_n \rightarrow \infty$, and $a_n = -\log(1 - \frac{\lambda}{n-k_n})$ for large n .

So in both cases of K_- and K_+ , in order to obtain Poissonian limits, a sequence (a_n) was defined by relating it to the local behavior in the neighborhood of α th

quantile of the df F of the exponential distribution, where $\alpha = \lim_{n \rightarrow \infty} k_n/n$. This observation will guide us in more general situations studied in Secs. 4, 5, and 6.

4. Approximations for K_- and K_+ when $\alpha = 0$

We first introduce the following quantities:

$$c_{n,k,j} = \frac{n!}{(n-k)!(k-j-1)!j!}, \quad n \geq 1, \quad k \in \{1, \dots, n\}, \quad j \in \{0, \dots, k-1\},$$

$$\beta(x, a) = \frac{1 - F(x)}{1 - F(x-a)}, \quad 0 < a \leq x,$$

which will be useful further. By $F^{\leftarrow}(x)$, we denote the generalized inverse function, i.e., $F^{\leftarrow}(x) = \inf\{s : F(s) \geq x\}$. Moreover, in this section we assume $l_F = 0$. Note that then $\beta(a, a) = 1 - F(a) = \bar{F}(a)$.

Before studying the asymptotic behavior of K_-, K_+ , we state a technical result which will be later of much use.

Lemma 4.1. *Let k be fixed, $k_n \rightarrow \infty$, $\alpha = 0$ and $0 < d < 1$. Then*

- a) $c_{n,k,j} \cdot d^{n-k} \xrightarrow{n \rightarrow \infty} 0$ for any $j \leq k$,
- b) $c_{n,k_n,j} \cdot d^{n-k_n} \xrightarrow{n \rightarrow \infty} 0$ for any j .

Proof. a) Observe that

$$0 \leq \frac{n!}{(n-k)!(k-j-1)!} d^{n-k} < \frac{1}{(k-j-1)!d^k} \cdot n^k d^n$$

and the right-hand side tends to 0 as $n \rightarrow \infty$. Thus a) follows.

b) Exploiting Stirling's approximation, we get for $\varepsilon > 0$

$$\begin{aligned} & \frac{n!}{(n-k_n)!(k_n-j-1)!j!} \cdot d^{n-k_n} \\ & \approx \frac{1}{\sqrt{2\pi}j!} \left(\frac{n}{n-k_n}\right)^n \left(\frac{n-k_n}{k_n}\right)^{k_n} k_n^{j+1/2} d^{n-k_n} \\ & \leq \frac{1}{\sqrt{2\pi}j!} \cdot \exp \left\{ k_n \log(e + \varepsilon) + k_n \log \left(\frac{n-k_n}{k_n}\right) \right. \\ & \quad \left. + \left(j + \frac{1}{2}\right) \log k_n - (n-k_n)(-\log d) \right\} \end{aligned}$$

where the inequality holds for n large enough. Now, the expression in the exponent can be written as

$$(n-k_n) \left[\frac{k_n}{n-k_n} \log(e + \varepsilon) - \frac{k_n}{n-k_n} \log \left(\frac{k_n}{n-k_n}\right) + \left(j + \frac{1}{2}\right) \frac{\log k_n}{n-k_n} - (-\log d) \right]$$

which apparently diverges to $-\infty$ as $n \rightarrow \infty$ since all elements of the sum inside the square brackets converge to zero except the last one which is negative. □

The next result shows that for a fixed $a > 0$ the rv's K_- and K_+ diverge in probability to ∞ as $n \rightarrow \infty$ and $k_n \rightarrow \infty$ ($\alpha = 0$).

Proposition 4.1. *Let X_1, X_2, \dots be iid rv's with a continuous df F such that $l_F = 0$. Let $k_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\alpha = 0$. Then for any fixed $a > 0$*

$$K_-(n, k_n, a) \xrightarrow{P} \infty, \quad K_+(n, k_n, a) \xrightarrow{P} \infty.$$

Proof. Observe that if $k_n \rightarrow \infty$ and $\alpha = 0$, then $P(X_{k_n:n} < a) \rightarrow 1$ for any fixed $a > 0$. This implies $P(K_-(n, k_n, a) = k_n - 1) \rightarrow 1$. Thus for any fixed $j \geq 0$ it follows that $P(K_-(n, k_n, a) = j) \rightarrow 0$.

We prove the second statement of this Proposition directly. Equality (5) can be written as

$$P(K_+(n, k, a) = j) = c_{n,k+j,j} \int_0^\infty [F(x+a) - F(x)]^j [1 - F(x+a)]^{n-k-j} [F(x)]^{k-1} dF(x).$$

Then, for any fixed $a > 0$, $0 \leq j \leq n - k$, $1 \leq k \leq n$

$$\begin{aligned} P(K_+(n, k, a) = j) &\leq c_{n,k+j,j} [1 - F(a)]^{n-k-j} \int_0^\infty [1 - F(x)]^j [F(x)]^{k-1} dF(x) \\ &\leq c_{n,k+j,j} [1 - F(a)]^{n-(k+j)}. \end{aligned}$$

Applying now Lemma 4.1, we get $P\{K_+(n, k_n, a) = j\} \rightarrow 0$ for $k_n \rightarrow \infty$, $\alpha = 0$, and any fixed $a > 0$ and $j \geq 0$. This, in fact, means $K_+(n, k_n, a) \xrightarrow{P} \infty$. \square

The second result of Proposition 4.1 can be commented in the following manner. The r.v. $X_{k_n:n}$ tends in probability to 0 as $n, k_n \rightarrow \infty$, and $\alpha = 0$. Consequently, for large enough n the behavior of $K_+(n, k_n, a)$ can be compared to a binomial rv $Y_n \sim b(n, p)$ with $p = F(a) - F(\varepsilon)$. And the sequence (Y_n) tends in probability to infinity. So, this may be an intuitive explanation why the random interval $(X_{k_n:n}, X_{k_n:n} + a)$ gets infinitively many observations as $n \rightarrow \infty$.

Now we have come to main results of Sec. 2. First a constant k is considered, while $a = a_n$ depends on n .

Theorem 4.1. *Let $k \geq 2$ be fixed, $n \rightarrow \infty$ and F be a continuous df which is invertible in a right neighborhood of 0. The sequence $a_n \rightarrow 0^+$ ($n \rightarrow \infty$) is defined by $a_n = F^{-1}(\lambda/n)$ for some $\lambda > 0$. If*

$$\lim_{(x,y) \rightarrow (0^+,0^+)} \frac{F(x+y) - F(x)}{F(y)} = 1, \tag{11}$$

then

$$K_-(n, k, a_n) \xrightarrow{d} \Lambda_{k-1} \sim \mathcal{P}_{k-1}(\lambda),$$

where the Poisson distribution swept at $k - 1$, $\mathcal{P}_{k-1}(\lambda)$, was defined in Sec. 3.

Proof. Since $l_F = 0$

$$\begin{aligned} P(K_-(n, k, a) = j) &= c_{n,k,j} \int_a^\infty [F(x-a)]^{k-j-1} [F(x) - F(x-a)]^j [\bar{F}(x)]^{n-k} dF(x) \\ &= c_{n,k,j} \int_a^\infty \left[1 - \frac{\bar{F}(x)}{\beta(x, a)}\right]^{k-j-1} \left[\frac{1}{\beta(x, a)} - 1\right]^j [\bar{F}(x)]^{n-k+j} dF(x) \end{aligned}$$

for $j = 0, 1, \dots, k-2$ and

$$P(K_-(n, k, a) = k-1) = 1 - \sum_{j=0}^{k-2} P(K_-(n, k, a) = j).$$

Since

$$\begin{aligned} P(n, k, a_n, j) &:= c_{n,k,j} \int_{a_n}^\infty \left[1 - \frac{\bar{F}(x)}{\bar{F}(a_n)}\right]^{k-j-1} \left[\frac{1}{\bar{F}(a_n)} - 1\right]^j [\bar{F}(x)]^{n-k+j} dF(x) \\ &= \binom{n-k+j}{j} [\bar{F}(a_n)]^{n-k+1} [1 - \bar{F}(a_n)]^j \\ &= \frac{(n-k+1)(n-k+2)\dots(n-k+j)}{n^j} \frac{1}{j!} \left[\left(1 - \frac{\lambda}{n}\right)^n\right]^{\frac{n-k+1}{n}} \lambda^j \\ &\xrightarrow{n \rightarrow \infty} \frac{\lambda^j}{j!} e^{-\lambda} \end{aligned}$$

for $j = 0, 1, \dots, k-2$, it suffices to show that

$$|P(K_-(n, k, a_n) = j) - P(n, k, a_n, j)| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } j = 0, 1, \dots, k-2.$$

For a fixed $x_0 > 0$ (to be chosen later on), $a_n < x_0$ for n large enough. Then we will show

$$|P(K_-(n, k, a_n) = j) - P(n, k, a_n, j)| \leq I_1 + I_2 + I_3 \xrightarrow{n \rightarrow \infty} 0,$$

where

$$\begin{aligned} I_1 &= I_1(n, k, j, a_n, x_0) \\ &= c_{n,k,j} \int_{x_0}^\infty [F(x-a_n)]^{k-j-1} [F(x) - F(x-a_n)]^j [\bar{F}(x)]^{n-k} dF(x), \\ I_2 &= I_2(n, k, j, a_n, x_0) \\ &= c_{n,k,j} \int_{x_0}^\infty \left[1 - \frac{\bar{F}(x)}{\bar{F}(a_n)}\right]^{k-j-1} \left[\frac{1}{\bar{F}(a_n)} - 1\right]^j [\bar{F}(x)]^{n-k+j} dF(x) \end{aligned}$$

and

$$\begin{aligned} I_3 &= I_3(n, k, j, a_n, x_0) \\ &= c_{n,k,j} \int_{a_n}^{x_0} |G_1(n, k, j, a_n, x) - G_2(n, k, j, a_n, x)| [\bar{F}(x)]^{n-k+j} dF(x), \end{aligned}$$

where

$$G_1(n, k, j, a_n, x) = \left[1 - \frac{\bar{F}(x)}{\beta(x, a_n)}\right]^{k-j-1} \left[\frac{1}{\beta(x, a_n)} - 1\right]^j,$$

$$G_2(n, k, j, a_n, x) = \left[1 - \frac{\bar{F}(x)}{\bar{F}(a_n)}\right]^{k-j-1} \left[\frac{1}{\bar{F}(a_n)} - 1\right]^j.$$

For estimating I_1 we use the inequality

$$I_1(n, k, j, a_n, x_0) \leq c_{n,k,j} [\bar{F}(x_0)]^{n-k} \int_{x_0}^{\infty} [F(x - a_n)]^{k-j-1} dF(x) < c_{n,k,j} [\bar{F}(x_0)]^{n-k}.$$

Applying Lemma 4.1 for the last expression when $n \rightarrow \infty$ we get $I_1(n, k, j, a_n, x_0) \rightarrow 0$.

Now we turn to I_2 . For $0 \leq j \leq k - 2$ we can write

$$I_2(n, k, j, a_n, x_0) = c_{n,k,j} [F(a_n)]^j [\bar{F}(a_n)]^{n-k+1} \int_0^{\frac{\bar{F}(x_0)}{\bar{F}(a_n)}} (1 - y)^{k-j-1} y^{n-k+j} dy$$

$$\leq c_{n,k,j} [F(a_n)]^j [\bar{F}(a_n)]^{n-k+1} \int_0^{\frac{\bar{F}(x_0)}{\bar{F}(a_n)}} y^{n-k+j} dy$$

$$= c_{n,k,j} [\bar{F}(x_0)]^{n-k+j+1} \frac{1}{n - k + j + 1} \left(\frac{\lambda}{n}\right)^j \left(1 - \frac{\lambda}{n}\right)^{-j} \rightarrow 0$$

by virtue of Lemma 4.1.

Let us now study I_3 . For $0 \leq j \leq k - 2$ the following inequality holds

$$I_3(n, k, j, a_n, x_0) \leq I_{31}(n, k, j, a_n, x_0) + I_{32}(n, k, j, a_n, x_0) + I_{33}(n, k, j, a_n, x_0),$$

where

$$I_{31}(n, k, j, a_n, x_0) = c_{n,k,j} \int_{a_n}^{x_0} \left| \left(1 - \frac{\bar{F}(x)}{\beta(x, a_n)}\right)^{k-j-1} - \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)}\right)^{k-j-1} \right|$$

$$\times \left| \left(\frac{1}{\beta(x, a_n)} - 1\right)^j - \left(\frac{1}{\bar{F}(a_n)} - 1\right)^j \right| [\bar{F}(x)]^{n-k+j} dF(x),$$

$$I_{32}(n, k, j, a_n, x_0) = c_{n,k,j} \int_{a_n}^{x_0} \left| \left(1 - \frac{\bar{F}(x)}{\beta(x, a_n)}\right)^{k-j-1} - \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)}\right)^{k-j-1} \right|$$

$$\times \left(\frac{1}{\bar{F}(a_n)} - 1\right)^j [\bar{F}(x)]^{n-k+j} dF(x)$$

and

$$I_{33}(n, k, j, a_n, x_0) = c_{n,k,j} \int_{a_n}^{x_0} \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)}\right)^{k-j-1} \cdot \left| \left(\frac{1}{\beta(x, a_n)} - 1\right)^j - \left(\frac{1}{\bar{F}(a_n)} - 1\right)^j \right|$$

$$\times [\bar{F}(x)]^{n-k+j} dF(x).$$

Let us first estimate I_{33} . Assumption (11) implies that

$$\lim_{\substack{(x, a_n) \rightarrow (0, 0) \\ x > a_n > 0}} \left(\frac{1}{\beta(x, a_n)} - 1 \right) \left(\frac{1}{\bar{F}(a_n)} - 1 \right)^{-1} = 1.$$

Thus

$$\forall_{\varepsilon > 0} \exists_{N_1} \exists_{\hat{x}_0} \forall_{n \geq N_1} \forall_{x \in (a_n, \hat{x}_0)} \left| \left[\left(\frac{1}{\beta(x, a_n)} - 1 \right) \left(\frac{1}{\bar{F}(a_n)} - 1 \right)^{-1} \right]^j - 1 \right| < \varepsilon \quad (12)$$

for $j = 0, 1, \dots, k - 2$. By (12) we can write for any $n \geq N_1$, $0 \leq j \leq k - 2$, and $0 < x_0 \leq \hat{x}_0$

$$\begin{aligned} I_{33}(n, k, j, a_n, x_0) &< c_{n, k, j} \cdot \varepsilon \cdot \left(\frac{1}{\bar{F}(a_n)} - 1 \right)^j \int_{a_n}^{x_0} \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)} \right)^{k-j-1} [\bar{F}(x)]^{n-k+j} dF(x) \\ &= c_{n, k, j} \cdot \varepsilon \cdot [F(a_n)]^j \cdot [\bar{F}(a_n)]^{n-k+1} \int_{\frac{\bar{F}(x_0)}{\bar{F}(a_n)}}^1 (1 - y)^{k-j-1} y^{n-k+j} dy \\ &\leq c_{n, k, j} \cdot \varepsilon \cdot [F(a_n)]^j \cdot [\bar{F}(a_n)]^{n-k+1} \int_0^1 (1 - y)^{k-j-1} y^{n-k+j} dy \\ &= \varepsilon \cdot \binom{n - k + j}{j} [F(a_n)]^j \cdot [\bar{F}(a_n)]^{n-k+1} = P(Z_n = j) \cdot \varepsilon \leq \varepsilon, \end{aligned}$$

where Z_n is a random variable with the negative binomial distribution $nb(n - k + 1, \bar{F}(a_n))$.

Let us show now that $I_{32} \leq \varepsilon$. First, we observe that

$$\left(1 - \frac{\bar{F}(x)}{\beta(x, a_n)} \right) \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)} \right)^{-1} = \bar{F}(a_n) \cdot \left[\frac{F(a_n + x - a_n) - F(a_n)}{F(x - a_n)} \right]^{-1}.$$

Since $\bar{F}(a_n) = 1 - \frac{\lambda}{n} \xrightarrow{n \rightarrow \infty} 1$, by assumption (11) applied to the expression in square brackets we can write

$$\forall_{\varepsilon > 0} \exists_{N_2} \exists_{\tilde{x}_0 > 0} \forall_{n \geq N_2} \forall_{x \in (a_n, \tilde{x}_0)} \left| \left[\left(1 - \frac{\bar{F}(x)}{\beta(x, a_n)} \right) \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)} \right)^{-1} \right]^{k-j-1} - 1 \right| < \varepsilon \quad (13)$$

for $j = 0, 1, \dots, k - 2$. Thus for any $n \geq N_2$, $0 \leq j \leq k - 2$, and $0 < x_0 \leq \tilde{x}_0$,

$$\begin{aligned} I_{32}(n, k, j, a_n, x_0) &< c_{n, k, j} \cdot \varepsilon \cdot \left(\frac{1}{\bar{F}(a_n)} - 1 \right)^j \int_{a_n}^{x_0} \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)} \right)^{k-j-1} [\bar{F}(x)]^{n-k+j} dF(x) \leq \varepsilon. \end{aligned}$$

The last inequality is identical to the one already used when dealing with I_{33} .

To show that $I_{31} < \varepsilon$ we use (12) and (13). Then for any $n \geq N = \max\{N_1, N_2\}$, $0 \leq j \leq k - 2$ and $x_0 = \min\{\hat{x}_0, \tilde{x}_0\}$

$$I_{31} < c_{n, k, j} \cdot \varepsilon^2 \cdot \left(\frac{1}{\bar{F}(a_n)} - 1 \right)^j \int_{a_n}^{x_0} \left(1 - \frac{\bar{F}(x)}{\bar{F}(a_n)} \right)^{k-j-1} [\bar{F}(x)]^{n-k+j} dF(x) \leq \varepsilon^2 < \varepsilon. \quad \square$$

Remark 4.1. Let F have non zero continuous right-hand side derivative at 0. Then the assumption (11) of Theorem 4.1 is satisfied.

Now we study the case of $k = k_n \rightarrow \infty$.

Theorem 4.2. Let $n, k_n \rightarrow \infty, \alpha = 0$, and F be a continuous df which is invertible in a right neighborhood of 0. Assume that the sequence $a_n \rightarrow 0^+ (n \rightarrow \infty)$ is such that $a_n = F^{\leftarrow}(\lambda/n)$ for some $\lambda > 0$. If

$$\lim_{(x, a_n) \rightarrow (0^+, 0^+)} \left[\frac{F(x + a_n) - F(a_n)}{F(x)} \right]^{k_n} = 1, \tag{14}$$

then

$$K_-(n, k_n, a_n) \xrightarrow{d} \Lambda \sim \mathcal{P}(\lambda).$$

Proof. We skip it since it follows on the same lines as the proof of Theorem 4.1. □

This section is concluded with the result on the Poisson limit law for the number of observations in the right vicinity of a low-order statistic.

Theorem 4.3. Let k_n be any sequence of integers satisfying the condition $\alpha = 0$ and let F be a continuous df which is invertible in a right neighborhood of 0. The sequence $(a_n), a_n \rightarrow 0^+$, is defined by $a_n = F^{\leftarrow}(\lambda/n)$ for some $\lambda > 0$. If condition (11) from Theorem 4.1 holds, then

$$K_+(n, k_n, a_n) \xrightarrow{d} \mathcal{P}(\lambda).$$

Proof. It follows from (5) and (6) that for $j = 0, 1, \dots, n - k_n$

$$\begin{aligned} P(K_+(n, k_n, a_n) = j) &= c_{n, k_n + j, j} \int_0^\infty [F(x)]^{k_n - 1} [\bar{F}(x)]^{n - k_n} [1 - \beta(x + a_n, a_n)]^j [\beta(x + a_n, a_n)]^{n - k_n - j} dF(x) \\ &= \int_0^{x_0} G(n - k_n, j, a_n, x) dF_{k_n:n}(x) + \int_{x_0}^\infty G(n - k_n, j, a_n, x) dF_{k_n:n}(x) = J_1 + J_2, \end{aligned}$$

where $x_0 > 0$ and

$$G(n - k_n, j, a_n, x) = \binom{n - k_n}{j} [1 - \beta(x + a_n, a_n)]^j [\beta(x + a_n, a_n)]^{n - k_n - j}.$$

Since $\beta(x + a_n, a_n) \in (0, 1)$, we have $G(n - k_n, j, a_n, x) = P(Y_n = j) < 1$, where Y_n is a random variable with the binomial distribution $b(n - k_n, 1 - \beta(x + a_n, a_n))$. As $X_{k_n:n} \xrightarrow{P} 0$ we get

$$0 \leq J_2 \leq \int_{x_0}^\infty dF_{k_n:n}(x) \leq P(X_{k_n:n} \geq x_0) \xrightarrow{n \rightarrow \infty} 0.$$

Now notice that since $\bar{F}(a_n) \xrightarrow{n \rightarrow \infty} 1$, condition (11) implies

$$\lim_{(x, a_n) \rightarrow (0^+, 0^+)} \frac{F(x + a_n) - F(x)}{F(a_n)\bar{F}(a_n)} = 1,$$

which means that

$$\forall_{\varepsilon > 0} \exists_{x_0 > 0} \exists_N \forall_{n \geq N} \forall_{x \in (0, x_0)} \left| \frac{F(x + a_n) - F(x)}{\bar{F}(a_n)} \cdot \frac{1}{F(a_n)} - 1 \right| < \varepsilon.$$

Hence for $x \in (0, x_0)$ and $n \geq N$ we have

$$\frac{\lambda(1 - \varepsilon)}{n} < 1 - \beta(x + a_n, a_n) < \frac{\lambda(1 + \varepsilon)}{n}.$$

It follows that

$$J_1 \leq \binom{n - k_n}{j} \left[\frac{\lambda(1 + \varepsilon)}{n} \right]^j \left[1 - \frac{\lambda(1 - \varepsilon)}{n} \right]^{n - k_n - j} \int_0^{x_0} dF_{k_n:n}(x) \xrightarrow{n \rightarrow \infty} \frac{[\lambda(1 + \varepsilon)]^j}{j!} e^{-\lambda(1 - \varepsilon)}$$

and

$$J_1 \geq \binom{n - k_n}{j} \left[\frac{\lambda(1 - \varepsilon)}{n} \right]^j \left[1 - \frac{\lambda(1 + \varepsilon)}{n} \right]^{n - k_n - j} \int_0^{x_0} dF_{k_n:n}(x) \xrightarrow{n \rightarrow \infty} \frac{[\lambda(1 - \varepsilon)]^j}{j!} e^{-\lambda(1 + \varepsilon)}$$

for $j = 0, 1, \dots$. Letting $\varepsilon \rightarrow 0$ we get $J_1 \xrightarrow{n \rightarrow \infty} \frac{\lambda^j}{j!} e^{-\lambda}$ for $j = 0, 1, \dots$, which completes the proof. \square

5. Approximations for K_-, K_+ when $\alpha \in (0, 1)$

Similarly as in the case $\alpha = 0$, we first observe that for a constant $a > 0$ both K_- and K_+ diverge in probability to ∞ , which intuitively is clear if only the df F is strictly increasing in a neighborhood of the α th quantile.

Proposition 5.1. *Let $\alpha \in (0, 1)$ (which implies $k_n \rightarrow \infty$) and there exists γ which satisfies $F(\gamma) = \alpha$ and F be a df continuous and strictly increasing in a neighborhood of γ . Then, for any fixed $a > 0$*

$$K_-(n, k_n, a) \xrightarrow{P} \infty, \quad K_+(n, k_n, a) \xrightarrow{P} \infty.$$

Proof. Without loss of generality we can assume that a is sufficiently small, i.e., $a + l_F < \gamma$. We rewrite (3) for $j = 0, 1, \dots, k_n - 1$ as

$$P(K_-(n, k_n, a) = j) = \int_{l_F + a}^{\infty} \binom{k_n - 1}{j} \left[\frac{F(x - a)}{F(x)} \right]^{k_n - j - 1} \left[1 - \frac{F(x - a)}{F(x)} \right]^j dF_{k_n:n}(x). \tag{15}$$

Let us denote

$$H(k_n, j, a, x) = \binom{k_n - 1}{j} \left[\frac{F(x - a)}{F(x)} \right]^{k_n - j - 1} \left[1 - \frac{F(x - a)}{F(x)} \right]^j.$$

Then we split (15) into three integrals:

$$P(K_-(n, k_n, a) = j) = \int_{l_F+a}^{x_1} H(k_n, j, a, x) dF_{k_n:n}(x) + \int_{x_1}^{x_2} H(k_n, j, a, x) dF_{k_n:n}(x) + \int_{x_2}^{\infty} H(k_n, j, a, x) dF_{k_n:n}(x) = L_1(j, n) + L_2(j, n) + L_3(j, n),$$

where $l_F + a < x_1 < \gamma < x_2$ and $j = 0, 1, \dots, k_n - 1$.

Note that for any $x > l_F + a$, $\frac{F(x-a)}{F(x)} \in (0, 1]$. Thus $H(k_n, j, a, x) = P(Y_n = j) \leq 1$, where Y_n is a rv with binomial distribution $b(k_n - 1, 1 - \frac{F(x-a)}{F(x)})$.

Since $X_{k_n:n} \xrightarrow{P} \gamma$, for any fixed $x_1 \in (l_F + a, \gamma)$ and $x_2 \in (\gamma, \infty)$ we have

$$0 \leq L_1(j, n) \leq \int_{l_F+a}^{x_1} dF_{k_n:n}(x) \leq P(X_{k_n:n} \leq x_1) \xrightarrow{n \rightarrow \infty} 0$$

and

$$0 \leq L_3(j, n) \leq \int_{x_2}^{\infty} dF_{k_n:n}(x) \leq P(X_{k_n:n} \geq x_2) \xrightarrow{n \rightarrow \infty} 0.$$

Now observe that $x_1 \leq x \leq x_2$ implies

$$\frac{F(x_1 - a)}{F(x_2)} \leq \frac{F(x - a)}{F(x)} \leq \frac{F(x_2 - a)}{F(x_1)}.$$

Consequently, for $j = 0, 1, \dots$

$$0 \leq L_2(j, n) \leq \binom{k_n - 1}{j} \left[\frac{F(x_2 - a)}{F(x_1)} \right]^{k_n - j - 1} \left[1 - \frac{F(x_1 - a)}{F(x_2)} \right]^j \int_{x_1}^{x_2} dF_{k_n:n}(x) \leq k_n^j \cdot d^{k_n} \xrightarrow{n \rightarrow \infty} 0,$$

where $d = \frac{F(x_2 - a)}{F(x_1)}$. In order to have $d < 1$ we choose $x_1 < \gamma < x_2$ in such a way that $x_2 - a < x_1$ and $x_2 - a$ is in the neighborhood of γ in which F is strictly increasing.

The result for $K_+(n, k_n, a)$ follows by the duality between K_- and K_+ indicated in Sec. 2. □

Theorem 5.1. *Let $\alpha \in (0, 1)$, there exists γ which satisfies $F(\gamma) = \alpha$ and F be continuous df, strictly increasing in a neighborhood of γ . The sequence $a_n \rightarrow 0^+ (n \rightarrow \infty)$ is defined by $a_n = \gamma - F^+(\alpha - \lambda/n)$ for some $\lambda > 0$. Assume that*

$$\lim_{(x,y) \rightarrow (\gamma, 0^+)} \frac{F(x) - F(x - y)}{F(\gamma) - F(\gamma - y)} = 1. \tag{16}$$

Then,

$$K_-(n, k_n, a_n) \xrightarrow{d} \mathcal{P}(\lambda).$$

Proof. Let us denote by V_γ a vicinity of γ in which F is strictly increasing. From the proof of Proposition 5.1, we know that for $j = 0, 1, \dots, k_n - 1$, and n

sufficiently large

$$P(K_-(n, k_n, a_n) = j) = \int_{l_F+a_n}^{x_1} H(k_n, j, a_n, x) dF_{k_n:n}(x) + \int_{x_1}^{x_2} H(k_n, j, a_n, x) dF_{k_n:n}(x) + \int_{x_2}^{\infty} H(k_n, j, a_n, x) dF_{k_n:n}(x) = L_1(j, n) + L_2(j, n) + L_3(j, n),$$

where $H(k_n, j, a_n, x)$ is defined in that proof, and $x_1, x_2 \in V_\gamma, l_F + a_n < x_1 < \gamma < x_2$, to be chosen later. As in the proof of Proposition 5.1 we get

$$L_1(j, n) \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad L_3(j, n) \xrightarrow{n \rightarrow \infty} 0.$$

Now we will show that

$$L_2(j, n) \xrightarrow{n \rightarrow \infty} \frac{\lambda^j}{j!} e^{-\lambda} \quad \text{for } j = 0, 1, \dots$$

First observe that by assumption (16)

$$\forall \varepsilon > 0 \exists (x_1, x_2) \subset V_\gamma, x_1 < \gamma < x_2 \exists N \forall n \geq N \forall x \in (x_1, x_2) \left| \frac{F(x) - F(x - a_n)}{F(\gamma) - F(\gamma - a_n)} \cdot \frac{F(\gamma)}{F(x)} - 1 \right| < \varepsilon,$$

which implies

$$\left| \frac{F(x - a_n)}{F(x)} - \frac{F(\gamma - a_n)}{F(\gamma)} \right| < \varepsilon \cdot \frac{\lambda}{\alpha n}.$$

Thus, for $x \in (x_1, x_2)$ and $n \geq N$, we have

$$1 - \frac{\lambda(1 + \varepsilon)}{\alpha n} \leq \frac{F(x - a_n)}{F(x)} \leq 1 - \frac{\lambda(1 - \varepsilon)}{\alpha n}.$$

Consequently for $j = 0, 1, \dots$,

$$L_2(j, n) \leq \binom{k_n - 1}{j} \left[1 - \frac{\lambda(1 - \varepsilon)}{\alpha n} \right]^{k_n - j - 1} \left[\frac{\lambda(1 + \varepsilon)}{\alpha n} \right]^j \int_{x_1}^{x_2} dF_{k_n:n}(x) \xrightarrow{n \rightarrow \infty} \frac{[\lambda(1 + \varepsilon)]^j}{j!} e^{-\lambda(1 + \varepsilon)}$$

and

$$L_2(j, n) \geq \binom{k_n - 1}{j} \left[1 - \frac{\lambda(1 + \varepsilon)}{\alpha n} \right]^{k_n - j - 1} \left[\frac{\lambda(1 - \varepsilon)}{\alpha n} \right]^j \int_{x_1}^{x_2} dF_{k_n:n}(x) \xrightarrow{n \rightarrow \infty} \frac{[\lambda(1 - \varepsilon)]^j}{j!} e^{-\lambda(1 - \varepsilon)}.$$

By letting $\varepsilon \rightarrow 0$ we obtain $L_2(j, n) \xrightarrow{n \rightarrow \infty} \frac{\lambda^j}{j!} e^{-\lambda}$ for $j = 0, 1, \dots$, and the proof is complete. □

Again by duality between K_- and K_+ , the respective result concerning the limit behavior of $K_+(n, k_n, a_n)$ follows directly from the preceding theorem.

Theorem 5.2. *Let $\alpha \in (0, 1)$, there exists γ such that $F(\gamma) = \alpha$ and F be a df continuous and strictly increasing in a neighborhood of γ . The sequence $a_n \rightarrow 0^+$ ($n \rightarrow \infty$) is defined by $a_n = F^{\leftarrow}(\alpha + \lambda/n) - \gamma$ for some $\lambda > 0$. Assume that*

$$\lim_{(x,y) \rightarrow (\gamma, 0^+)} \frac{F(x+y) - F(x)}{F(\gamma+y) - F(\gamma)} = 1. \tag{17}$$

Then,

$$K_+(n, k_n, a_n) \xrightarrow{d} \mathcal{P}(\lambda).$$

Remark 5.1. If we assume that $F'(x) = p(x)$ exists in a vicinity of γ , $p(\gamma) \neq 0$ and $p(x)$ is continuous at γ , then conditions (16) and (17) are satisfied.

6. Approximations for K_+ when $\alpha = 1$

As it was mentioned in the introduction, the limiting behavior of $K_-(n, n - k + 1, a_n)$ has been intensively studied in the literature. We failed to obtain analogs of our earlier results for K_- in the case $\alpha = 1$, sequences (k_n) different from $(n - k + 1)$ and $r_F = \infty$. Note that for $r_F < \infty$ the respective results for both K_- and K_+ can be obtained by duality from theorems we proved in Sec. 3. The only result for $\alpha = 1$ and $r_F = \infty$ needs a somewhat different definition (as it was also in the exponential example, Sec. 3) of the sequence (a_n) than the one we exploited up to now.

Theorem 6.1. *Let the df F be continuous, $r_F = \infty$, $\alpha = 1$ and $m_n := n - k_n \rightarrow \infty$. Let (a_n) be a sequence of positive numbers satisfying the condition*

$$\lim_{x \rightarrow \infty} m_n \left(1 - \frac{1 - F(x + a_n)}{1 - F(x)} \right) = \lambda$$

uniformly in n . Assume that

$$\lim_{(x,y) \rightarrow (\infty, 0^+)} \frac{1 - F(x+y)}{1 - F(x)} = 1.$$

Then,

$$K_+(n, k_n, a_n) \xrightarrow{d} \mathcal{P}(\lambda).$$

Proof. By (6) we have

$$\begin{aligned} P(K_+(n, k_n, a_n) = j) &= \int_{I_F} \binom{n - k_n}{j} \left[1 - \frac{1 - F(x + a_n)}{1 - F(x)} \right]^j \left[\frac{1 - F(x + a_n)}{1 - F(x)} \right]^{n - k_n - j} dF_{k_n, n}(x) \\ &\quad + \int_{x_0}^{\infty} \binom{n - k_n}{j} \left[1 - \frac{1 - F(x + a_n)}{1 - F(x)} \right]^j \left[\frac{1 - F(x + a_n)}{1 - F(x)} \right]^{n - k_n - j} dF_{k_n, n}(x) \\ &= M_1 + M_2 \end{aligned}$$

for $j = 0, 1, \dots, n - k_n$ and for any $x_0 > l_F$. We can now proceed analogously to the proof of Theorem 4.3 or 5.1 showing that $\forall_{\epsilon > 0} \exists_{x_0}$ large enough and such that

$$0 \leq M_1 \leq \int_{l_F}^{x_0} dF_{k_n:n}(x) \leq P(X_{k_n:n} \leq x_0) \xrightarrow{n \rightarrow \infty} 0,$$

$$M_2 \leq \binom{m_n}{j} \left[\frac{\lambda(1+\epsilon)}{m_n} \right]^j \left[1 - \frac{\lambda(1-\epsilon)}{m_n} \right]^{m_n-j} \int_{x_0}^{\infty} dF_{k_n:n}(x) \xrightarrow{n \rightarrow \infty} \frac{[\lambda(1+\epsilon)]^j}{j!} e^{-\lambda(1+\epsilon)}$$

and

$$M_2 \geq \binom{m_n}{j} \left[\frac{\lambda(1-\epsilon)}{m_n} \right]^j \left[1 - \frac{\lambda(1+\epsilon)}{m_n} \right]^{m_n-j} \int_{x_0}^{\infty} dF_{k_n:n}(x) \xrightarrow{n \rightarrow \infty} \frac{[\lambda(1-\epsilon)]^j}{j!} e^{-\lambda(1+\epsilon)}$$

for $j = 0, 1, \dots$. Letting $\epsilon \rightarrow 0$ ends the proof. \square

Remark 6.1. Note that if $k_n/n \rightarrow \alpha \in (0, 1)$ then $m_n \rightarrow \infty$. For γ such that $F(\gamma) = \alpha$ (assuming F invertible in a neighborhood of γ) define (a_n) by

$$\lim_{x \rightarrow \gamma} \frac{1 - F(x + a_n)}{1 - F(x)} = 1 - \frac{\lambda}{m_n}.$$

Then the condition

$$\lim_{(x,y) \rightarrow (\gamma, 0^+)} \frac{1 - F(x+y)}{1 - F(x)} = 1$$

implies that $K_+(n, k_n, a_n)$ converges to the Poisson distribution $\mathcal{P}(\lambda)$. The argument is analogous to the one given above or in Sec. 5.

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