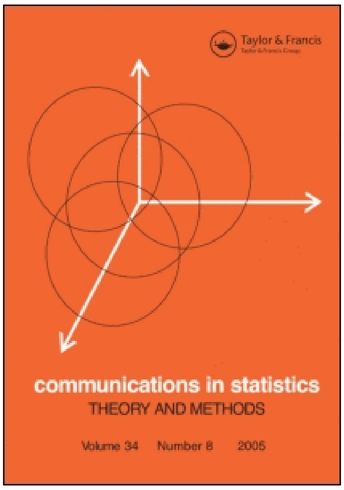

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Characterization Problems Related to the Shepp Property

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This article studies the relationship between the distribution of $Z = \frac{2XY}{\sqrt{X^2+Y^2}}$ and the common distribution of X, Y when X, Y are i.i.d. random variables. Results concern the identifiability of the distribution of Z by that of X, and the possibility that Z and X have the same distribution.

Keywords Characterization of distributions; Functional equations for characteristic functions; Normal distribution.

Mathematics Subject Classification Primary 62E10; Secondary 60E10.

1. Introduction

Let ψ be the transformation of the plane into itself that doubles the polar angle θ, and keeps the radius r unchanged, i.e., ψ is defined by $(r, \theta) \to (r, 2\theta)$. In terms of the variables $(x = r\cos(\theta), y = r\sin(\theta))$ we have

$$\psi(x, y) = \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}\right),$$

with $\psi(0, 0) = (0, 0)$ defined by continuity. Let (X, Y) be a random vector. We use ψ to define a new random vector (W, Z) by $(W, Z) = \psi(X, Y)$, that is

$$W := \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}, \quad Z := \frac{2XY}{\sqrt{X^2 + Y^2}}. \tag{1}$$

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Shepp (1964) observed that if X, Y are independent identically distributed (i.i.d.) standard normal ($\mathcal{N}(0, 1)$) random variables (r.v's) then W, Z are also independent standard normal. For future reference we name this fact the Shepp property.

Note that the Shepp property follows easily from the fact that if X and Y are i.i.d. standard normal, then the random polar angle Θ has a uniform distribution on $(0, 2\pi)$ and then $2\Theta(\mathrm{mod}\,(2\pi))$ is also uniform in $(0, 2\pi)$.

For related converse results see, for instance, Beer and Lukacs (1973), Bansal et al. (1999), and Hamedani and Volkmer (2001). In particular, Bansal et al. (1999) used the fact Z alone is standard normal to characterize the normal distribution. The proof of their result is based on the observation that

$$\frac{4}{Z^2} = \frac{1}{X^2} + \frac{1}{Y^2}$$

which relates the distribution of Z to a sum of i.i.d r.v's. An investigation of the distribution of W seems to be more difficult since apparently there is no such a connection. Moreover, W is defined in terms of squares of X and Y only, so there is no way to extract the distribution of X or Y from that of W without additional knowledge.

In Bansal et al. (1999), Corollary 2.4, it is claimed that X, Y i.i.d. r.v's and $\alpha W + \beta Z \sim N(0, 1)$ for arbitrary and fixed $\alpha \geq 0$, $\beta \geq 0$, $\alpha^2 + \beta^2 = 1$ imply that $X \sim N(0, 1)$. In view of the above observation, this statement is false when $\alpha = 1$. We conjecture that it is false always except for the case $\beta = 1$.

On the other hand, Misiewicz and Wesołowski (2005) have proved recently that, essentially, for any random vector (X, Y) (the components are not necessarily independent or identically distributed) the distribution of (X, Y) is invariant under ψ iff it is rotationally invariant. Note that under additional assumption of independence of X and Y it implies that X and Y are standard normal. Also, it is shown in that article that independence of X and Y and independence of W and Z, under some technical smoothness assumptions, imply that X and Y are standard normal.

In this article we study properties of Z alone. In Sec. 2 we present an identifiability result being a straightforward generalization of the main result of Bansal et al. (1999). In Sec. 3, analytical approach is developed which is helpful in studying the structure of the distribution of X and Y in more depth. The tools developed in Section 3 are exploited intensively in Sec. 4 where the property of identical distribution of X and Z is investigated.

2. Identifiability

In this section we prove that any symmetric distribution of Z, not just the standard normal one, uniquely determines the distribution of the parent variables X and Y. Moreover, if the symmetry assumption of the distribution of Z is dropped, then the distribution of X is determined by that of Z up to reflection about zero.

Theorem 2.1. *Let X and Y be i.i.d. r.v's. Then the distribution of Z determines the distribution of X up to reflection about* 0.

Proof. Let $U = X^{-2} + Y^{-2}$ if $XY \neq 0$ and $U = 0$ if $XY = 0$. For any $t > 0$,

$$P(\text{sgn}(XY) = 1, 0 < U < t^2) = P(2/t < Z < \infty) = 1 - F(2/t) \tag{2}$$

and

$$P(\text{sgn}(XY) = -1, 0 < U < t^2) = P(Z < -2/t) = F(-2/t^-), \tag{3}$$

where F is the distribution function of Z. Thus the distribution of $(\text{sgn}(XY), U)I_{\mathbf{R}\setminus\{0\}}(XY)$ is defined in terms of F (I_A denotes the indicator function of the set A). Consequently, the function

$$h(k, x) = E\big([\text{sgn}(XY)]^k e^{xU} I_{\mathbf{R}\setminus\{0\}}(XY)\big), \quad k = 0, 1, \ldots; \; x \leq 0,$$

is uniquely determined by F. But the independence assumption implies

$$h(k, x) = H(k, x)^2, \tag{4}$$

where

$$H(k, x) := E\big([\text{sgn}(X)]^k e^{x/X^2} I_{\mathbf{R}\setminus\{0\}}(X)\big).$$

Thus, for any $k = 0, 1, \ldots,$ $x \leq 0$, the value of the function $H(k, x)$ is defined up to a sign. Note that we need to consider only $k = 0, 1$, since $H(0, x)$ is positive, then unique. Since $H(1, \cdot)$ as a function of x is analytic for $x \leq 0$, it follows that there are only two versions of $H(1, \cdot)$, one for X and the other one for $-X$. Moreover, H determines uniquely the joint distribution of $(\text{sgn}(X), \exp(-X^{-2}))I_{\mathbf{R}\setminus\{0\}}(X)$ by identification of joint moments. Thus, H uniquely determines the joint distribution of $(\text{sgn}(X), |X|)I_{\mathbf{R}\setminus\{0\}}(X)$. Finally, we conclude that $(\text{sgn}(X), |X|)I_{\mathbf{R}\setminus\{0\}}(X)$ or $(\text{sgn}(-X), |-X|)I_{\mathbf{R}\setminus\{0\}}(-X)$ is identified by F.

Note that

$$P(\text{sgn}(XY) = 0) = P(Z = 0) = F(0) - F(0^-).$$

On the other hand,

$$P(\text{sgn}(XY) = 0) = 2P(X = 0) - (P(X = 0))^2.$$

Consequently,

$$P(X = 0) = 1 - \sqrt{1 - F(0) + F(0^-)}. \tag{5}$$

Finally, we conclude that the distribution of $(\text{sgn}(X), |X|)$ or $(\text{sgn}(-X), |-X|)$ is determined by F, implying that the distribution of X or $-X$ is unique. □

The characterization of the normal law from Bansal et al. (1999), we referred to in Sec. 1, follows now directly from Theorem 2.1.

Theorem 2.2 (NC1, Bansal et al., 1999)**.** *The random variable Z has the standard normal distribution iff X has the standard normal distribution.*

Proof. We need to prove sufficiency only. This follows immediately from Theorem 2.1 using the necessity part and the fact that the standard normal distribution is symmetric. □

It is also an immediate consequence of the following identifiability result which exploits symmetry of Z additionally.

Theorem 2.3. *Let X and Y be i.i.d. r.v's. If the distribution of Z is symmetric and has no atom at* 0, *then it uniquely determines the distribution of X.*

Proof. We assume that Z is symmetric and use (2) and (3) of the proof of Theorem 2.1. By symmetry it follows that

$$P(\operatorname{sgn}(XY) = 1, U < t^2) = 1 - F(2/t) = P(\operatorname{sgn}(XY) = -1, U < t^2), \quad t > 0. \tag{6}$$

Taking $t \to \infty$ in (6), we have $P(\operatorname{sgn}(XY) = \pm 1) = 1 - F(0) = 1/2$. Moreover, from (6) it follows that for any $t > 0$

$$P(U < t^2) = P(\operatorname{sgn}(XY) = 1, U < t^2) + P(\operatorname{sgn}(XY) = -1, U < t^2) = 2(1 - F(2/t)).$$

Thus $\operatorname{sgn}(XY)$ and U are independent.

For the function h, introduced in the proof of Theorem 2.1, this independence property implies $h(k, x) = h(k, 0)h(0, x)$ for any $k = 0, 1, \ldots, x \leq 0$. Note that by (5) the distribution of X also has no atom at 0. Then (4) yields independence of $\operatorname{sgn}(X)$ and $|X|$. Consequently, X is symmetric. Together with the conclusion of Theorem 2.1 it proves the result. □

3. Analytical Approach

The distribution of X determines the distribution of Z. This defines a mapping $T : \mathscr{M} \to \mathscr{M}$, where $\mathscr{M}$ is the set of all probability measures μ on the σ-algebra $\mathscr{B}$ of Borel subsets of $\mathbf{R}$ with $\mu\{0\} = 0$. (We may also consider measures with $\mu\{0\} > 0$ but then the following becomes a little more technical.) It means that for any $B \in \mathscr{B}$

$$T\mu(B) = \mu \times \mu\left(\left\{(x, y) \in \mathbf{R}^2 : \frac{2xy}{\sqrt{x^2 + y^2}} \in B\right\}\right).$$

We want to study properties of this mapping T.

If X is a random variable with symmetric distribution $\mu \in \mathscr{M}$ then we let $\mu^\dagger$ be the distribution of X^{-2}, that is,

$$\mu^\dagger(B) = 2\mu\{x > 0 : x^{-2} \in B\}, \quad B \in \mathscr{B}.$$

Note that $\mu^\dagger(-\infty, 0] = 0$. We extend this definition to all finite signed measures μ on $\mathscr{B}$ with $\mu\{0\} = 0$. If μ has a density $f : \mathbf{R} \to \mathbf{R}$, then $\mu^\dagger$ has density

$$f^\dagger(u) := \begin{cases} 2u^{-3/2} f(u^{-1/2}) & \text{if } u > 0, \\ 0 & \text{if } u \leq 0. \end{cases}$$

For $\mu \in \mathscr{M}$, we introduce their symmetric and anti-symmetric parts

$$\mu_1(B) = \frac{1}{2}(\mu(B) + \mu(-B)), \quad \mu_2(B) = \frac{1}{2}(\mu(B) - \mu(-B)).$$

It is clear that μ_1 is also a probability measure but, in general, μ_2 will be a signed measure.

Theorem 3.1. *Let $\nu = T\mu$ with $\mu \in \mathscr{M}$. Let ϕ_j be the Fourier transform of $\mu_j^\dagger$ and ψ_j the Fourier transform of $\nu_j^\dagger$, $j = 1, 2$. Then*

$$\psi_j(4s) = \left(\phi_j(s)\right)^2 \quad \textit{for } \operatorname{Im} s \geq 0, \ \ j = 1, 2,$$

where $\operatorname{Im} s$ *denotes the imaginary part of s.*

In particular, $T\mu = \mu$ if and only if

$$\phi_j(4s) = \phi_j^2(s) \ \ \textit{for } \operatorname{Im} s \geq 0, \ \ j = 1, 2. \tag{7}$$

Proof. For $u > 0$, let

$$D_u = \left\{(x, y) : x > 0, y > 0, 2xy > u\sqrt{x^2 + y^2}\right\},$$
$$E_u = \left\{(x, y) : x < 0, y > 0, 2xy < -u\sqrt{x^2 + y^2}\right\}.$$

Then

$$\nu(u, \infty) = \mu \times \mu(D_u \cup (-D_u)),$$
$$\nu(-\infty, -u) = \mu \times \mu(E_u \cup (-E_u)).$$

Note that the symmetric and anti-symmetric parts of $\mu \times \mu$ are $\mu_1 \times \mu_1 + \mu_2 \times \mu_2$ and $\mu_1 \times \mu_2 + \mu_2 \times \mu_1$, respectively. Therefore, we have

$$\begin{aligned} \nu(u, \infty) &= 2(\mu_1 \times \mu_1 + \mu_2 \times \mu_2)(D_u), \\ \nu(-\infty, -u) &= 2(\mu_1 \times \mu_1 + \mu_2 \times \mu_2)(E_u) \\ &= 2(\mu_1 \times \mu_1 - \mu_2 \times \mu_2)(D_u). \end{aligned}$$

Adding and subtracting, we obtain

$$\nu_1(u, \infty) = 2(\mu_1 \times \mu_1)(D_u),$$
$$\nu_2(u, \infty) = 2(\mu_2 \times \mu_2)(D_u).$$

Using

$$D_u = \left\{(x, y) : x, y > 0, x^{-2} + y^{-2} < 4u^{-2}\right\},$$

we obtain

$$2\nu_j(u, \infty) = (\mu_j^\dagger * \mu_j^\dagger)(-\infty, 4u^{-2}), \quad j = 1, 2,$$

where the asterisk denotes convolution of measures. This implies

$$\nu_j^\dagger(-\infty, \tfrac{1}{4}v) = (\mu_j^\dagger * \mu_j^\dagger)(-\infty, v), \quad v > 0, \ \ j = 1, 2.$$

Applying the Fourier transform gives the desired formulas. □

For example, consider the normal distribution μ with probability density function (pdf)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \sigma > 0. \tag{8}$$

Then $\mu_1 = \mu$, $\mu_2 = 0$ and

$$\phi_1(s) = \exp\left(-\frac{1}{\sigma}\sqrt{-2is}\right)$$

which satisfies (7). Therefore, $T\mu = \mu$ by Theorem 3.1.

Remark 3.1. Using the analytical tools developed above we can reprove Theorem 2.1 (under the assumption that Z has no atom at 0.) Let $\mu, \rho \in \mathscr{M}$ be such that $T\mu = T\rho$. Let ϕ_j be the Fourier transform of $\mu_j^\dagger$, and let χ_j be the Fourier transform of $\rho_j^\dagger$, $j = 1, 2$. By Theorem 3.1, we have $\phi_j^2(s) = \chi_j^2(s)$ for $\operatorname{Im} s \geq 0$ and $j = 1, 2$. Since ϕ_j, χ_j are analytic in the half-plane $\operatorname{Im} s > 0$, we obtain that $\phi_j = \chi_j$ or $\phi_j = -\chi_j$. If $j = 1$, then $\phi_1(0) = \chi_1(0) = 1$ so $\phi_1 = \chi_1$. If $\phi_2 = \chi_2$, then $\mu = \rho$, and if $\phi_2 = -\chi_2$, then $\mu(B) = \mu_1(B) + \mu_2(B) = \rho_1(B) - \rho_2(B) = \rho(-B)$.

Remark 3.2. If $T\mu$ is symmetric, then Theorem 3.1 shows that μ is symmetric, so $\mu = \rho$ by Remark 3.1. This gives an alternative proof of Theorem 2.3.

Similarly, using Theorem 3.1 one can prove the following results revealing more about the relations between the distributions of Z and X.

Theorem 3.2. *Let X, Y be i.i.d. random variables.*

(a) *The symmetric part of the distribution of X determines the symmetric part of the distribution of Z and conversely.*
(b) *The anti-symmetric part of the distribution of X determines the anti-symmetric part of the distribution of Z. Conversely, the anti-symmetric part of the distribution of Z determines the anti-symmetric part of the distribution of X up to reflection about* 0.

4. Additional Results on the Invariance Property

It is interesting to note that there are i.i.d. rv's X, Y which share their distribution with Z but which are not normally distributed. For example, let $\rho \in \mathscr{M}$ be defined by the density

$$g(x) = \begin{cases} 2f(x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

where f is the density (8) of the standard normal measure μ. Then $\rho_1 = \mu$ and $\rho_2(B) = \mu(B)$ if $B \subset (0, \infty)$. Therefore, $\rho_j^\dagger = \mu^\dagger$ for $j = 1, 2$ and so $T\rho = \rho$ by the second part of Theorem 3.1. In other words, if X, Y are i.i.d. rv's with distribution ρ then Z also has distribution ρ.

In the following theorem we describe all i.i.d. rv's X, Y which share their distribution with Z.

Theorem 4.1. *Let X, Y be i.i.d. symmetric rv's. Then X and $\frac{2XY}{\sqrt{X^2+Y^2}}$ have the same distribution iff the characteristic function ϕ of X^{-2} has the form*

$$\ln \phi(s) = \int_0^\infty (e^{isx} - 1)dM(x) \quad \textit{for all } s \in \mathbf{R},$$

where $\lambda : \mathbf{R} \to \mathbf{R}$ is left-continuous with period $\rho = \ln 4$ and

$$M(x) = -\lambda(-\ln x)x^{-1/2}$$

is non decreasing for $x > 0$.

Proof. Applying Theorem 3.1, our task consists in finding all complex-valued functions $\phi : \mathbf{R} \to \mathbf{C}$ with the following properties:

(A) ϕ is a characteristic function;
(B) $\phi(4s) = \phi^2(s)$ for all $s \in \mathbf{R}$;
(C) the distribution belonging to ϕ is supported on $[0, \infty)$.

We will use results due to Lukacs (1970) and Shimizu (1968) to show that ϕ satisfies (A), (B), (C) iff it is of the form given in the theorem. Characteristic functions satisfying (B) are infinitely divisible and thus have a canonical Levy representation. Using this representation, Shimizu (1968) found all characteristic functions ϕ that solve the functional equation $\phi(4s) = \phi^2(s)$. Actually, Shimizu (1968) considered a more general functional equation

$$\phi(s) = \phi(a_1 s) \dots \phi(a_p s)\phi(-a_{p+1} s) \dots \phi(-a_n s),$$

where $a_1, \dots, a_n$ lie in $(0, 1)$. We set $p = n = 2$, $a_1 = a_2 = \frac{1}{4}$ to specialize to our equation. Shimizu defines α as the solution of

$$a_1^\alpha + \dots + a_n^\alpha = 1.$$

In our case, we have $\alpha = \frac{1}{2}$. Shimizu showed that ϕ satisfies (A), (B) if and only if

$$\ln \phi(s) = \int_0^\infty (e^{isx} - 1)dM(x) + \int_{-\infty}^0 (e^{isx} - 1)dN(x),$$

where

$$M(x) = -\lambda(-\ln x)x^{-1/2}, \quad x > 0,$$
$$N(x) = \mu(-\ln |x|)|x|^{-1/2}, \quad x < 0,$$

M and N are non decreasing and $\lambda, \mu : \mathbf{R} \to \mathbf{R}$ are left-continuous functions with period ρ. So far we have used only (A) and (B). It follows from Theorem 11.2.2 in Lukacs (1970) that the distribution corresponding to ϕ is supported in $[0, \infty)$ (condition (C)) if and only if N is constant. This gives us the statement of the theorem. □

Below we present some discussion of the above result.

1. We see that $M(x) \to 0$ as $x \to \infty$. Thus, $M(x) \le 0$ for all $x > 0$. If $M(x_0) = 0$ for some $x_0 > 0$ then $M(x) = 0$ for $x \ge x_0$ which implies that $\lambda(u) = 0$ for all $u \in \mathbf{R}$ and $M(x) = 0$ for all $x > 0$. But then $\phi(s) = 1$ for all s and the corresponding distribution is concentrated at 0. All other distributions are supported on $(0, \infty)$. If we exclude $M = 0$ then there is $\delta > 0$ such that $\lambda(u) \ge \delta$ for all $u \in \mathbf{R}$ and $\lambda(u)$ is of bounded variation over each period. The latter follows from

$$\ln \lambda(u) = -\frac{1}{2}u + \ln(-M(e^{-u})), \quad u \in \mathbf{R}$$

which represents $\ln \lambda$ as a difference of non decreasing functions.

2. If we choose $\lambda(u) = c > 0$ constant we obtain

$$\begin{aligned}\ln \phi(s) &= c \int_0^\infty (e^{isx} - 1)\, d(-x^{-1/2}) \\ &= c\frac{1}{2}\int_0^\infty (e^{ixs} - 1)x^{-3/2}dx \\ &= -\sqrt{\sigma}\sqrt{\pi}c\end{aligned}$$

where $s = i\sigma$. We obtain the familiar characteristic function

$$\phi(s) = e^{-d\sqrt{\sigma}}, \quad d = c\sqrt{\pi}$$

which leads to the case of normal X and Y (see the paragraph preceding Remark 3.1.)

3. A possible nontrivial choice for λ is

$$\lambda(u) = 10 + \sin\frac{2\pi u}{\rho}.$$

One can easily check that the corresponding function M is non decreasing. However, we do not get an explicit formula for ϕ.

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