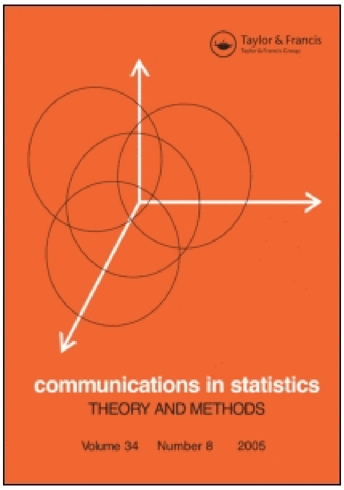

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Identification of Product Measures by Random Choice of Marginals

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Three independent random variables are transformed into a bivariate vector by choosing at random one of the variables from two pairs. It appears that such a transformation preserves all information about the parent product measure and the random choice mechanism. Moreover, the original distribution can be explicitly identified. Also identifiability under combinations of random choice with convolution, minimum, and maximum is considered.

Keywords Characterizations of probability distributions; Identifiability of statistical models; Random choice.

Mathematics Subject Classification Primary 62E10; Secondary 60E05.

1. Introduction

Consider independent random variables (rv's) X_0, X_1, X_2 (input), which are transformed into a bivariate vector by a mapping $\psi : \mathbf{R}^3 \to \mathbf{R}^2$, defined by coding functions ϕ_1 and ϕ_2 as $\psi(x_0, x_1, x_2) = (\phi_1(x_0, x_1), \phi_2(x_0, x_2))$. Thus, only the pair $(Y_1, Y_2) = \psi(X_0, X_1, X_2)$ (output) is observed. The first element of the pair depends only on X_0 and X_1, while the second depends only on X_0 and X_2. The problem, we are interested in, is to identify the distributions of the rv's X_0, X_1 and X_2 knowing that of (Y_1, Y_2). Obviously, such identification is not always possible. Usually it

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needs particular coding functions and some additional properties of the output distribution.

Note that the scheme described above fits also the bivariate latent variable model since the rv's Y_1 and Y_2 are conditionally independent given the "latent" variable X_0. Consequently, the question, we study here, can be viewed as an identifiability problem for a class of latent variable models.

Problems of this nature were considered mostly within the framework of characterization of probability distributions. Here the main contributor was I. Kotlarski. Among other authors we recall here Yu. Prokhorov, C.R. Rao, B.L.S. Prakasa Rao, and L. Klebanov. The results up to the early 1990s are thoroughly reviewed in the monograph Prakasa Rao (1992), especially in Chs. 2 and 3, which is also a source of valuable references. All these considerations were restricted to the following types of coding functions $\phi_i(x, y)$: $x + y$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, or xy, $i = 1, 2$, and their combinations. All of them fall into a semigroup scheme described in Kotlarski and Sasvari (1992), where the authors developed a general approach for the identifiability problem within these schemes. Since that time there has been no progress in the area, which resulted in an impression that identifiability for models with other coding functions could be impossible.

In the present article, we show that this is not the case. We consider a new coding scheme based on a simple random choice mechanism, which is outside the semigroup family considered in Kotlarski and Sasvari (1992): choose at random one of the rv's X_0 and X_1 for the first component of the output and, for the second component of the output, choose at random one of X_0 and X_2. We will show how to identify the original distribution of the input variables in this scheme in Sec. 2. A random choice setup will be combined with the standard coding functions, given above, in Sec. 3.

It is worth mentioning that a related problem of identifiability of random vectors with independent components by a random choice of one of the vectors has been studied recently in Hall and Zhou (2003) in the context of estimation of components in multivariate finite mixtures. They showed that in the bivariate case the model can be identified up to two-parameter family, while higher dimensional models are completely identifiable (under mild assumptions). They assumed additionally that the random choice mechanism is known, which is not the case in our setting. A similar problem was treated earlier in the framework of a latent variable model in Luboińska and Niemiro (1991).

2. Random Choice Coding for Both Components

This section is devoted to the situation in which the first coding function represents a random choice between X_0 and X_1 and the second coding function represents a random choice between X_0 and X_2. The random choice can be described in terms of Z_1 and Z_2 which are independent Bernoulli rv's, i.e., $Z_i \sim b(1, p)$, $i = 1, 2$, with $p \in (0, 1)$. We assume also that (Z_1, Z_2) and the input vector (X_0, X_1, X_2) are independent. Formally, the output of the coding is

$$(Y_1, Y_2) = (Z_1X_0 + (1 - Z_1)X_1,\ Z_2X_0 + (1 - Z_2)X_2). \tag{1}$$

Let H be the df of (Y_1, Y_2) and let F_i be the df of X_i, $i = 0, 1, 2$. Conditioning with respect to (Z_1, Z_2) we can express H as

$$H(x, y) = p^2 F_0(x \wedge y) + p(1-p)F_0(x)F_2(y) + p(1-p)F_1(x)F_0(y) + (1-p)^2 F_1(x)F_2(y) \tag{2}$$

for any real x and y.

Thus the marginal df's of Y_1 and Y_2 are, respectively,

$$H(x, \infty) = pF_0(x) + (1-p)F_1(x), \quad x \in \mathbf{R}, \tag{3}$$

and

$$H(\infty, y) = pF_0(y) + (1-p)F_2(y), \quad y \in \mathbf{R}. \tag{4}$$

We assume that the df H of (Y_1, Y_2) is known. In the next theorem we discuss the possibility of recovering from H the input df's F_i, $i = 0, 1, 2$, and the random choice probability p. A key role is played by the function G defined by

$$G(x, y) = H(x, y) - H(x, \infty)H(\infty, y), \quad x, y \in \mathbf{R} \tag{5}$$

and the following quantities $G(a, a) = \lim_{x \to a^+} G(x, x)$, $G(b^-, b^-) = \lim_{x \to b^-} G(x, x)$, and $G(a, b^-) = \lim_{(x,y) \to (a^+, b^-)} G(x, y)$.

Theorem 2.1. *Let X_0, X_1, X_2, Z_1 and Z_2 be non degenerate independent rv's and $Z_i \sim b(1, p)$, $i = 1, 2$ with unknown $p \in (0, 1)$ and let (1) hold. Let*

$$a = \inf\{x : G(x, x) > 0\} \quad \text{and} \quad b = \sup\{x : G(x, x) > 0\}.$$

Then $a < b$ are, respectively, the lower and upper end of the support of X_0. Moreover, the distributions of X_0, X_1, X_2, Z_1 and Z_2 are uniquely determined by the distribution of (Y_1, Y_2) in the following way:

(i) *if $G(a, a) = 0$ then $F_0(a) = 0$ and*

$$F_0(x) = 1 - \lim_{y \to a^+} \frac{G(x, y)}{G(y, y)}, \quad x \in (a, b),$$

(ii) *if $G(b^-, b^-) = 0$ then $F_0(b^-) = 1$ and*

$$F_0(x) = \lim_{y \to b^-} \frac{G(x, y)}{G(y, y)}, \quad x \in (a, b),$$

(iii) *if $G(a, a)G(b^-, b^-) \neq 0$ and $P(X_0 \in \{a, b\}) \neq 1$ then $F_0(a) > 0$, $F_0(b^-) < 1$ and*

$$F_0(x) = \frac{1 - \gamma_1}{1 - \gamma_1\gamma_2} \frac{G(x, b^-)}{G(b^-, b^-)} = 1 - \frac{1 - \gamma_2}{1 - \gamma_1\gamma_2} \frac{G(x, a)}{G(a, a)}, \quad x \in (a, b)$$

with

$$\gamma_1 = \frac{G(a, b^-)}{G(a, a)} \quad \text{and} \quad \gamma_2 = \frac{G(a, b^-)}{G(b^-, b^-)}.$$

In any of the three previous cases,

$$p = \sqrt{\frac{G(x, x)}{F_0(x)(1 - F_0(x))}} = const, \quad \forall x \in (a, b),$$

$$F_1(x) = \frac{H(x, \infty) - pF_0(x)}{1 - p}, \quad F_2(x) = \frac{H(\infty, x) - pF_0(x)}{1 - p}, \ x \in \mathbf{R}.$$

Proof. Solve (3) for $F_1(x)$ and (4) for $F_2(y)$ and substitute these quantities into (2). After some easy algebra we have

$$p^2 [F_0(x \wedge y) - F_0(x)F_0(y)] = G(x, y), \quad x, y \in \mathbf{R}. \tag{6}$$

Inserting $y = x$ in (6) we get

$$p^2 F_0(y)(1 - F_0(y)) = G(y, y), \quad y \in \mathbf{R}. \tag{7}$$

As X_0 is non degenerate, it follows from (7) that $a < b$ and also that $G(y, y) > 0$ for $y \in (a, b)$. Additionally, we conclude that a and b are lower and upper end of the support of the distribution of X_0, respectively.

For $a < x < y < b$ we get from (6) that

$$p^2 F_0(x)(1 - F_0(y)) = G(x, y). \tag{8}$$

Combining (7) with (8) we get

$$F_0(x) = F_0(y)\frac{G(x, y)}{G(y, y)} \quad \text{for } a < x < y < b. \tag{9}$$

Similarly, for $a < y < x < b$ we obtain from (6)

$$p^2 F_0(y)(1 - F_0(x)) = G(x, y), \tag{10}$$

and dividing (10) by (7), after easy algebra we get

$$F_0(x) = 1 - (1 - F_0(y))\frac{G(x, y)}{G(y, y)} \quad \text{for } a < y < x < b. \tag{11}$$

We consider now three different cases:

(i) Suppose $G(a, a) = 0$. From (7) it follows that $F_0(a) = 0$. Taking the limit as $y \to a^+$ in (11) we get

$$F_0(x) = 1 - \lim_{y \to a^+} \frac{G(x, y)}{G(y, y)} \quad \text{for } x \in (a, b).$$

(ii) Suppose $G(b^-, b^-) = 0$. From (7) it follows that $F_0(b^-) = 1$ and taking the limit as $y \to b^-$ in (9) we obtain

$$F_0(x) = \lim_{y \to b^-} \frac{G(x, y)}{G(y, y)} \quad \text{for } x \in (a, b).$$

(iii) Suppose $G(a,a)G(b^-,b^-) \neq 0$ and $P(X_0 \in \{a,b\}) < 1$. Then from (7) we have

$$G(a,a) = \lim_{x \to a^+} G(x,x) = p^2 F_0(a)(1 - F_0(a)) \neq 0 \tag{12}$$

and

$$G(b^-,b^-) = \lim_{y \to b^-} G(y,y) = p^2 F_0(b^-)(1 - F_0(b^-)) \neq 0\,. \tag{13}$$

Expressions (12) and (13) imply that F_0 has jumps at the upper and lower ends of the support. From (6) we have

$$G(a,b^-) = \lim_{(x,y) \to (a^+,b^-)} G(x,y) = p^2 F_0(a)(1 - F_0(b^-)) \neq 0\,. \tag{14}$$

Dividing (14) by (12) and separately by (13) we, respectively, get

$$\gamma_1 = \frac{G(a,b^-)}{G(a,a)} = \frac{1 - F_0(b^-)}{1 - F_0(a)} \quad \text{and} \quad \gamma_2 = \frac{G(a,b^-)}{G(b^-,b^-)} = \frac{F_0(a)}{F_0(b^-)}. \tag{15}$$

Since $P(X_0 \in \{a,b\}) < 1$ then $F(a) \neq F(b^-)$ and thus $\gamma_1, \gamma_2 \in (0,1)$. From (15) we obtain

$$F_0(a) = \gamma_2 \frac{1 - \gamma_1}{1 - \gamma_1\gamma_2} \quad \text{and} \quad F(b^-) = \frac{1 - \gamma_1}{1 - \gamma_1\gamma_2}.$$

Taking the limit as $y \to a^+$ in (11) we get

$$F_0(x) = 1 - \frac{1 - \gamma_2}{1 - \gamma_1\gamma_2} \frac{G(x,a)}{G(a,a)}, \quad x \in (a,b).$$

Alternatively, taking the limit as $y \to b^-$ in (9) we have

$$F_0(x) = \frac{1 - \gamma_1}{1 - \gamma_1\gamma_2} \frac{G(x,b^-)}{G(b^-,b^-)}, \quad x \in (a,b).$$

Thus the form of F_0 is as in the formulation of the result. The expression for p follows now from (7). Finally, the df's F_1 and F_2 are recovered from (3) and (4), respectively. □

Let $P(X_0 \in \{a,b\}) = 1$. Note that (12) implies $G(a,a) \leq p^2/4$. Moreover, solving (12) for $F_0(a)$ we get

$$F_0(a) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{p^2} G(a,a)}\right) = F_0(x) = F_0(b^-) < 1, \quad x \in (a,b).$$

However, the parameter p cannot be identify.

Below we provide an example to illustrate the fact that though the formulas in Theorem 1 look somewhat complicated, in special cases they can work nicely.

Example 2.1. Let

$$H(x, y) = \begin{cases} 0, & (x, y) \notin [0, \infty)^2, \\ (1 - e^{-x})\left(1 - \frac{3}{4}e^{-y}\right), & 0 < x \le y, \\ (1 - e^{-y})\left(1 - \frac{3}{4}e^{-x}\right), & 0 < y \le x. \end{cases}$$

Then $H(x, \infty) = H(\infty, x) = (1 - e^{-x})I_{(0,\infty)}(x)$ and

$$G(x, y) = \frac{1}{4}\big(1 - e^{-(x\wedge y)}\big)e^{-(x\vee y)}I_{(0,\infty)^2}(x, y).$$

Consequently,

$$G(x, x) = \frac{1}{4}e^{-x}(1 - e^{-x})I_{(0,\infty)}(x).$$

Thus $a = 0$, $b = \infty$ and both the assumptions (i) and (ii) are satisfied. For instance, using (ii), we get

$$F_0(x) = \lim_{y\to\infty} \frac{G(x, y)}{G(x, x)} = 1 - e^{-x}$$

for any $x > 0$. Hence, X_0 is an exponential rv with the mean 1. Further,

$$p = \sqrt{\frac{\frac{1}{4}e^{-x}(1 - e^{-x})}{(1 - e^{-x})e^{-x}}} = \frac{1}{2}$$

for any $x > 0$. Finally,

$$F_1(x) = F_2(x) = 2H(\infty, x) - F_0(x) = (1 - e^{-x})I_{(0,\infty)}(x).$$

Assume now that in the scheme considered in Theorem 2.1 the rv's Z_1 and Z_2 are independent binomial, but with different success probabilities, i.e., $Z_1 \sim b(1, p_1)$ and $Z_2 \sim b(1, p_2)$ with p_1, p_2 are unknown. Then (6) changes into

$$p_1p_2\,[F_0(x \wedge y) - F_0(x)F_0(y)] = G(x, y), \quad x, y \in \mathbf{R}.$$

Hence, repeating the first part of the argument from the proof above we get the same formula for F_0 as in Theorem 2.1. However, now p_1 and p_2 cannot be separated, i.e., we can identify only the product

$$p_1p_2 = \frac{G(x, x)}{F_0(x)(1 - F_0(x))} = const.$$

Consequently, also the df's F_1 and F_2 are not identifiable.

3. Combining Random Choice with Standard Transformations

In this section we consider transformations of the following type: for the first element of the output we take the random choice as defined in Sec. 2, while for the second element of the output we take separately each of the three standard semigroup transformations $x + y$, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

We begin with the coding function $x + y$ for the second component. Then the distributions of X_0, X_1, and X_2 will be identified through their characteristic functions (chf's).

Theorem 3.1. *Assume that X_0, X_1, X_2, and Z are independent non degenerate rv's and $Z \sim b(1, p)$, where $p \in (0, 1)$ is unknown. Denote by ϕ_i the chf of X_i, $i = 1, 2$. Let*

$$(Y_1, Y_2) = (ZX_0 + (1 - Z)X_1,\ X_0 + X_2). \tag{16}$$

Denote by ψ the chf of the random vector (Y_1, Y_2) and assume that

$$\psi(s, t) - \psi(s, 0)\psi(0, t) \neq 0 \tag{17}$$

for any $s, t \neq 0$.

Then the distributions of X_0, X_1, X_2, and Z are uniquely determined by the distribution of (Y_1, Y_2). More precisely:

The limit

$$g(t) = \lim_{s\to 0} \frac{\psi(t, s) - \psi(t, 0)\psi(0, s)}{\psi(s, t) - \psi(s, 0)\psi(0, t)}$$

exists for any $t \neq 0$ and

$$\phi_0(t) = \psi(0, t)g(t), \quad t \in \mathbf{R},$$

with $g(0) = 1$. Moreover,

$$p = g(t)\frac{\psi(s, t) - \psi(s, 0)\psi(0, t)}{\phi_0(s + t) - \phi_0(s)\phi_0(t)} = const$$

for any $s, t \neq 0$. Finally,

$$\phi_1(t) = \frac{\psi(t, 0) - p\phi_0(t)}{1 - p}, \quad \psi_2(t) = \frac{1}{g(t)}, \quad t \in \mathbf{R}.$$

Proof. Using conditioning with respect to Z and independence we can rewrite (16) in terms of chf's as

$$\psi(s, t) = p\phi_0(s + t)\phi_2(t) + (1 - p)\phi_0(t)\phi_1(s)\phi_2(t), \quad s, t \in \mathbf{R}. \tag{18}$$

Plugging $s = 0$ and $t = 0$ in (18) we obtain, respectively,

$$\phi_0(t)\phi_2(t) = \psi(0, t), \quad t \in \mathbf{R}, \tag{19}$$

and

$$p\phi_0(s) + (1-p)\phi_1(s) = \psi(s, 0), \quad s \in \mathbf{R}. \tag{20}$$

Substituting (19) and (20) into (18), after some elementary algebra, we get

$$\phi_0(t)\,[\psi(s,t) - \psi(s,0)\psi(0,t)] = p\,[\phi_0(s+t) - \phi_0(s)\phi_0(t)]\,\psi(0,t) \tag{21}$$

for any real s and t. Exchanging the role of s and t in (21) gives together with (21) a system of equations which leads to

$$\phi_0(t)\psi_0(0,s) = \phi_0(s)\psi(0,t)\frac{\psi(t,s) - \psi(t,0)\psi(0,s)}{\psi(s,t) - \psi(s,0)\psi(0,t)}, \quad s, t \in \mathbf{R}. \tag{22}$$

Note that the limit of the left-hand side of (22) as $s \to 0$ is $\phi_0(t)$. Consequently the limit of the right-hand side of (22) exists, which further means that g is correctly defined in Theorem 3.1 and the suitable representation for ϕ_0 holds.

The formula for p follows now immediately from (21). The chf's ϕ_1 and ϕ_2 are recovered directly from (20) and (19), respectively. □

To illustrate, how one can actually use Theorem 3.1 to decipher the distribution of the input variables, we provide the following example.

Example 3.1. Let the chf of (Y_1, Y_2) be of the form

$$\psi(s,t) = \frac{1}{2}\exp\left(-\frac{s^2}{2} - t^2\right)(\exp(-st) + 1), \quad s, t \in \mathbf{R}.$$

Then $\psi(s,0) = \exp(-s^2/2)$, $\psi(0,t) = \exp(-t^2)$ and

$$\psi(s,t) - \psi(s,0)\psi(0,t) = \frac{1}{2}\exp\left(-\frac{s^2}{2} - t^2\right)(\exp(-st) - 1) \neq 0, \quad s, t \in \mathbf{R}\backslash\{0\}.$$

And so the assumptions of Theorem 3.1 are satisfied. Thus

$$g(t) = \lim_{s\to 0}\frac{\exp(-s^2 - t^2/2)}{\exp(-t^2 - s^2/2)} = e^{t^2/2}.$$

Now,

$$\phi_0(t) = e^{-t^2}g(t) = e^{-t^2/2}, \quad t \in \mathbf{R},$$

i.e., X_0 is a standard normal rv. Moreover,

$$p = e^{t^2/2}\frac{\frac{1}{2}\exp\left(-\frac{s^2}{2} - t^2\right)(\exp(-st) - 1)}{\exp(-(s+t)^2/2) - \exp(-s^2/2)\exp(-t^2/2)} = \frac{1}{2}\,.$$

Finally,

$$\phi_1(t) = \frac{\exp(-t^2/2) - \frac{1}{2}\exp(-t^2/2)}{1/2} = e^{-t^2/2}\ , \quad \phi_2(t) = \frac{1}{\exp(t^2/2)} = e^{-t^2/2},$$

i.e., X_1 and X_2 are also standard normal.

In the next case we consider the maximum as the coding function for the second component of the output. Let X_0, X_1, X_2, and Z be non degenerate independent rv's and $Z \sim b(1, p)$ with unknown $p \in (0, 1)$. We will be able to identify the df's of X_i's, however to identify X_2 we need an additional assumption that the lower end points of the supports of X_0 and X_2 coincide, otherwise the df of X_2 can be reconstructed only partially. Formally, as before, denote by F_i the df of X_i, $i = 0, 1, 2$. Let

$$(Y_1, Y_2) = (ZX_0 + (1 - Z)X_1, X_0 \vee X_2) \tag{23}$$

with joint distribution function

$$H(x, y) = pF_0(x \wedge y)F_2(y) + (1 - p)F_0(y)F_1(x)F_2(y), \quad x, y \in \mathbf{R}. \tag{24}$$

Thus the marginal df's of Y_1 and Y_2 are, respectively,

$$H(x, \infty) = pF_0(x) + (1 - p)F_1(x), \quad x \in \mathbf{R} \tag{25}$$

and

$$H(\infty, y) = F_0(y)F_2(y) \quad y \in \mathbf{R}. \tag{26}$$

Consider also

$$G(x, y) = H(x, y) - H(x, \infty)H(\infty, y). \tag{27}$$

Theorem 3.2. *Let X_0, X_1, X_2, and Z be non degenerate independent rv's and $Z \sim b(1, p)$ with unknown $p \in (0, 1)$ and let (23) hold. Assume that the distributions of X_0 and X_2 have the same lower end points of their supports. Let*

$$a = \inf\{x : H(\infty, x) > 0\} \quad and \quad b = \sup\{y : G(y, y) > 0\}.$$

Then $a < b$ are, respectively, the lower and the upper end points of the support of X_0. Moreover, the distributions of X_0, X_1, X_2, and Z are uniquely determined by the distribution of (Y_1, Y_2) in the following way:

(i) *If $G(a, a) = 0$ then $F_0(a) = 0$ and*

$$F_0(x) = 1 - \lim_{y \to a^+} \frac{G(x, y)}{G(y, y)}.$$

(ii) *If $G(b^-, b^-) = 0$ then $F_0(b^-) = 1$ and*

$$F_0(x) = \lim_{y \to b^-} \frac{G(x, y)}{G(y, y)}.$$

(iii) *If* $G(a,a)G(b^-,b^-) \neq 0$ *and* $P(X_0 \in \{a,b\}) \neq 1$ *then* $F_0(a) > 0$, $F_0(b^-) < 1$ *and*

$$F_0(x) = \frac{1-\gamma_1}{1-\gamma_1\gamma_2}\frac{G(x,b^-)}{G(b^-,b^-)} = 1 - \frac{1-\gamma_2}{1-\gamma_1\gamma_2}\frac{G(x,a)}{G(a,a)}, \quad x \in (a,b),$$

where

$$\gamma_1 = \frac{G(b^-,a)}{G(a,a)} \quad \textit{and} \quad \gamma_2 = \frac{G(a,b^-)}{G(b^-,b^-)}.$$

In any of the three previous cases,

$$p = \frac{G(x,x)}{H(\infty,x)(1-F_0(x))} = const, \quad x \in (a,b)$$

$$F_1(x) = \frac{H(x,\infty) - pF_0(x)}{1-p}, \quad x \in \mathbf{R} \quad \textit{and} \quad F_2(x) = \frac{H(\infty,x)}{F_0(x)}, \quad x > a.$$

Proof. From (26), it is clear that $a = \inf\{x : H(\infty,x) > 0\}$ is the lower endpoint of F_0 (and of F_2). Using expressions (24), (25), and (26), after some algebra, it can be checked that:

$$G(x,y)F_0(y) = pH(\infty,y)\{F_0(x \wedge y) - F_0(x)F_0(y)\}, \quad x,y \in \mathbf{R}. \tag{28}$$

Expression (28) for $y = x$ is

$$G(y,y)F_0(y) = pH(\infty,y)F_0(y)(1-F_0(y)), \quad y \in \mathbf{R}. \tag{29}$$

Note that $F_0(y) > 0$, for $y > a$, then from (29)

$$G(y,y) = pH(\infty,y)(1-F_0(y)), \quad y > a. \tag{30}$$

Note also that $H(\infty,y) > 0$ for $y > a$, then from (30) it follows that

$$b = \sup\{y > a : G(y,y) > 0\} = \sup\{y > a : F_0(y) < 1\},$$

hence b is the upper end of the distribution of X_0 and for all $x \in (a,b)$, $F_0(x) \neq 0$.

For $a < y < x < b$ from the expression (28) we get

$$G(x,y) = pH(\infty,y)(1-F_0(x)). \tag{31}$$

Divide (31) by (30) and obtain after easy manipulations

$$F_0(x) = 1 - (1-F_0(y))\frac{G(x,y)}{G(y,y)} \quad \text{for } a < y < x < b$$

now, let $y \to a^+$ in the previous expression to get

$$F_0(x) = 1 - (1-F_0(a))\lim_{y\to a^+}\frac{G(x,y)}{G(y,y)} \quad \text{for } a < x < b. \tag{32}$$

For $a < x < y < b$, expression (28) gives

$$G(x, y)F_0(y) = pH(\infty, y)F_0(x)\,(1 - F_0(y))\,. \tag{33}$$

Divide (33) by (30) to obtain

$$F_0(x) = F_0(y)\frac{G(x, y)}{G(y, y)} \quad \text{for } a < x < y < b$$

and taking limits as $y \to b^-$ in the previous relation, we obtain

$$F_0(x) = F_0(b^-)\lim_{y\to b^-}\frac{G(x, y)}{G(y, y)} \quad \text{for } a < x < b. \tag{34}$$

Now we consider the different cases:

(i) Take limits $x \to a^+$ in (33) and obtain

$$G(a, y)F_0(y) = pH(\infty, y)F_0(a)(1 - F_0(y)), \quad \text{for all } y \in (a, b). \tag{35}$$

We claim here that $G(a, a) = 0$ iff there exists $y \in (a, b)$ such that $G(a, y) = 0$ iff for all $y \in (a, b)$, $G(a, y) = 0$.

To prove the *if* statements of this claim, assume that $G(a, a) = 0$. By (30) it follows that $H(\infty, a) = 0$, so by (26) we obtain $F_0(a) = 0$, then from (35) we conclude that $G(a, y) = 0$, for all $y \in (a, b)$. To prove the *only if* statements suppose that there exists $y \in (a, b)$ such that $G(a, y) = 0$. From (35) it follows that $F_0(a) = 0$ and again from the same formula we conclude that $G(a, y) = 0$ for all $y \in (a, b)$, and this implies that $G(a, a) = 0$ (just take the limit as $y \to a^+$).

It is now clear that $G(a, a) = 0$ is equivalent to $F_0(a) = 0$ and in this case, from (32)

$$F_0(x) = 1 - \lim_{y\to a^+}\frac{G(x, y)}{G(y, y)} \quad \text{for } x \in (a, b).$$

(ii) From (30),

$$G(b^-, b^-) = pH(\infty, b^-)\,(1 - F_0(b^-))$$

and as $H(\infty, b^-) > 0$, we conclude that $F_0(b^-) = 1$ if and only if $G(b^-, b^-) = 0$, and from (34)

$$F_0(x) = \lim_{y\to b^-}\frac{G(x, y)}{G(y, y)}.$$

(iii) In this case, we have $0 < F_0(a) < F_0(b^-) < 1$. Take limits $x \to a^+$ in (34) to obtain

$$F_0(a) = F_0(b^-)\gamma_1 \tag{36}$$

and also let $x \to b^-$ in (32) to get

$$F_0(b^-) = 1 - (1 - F_0(a))\gamma_2 \tag{37}$$

now, from (36) and (37) we have

$$F_0(a) = \gamma_2 \frac{1-\gamma_1}{1-\gamma_1\gamma_2} \quad \text{and} \quad F_0(b^-) = \frac{1-\gamma_1}{1-\gamma_1\gamma_2}$$

and using (32) and (34) we get

$$F_0(x) = \frac{1-\gamma_1}{1-\gamma_1\gamma_2}\frac{G(x,b^-)}{G(b^-,b^-)} = 1 - \frac{1-\gamma_2}{1-\gamma_1\gamma_2}\frac{G(x,a)}{G(a,a)}, \quad x \in (a,b).$$

In any case, once F_0 is recovered, the expressions for p, F_1 and F_2 follow easily from (30), (25), and 26), respectively. □

Note that if it is not assumed that the distributions of X_0 and X_2 have the same lower end points of the supports then the df F_2 can be identified only for the arguments in (a,∞). To see this observe that $F_2(y)$ appears in (24) multiplied by $F_0(x \wedge y)$ and by $F_0(y)$ and for $y \le a$ both these expressions are zero.

Again we provide an example of identification of particular distribution of the output in the model we consider here.

Example 3.2. Let the df H of (Y_1, Y_2) be of the form

$$H(x,y) = \begin{cases} 0, & (x,y) \notin [0,\infty)^2, \\ \frac{1}{2}(x \wedge y)y(1 + x \vee y), & (x,y) \in [0,1]^2, \\ x, & 0 \le x < 1 \le y, \\ y^2, & 0 \le y < 1 \le x, \\ 1, & (x,y) \in [1,\infty)^2. \end{cases}$$

Then $H(x,\infty) = x$ if $x \in [0,1]$, is 0 for negative x's and is 1 if $x > 1$; $H(\infty, y) = y^2$ if $y \in [0,1]$, is 0 for negative y's and is 1 if $y > 1$. So,

$$G(x,y) = \frac{1}{2} y\,(x \wedge y - xy)\, I_{(0,1)^2}(x,y)$$

and

$$G(y,y) = \frac{1}{2} y^2 (1-y) I_{(0,1)}(y).$$

Hence, by the definition, $a = 0$ and $b = 1$. It holds that $G(1^-, 1^-) = 0$, then by Theorem 3.2(ii), we conclude that

$$F_0(x) = \lim_{y \to 1^-} \frac{G(x,y)}{G(y,y)} = x, \quad x \in (0,1)$$

$$p = \frac{1}{2},$$

and for $x \in (0, 1)$

$$F_0(x) = 1 - 2\frac{1-x}{2} = x, \quad F_1(x) = \frac{x - x/2}{1/2} = x, \quad F_2(x) = \frac{x^2}{x} = x,$$

i.e., all the X_i's have the uniform distribution on [0, 1].

Finally, we will consider the minimum coding function for the second element of the output in our scheme. Similarly as in the preceding case, we need an extra condition on the support of X_2 to be able to pin down the df of this rv. In this situation, it is more convenient to work with the functions $\overline{H}(x, y) = P(Y_1 > x, Y_2 > y)$ and $\overline{G}(x, y) = \overline{H}(x, y) - \overline{H}(x, \infty)\overline{H}(\infty, y)$, $x, y \in \mathbf{R}$. Also we denote $\bar{F}_i = 1 - F_i$, where F_i is the df of X_i, $i = 0, 1, 2$. Since the proof follows the lines of the proof of Theorem 3.2 it will be omitted. We state only the result and then give an example.

Theorem 3.3. *Let X_0, X_1, X_2, and Z be non degenerate independent rv's and $Z \sim b(1, p)$ with unknown $p \in (0, 1)$. Assume that the upper ends of the supports of the distributions of X_0 and X_2 coincide. Let*

$$(Y_1, Y_2) = (ZX_0 + (1 - Z)X_1, X_1 \wedge X_2). \tag{38}$$

Let

$$a = \inf\{x : \overline{G}(x, x) > 0\} \quad \text{and} \quad b = \sup\{y : \overline{H}(-\infty, y) > 0\}.$$

Then $a < b$ are, respectively, the lower and upper ends of the support of X_0 and the distributions of X_0, X_1, X_2, and Z are uniquely determined by the distribution of (Y_1, Y_2) in the following way:

(i) *If $\overline{G}(a, a) = 0$ then $F_0(a) = 0$ and*

$$F_0(x) = \lim_{y \to a^+} \frac{\overline{G}(x, y)}{\overline{G}(y, y)}, \quad x \in (a, b),$$

(ii) *If $\overline{G}(b^-, b^-) = 0$ then $F_0(b^-) = 1$ and*

$$F_0(x) = 1 - \lim_{y \to b^-} \frac{\overline{G}(x, y)}{\overline{G}(y, y)}, \quad x \in (a, b),$$

(iii) *If $\overline{G}(a, a)\overline{G}(b^-, b^-) \neq 0$ and $P(X_0 \in \{a, b\}) \neq 1$ then $F_0(a) > 0$, $F_0(b^-) < 1$ and*

$$F_0(x) = \frac{1 - \gamma_1}{1 - \gamma_1\gamma_2} \frac{\overline{G}(x, b^-)}{\overline{G}(b^-, b^-)} = 1 - \gamma_1 \frac{1 - \gamma_2}{1 - \gamma_1\gamma_2} \frac{\overline{G}(x, a)}{\overline{G}(a, a)}, \quad x \in (a, b).$$

In any case,

$$p = \frac{\overline{G}(x, x)}{\overline{H}(-\infty, x)F_0(x)} = const, \quad x \in (a, b),$$

$$\overline{F}_1(x) = \frac{\overline{H}(x, -\infty) - p\overline{F}_0(x)}{1 - p}, \quad x \in \mathbf{R} \quad \text{and} \quad \overline{F}_2(x) = \frac{\overline{H}(-\infty, x)}{\overline{F}_0(x)}, \quad x > a.$$

Example 3.3. Let (Y_1, Y_2) be a random vector such that

$$\overline{H}(x, y) = \begin{cases} \dfrac{1 + (x \wedge y)^{-1}}{2y(x \vee y)}, & x, y \geq 1, \\ y^{-2}, & x \leq 1 \leq y, \\ x^{-1}, & y \leq 1 \leq x, \\ 1, & (x, y) \in (-\infty, 1]^2. \end{cases}$$

Then $\overline{H}(-\infty, y) = y^{-2}$ if $y \geq 1$ and is 1 otherwise, $\overline{H}(x, -\infty) = x^{-1}$ if $x \geq 1$ and is 1 otherwise. Then,

$$\overline{G}(x, y) = \frac{x \wedge y - 1}{2xy^2} I_{(1,\infty)^2}(x, y)$$

and

$$\overline{G}(y, y) = \frac{y - 1}{2y^3} I_{(1,\infty)}(y).$$

So, $a = 1$ and $b = \infty$. Observe that $\overline{G}(1, 1) = 0$, then according to Theorem 3.3(i)

$$F_0(x) = 1 - \lim_{y \to 1} \frac{\overline{G}(x, y)}{\overline{G}(y, y)} = 1 - 1/x, \quad x \geq 1,$$
$$p = \frac{1}{2},$$

and for $x \geq 1$

$$F_1(x) = 1 - \frac{1/x - 1/(2x)}{1/2} = 1 - 1/x, \quad F_2(x) = 1 - \frac{1/x^2}{1/x} = 1 - 1/x,$$

i.e., X_0, X_1, and X_2 have the same Pareto distribution.

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