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Hitting times of Brownian motion and the Matsumoto–Yor property on trees

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Abstract

The Matsumoto–Yor property in the bivariate case was originally defined through properties of functionals of the geometric Brownian motion. A multivariate version of this property was described in the language of directed trees and outside of the framework of stochastic processes in Massam and Wesołowski [H. Massam, J. Wesołowski, The Matsumoto–Yor property on trees, Bernoulli 10 (2004) 685–700]. Here we propose its interpretation through properties of hitting times of Brownian motion, extending the interpretation given in the bivariate case in Matsumoto and Yor [H. Matsumoto, M. Yor, Interpretation via Brownian motion of some independence properties between GIG and gamma variables, Statist. Probab. Lett. 61 (2003) 253–259].

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1. Introduction and preliminaries

An interpretation of the bivariate Matsumoto–Yor property through hitting times of Brownian motion (BM) was originally given in Matsumoto and Yor [8]. We start by rephrasing it in a symmetric way which seems to be more suitable for generalizations to the multivariate situation which is our main object here.

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Let $q \in \mathbb{R}$ and $p, a, b > 0$. The gamma, $\gamma(p, a)$, and the generalized inverse Gaussian, GIG(q, a, b), distributions are defined, respectively, by the following densities:

$$f(x) = \frac{a^p}{\Gamma(p)} \cdot x^{p-1} \mathrm{e}^{-ax} \cdot I_{(0,\infty)}(x)$$

and

$$g(x) = \left(\frac{a}{b}\right)^{\frac{q}{2}} \frac{1}{2K_q(2\sqrt{ab})} \cdot x^{q-1} \exp\left[-ax - \frac{b}{x}\right] \cdot I_{(0,\infty)}(x),$$

where K_q is the modified Bessel function of the third kind with index q. It is easy to check that GIG(q, a, b) converges weakly to $\gamma(q, a)$ as $b \to 0$, if $q > 0$. Moreover, if X follows the GIG(q, a, b) distribution then $\frac{1}{X}$ follows the GIG$(-q, b, a)$ distribution.

Let B be a BM and $a, b > 0$. For the process B, we define the first hitting time of the line $b - at$ by

$$\tau_b^a(B) = \inf\{t > 0 : B_t = b - at\}.$$

It is well known that $\tau_b^a(B)$ is a finite stopping time and follows the GIG $\left(-\frac{1}{2}, \frac{a^2}{2}, \frac{b^2}{2}\right)$ distribution. This can be seen from the optional stopping theorem (cf. [9]) or from the reflection principle and Girsanov's theorem (cf. [2]). Moreover, $\tau_b^a(B)$ can also be viewed as the first hitting time of a level b for a Brownian motion with drift $a > 0$, i.e.

$$\tau_b^a(B) = \inf\left\{t > 0 : \ B_t^{(a)} = b\right\}, \quad B_t^{(a)} := B_t + at.$$

We also define the last hitting time of the line $b - at$ by

$$\sigma_b^a(B) = \sup\{t > 0 : B_t = b - at\}.$$

Thanks to the assumption $a, b > 0$, $\sigma_b^a(B)$ is a finite random variable. Note that $\sigma_b^a(B)$ is not a stopping time. It is also well known that $\sigma_b^a(B)$ follows the GIG $\left(\frac{1}{2}, \frac{a^2}{2}, \frac{b^2}{2}\right)$ distribution. In particular, if $a > 0$ then $\sigma_0^a(B) = \sup\{t > 0 : B_t = -at\}$ follows the $\gamma\left(\frac{1}{2}, \frac{a^2}{2}\right)$ distribution.

In this paper we use the following convention: for any BM X the process $\widetilde{X}$ is defined by

$$\widetilde{X}_t = \begin{cases} -t \cdot X_{1/t}, & t > 0 \\ 0, & t = 0. \end{cases} \tag{1}$$

It is well known that $\widetilde{X}$ is also a BM.

A simple calculation shows that the following identity holds:

$$\sigma_b^a(B) = \frac{1}{\tau_a^b\left(\widetilde{B}\right)}. \tag{2}$$

This relationship leads immediately to the distribution of $\sigma_b^a(B)$ as given above.

The following version of the strong Markov property for BM is fundamental for this paper:

Let B be a standard $\{\mathcal{F}_t\}$-Brownian motion and τ be a finite stopping time with respect to the same filtration $\{\mathcal{F}_t\}$. Then the process W defined by

$$W_t = B_{\tau+t} - B_\tau \quad \textit{for } t \geq 0$$

is a standard $\{\mathcal{F}_{\tau+t}\}$-Brownian motion independent of $\mathcal{F}_\tau$.

Now, let us mention a simple consequence of the strong Markov property. If $a, b > 0$ then the random variable $\sigma_b^a(B) - \tau_b^a(B)$ follows the $\gamma\left(\frac{1}{2}, \frac{a^2}{2}\right)$ distribution. Moreover, the random variables $\tau_b^a(B)$ and $\sigma_b^a(B) - \tau_b^a(B)$ are independent. Hence

$$\left(\tau_b^a(B), \sigma_b^a(B) - \tau_b^a(B)\right) \sim \mathrm{GIG}\left(-\frac{1}{2}, \frac{a^2}{2}, \frac{b^2}{2}\right) \otimes \gamma\left(\frac{1}{2}, \frac{a^2}{2}\right). \tag{3}$$

For a proof, it suffices to observe that

$$\sigma_b^a(B) - \tau_b^a(B) = \sup\{t > 0 : W_t = -at\} = \sigma_0^a(W) \sim \gamma\left(\frac{1}{2}, \frac{a^2}{2}\right),$$

where the process W is defined by $W_t = B_{\tau_b^a(B)+t} - B_{\tau_b^a(B)}$ for $t \geq 0$.

While studying properties of exponential Brownian motion, Matsumoto and Yor [7] discovered an interesting identity in law involving GIG and gamma variables. Their result was later developed in Letac and Wesołowski [4] for univariate and matrix variate variables. For real valued random variables the Matsumoto–Yor (MY) property (see also Stirzaker [10], p. 43) reads: if X and Y are independent and follow the $\mathrm{GIG}(-q, a, b)$ and $\gamma(q, a)$ distributions respectively, then the random variables U and V given by

$$U = \frac{1}{X} - \frac{1}{X+Y}, \qquad V = \frac{1}{X+Y}$$

are also independent and follow the $\gamma(q, b)$ and $\mathrm{GIG}(-q, b, a)$ distributions, respectively.

We prefer to rephrase the MY property in an equivalent but more symmetric way:

Let $K = (K_1, K_2)$ be a random vector. Then

$$\left(K_1, K_2 - \frac{1}{K_1}\right) \sim \mathrm{GIG}(q, b, a) \otimes \gamma(q, a)$$

iff

$$\left(K_1 - \frac{1}{K_2}, K_2\right) \sim \gamma(q, b) \otimes \mathrm{GIG}(q, a, b).$$

Matsumoto and Yor [8] interpreted the property for $q = \frac{1}{2}$ using the hitting times of BM. Their interpretation for the symmetric statement given above can be formulated as follows:

Let B be a BM. Then

$$\begin{aligned} &\left(\frac{1}{\tau_b^a(B)}, \sigma_b^a(B) - \tau_b^a(B)\right) \sim \mathrm{GIG}\left(\frac{1}{2}, \frac{b^2}{2}, \frac{a^2}{2}\right) \otimes \gamma\left(\frac{1}{2}, \frac{a^2}{2}\right) \\ &\left(\frac{1}{\tau_b^a(B)} - \frac{1}{\sigma_b^a(B)}, \sigma_b^a(B)\right) \sim \gamma\left(\frac{1}{2}, \frac{b^2}{2}\right) \otimes \mathrm{GIG}\left(\frac{1}{2}, \frac{a^2}{2}, \frac{b^2}{2}\right). \end{aligned} \tag{4}$$

The main object of the present paper is to extend this interpretation to the MY property on trees which was introduced in Massam and Wesołowski [5]. This is done through our two main results.

First, in Section 2, we establish a property of BM, which is a multivariate version of (4). This is achieved by considering first and last hitting times for a family of BM's defined in terms of a class of transformations of the original BM B.

Second, in Section 3, we show that this property is equivalent to the MY property on trees for $q = \frac{1}{2}$. This is done by relating to the tree structures the class of transformations of the original BM B, which leads to the multivariate version of (4).

To have a clear understanding of our task and to prepare for considerations of Section 3, we explain now, following [5], the general MY property on trees. Let G_n be a tree of size n, where $V(G_n) = \{1, \dots, n\}$ is a set of vertices and $E(G_n)$ is a set of unordered edges $\{u, v\}$, i.e. the distinct vertices u and v are linked in G_n. Let $L(G_n)$ denote the set of leaves of G_n. From an undirected tree G_n we can create a directed tree $G_{n,(r)}$ by choosing a single root $r \in V(G_n)$. In this paper directed trees have only one root. Let (u, v) denote a directed edge going from vertex u to v in the directed tree $G_{n,(r)}$. We then say that the vertex u is a parent of v and the vertex v is a child of u. Each vertex u has at most one child, which is denoted by $c_r(u)$. We write $c_r(r) = \emptyset$. Moreover, each vertex v may have several parents. The set of parents of v in the directed tree $G_{n,(r)}$ is denoted by $p_r(v)$. If v is a leaf and $v \neq r$ then $p_r(v) = \emptyset$.

Let $\mathcal{V}_n^+$ be the cone of $n \times n$ positive definite symmetric matrices. For a tree G_n let

$$K_{G_n} = \left\{k_{ij} \in \mathbb{R} : k_{ij} \neq 0, \{i, j\} \in E(G_n),\ k_{ij} = 0, \{i, j\} \notin E(G_n)\right\}$$

be a given set. Then, we define a set $M(G_n, K_{G_n})$ as

$$M(G_n, K_{G_n}) = \left\{(k_1, \dots, k_n) \in \mathbb{R}^n : \bar{k} = [k_{ij}] \in \mathcal{V}_n^+, k_{ii} = k_i;\ k_{ij} \in K_{G_n}, i \neq j\right\}.$$

Thus, K_{G_n} is a fixed set of off-diagonal entries for matrices $\bar{k} = [k_{ij}]$, which can differ only on the diagonal. We attach to the directed tree $G_{n,(r)}$ the mapping ψ_r defined by

$$\psi_r(k_1, \dots, k_n) = \left(k_{1,(r)}, \dots, k_{n,(r)}\right),$$

where

$$k_{i,(r)} = \begin{cases} k_i, & i \in L(G_n) \setminus \{r\} \\ k_i - \displaystyle\sum_{j \in p_r(i)} \frac{k_{ij}^2}{k_{j,(r)}}, & \text{otherwise.} \end{cases}$$

In this definition, we start from the leaves and move to the root along the directed paths. For any $r \in V(G_n)$ the mapping ψ_r is a bijection from $M(G_n, K_{G_n})$ onto $\mathbb{R}_+^n$ — see Lemmas 2.1–2.4 in [5]. In particular the positive definiteness of $\bar{k}$ and the identity $\det(\bar{k}) = \prod_{i \in G_n} k_{i,(r)}$ (see (2.8) in [5]) imply that the $k_{i,(r)},\ i \in G_n$, are positive.

For a given $q > 0$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$, the $W_{G_n}(q, K_{G_n}, \mathbf{a})$ distribution is defined by the density

$$f(\mathbf{k}) \propto |\bar{k}|^{q-1} \exp(-(\mathbf{a}, \mathbf{k})) \cdot \mathbb{I}_{M(G_n, K_{G_n})}(\mathbf{k}), \quad \mathbf{k} = (k_1, \dots, k_n).$$

Now we are in a position to formulate the MY property on trees as follows:

Let G_n be a tree of size $n \geq 2$. Let $\mathbf{K} = (K_1, \dots, K_n)$ be a random vector following the $W_{G_n}(q, K_{G_n}, \mathbf{a})$ distribution with $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$ and $q > 0$. *Define $\mathbf{K}_r = \psi_r(\mathbf{K})$ for $r \in V(G_n)$. Then for all $r \in V(G_n)$ the components of $\mathbf{K}_r = \left(K_{1,(r)}, \dots, K_{n,(r)}\right)$ are mutually independent. Moreover,*

$$K_{r,(r)} \sim \gamma(q, a_r) \quad \textit{and} \quad K_{i,(r)} \sim \mathrm{GIG}\left(q, a_i, k_{ic_r(i)}^2 a_{c_r(i)}\right), \quad i \in V(G_n) \setminus \{r\}.$$

It is clear that since the ψ_r are bijections the distribution of $\mathbf{K}_{r_0}$ for an arbitrary fixed $r_0 \in G_n$, as given above, uniquely determines the distribution of $\mathbf{K}_r$ for any $r \in G_n$ to be also as given above.

As has already been mentioned, in Section 3 we show that the multivariate version of (4) obtained in Section 2 is equivalent to the above property for $q = \frac{1}{2}$ and

$$K_{G_n} = \left\{k_{ij} = 1, \{i, j\} \in E(G_n); k_{ij} = 0, \{i, j\} \notin E(G_n)\right\}.$$

To have a more complete picture of the area related to the MY property we recall that characterizations of the gamma and GIG laws related to the MY independence are available in univariate, multivariate and matrix variate cases; see for instance: Letac and Wesołowski [4], Massam and Wesołowski [5,6], Wesołowski [11], Bobecka and Wesołowski [1]. Moreover it was shown in Massam and Wesołowski [6] that the matrix version of the MY property can be read out from the conditional structure of Wishart matrices. Finally it is worth mentioning that recently Koudou [3] pointed out a link between the bivariate MY property and properties of electrical networks with independent (reciprocal) inverse Gaussian resistances, which were expressed in the language of trees.

2. Independence properties of hitting times of Brownian motion

Let $n \in \mathbb{N}$ and $n \geq 2$. We define a discrete function

$$c : \{1, \ldots, n-1\} \to \{2, \ldots, n\}$$

satisfying

$$i < c(i) < n \quad \text{for } i = 1, \ldots, n-2 \quad \text{and} \quad c(n-1) = n. \tag{5}$$

Note that for a given function c and fixed $r \in \{1, \ldots, n-1\}$ there exists one and only one sequence $(i_1, \ldots, i_s)$ such that $i_1 = r$, $i_s = n$ and

$$i_{k+1} = c(i_k) \quad \text{for } k = 1, \ldots, s-1.$$

Thus we may define two subsets $I_r(c)$ and $J_r(c)$ of the set $\{1, \ldots, n\}$ as follows:

$$I_r(c) = \{i_1, \ldots, i_s\} \quad \text{and} \quad J_r(c) = \{1, \ldots, n\} \setminus I_r(c).$$

Moreover, $i_1 < \cdots < i_s$ and $i_{s-1} = n-1$. For $r = n$ we simply put $I_n(c) = \{i_1 = n\}$ and $J_n(c) = \{1, \ldots, n-1\}$. Note that the sets $I_r(c)$ and $J_r(c)$ are uniquely determined by the function c and $r \in \{1, \ldots, n\}$.

Let B be a BM, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ be a given positive vector and c be a function considered above. We define a family of n Brownian motions $\mathcal{B}_n(B, \mathbf{a}, c) = \{B^{(i)} : i = 1, \ldots, n\}$ as follows:

$$B_t^{(1)} = B_t \quad \text{for } t \geq 0,$$
$$B_t^{(i+1)} = B^{(i)}_{\tau_{a_i}^{a_{c(i)}}(B^{(i)})+t} - B^{(i)}_{\tau_{a_i}^{a_{c(i)}}(B^{(i)})} \quad \text{for } t \geq 0,\ i = 1, \ldots, n-1.$$

By the strong Markov property, the processes

$$\left\{B_t^{(1)} :\ t \leq \tau_{a_1}^{a_{c(1)}}\left(B^{(1)}\right)\right\}, \ldots, \left\{B_t^{(n-1)} : t \leq \tau_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right)\right\}, \left\{B_t^{(n)} :\ t \geq 0\right\}$$

are mutually independent. Moreover, for any $i \in \{1, \ldots, n-1\}$ the stopped process $\left\{B_t^{(i)} : t \leq \tau_{a_i}^{a_{c(i)}}\left(B^{(i)}\right)\right\}$ is independent of all the processes $B^{(i+1)}, \ldots, B^{(n)}$. Hence, since

$\sigma_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right) - \tau_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right) = \sigma_0^{a_{c(n-1)}}\left(B^{(n)}\right)$, the random variables

$$\tau_{a_1}^{a_{c(1)}}\left(B^{(1)}\right), \ldots, \tau_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right), \sigma_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right) - \tau_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right)$$

are mutually independent.

For a given function $c : \{1, \ldots, n-1\} \to \{2, \ldots, n\}$ satisfying (5) and $r \in \{1, \ldots, n\}$, we define a mapping $\phi_r^{(c)}$ given by

$$\phi_r^{(c)}(x_1, \ldots, x_n) = \left(x_{1,(r)}, \ldots, x_{n,(r)}\right), \tag{6}$$

where, assuming that $x_{i_0} = x_{i_{s+1,(r)}} = \infty$, we have

$$x_{i,(r)} = x_i \quad \text{for } i \in J_r(c),$$
$$x_{i_k,(r)} = x_{i_k} + \frac{1}{x_{i_{k-1}}} - \frac{1}{x_{i_{k+1,(r)}}} \quad \text{for } k = 1, \ldots, s$$

and $\{i_1, \ldots, i_s\} = I_r(c)$. Computing the explicit form of (6), we should start from $x_{i_s,(r)}$, i.e.

$$x_{i_s,(r)} = x_{i_s} + \frac{1}{x_{i_{s-1}}}, \qquad x_{i_{s-1},(r)} = x_{i_{s-1}} + \frac{1}{x_{i_{s-2}}} - \frac{1}{x_{i_s} + \frac{1}{x_{i_{s-1}}}}, \text{ etc.}$$

Note that, since $\# I_n(c) = 1$, $\phi_n^{(c)}$ is an identity mapping. Define $\Phi^{(c)} = \{\phi_r^{(c)} : r = 1, \ldots, n\}$.

Let us now define the random vector $\mathbf{X} = (X_1, \ldots, X_n)$, using the first and last hitting times of BM, for a given family $\mathcal{B}_n(B, \mathbf{a}, c)$. More precisely, let

$$X_i = \begin{cases} \dfrac{1}{\tau_{a_i}^{a_{c(i)}}\left(B^{(i)}\right)}, & i = 1, \ldots, n-1 \\ \sigma_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right) - \tau_{a_{n-1}}^{a_{c(n-1)}}\left(B^{(n-1)}\right), & i = n. \end{cases} \tag{7}$$

Note that the components of $\mathbf{X}$ are mutually independent. Then, for $i \in J_r(c)$ we simply have

$$X_{i,(r)} = X_i = \frac{1}{\tau_{a_i}^{a_{c(i)}}\left(B^{(i)}\right)}.$$

For $i \in I_r(c) = \{i_1, \ldots, i_s\}$ and $r \neq n$ the random variables $X_{i,(r)}$ satisfy the following recursion:

$$X_{i_k,(r)} = \begin{cases} \sigma_{a_{i_{s-1}}}^{a_{c(i_{s-1})}}\left(B^{(i_{s-1})}\right), & k = s \\ \dfrac{1}{\tau_{a_{i_k}}^{a_{c(i_k)}}\left(B^{(i_k)}\right)} + \tau_{a_{i_{k-1}}}^{a_{c(i_{k-1})}}\left(B^{(i_{k-1})}\right) - \dfrac{1}{X_{i_{k+1},(r)}}, & k = 2, \ldots, s-1 \\ \dfrac{1}{\tau_{a_{i_1}}^{a_{c(i_1)}}\left(B^{(i_1)}\right)} - \dfrac{1}{X_{i_2,(r)}}, & k = 1. \end{cases}$$

Theorem 1. *Let $n \geq 2$ be an integer. Let B be a BM, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ be a given vector and $c : \{1, \ldots, n-1\} \to \{2, \ldots, n\}$ be a function satisfying (5). Define for $r \in \{1, \ldots, n\}$ the sets $I_r(c)$ and $J_r(c)$ and the family $\mathcal{B}_n(B, \mathbf{a}, c)$ as described above. Let $\mathbf{X}$ be a random vector given by (7) and $\phi_r^{(c)}$ for $r \in \{1, \ldots, n\}$ be the mappings defined by (6).*

Then

(i)

$$\phi_n^{(c)}(\mathbf{X}) = \mathbf{X} \sim \left[\bigotimes_{k=1}^{n-1} \mathrm{GIG}\left(\frac{1}{2}, \frac{a_k^2}{2}, \frac{a_{c(k)}^2}{2}\right)\right] \otimes \gamma\left(\frac{1}{2}, \frac{a_n^2}{2}\right),$$

(ii) *for any* $r \in \{1, \ldots, n-1\}$ *the components of the random vector* $\phi_r^{(c)}(\mathbf{X}) = \left(X_{1,(r)}, \ldots, X_{n,(r)}\right)$ *are mutually independent; moreover,*

$$X_{i,(r)} \sim \mathrm{GIG}\left(\frac{1}{2}, \frac{a_i^2}{2}, \frac{a_{c(i)}^2}{2}\right) \quad \textit{for } i \in J_r(c)$$

and

$$X_{i_1,(r)} \sim \gamma\left(\frac{1}{2}, \frac{a_{i_1}^2}{2}\right), \qquad X_{i_k,(r)} \sim \mathrm{GIG}\left(\frac{1}{2}, \frac{a_{i_k}^2}{2}, \frac{a_{i_{k-1}}^2}{2}\right) \quad \textit{for } k = 2, \ldots, s,$$

where $\{i_1, \ldots, i_s\} = I_r(c)$.

Remark 1. This theorem shows that the mappings $\Phi^{(c)}$ are independence preserving. Moreover, for any $r \in \{1, \ldots, n-1\}$ the random vector $\phi_r^{(c)}(\mathbf{X})$ still follows the product of $n-1$ GIG distributions and one gamma distribution.

Remark 2. For $n = 2$ we obtain the interpretation of the classical MY property given by Matsumoto and Yor [8] (compare to (4)).

Proof of Theorem 1. (i) It is a consequence of the strong Markov property and the facts introduced in Section 1.

(ii) For simplicity of notation, we write I_r (J_r) instead of $I_r(c)$ ($J_r(c)$), since function c is fixed. Fix $r \in \{1, \ldots, n-1\}$ and define $I_r = \{i_1, \ldots, i_s\}$. The proof will be divided into three steps. In the first step we show that the random vectors $\left(X_{i,(r)},\ i \in J_r\right)$ and $\left(X_{i,(r)},\ i \in I_r\right)$ are independent. In the second and third steps we compute the distributions of $\left(X_{i,(r)}, i \in J_r\right)$ and $\left(X_{i,(r)}, i \in I_r\right)$, respectively.

Step 1. From (6) we see that the random variables $X_{i,(r)},\ i \in I_r$, are functions of $X_{i_1}, \ldots, X_{i_s}$ and thus do not depend on $\{X_j : j \in J_r\}$. Moreover, $X_{i,(r)} = X_i$ for any $i \in J_r$. Then, since $\mathbf{X}$ has mutually independent components, the random vectors $\left(X_{i,(r)}, i \in I_r\right)$ and $\left(X_{i,(r)}, i \in J_r\right)$ are independent.

Step 2. Since

$$X_{i,(r)} = X_i = \frac{1}{\tau_{a_i}^{a_{c(i)}}\left(B^{(i)}\right)} \quad \text{for } i \in J_r,$$

the strong Markov property shows that for $i \in J_r$ the random variables $X_{i,(r)}$ are mutually independent. Hence $\left(X_{i,(r)},\ i \in J_r\right)$ follows the product of $n-s$ GIG distributions:

$$\left(X_{i,(r)},\ i \in J_r\right) \sim \bigotimes_{i \in J_r} \mathrm{GIG}\left(\frac{1}{2}, \frac{a_i^2}{2}, \frac{a_{c(i)}^2}{2}\right).$$

Step 3. Now, our goal is to prove that

$$\left(X_{i_1,(r)}, \ldots, X_{i_s,(r)}\right) \sim \gamma\left(\frac{1}{2}, \frac{a_{i_1}^2}{2}\right) \otimes \left[\bigotimes_{k=1}^{s-1} \mathrm{GIG}\left(\frac{1}{2}, \frac{a_{i_{k+1}}^2}{2}, \frac{a_{i_k}^2}{2}\right)\right]. \tag{8}$$

To show this we use the following, more general, result.

Lemma 1. *Let m be any integer such that $m \geq 2$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}_+^m$ be a given vector. Let $\mathcal{W}_m(\mathbf{b}) = \left\{W^{(i)}, i = 1, \ldots, m-1\right\}$ be a family of $m-1$ Brownian motions such that for any $i = 2, \ldots, m-1$ the processes*

$$\left\{W_t^{(1)}, t \leq \tau_{b_1}^{b_2}\left(W^{(1)}\right)\right\}, \ldots, \left\{W_t^{(i-1)}, t \leq \tau_{b_{i-1}}^{b_i}\left(W^{(i-1)}\right)\right\}, \left\{W_t^{(i)}, t \geq 0\right\}$$

are mutually independent. Let $\mathbf{Y} = (Y_1, \ldots, Y_m)$ be a random vector satisfying

$$Y_k = \begin{cases} \sigma_{b_{m-1}}^{b_m}\left(W^{(m-1)}\right), & k = m \\ \dfrac{1}{\tau_{b_k}^{b_{k+1}}\left(W^{(k)}\right)} + \tau_{b_{k-1}}^{b_k}\left(W^{(k-1)}\right) - \dfrac{1}{Y_{k+1}}, & k = 2, \ldots, m-1 \\ \dfrac{1}{\tau_{b_1}^{b_2}\left(W^{(1)}\right)} - \dfrac{1}{Y_2}, & k = 1. \end{cases} \tag{9}$$

Then

$$(Y_1, \ldots, Y_m) \sim \gamma\left(\frac{1}{2}, \frac{b_1^2}{2}\right) \otimes \left[\bigotimes_{k=1}^{m-1} \mathrm{GIG}\left(\frac{1}{2}, \frac{b_{k+1}^2}{2}, \frac{b_k^2}{2}\right)\right].$$

Proof of Lemma 1. We will proceed by induction with respect to m. Let $m = 2$. Therefore, by (2),

$$Y_2 = \sigma_{b_1}^{b_2}\left(W^{(1)}\right) = \frac{1}{\tau_{b_2}^{b_1}\left(\widetilde{W}^{(1)}\right)},$$

$$Y_1 = \frac{1}{\tau_{b_1}^{b_2}\left(W^{(1)}\right)} - \frac{1}{\sigma_{b_1}^{b_2}\left(W^{(1)}\right)} = \sigma_{b_2}^{b_1}\left(\widetilde{W}^{(1)}\right) - \tau_{b_2}^{b_1}\left(\widetilde{W}^{(1)}\right).$$

By (3), we obtain that

$$(Y_1, Y_2) \sim \gamma\left(\frac{1}{2}, \frac{b_1^2}{2}\right) \otimes \mathrm{GIG}\left(\frac{1}{2}, \frac{b_2^2}{2}, \frac{b_1^2}{2}\right).$$

This completes the proof for the case $m = 2$.

Let us now assume that the lemma holds for $m-1$, for any $\mathbf{b} \in \mathbb{R}_+^{m-1}$, $\mathcal{W}_{m-1}(\mathbf{b})$ and for any $\mathbf{Y} = (Y_1, \ldots, Y_{m-1})$ satisfying (9). By (2), we have

$$Y_m = \sigma_{b_{m-1}}^{b_m}\left(W^{(m-1)}\right) = \frac{1}{\tau_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right)},$$

$$\begin{aligned} Y_{m-1} &= \frac{1}{\tau_{b_{m-1}}^{b_m}\left(W^{(m-1)}\right)} + \tau_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right) - \frac{1}{Y_m} \\ &= \tau_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right) + \sigma_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right) - \tau_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right), \end{aligned}$$

$$Y_k = \frac{1}{\tau_{b_k}^{b_{k+1}}\left(W^{(k)}\right)} + \tau_{b_{k-1}}^{b_k}\left(W^{(k-1)}\right) - \frac{1}{Y_{k+1}} \quad \text{for } k = 2, \ldots, m-2$$

$$Y_1 = \frac{1}{\tau_{b_1}^{b_2}\left(W^{(1)}\right)} - \frac{1}{Y_2}.$$

By assumption on $\mathcal{W}_m(\mathbf{b})$ and (3), since $\widetilde{W}^{(m-1)}$ is created from $W^{(m-1)}$, we conclude that the random variables

$$\tau_{b_1}^{b_2}\left(W^{(1)}\right), \ldots, \tau_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right), \tau_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right), \sigma_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right) - \tau_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right)$$

are mutually independent. Hence, the random variable Y_m is independent of $(Y_1, \ldots, Y_{m-1})$.

Define random vectors U and V as follows:

$$U = \left(\tau_{b_1}^{b_2}\left(W^{(1)}\right), \ldots, \tau_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right), \sigma_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right) - \tau_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right)\right),$$
$$V = \left(\tau_{b_1}^{b_2}\left(W^{(1)}\right), \ldots, \tau_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right), \sigma_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right) - \tau_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right)\right).$$

Note that, by assumption on $\mathcal{W}_m(\mathbf{b})$, U and V have independent components. Moreover, the distribution of the random variable $\sigma_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right) - \tau_{b_m}^{b_{m-1}}\left(\widetilde{W}^{(m-1)}\right)$ does not depend on b_m. We thus get $U \stackrel{d}{=} V$.

Let $\xi : \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$ be a mapping such that

$$\xi(U) = (Y_1, \ldots, Y_{m-1})$$

and consider the random vector $\widetilde{\mathbf{Y}} = \left(\widetilde{Y}_1, \ldots, \widetilde{Y}_{m-1}\right)$, where

$$\widetilde{Y}_{m-1} = \sigma_{b_{m-2}}^{b_{m-1}}\left(W^{(m-2)}\right),$$
$$\widetilde{Y}_i = \frac{1}{\tau_{b_i}^{b_{i+1}}\left(W^{(i)}\right)} + \tau_{b_{i-1}}^{b_i}\left(W^{(i-1)}\right) - \frac{1}{\widetilde{Y}_{i+1}}, \quad \text{for } i = 2, \ldots, m-2$$
$$\widetilde{Y}_1 = \frac{1}{\tau_{b_1}^{b_2}\left(W^{(1)}\right)} - \frac{1}{\widetilde{Y}_2}.$$

It is easy to check that $\xi(V) = \left(\widetilde{Y}_1, \ldots, \widetilde{Y}_{m-1}\right)$. Since $\xi(U) \stackrel{d}{=} \xi(V)$,

$$(Y_1, \ldots, Y_{m-1}) \stackrel{d}{=} \left(\widetilde{Y}_1, \ldots, \widetilde{Y}_{m-1}\right).$$

Note that $\widetilde{\mathbf{Y}}$ satisfies the assumptions of the lemma with $\widetilde{\mathbf{b}} = (b_1, \ldots, b_{m-1})$ and $\mathcal{W}_{m-1}\left(\widetilde{\mathbf{b}}\right) = \left\{W^{(i)} :\ i = 1, \ldots, m-2\right\}$. Then, by the induction assumption,

$$\left(\widetilde{Y}_1, \ldots, \widetilde{Y}_{m-1}\right) \sim \gamma\left(\frac{1}{2}, \frac{b_1^2}{2}\right) \otimes \left[\bigotimes_{k=1}^{m-2} \mathrm{GIG}\left(\frac{1}{2}, \frac{b_{k+1}^2}{2}, \frac{b_k^2}{2}\right)\right].$$

Since $Y_m \sim \mathrm{GIG}\left(\frac{1}{2}, \frac{b_m^2}{2}, \frac{b_{m-1}^2}{2}\right)$ is independent of $(Y_1, \ldots, Y_{m-1})$, the lemma follows. $\square$

Now we can finish *Step* 3 of the proof of Theorem 1. Since $c(i_k) = i_{k+1}$, the random vector $\left(X_{i_1,(r)}, \ldots, X_{i_s,(r)}\right)$ satisfies (9) with $\mathbf{b} = (a_{i_1}, \ldots, a_{i_s})$ and $\mathcal{W}_s(\mathbf{b}) = \left\{B^{(i_k)} :\ k = 1, \ldots, s-1\right\}$. Of course, since $i_1 < \cdots < i_s$, the family $\left\{B^{(i_k)} :\ k = 1, \ldots, s-1\right\}$ satisfies the assumptions of the above lemma, which gives (8).

Combining the above three steps, we obtain (ii) and the proof is complete. $\square$

3. Interpretation of the MY property on trees through BM hitting times

In this section we show the correspondence between the mappings $\Phi^{(c)}$ and the multivariate MY property on trees for $q = \frac{1}{2}$ and $K_{G_n} = \left\{k_{ij} = 1, \{i, j\} \in E(G_n); k_{ij} = 0, \{i, j\} \notin E(G_n)\right\}$.

Let G_n be a tree of size n, i.e. $V(G_n) = \{1, \dots, n\}$. Let $(i_1, \dots, i_p)$ be a path from vertex u to v in the tree G_n, i.e. $i_1 = u$, $i_p = v$ and $\{i_j, i_{j+1}\} \in E(G_n)$ for $j = 1, \dots, p-1$. The distance from u to v is given by $d(u, v) = p - 1$. Since for all the vertices $u, v \in V(G_n)$ there exists a unique path from u to v, the distance d is well defined.

We can direct G_n uniquely by choosing a single root $r \in V(G_n)$. Recall that $c_r(v)$ denotes the child of v and $p_r(v)$ denotes the set of parents of v in the directed tree $G_{n,(r)}$. If $v \neq r$ is a leaf then $p_r(v) = \emptyset$ and $\# c_r(v) = 1$.

Since each tree has at least two leaves, without loss of generality we assume that the vertex n is a leaf and for any vertices $u, v \in V(G_n)$

$$d(u, n) < d(v, n) \Rightarrow u > v. \tag{10}$$

Note that this numeration is not unique and the vertex numbers decrease along with the distance from the vertex n. Moreover,

$$\{v : d(v, n) = 1\} = \{n - 1\} \quad \text{and} \quad c_n(n-1) = n, \qquad p_n(n) = \{n - 1\}.$$

Since each vertex (except a root) has one and only one child, (10) shows that the function $c_n : \{1, \dots, n-1\} \to \{2, \dots, n\}$, satisfies (5).

Recall that, in our case, the mappings ψ_r, $r \in V(G_n)$, are given by

$$\psi_r(k_1, \dots, k_n) = \left(k_{1,(r)}, \dots, k_{n,(r)}\right), \tag{11}$$

where

$$k_{i,(r)} = \begin{cases} k_i, & i \in L(G_n) \setminus \{r\} \\ k_i - \sum\limits_{j \in p_r(i)} \dfrac{1}{k_{j,(r)}}, & \text{otherwise.} \end{cases}$$

The following lemma shows the correspondence between the mappings $\Psi = \{\psi_r : r \in V(G_n)\}$ satisfying (11) and the mappings $\Phi^{(c_n)} = \{\phi_r^{(c_n)} : r = 1, \dots, n\}$ considered in the previous section. Note that we use a *child*-function c_n as a function c.

Lemma 2. *Let G_n be a tree of size n described above, i.e. vertex $n \in L(G_n)$, (10) holds for any vertices $u, v \in V(G_n)$ and c_n is the child-function.*

Then, for any $r \in V(G_n) = \{1, \dots, n\}$,

$$\psi_r(k_1, \dots, k_n) = \phi_r^{(c_n)}\left(k_{1,(n)}, \dots, k_{n,(n)}\right),$$

where the ψ_r are defined by (11) and the $\phi_r^{(c_n)}$ are defined by (6).

Proof. Since $\phi_n^{(c_n)}$ is an identity mapping, it remains to show the lemma for $r \neq n$. Fix $r \in \{1, \dots, n-1\}$. Then $I_r(c_n) = \{i_1, \dots, i_s\}$, where $(i_1, \dots, i_s)$ is a path from vertex r to n in the directed tree $G_{n,(n)}$, i.e. $i_1 = r$, $i_s = n$ and $(i_j, i_{j+1}) \in E\left(G_{n,(n)}\right)$ for $j = 1, \dots, s-1$. We recall that (u, v) denotes a directed edge from vertex u to v, i.e. v is a child of u. Since n is a root in the tree $G_{n,(n)}$, for any $r \in \{1, \dots, n-1\}$ there exists such a path. To shorten notation, we write I_r (J_r) instead of $I_r(c_n)$ ($J_r(c_n)$). Note that

$$p_n(i_j) \cap I_r = \begin{cases} \emptyset, & j = 1 \\ \{i_{j-1}\}, & j = 2, \ldots, s-1 \end{cases}, \tag{12}$$

$$p_n(i_s) = \{i_{s-1}\}. \tag{13}$$

Since ψ_n is bijective, then by the definition of ψ_n, (12) and (13), we get

$$\mathbf{k} = (k_1, \ldots, k_n) = \psi_n^{-1}(k_{1,(n)}, \ldots, k_{n,(n)}),$$

where

$$k_{i_1} = k_{i_1,(n)} + \sum_{j \in p_n(i_1)} \frac{1}{k_{j,(n)}} = k_{i_{1,(n)}} + \sum_{j \in p_n(i_1) \setminus I_r} \frac{1}{k_{j,(n)}}, \tag{14}$$

$$\begin{aligned} k_{i_j} &= k_{i_j,(n)} + \sum_{l \in p_n(i_j)} \frac{1}{k_{l,(n)}} = k_{i_{j,(n)}} + \frac{1}{k_{i_{j-1},(n)}} + \sum_{l \in p_n(i_j) \setminus I_r} \frac{1}{k_{l,(n)}} \\ &\quad \text{for } j = 2, \ldots, s-1, \end{aligned} \tag{15}$$

$$k_{i_s} = k_{i_s,(n)} + \sum_{j \in p_n(i_s)} \frac{1}{k_{j,(n)}} = k_{i_s,(n)} + \frac{1}{k_{i_{s-1},(n)}}. \tag{16}$$

If we now move the root from the vertex n to r, all the edges from the path $(i_1, \ldots, i_s)$ change their direction. The rest of the edges do not change. Therefore the set of the edges in $G_{n,(r)}$ is the following:

$$\begin{aligned} E\left(G_{n,(r)}\right) = E\left(G_{n,(n)}\right) &\setminus \left\{(i_j, i_{j+1}), j = 1, \ldots, s-1\right\} \\ &\cup \left\{(i_{j+1}, i_j), j = 1, \ldots, s-1\right\}. \end{aligned}$$

From the above it follows that

$$p_n(i) = p_r(i) \quad \text{for } i \in J_r, \tag{17}$$

$$p_n(i) \setminus I_r = p_r(i) \setminus I_r \quad \text{for } i \in I_r. \tag{18}$$

Moreover,

$$p_r(i_j) \cap I_r = \begin{cases} \{i_{j+1}\}, & j = 1, \ldots, s-1 \\ \emptyset, & j = s. \end{cases} \tag{19}$$

We are now in the position to express $\psi_r(\mathbf{k}) = \left(k_{1,(r)}, \ldots, k_{n,(r)}\right)$ in terms of $\{k_{i,(n)}, i = 1, \ldots, n\}$. We observe that $p_r(i) \cap I_r = \emptyset$ for $i \in J_r$. Thus, by (17),

$$k_{i,(r)} = k_{i,(n)} \quad \text{for } i \in J_r. \tag{20}$$

Since i_s is a leaf in the tree $G_{n,(r)}$,

$$k_{i_s,(r)} = k_{i_s}. \tag{21}$$

From (19), (18) and (20) we conclude that

$$\begin{aligned} k_{i_j,(r)} &= k_{i_j} - \sum_{l \in p_r(i_j)} \frac{1}{k_{l,(r)}} = k_{i_j} - \frac{1}{k_{i_{j+1},(r)}} - \sum_{l \in p_r(i_j) \setminus I_r} \frac{1}{k_{l,(r)}} \\ &= k_{i_j} - \frac{1}{k_{i_{j+1},(r)}} - \sum_{l \in p_n(i_j) \setminus I_r} \frac{1}{k_{l,(n)}} \quad \text{for } j = 1, \ldots, s-1. \end{aligned} \tag{22}$$

Combining (14)–(16) with (20)–(22) we get

$$\begin{aligned}
&k_{i,(r)} = k_{i,(n)} \quad \text{for } i \in J_r(c_n),\\
&k_{i_s,(r)} = k_{i_s,(n)} + \frac{1}{k_{i_{s-1},(n)}},\\
&k_{i_j,(r)} = k_{i_j,(n)} + \frac{1}{k_{i_{j-1},(n)}} - \frac{1}{k_{i_{j+1},(r)}} \quad \text{for } j = 2, \ldots, s-1,\\
&k_{i_1,(r)} = k_{i_1,(n)} - \frac{1}{k_{i_2,(r)}},
\end{aligned}$$

where $\{i_1, \ldots, i_s\} = I_r(c_n)$. Comparing the above equations with (6), we finally obtain

$$\psi_r(\mathbf{k}) = \phi_r^{(c_n)}\left(k_{1,(n)}, \ldots, k_{n,(n)}\right) \quad \text{for } r \in \{1, \ldots, n-1\}. \quad \square$$

Now we are ready to give the interpretation of the MY property on trees through properties of hitting times of BM, which extends the result of Matsumoto and Yor [8].

Let G_n be a tree of size n such that vertex $n \in L(G_n)$ and (10) holds for any vertices $u, v \in V(G_n)$. Let B be a BM and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ be a given vector. Define a family of n Brownian motions $\mathcal{B}_n(B, \mathbf{a}, c_n)$ as follows:

$$\begin{aligned}
&B_t^{(1)} = B_t \quad \text{for } t \geq 0,\\
&B_t^{(i+1)} = B^{(i)}_{\tau_{a_i}^{a_{c_n(i)}}(B^{(i)})+t} - B^{(i)}_{\tau_{a_i}^{a_{c_n(i)}}(B^{(i)})} \quad \text{for } t \geq 0,\ i = 1, \ldots, n-1.
\end{aligned}$$

Let the ψ_r, for $r \in V(G_n)$, be the mappings defined by (11) and $\mathbf{K} = (K_1, \ldots, K_n)$ be a random vector satisfying

$$K_{i,(n)} = \begin{cases} \dfrac{1}{\tau_{a_i}^{a_{c_n(i)}}\left(B^{(i)}\right)}, & i = 1, \ldots, n-1\\ \sigma_{a_{n-1}}^{a_{c_n(n-1)}}\left(B^{(n-1)}\right) - \tau_{a_{n-1}}^{a_{c_n(n-1)}}\left(B^{(n-1)}\right), & i = n \end{cases}$$

where $\left(K_{1,(n)}, \ldots, K_{n,(n)}\right) = \psi_n(\mathbf{K})$. Then, of course,

$$\left(K_{1,(n)}, \ldots, K_{n,(n)}\right) \sim \left[\bigotimes_{k=1}^{n-1} \mathrm{GIG}\left(\frac{1}{2}, \frac{a_k^2}{2}, \frac{a_{c_n(k)}^2}{2}\right)\right] \otimes \gamma\left(\frac{1}{2}, \frac{a_n^2}{2}\right).$$

Since $I_r(c_n) = \{i_1, \ldots, i_s\}$, where $(i_1, \ldots, i_s)$ is a path from vertex r to n in the directed tree $G_{n,(n)}$, we have for $r \in \{1, \ldots, n-1\}$

$$c_n(i) = c_r(i) \quad \text{for } i \in J_r(c_n)$$

and

$$i_{k-1} = c_r(i_k) \quad \text{for } k = 2, \ldots, s,\ i_1 = r.$$

Hence, by Lemma 2 we obtain for $r \in \{1, \ldots, n-1\}$

$$\psi_r(\mathbf{K}) = \phi_r^{(c_n)}\left(K_{1,(n)}, \ldots, K_{n,(n)}\right)$$

and then by Theorem 1

$$K_{r,(r)} \sim \gamma\left(\frac{1}{2}, \frac{a_r^2}{2}\right), \qquad K_{i,(r)} \sim \mathrm{GIG}\left(\frac{1}{2}, \frac{a_i^2}{2}, \frac{a_{c_r(i)}^2}{2}\right) \quad \text{for } i \in V(G_n) \setminus \{r\},$$

where $\left(K_{1,(r)}, \ldots, K_{n,(r)}\right) = \psi_r(\mathbf{K})$. Moreover, the components of the random vector $\left(K_{1,(r)}, \ldots, K_{n,(r)}\right)$ are mutually independent.

This is exactly the multivariate MY property on trees (see Section 1) for $q = \frac{1}{2}$ and

$$K_{G_n} = \left\{k_{ij} = 1, \{i, j\} \in E(G_n); k_{ij} = 0, \{i, j\} \notin E(G_n)\right\}.$$

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