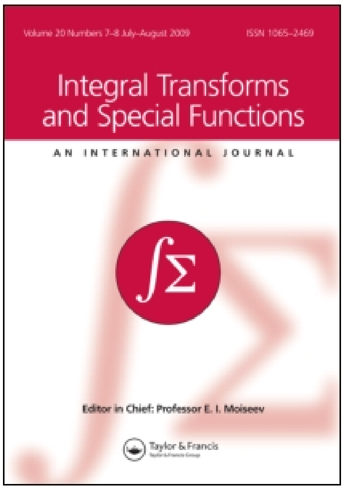

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Lauricella and Humbert functions through probabilistic tools

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New integral formulas for the Humbert Φ_2 function and for the Lauricella F_A and F_D functions are derived. The basic tools are neutrality properties of the probability Dirichlet distribution, the Laplace transform of which is the Humbert Φ_2 function.

Keywords: Lauricella functions; Humbert function; Dirichlet distribution; integral identities; neutrality; independence; beta distribution

AMS 2000 Subject Classification: Primary: 33C65, 60E10; Secondary: 33C15, 33C70

1. Introduction

Since 1941, with papers by Feldheim [4] to 2004, and with a paper by Lijoi and Regazzini [13], more than 60 years have been spent during which the corpus of results on multidimensional hypergeometric functions have been widely used as a tool in the field of probability theory, giving rise to *ad hoc* developments of new specific results on special functions to be applied in the probability domain, for instance, see [10]. However, much more rarely the intrinsic results of the probability theory have been used to give alternative proofs to known results [15] or to provide new extensions to results in the field of multidimensional hypergeometric functions. The aim of this paper is to try to reach that last goal in a restricted area: the closed form calculation of Dirichlet integrals of Humbert and Lauricella hypergeometric functions.

2. Dirichlet distribution and neutrality

We say that a random vector $(X_1, \ldots, X_n)$ has the Dirichlet distribution, $\mathrm{Dir}(\mathbf{a}; b)$, where $\mathbf{a} = (a_1, \ldots, a_n)$ is a vector of positive numbers and $b > 0$, if its distribution is absolutely continuous

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with respect to the Lebesgue measure on $\mathbf{R}^n$ and the density has the form

$$D_n(\mathbf{a}; b; \mathbf{x}) = C\left(1 - \sum_{i=1}^{n} x_i\right)^{b-1} \prod_{i=1}^{n} x_i^{a_i - 1} I_{T_n}(\mathbf{x}), \quad \forall\, \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n, \tag{1}$$

where the constant C has the form

$$C = \frac{\Gamma(b + \sum_{i=1}^{n} a_i)}{\Gamma(b) \prod_{i=1}^{n} \Gamma(a_i)}$$

and T_n is an open unit simplex in $\mathbf{R}^n$, i.e.

$$T_n = \left\{(x_1, \ldots, x_n) : \ x_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i < 1\right\}.$$

If $(X_1, \ldots, X_n)$ is a Dirichlet random vector then, with $X_{n+1} = 1 - \sum_{i=1}^{n} X_1$ we define a random vector of Dirichlet probabilities as $(X_1, \ldots, X_n, X_{n+1})$. If $(X_1, \ldots, X_n) \sim \mathrm{Dir}_n(\mathbf{a}; a_{n+1})$, then we write $(X_1, \ldots, X_{n+1}) \sim \widetilde{\mathrm{Dir}}_{n+1}(\tilde{\mathbf{a}})$, where $\tilde{\mathbf{a}} = (a_1, \ldots, a_{n+1})$.

It is well known that if $(X_1, \ldots, X_{n+1}) \sim \widetilde{\mathrm{Dir}}_{n+1}(\tilde{\mathbf{a}})$, then any sub-vector $(X_{i_1}, \ldots, X_{i_k})$, $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n+1\}$ is Dirichlet

$$(X_{i_1}, \ldots, X_{i_k}) \sim \mathrm{Dir}_k\left((a_{i_1}, \ldots, a_{i_k}); \sum_{i \in \{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_k\}} a_i\right). \tag{2}$$

Let $T = \sum_{l=1}^{k} X_{i_l}$. Then

$$\left(\frac{X_{i_1}}{T}, \ldots, \frac{X_{i_k}}{T}\right) \sim \widetilde{\mathrm{Dir}}_k(a_{i_1}, \ldots, a_{i_k}). \tag{3}$$

Let $(B_1, \ldots, B_K)$ be an arbitrary partition of the set $\{1, \ldots, n+1\}$, and let $S_j = \sum_{i \in B_j} X_i$, $j = 1, \ldots, K$. Then

$$(S_1, \ldots, S_K) \sim \widetilde{\mathrm{Dir}}_K\left(\sum_{i \in B_1} a_i, \ldots, \sum_{i \in B_K} a_i\right). \tag{4}$$

Additionally, the random vectors

$$(S_1, \ldots, S_K), \left(\frac{X_i}{S_1}; i \in B_1\right), \ldots, \left(\frac{X_i}{S_K}; i \in B_K\right) \tag{5}$$

are independent. This property is referred to as neutrality of $(X_1, \ldots, X_{n+1})$ with respect to the partition $(B_1, \ldots, B_K)$ of the set of indices $\{1, \ldots, n+1\}$. It is a well known (see, for instance [6] or [3]) consequence of the representation of the vector of Dirichlet probabilities through independent gamma variables:

$$(X_1, \ldots, X_n, X_{n+1}) = \frac{(U_1, \ldots, U_n, U_{n+1})}{\sum_{i=1}^{n+1} U_i},$$

where $U_1, \ldots, U_n, U_{n+1}$ are independent gamma $G(\sigma, a_i)$ random variables, that is U_i have the density

$$f_i(x) = \frac{\sigma^{a_i}}{\Gamma(a_i)} x^{a_i - 1} \mathrm{e}^{-\sigma x} I_{(0,\infty)}(x)$$

with positive σ and a_i, $i = 1, \ldots, n+1$. Note that for gamma variables, the vector $(U_1, \ldots, U_n, U_{n+1})/\sum_{i=1}^{n+1} U_i$ and the random variable $\sum_{i=1}^{n+1} U_i$ are independent.

The neutrality property plays an important role in the Bayesian approach to nonparametric statistics (see, for instance, the monograph [8]) as well as it is exploited in characterizations of the Dirichlet distribution (and process), see, e.g. [1,7,11]. In this paper, it will be used to derive new identities for Humbert and Lauricella functions.

3. Humbert and Lauricella functions

The $\Phi_2^{(n)}$ function was introduced by Humbert [9] and Exton [4, 2.1.1.2. page 42]. It appears that it is the Laplace transform of the Dirichlet distribution. That is if $\mathbf{X} = (X_1, \ldots, X_n) \sim \mathrm{Dir}_n(\mathbf{a}; b)$, then

$$\Phi_2^{(n)}(\mathbf{a}; b; \mathbf{t}) = E(\exp(< \mathbf{t}, \mathbf{X} >)), \quad \forall\, \mathbf{t} = (t_1, \ldots, t_n) \in \mathbf{R}^n. \tag{6}$$

Thus, it has the integral representation as follows:

$$\Phi_2^{(n)}(\mathbf{a}; b; \mathbf{t}) = \frac{\Gamma(b + \sum_{i=1}^n a_i)}{\Gamma(b)\prod_{i=1}^n \Gamma(a_i)} \int_{T_n} \mathrm{e}^{<\mathbf{x},\mathbf{t}>} \left(1 - \sum_{i=1}^n x_i\right)^{b-1} \prod_{i=1}^n x_i^{a_i - 1} \mathrm{d}x_1, \ldots, \mathrm{d}x_n. \tag{7}$$

Note that $\Phi_2^{(n)}$ can be viewed as a multivariate version of the hypergeometric function ${}_1F_1$. It is known that ${}_1F_1(a; a+b, \cdot)$ is the Laplace transform of the beta $B_I(a, b)$ distribution. But $B_I(a, b) = \mathrm{Dir}_1(a, b)$. Thus Equation (7) implies

$$\Phi_2^{(1)}(a; b; t) = {}_1F_1(a; a+b; t). \tag{8}$$

Let Z be a random variable with the gamma distribution $G(1, c)$. If $\mathbf{X} \sim \mathrm{Dir}_n(\mathbf{a}; b)$ and $\mathbf{X}$ and Z are independent, then define the random vector $\mathbf{Y} = Z\mathbf{X}$. It appears that its Laplace transform is the Lauricella $F_D^{(n)}$ function, i.e.

$$F_D^{(n)}(c, \mathbf{a}; b; \mathbf{t}) = E\left(\exp(\langle \mathbf{t}, \mathbf{Y} \rangle)\right), \quad \mathbf{t} \in \mathbf{R}^n.$$

Thus, conditioning with respect to Z, we get the following integral representation of $F_D^{(n)}$:

$$F_D^{(n)}(c, \mathbf{a}; b; \mathbf{t}) = \frac{1}{\Gamma(c)} \int_0^\infty z^{c-1} \mathrm{e}^{-z} \Phi_2^{(n)}(\mathbf{a}; b; \mathbf{t}z)\, \mathrm{d}z. \tag{9}$$

On the other hand, conditioning with respect to $\mathbf{X}$ we get another integral representation with the Dirichlet density D_n defined in Equation (1):

$$F_D^{(n)}(c, \mathbf{a}; b; \mathbf{t}) = \int_{T_n} (1 - \langle \mathbf{t}, \mathbf{x} \rangle)^{-c} D_n(\mathbf{a}; b; \mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{10}$$

Since the hypergeometric function ${}_2F_1 = F_D^{(1)}$, it appears that ${}_2F_1(c, a; b; \cdot)$ is a Laplace transform of the product of independent beta $B_I(a, b)$ and gamma $G(c)$ random variables.

For any set $B \subset \{1, \ldots, n\}$, $n > \#(B) = k > 0$, define operators:

(1) $\Xi_B^{(a)}$, such that for any $\mathbf{x} \in \mathbf{R}^n$ the vector $\Xi_B^{(a)}(\mathbf{x})$ changes x_i into a for $i \notin B$, while other components of $\mathbf{x}$ remain the same.
(2) Ψ_B, such that for any $\mathbf{x} \in \mathbf{R}^n$ the k-dimensional vector $\Psi_B(\mathbf{x})$ is created from $\mathbf{x}$ by deleting x_i for $i \notin B$, and shifting the remaining components to the left.

Note that by Equation (2) we immediately get

$$\Phi_2^{(n)}\left(\mathbf{a};\, b;\, \Xi_B^{(0)}(\mathbf{t})\right) = \Phi_2^{(k)}\left(\Psi_B(\mathbf{a});\, b + \sum_{i\in B^c} a_i;\, \Psi_B(\mathbf{t})\right). \tag{11}$$

Similarly, Equation (4) implies

$$\Phi_2^{(n)}\left(\mathbf{a};\, b;\, \Xi_B^{(t)}(\mathbf{t})\right) = \Phi_2^{(k+1)}\left(\left(\Psi_B(\mathbf{a}),\, \sum_{i\in B^c} a_i\right);\, b;\, (\Psi_B(\mathbf{t}), t)\right), \tag{12}$$

where $B^c = \{1, \dots, n\} \setminus B$.

Due to the representation (9), the analogous principles hold for the Lauricella function $F_D^{(n)}$, i.e.

$$F_D^{(n)}\left(c, \mathbf{a};\, b;\, \Xi_B^{(0)}(\mathbf{t})\right) = F_D^{(k)}\left(c,\, \Psi_B(\mathbf{a});\, b + \sum_{i\in B^c} a_i;\, \Psi_B(\mathbf{t})\right), \tag{13}$$

$$F_D^{(n)}\left(c, \mathbf{a};\, b;\, \Xi_B^{(t)}(\mathbf{t})\right) = F_D^{(k+1)}\left(c,\, \left(\Psi_B(\mathbf{a}),\, \sum_{i\in B^c} a_i\right);\, b;\, (\Psi_B(\mathbf{t}), t)\right). \tag{14}$$

Our main result is a rather general integral identity involving the Humbert Φ_2 functions. It is a consequence of the neutrality property of the Dirichlet distribution. Together with the reduction and transformation principles formulated above, it will be a source of many other integral relations, some of them being new and some of them being already known. Anyway they will all follow from one basic identity. This identity will be conveniently written in terms of somewhat transformed $\Phi_2^{(n)}$ functions. Namely, for any positive integer m, any $j \in \{1, \dots, m\}$, any $\mathbf{a} \in (0, \infty)^{m+1}$ and any $\mathbf{t} \in \mathbf{R}^{m+1}$, we define

$$\Upsilon(j; m+1; \mathbf{a}; \mathbf{t}) = \mathrm{e}^{t_j}\, \Phi_2^{(m)}\left(\tilde{\Psi}_j(\mathbf{a});\, a_j;\, (\tilde{\Psi}_j(\mathbf{t}) - t_j \mathbf{1}_m)\right), \tag{15}$$

where $\mathbf{1}_m$ denotes the m-dimensional vector of 1's, and $\tilde{\Psi}_j(\mathbf{x})$ suppresses the $(m+1)$ dimensional vector $\mathbf{x}$ into the m-dimensional vector by deleting its jth component. Integrating both sides of Equation (15) with respect to the gamma density, we introduce

$$H(j; m+1; c, \mathbf{a}; \mathbf{t}) = \frac{1}{\Gamma(c)} \int_0^\infty z^{c-1} \mathrm{e}^{-z}\, \Upsilon(j; m+1; \mathbf{a}; z\mathbf{t})\, \mathrm{d}z. \tag{16}$$

Thus we have

$$H(j; m+1; c, \mathbf{a}; \mathbf{t}) = \frac{1}{\Gamma(c)} \int_0^\infty z^{c-1} \mathrm{e}^{-z(1-t_j)}\, \Phi_2^{(m)}\left(\tilde{\Psi}_j(\mathbf{a});\, a_j;\, z(\tilde{\Psi}_j(\mathbf{t}) - t_j \mathbf{1}_m)\right) \mathrm{d}z, \tag{17}$$

and then

$$H(j; m+1; c, \mathbf{a}; \mathbf{t}) = \frac{1}{(1-t_j)^c}\, F_D^{(m)}\left(c,\, \tilde{\Psi}_j(\mathbf{a});\, a_j;\, \frac{\tilde{\Psi}_j(\mathbf{t}) - t_j \mathbf{1}_m}{1 - t_j}\right). \tag{18}$$

The above formula can be simplified further.

LEMMA 1 *Let* $a_j^* = a_j - \sum_{i=1, i\neq j}^{m+1} a_i > 0$. *Then,*

$$H(j; m+1; c, \mathbf{a}; \mathbf{t}) = F_D^{(m+1)}\left(c,\, \mathbf{a}_{(j)}^{(*)};\, a_j;\, \mathbf{t}\right), \tag{19}$$

where the modified sequence of parameters is the following: $\mathbf{a}_{(j)}^{(*)} = a_1, \dots, a_j^*, \dots, a_{m+1}$.

Proof From the integral representation of F_D, see [4, 2.3.6, page 49], one can write

$$F_D^{(m)}\left(c, \tilde{\Psi}_j(\mathbf{a}); a_j; \frac{\tilde{\Psi}_j(\mathbf{t}) - t_j \mathbf{1}_m}{1 - t_j}\right) = F_D^{(m+1)}\left(c, \tilde{\Psi}_j(\mathbf{a}), a_j^*; a_j; \frac{\tilde{\Psi}_j(\mathbf{t}) - t_j \mathbf{1}_m}{1 - t_j}, \frac{-t_j}{1 - t_j}\right) \tag{20}$$

and from Exton [4, A 2.2, page 286] we get the transformation

$$F_D^{(m+1)}\left(c, \mathbf{a}_{(j)}^{(*)}; a_j; \mathbf{t}\right) = \frac{1}{(1 - t_j)^c} F_D^{(m+1)}\left(c, \tilde{\Psi}_j(\mathbf{a}), a_j^*; a_j; \frac{\tilde{\Psi}_j(\mathbf{t}) - t_j \mathbf{1}_m}{1 - t_j}, \frac{-t_j}{1 - t_j}\right). \tag{21}$$

■

COROLLARY 1 *Let* $a_j^* = a_j - \sum_{i=1, i \neq j}^{m+1} a_i > 0$. *Then,*

$$\Upsilon(j; m+1; \mathbf{a}; \mathbf{t}) = \Phi_2^{(m+1)}\left(\mathbf{a}_{(j)}^{(*)}; a_j; \mathbf{t}\right), \tag{22}$$

where the modified sequence of parameters is the following: $\mathbf{a}_{(j)}^{(*)} = (a_1, \ldots, a_j^*, \ldots, a_{m+1})$.

The proof, analogous to the previous one, uses the Erdelyi transform, see [4, 5.10.15, page 177]. The case $m = 1$ gives rise to the Burchnall and Chaundy relation, see [4, 4.12.2, page 117],

$$\Phi_2^{(2)}(a_1, a_2 - a_1; a_2; t_1, t_2) = \mathrm{e}^{t_2} {}_1F_1(a_1; a_2; t_1 - t_2). \tag{23}$$

4. Main identity and its consequences

Our main result is formulated as an integral identity involving the Dirichlet integral of the function Υ:

THEOREM 1 *Let* $(B_j)_{j=1,\ldots,K}$ *be a partition of the set* $\{1, \ldots, n+1\}$.
Let $B_j = \{i_1^{(j)}, \ldots, i_{m_j}^{(j)}\}$, $j = 1, \ldots, K$. *Let* $\mathbf{a} \in (0, \infty)^{n+1}$ *and* $J \in \{1, \ldots, n+1\}$.
Then for any $\mathbf{t} \in \mathbf{R}^{n+1}$ *and for any* $r_j \in B_j$, $j = 1, \ldots, K$,

$$\Upsilon(J; n+1; \mathbf{a}; \mathbf{t}) = \int_{T_{K-1}} \left[\prod_{j=1}^{K} \Upsilon\left(r_j; m_j; \Psi_{B_j}(\mathbf{a}); y_j \Psi_{B_j}(\mathbf{t})\right)\right] D_{K-1}(A_1, \ldots, A_{K-1}; A_K; \mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{24}$$

where $y_K = 1 - \sum_{i=1}^{K-1} y_i$ *and* $A_j = \sum_{i \in B_j} a_i$, $j = 1, \ldots, K$.

Proof Consider a random vector of Dirichlet probabilities $\mathbf{X} = (X_1, \ldots, X_{n+1}) \sim \widetilde{\mathrm{Dir}}_{n+1}(\mathbf{a})$. Then Equation (2) implies that its sub-vector $\tilde{\Psi}_J(\mathbf{X})$ is Dirichlet $\mathrm{Dir}_n(\tilde{\Psi}_J(\mathbf{a}); a_J)$. Since $\langle \mathbf{X}, \mathbf{1}_{n+1} \rangle = 1$, then the Laplace transform of $\mathbf{X}$ has the form

$$L_{\mathbf{X}}(\mathbf{t}) = E\left(\mathrm{e}^{\langle \mathbf{t}, \mathbf{X} \rangle}\right) = \mathrm{e}^{t_J} E\left(\mathrm{e}^{\langle \tilde{\Psi}_J(\mathbf{t}) - t_J \mathbf{1}_n, \tilde{\Psi}_J(\mathbf{X}) \rangle}\right) = \mathrm{e}^{t_J} \Phi_2^{(n)}\left(\tilde{\Psi}_J(\mathbf{a}); a_J; \tilde{\Psi}_J(\mathbf{t}) - t_j \mathbf{1}_n\right),$$

and thus

$$L_{\mathbf{X}}(\mathbf{t}) = \Upsilon(J; n+1; \mathbf{a}; \mathbf{t}), \tag{25}$$

which is the left-hand side of Equation (24). On the other hand, Equation (25) implies that $\Upsilon(J; n+1; \mathbf{a}; \cdot)$ is a Laplace transform of the Dirichlet probabilities vector for any $J \in \{1, \ldots, n+1\}$.

Note that the neutrality of the Dirichlet distribution, see Equation (5), implies that

$$\left(\frac{X_i}{S_1}; i \in B_1\right), \dots, \left(\frac{X_i}{S_K}; i \in B_K\right)$$

are conditionally independent given $(S_1, \dots, S_K)$, and thus

$$L_{\mathbf{X}}(\mathbf{t}) = E\left[\exp\left(\sum_{j=1}^{K} S_j \sum_{i \in B_j} t_i \frac{X_i}{S_j}\right)\right] = E\left[\prod_{j=1}^{K} E\left(\exp\left(S_j \sum_{i \in B_j} t_i \frac{X_i}{S_j}\right)\middle| S_j\right)\right].$$

Moreover Equation (3) together with Equation (25) imply

$$E\left(\exp\left(S_j \sum_{i \in B_j} t_i \frac{X_i}{S_j}\right)\middle| S_j\right) = \Upsilon\left(r_j; m_j; \Psi_{B_j}(\mathbf{a}); S_j \Psi_{B_j}(\mathbf{t})\right) \quad j = 1, \dots, K.$$

Now by Equation (4) we obtain that $L_{\mathbf{X}}(\mathbf{t})$ is equal to the right-hand side of Equation (24), which in view of Equation (25) proves the result. ■

Corollary 2 *Let $A = (a_{ij})_{j=1,\dots,m_i, i=1,\dots,K}$ be a triangular array of positive numbers and $(b_l)_{l=1,\dots,K}$ be a vector with positive components. Let $N = \sum_{i=1}^{K} m_i$. Then for any $T = (t_{ij})_{j=1,\dots,m_i, i=1,\dots,K}$,*

$$\Phi_2^{(N)}(A; b; T) = \int_{T_{K-1}} \left(\prod_{i=1}^{K} \Phi_2^{(m_i)}(\mathbf{a}_i; b_i; y_i \mathbf{t}_i)\right) \times D_{K-1}\left((a_{1\cdot} + b_1, \dots, a_{(K-1)\cdot} + b_{K-1}); a_{K\cdot} + b_K; \mathbf{y}\right) \mathrm{d}\mathbf{y}, \tag{26}$$

where $y_K = 1 - \sum_{i=1}^{K-1} y_i$, $\mathbf{a}_i = (a_{i1}, \dots, a_{im_i})$, $\mathbf{t}_i = (t_{i1}, \dots, t_{im_i})$, and $a_{i\cdot} = \sum_{j=1}^{m_i} a_{ij}$, $i = 1, \dots, K$, and $b = \sum_{i=1}^{K} b_i$.

Proof The result follows from Theorem 1 by specifying the set of indices $\{1, \dots, n+1\} = \{(i, j) : j = 1, \dots, m_i + 1, i = 1, \dots, K\}$ with the partition defined by blocks $B_i = \{(i, 1), \dots, (i, m_i + 1)\}$, $i = 1, \dots, K$, and taking $\mathbf{a} = \{\alpha_{i,j} : \alpha_{i,j} = a_{ij}, j = 1, \dots, m_i, \alpha_{i,m_i+1} = b_i, : i = 1, \dots, K\}$. With these parameters (24) for $t_{i,m_i+1} = 0$ and $J = (1, m_1 + 1)$ gives Equation (26). ■

The case $K = 2$, by integrating both sides of Equation (26) in the Laplace fashion, is related to the function ${}^{(k)}_{(1)}E_D^{(N)}$ introduced by Exton [4, 3.4.4.1, page 95] with the integral representation given in [4, 3.4.2.4, page 93]. Its multivariate extension has the form

$${}^{(m_1,\dots,m_K)}_{(K)}E_D^{(N)}(c, \mathbf{a}_i; \mathbf{b}; \mathbf{t}_i) = \frac{1}{\Gamma(c)} \int_0^\infty z^{c-1} \mathrm{e}^{-z} \left(\prod_{i=1}^{K} \Phi_2^{(m_i)}(\mathbf{a}_i; b_i; z\mathbf{t}_i)\right) \mathrm{d}z, \tag{27}$$

where $\mathbf{b} = (b_1, \dots, b_K)$.

Corollary 3 *Let $A = (a_{ij})_{j=1,\dots,m_i, i=1,\dots,K}$ be a triangular array of positive numbers and $(b_l)_{l=1,\dots,K}$ be a vector with positive components. Let $N = \sum_{i=1}^{K} m_i$. Then for any*

$T = (t_{ij})_{j=1,\dots,m_i,i=1,\dots,K}$,

$$F_D^{(N)}(c, A; b; T) = \int_{T_{K-1}} \left({}^{(m_1,\dots,m_K)}_{(K)} E_D^{(N)}(c, \mathbf{a}_i; \mathbf{b}; y_i\mathbf{t}_i) \right) \times D_{K-1}((a_{1\cdot} + b_1, \dots, a_{K\cdot} + b_K); b; \mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{28}$$

where $y_K = 1 - \sum_{i=1}^{K-1} y_i$, $\mathbf{a}_i = (a_{i1}, \dots, a_{im_i})$, $\mathbf{t}_i = (t_{i1}, \dots, t_{im_i})$, *and* $a_{i\cdot} = \sum_{j=1}^{m_i} a_{ij}$, $i = 1, \dots, K$, *and* $b = \sum_{i=1}^{K} b_i$.

Proof The right-hand side of Equation (28), due to the definition (27) is just the integral of the right-hand side of Equation (26) with respect to the gamma density. Now the result follows by Equation (9). ∎

COROLLARY 4 *Let* $\mathbf{a} = (a_1, \dots, a_K)$ *and* $\mathbf{b} = (b_1, \dots, b_K)$ *be constant vectors with positive components, and let* $b = \sum_{k=1}^{K} b_k$. *Then*

$$\Phi_2^{(K)}(\mathbf{a}; b; \mathbf{t}) = \int_{T_{K-1}} \left(\prod_{i=1}^{K} {}_1F_1(a_i; a_i + b_i; y_i t_i) \right) D_{K-1}(\mathbf{a} + \mathbf{b}; \mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{29}$$

where $y_K = 1 - \sum_{i=1}^{K-1} y_i$ *and* $\mathbf{a} + \mathbf{b}$ *as a parameter in* D_{K-1} *should be read as* $(a_1 + b_1, \dots, a_{K-1} + b_{K-1}); a_K + b_K$.

Proof It follows from Equation (26) by taking $m_i = 1$ for $i = 1, \dots, K$. ∎

This gives the fourth way to solve the problem 11000 of American Mathematical Monthly [12]:

$$\int_0^{\pi/2} \operatorname{erf}(\sqrt{t}\cos\theta) \operatorname{erf}(\sqrt{t}\sin\theta) \sin(2\theta)\, \mathrm{d}\theta = 1 - \frac{1 - \mathrm{e}^{-t}}{t} \tag{30}$$

by choosing the parameters $L = 2$, $a_1 = a_2 = 1$, $b_1 = b_2 = 1/2$, $t_1 = t_2 = t$ in Equation (29). It appears, see [2], that Equation (30) is equivalent to

$$\frac{8}{\pi} \int_0^1 \sqrt{y(1-y)}\, {}_1F_1\left(1; \frac{3}{2}; ty\right) {}_1F_1\left(1; \frac{3}{2}; t(1-y)\right) \mathrm{d}y = {}_1F_1(2; 3; t). \tag{31}$$

To see this, one can use the representation of the error function in terms of function ${}_1F_1$

$$\operatorname{erf}(\xi) = \left(\frac{2\xi}{\sqrt{\pi}} \mathrm{e}^{-\xi^2} \right) {}_1F_1\left(1; \frac{3}{2}; \xi^2\right). \tag{32}$$

Formula (29) can be found in [4, 2.7.11, page 61]. Note that the product of Kummer confluent hypergeometric functions ${}_1F_1$ is known to be represented by a Kampé de Fériet function (see [14, 1.5, formula 31]) and from formula (29) we get the identity:

$$\Phi_2^{(K)}(\mathbf{a}; b; \mathbf{t}) = \int_{T_{K-1}} F^{0:1;.;1}_{0:1;.;1} \left[\begin{matrix} - : & a_1; .; & a_K; & \\ & & & y_1 t_1, ., y_K t_K \\ - : & a_1 + b_1; .; & a_K + b_K; & \end{matrix} \right] D_{K-1}(\mathbf{a} + \mathbf{b}; \mathbf{y})\, \mathrm{d}\mathbf{y} \tag{33}$$

such that the Laplace integral gives a Lauricella function F_D on the left-hand side and a F_A in the integrand of the right-hand side. In this way, from the above result, we get the identity given in [4, 2.7.9, page 60].

The Lauricella F_A function is a Laplace transform of the product of a gamma random variable by an independent random vector having as its components independent beta random variables. More precisely, let Y be a gamma $G(a_0)$ random variable and let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector with independent components such that X_i is beta $B_I(a_i, b_i)$, i.e. its density has the form

$$B_{a_i,b_i}(x) = \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)\Gamma(b_i)} x^{a_i-1}(1-x)^{b_i-1} I_{(0,1)}(x),$$

and let Y and $\mathbf{X}$ be independent. Then the Lauricella $F_A^{(n)}$ function is defined as

$$F_A^{(n)}(c; \mathbf{a}; \mathbf{b}; \mathbf{t}) = E(\exp\langle \mathbf{t}, Y\mathbf{X}\rangle).$$

Consequently, its integral representations are as follows:

$$F_A^{(n)}(c; \mathbf{a}; \mathbf{b}; \mathbf{t}) = \frac{1}{\Gamma(c)} \int_0^\infty y^{c-1} \mathrm{e}^{-y} \prod_{i=1}^{K} {}_1F_1(a_j, a_j + b_j, t_j y)\, \mathrm{d}y \tag{34}$$

or

$$F_A^{(n)}(c; \mathbf{a}; \mathbf{b}; \mathbf{t}) = \int_{[0,1]^K} (1 - \langle \mathbf{t}, \mathbf{x}\rangle)^{-c} \prod_{i=1}^{K} B_{a_i,b_i}(\mathrm{d}x_i). \tag{35}$$

Corollary 5 *Let* $\mathbf{a} = (a_1, \ldots, a_K)$ *and* $\mathbf{b} = (b_1, \ldots, b_K)$ *be constant vectors with positive components, and let* $b = \sum_{i=1}^{K} b_i$. *Then*

$$F_D^{(K)}(c, \mathbf{a}; b; \mathbf{t}) = \int_{T_{K-1}} F_A^{(K)}(c, \mathbf{a}; \mathbf{b}; \mathbf{yt}) D_{K-1}(\mathbf{a} + \mathbf{b}; \mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{36}$$

where $\mathbf{yt} = (y_1 t_1, \ldots, y_K t_K)$, $y_K = 1 - \sum_{i=1}^{K-1} y_i$ *and* $\mathbf{a} + \mathbf{b}$ *as a parameter in* D_{K-1} *should be read as* $(a_1 + b_1, \ldots, a_{K-1} + b_{K-1}); a_K + b_K$.

Proof Integrate $\Phi_2^{(K)}(\mathbf{a}; b; z\mathbf{t})$ as a function of z with respect to the gamma density. Then one gets by Equation (9) the right-hand side of Equation (36). The left-hand side follows then by Equations (29) and (34). ■

Corollary 6 *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be a constant vector with positive components and* b, c *are positive constants. Then*

$$\Phi_2^{(r)}(\mathbf{a}; b + c; \mathbf{t}) = \int_0^1 \Phi_2^{(r)}(\mathbf{a}; b; y\mathbf{t}) B_I(a + b; c; y)\, \mathrm{d}y, \tag{37}$$

where $a = \sum_{i=1}^{r} a_i$.

Proof This identity follows by taking $t_{ij} = 0$ for $j = 1, \ldots, m_i$, $i = 2, \ldots, K$, and $r = m_1$ in Equation (26) and denoting $a = \sum_{j=1}^{m_1} a_{1j}$, $b = b_1$ and $c = \sum_{i=2}^{K} (b_i + \sum_{j=1}^{m_i} a_{ij})$. Of course, in the above formula, we used the fact that univariate marginal distributions of Dirichlet are beta according to the general rule (2). ■

Note that as a particular case of Equation (37) (for $r = 1$) by Equation (8), we have

$${}_1F_1(a; a + b + c; t) = \int_0^1 {}_1F_1(a; a + b; yt) B_I(a + b; c; y)\, \mathrm{d}y, \tag{38}$$

which is responsible for the well-known fact that the product of independent beta $B_I(a, b)$ and $B_1(a + b, c)$ random variables is again beta $B_I(a, b + c)$.

COROLLARY 7 *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *be* K*-dimensional vectors with positive components and let* a_0 *be a positive number. Then for any* $\mathbf{t}$,

$$F_A^{(K)}(a_0; \mathbf{a}; \mathbf{b}+\mathbf{c}; \mathbf{t}) = \int_{(0,1)^K} F_A^{(K)}(a_0; \mathbf{a}; \mathbf{b}; \mathbf{yt}) \prod_{i=1}^{K} B_I(a_i+b_i; c_i; y_i)\,\mathrm{d}y_1, \ldots, \mathrm{d}y_K. \quad (39)$$

Proof First we take a product of each side of Equation (38) written for $a = a_i, b = b_i, c = c_i, t = zt_i$, for $i = 1, \ldots, K$. Then we integrate both sides with respect to the variable z using the gamma density. Then the result follows by the representation (34) of F_A. ■

COROLLARY 8 *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be a constant vector with positive components and* a_0, b, c *are positive constants. Then*

$$F_D^{(r)}(a_0; \mathbf{a}; b+c; \mathbf{t}) = \int_0^1 F_D^{(r)}(a_0; \mathbf{a}; b; y\mathbf{t}) B_I(a+b; c; y)\,\mathrm{d}y, \quad (40)$$

where $a = \sum_{i=1}^{r} a_i$.

Proof This follows by integrating with respect to the variable z the function $\Phi_2^{(r)}(\mathbf{a}; b+c; z\mathbf{t})$ using the gamma density with the parameter a_0. Then we get Equation (40) by Equation (9) and the identity (37). ■

Note that since the sum of components of the Dirichlet distribution $\mathrm{Dir}_n(\mathbf{a}; b)$ is beta $B_I(a, b)$ with $a = \sum_i a_i$, we have

$$\Phi_2^{(n)}(\mathbf{a}; b; (t, \ldots, t)) = {}_1F_1(a; a+b; t).$$

Consequently, by Equation (29), we have

$${}_1F_1(a; a+b; t) = \int_{T_{K-1}} \left(\prod_{i=1}^{K} {}_1F_1(a_i; a_i+b_i; y_i t) \right) D_{K-1}(\mathbf{a}+\mathbf{b}; \mathbf{y})\,\mathrm{d}\mathbf{y}. \quad (41)$$

We add a superscript (j), in the above formula taken for t changed into $t_j z$, $j = 1, \ldots, M$. Then we take product of both sides with respect to j. Finally, we integrate with respect to z and the gamma density. These leads to a new identity involving Lauricella F_A functions of different dimensions.

COROLLARY 9 *Let* $\mathbf{a} = [a_{ij}]$ *and* $\mathbf{b} = [b_{ij}]$ *be two* $K \times M$ *matrices with positive entries and let* $a_0 > 0$. *Then*

$$F_A^{(M)}(a_0; \mathbf{a}_{\cdot}; \mathbf{b}_{\cdot},; (t_1, \ldots, t_M)) = \int_{T_K^M} F_A^{(KM)}(a_0; \mathbf{a}; \mathbf{b}; (t_1\mathbf{y}_1, \ldots, t_M\mathbf{y}_M)) \times \prod_{j=1}^{M} D_{K-1}((a_{ij}+b_{ij}, i = 1, \ldots, K); \mathbf{y}_j)\,\mathrm{d}\mathbf{y}_1 \ldots \mathrm{d}\mathbf{y}_M, \quad (42)$$

where $\mathbf{a}_{\cdot} = (\mathbf{a}_{\cdot,j}, j = 1, \ldots, M)$ *with* $\mathbf{a}_{\cdot,j} = \sum_{i=1}^{K} a_{ij}$ *and* $\mathbf{b}_{\cdot} = (\mathbf{b}_{\cdot,j}, j = 1, \ldots, M)$ *with* $\mathbf{b}_{\cdot,j} = \sum_{i=1}^{K} b_{ij}$.

Note that in the above formulation $\mathbf{y}_j = (y_{1,j}, \ldots, y_{K,j})$ and $y_{K,j} = 1 - \sum_{i=1}^{K-1} y_{i,j}$; moreover $\mathrm{d}\mathbf{y}_j$ denotes $\mathrm{d}y_{1,j} \ldots \mathrm{d}y_{(K-1),j}$, $j = 1, \ldots, M$, and the integral is over the cartesian product of M K-dimensional unit simplexes.

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