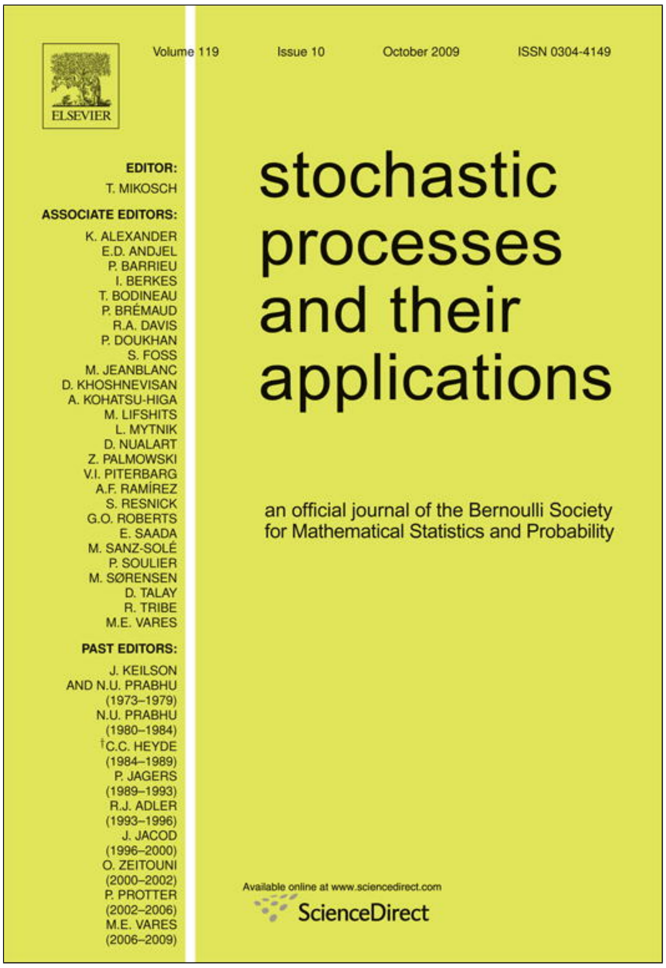
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Tree structured independence for exponential Brownian functionals☆

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Abstract

The product of GIG and gamma distributions is preserved under the transformation $(x, y) \mapsto ((x+y)^{-1}, x^{-1} - (x+y)^{-1})$. It is also known that this independence property may be reformulated and extended to an analogous property on trees. The purpose of this article is to show the independence property on trees, which was originally derived outside the framework of stochastic processes, in terms of a family of exponential Brownian functionals.

MSC: 60J65; 60E05

Keywords: Brownian motion; Exponential Brownian functionals; Generalized inverse Gaussian distribution; Gamma distribution; Independence properties; Initial enlargements of filtrations; Directed and undirected trees

1. Introduction

In this article we are concerned with the following two probability distributions on $(0, \infty)$. For, $q, a > 0$, we denote by $\gamma(q, a)$ the gamma distribution with the density function

$$f(x) = \frac{a^q}{\Gamma(q)} x^{q-1} \mathrm{e}^{-ax}, \quad x > 0.$$

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For $q \in \mathbb{R}$, $a, b > 0$, we denote by GIG$(q; a, b)$ the GIG (generalized inverse Gaussian) distribution whose density function is given by

$$g(x) = \left(\frac{a}{b}\right)^{q/2} \frac{1}{2K_q\left(2\sqrt{ab}\right)} x^{q-1} \exp\left(-ax - \frac{b}{x}\right), \quad x > 0,$$

where K_q denotes the modified Bessel function of the third kind with index q. Note that GIG$(q; a, 0)$ for positive q may be identified with $\gamma(q, a)$.

Let X and Y be independent random variables which obey GIG$(-q; a, b)$ and $\gamma(q, a)$, respectively, for some $q, a, b > 0$. Then it is well known and is easy to show that the distribution of the sum $X + Y$ is GIG$(q; a, b)$. It is also easy to see $(X + Y)^{-1} \sim$ GIG$(-q; b, a)$. For more properties of GIG distributions, see [7,19].

During a study of some exponential type functionals of Brownian motion, Matsumoto and Yor [15] have shown that, for the above mentioned variables X and Y, the random variables U and V given by

$$U = \frac{1}{X} - \frac{1}{X+Y}, \qquad V = \frac{1}{X+Y}$$

are also independent and their distributions are again a gamma and a GIG distributions. Precisely, the distribution of U is $\gamma(q, b)$ and, of course, that of V is GIG$(-q; b, a)$ (see also [20], p. 43). In fact, the special case $a = b$ was considered in [15] and the result was extended in Letac and Wesołowski [9], whose main result says that we have such an independence property only for this pair of probability distributions (a regression version of this characterization was shown in [21]). The mentioned independence property and its converse were also considered in a matrix setting involving a matrix variate GIG and Wishart distributions (see [9,12,21]).

It is well known that the GIG distributions appear in a study of Brownian motion. If we consider the first and the last passage times of one-dimensional Brownian motion with constant drift, their distributions are GIG$(-1/2; a, b)$ and GIG$(1/2; a, b)$ for some $a > 0$ and $b \geq 0$, respectively. In fact, in [17], it is shown that, when $q = 1/2$, the above mentioned independence property can be interpreted through these random times. Moreover, in [17], a better understanding in the case of $a = b$ has been obtained by using the exponential type functionals of Brownian motion.

On the other hand, the independence property and its converse for the pairs of these two distributions has been extended to the multivariate situation by Massam and Wesołowski [11] in terms of directed trees, where the result was derived outside the framework of stochastic processes (see also [8]). In [22], this multivariate independence property for the special case of $q = 1/2$ was studied and rederived by using the hitting times of a family of Brownian motions built upon a single Brownian motion.

In this article, considering the general case of parameters in the multivariate tree setting, we give a complete description of the independence property of the GIG and gamma distributions in terms of properties of exponential functionals of Brownian motion.

This paper is organized as follows. Below in this section, several basic facts on the exponential type functionals of Brownian motion are presented and discussed for families of Brownian motions and their functionals constructed from a single Brownian motion. These families play important roles in Section 3, where the main result on the conditional independence of functionals of these families is derived. First, in Section 2, a separate derivation of the original bivariate independence property is presented mostly in order to introduce some ideas which we develop in the general multivariate case in Section 3. This derivation is somewhat different than the original one given in [17], mostly due to the fact that we consider a symmetric version of the independence property. In the final Section 4, we explain how the conditional independence derived in Section 3 is connected to the independence property on trees.

Let $B = \{B_s\}_{s\ge 0}$ be a Brownian motion with drift $-\mu$ starting from 0, $\mu \in \mathbb{R}$. Since μ will be fixed throughout the paper, we use this notation only to shorten the formulae and thus the dependence on μ will not be indicated. We denote by $\{\mathcal{B}_s\}$ the natural filtration of B. Let us consider the exponential Brownian functionals $A_{s,t}$, $0 \le s < t < \infty$, given by

$$A_{s,t} = \int_s^t \exp(2(B_u - B_s))\mathrm{d}u.$$

In particular, we denote $A_{0,t}$ by A_t.

If $\mu > 0$, $A_{s,t}$ converges almost surely as $t \to \infty$, and we define for $s \ge 0$

$$A_{s,\infty} := \lim_{t\to\infty} A_{s,t} = \int_s^\infty \exp(2(B_u - B_s))\mathrm{d}u.$$

By the Markov property of Brownian motion, $A_{s,\infty}$ is independent of $\mathcal{B}_s$. These exponential Wiener functionals have been widely investigated in many fields, including mathematical finance (see e.g. [4,23]), hyperbolic spaces (see e.g. [1,5]) or diffusion processes in random environment (see e.g. [2]). It is worth to mention that the joint law of (B_t, A_t) is quite complicated and it can be expressed through certain special functions. We refer to [5,23] for examples of description of this law. It is well known (see [3,26]) that for $\mu > 0$ the reciprocal of the random variable $A_\infty := A_{0,\infty}$ obeys $\gamma(\mu, 1/2)$ distribution. Moreover, $A_{s,\infty} \stackrel{d}{=} A_\infty$ for any $s \ge 0$. We also refer to [18,25], where several topics on these exponential type functionals of Brownian motion are gathered. We would like to emphasize that our notation is slightly different from that used in mentioned above literature.

We also consider the stochastic processes $e = \{e_s\}_{s\ge 0}$ and $Z = \{Z_s\}_{s\ge 0}$ given by

$$e_s = \exp(B_s) \quad \text{and} \quad Z_s = (e_s)^{-1} A_s. \tag{1.1}$$

The process Z has been investigated by Matsumoto and Yor in a series of papers [13–15]. They showed that the conditional law $\mathcal{L}(e_t | Z_{[0,t]} = z)$ of e_t is $\mathrm{GIG}(-\mu; 1/2z_t, 1/2z_t)$ for any $t > 0$ and that Z is a diffusion process with respect to its own filtration $\{\mathcal{Z}_s\}$, where, for a stochastic process $X = \{X_s\}_{s\ge 0}$, $X_{[0,t]}$ denotes the original process X restricted to the interval $[0, t]$ and z is a continuous path defined on $[0, t]$. Here and in the following, we understand by $\mathcal{L}(U|Y = y)$ a version of regular conditional distribution of a random variable U given $Y = y$.

By definition it is obvious that $\mathcal{Z}_s \subseteq \mathcal{B}_s$ for any $s \ge 0$. More precisely, it is known (see Lemma 10.1, [15]) that

$$\mathcal{B}_s = \mathcal{Z}_s \vee \sigma\{A_s\} = \mathcal{Z}_s \vee \sigma\{B_s\}.$$

Moreover, when $\mu > 0$, the stochastic process Z and the random variable A_∞ are independent.

The following identity (1.2) (see Proposition 3.1, [16]) plays an important role in this article. Its proof is based on the theory of the initial enlargements of filtrations. Fix $\mu > 0$ and let $\{\widehat{\mathcal{B}_s}\}$ be the initially enlarged filtration of $\{\mathcal{B}_s\}$ given by $\widehat{\mathcal{B}_s} = \mathcal{B}_s \vee \sigma\{A_\infty\}$. Applying the enlargement formula from Theorem 12.1, [24], it can be shown that there exists a Brownian motion

$B^* = \{B_s^*\}_{s\geq 0}$ with drift $+\mu$ with respect to $\{\widehat{\mathcal{B}}_s\}$ such that

$$B_s = B_s^* - \int_0^s \frac{\exp(2B_u)}{A_\infty - A_u}\mathrm{d}u.$$

The Brownian motion B^* and A_∞ are independent, since $\widehat{\mathcal{B}}_0 = \sigma\{A_\infty\}$. To have a more complete picture of this subject we refer to Jeulin [6] and the monograph by Mansuy and Yor [10]. Therefore, B_s is the solution of

$$z_s = B_s^* - \int_0^s \frac{\exp(2z_u)}{A_\infty - \int_0^u \exp(2z_w)\mathrm{d}w}\mathrm{d}u.$$

This equation, considered as an ordinary equation, has a unique solution

$$z_s = B_s^* - \log\left(1 + \frac{A_s^*}{A_\infty}\right),$$

where $A_s^* = \int_0^s \exp(2B_u^*)\mathrm{d}u$ (see Section 3, [16]). Moreover, some algebra yields

$$\left(1 + \frac{A_s^*}{A_\infty}\right)\left(1 - \frac{A_s}{A_\infty}\right) = 1.$$

To summarize the above, we give the following

Proposition 1.1 (*see Proposition 3.1, [16]*). *Fix $\mu > 0$ and let B a Brownian motion with drift $-\mu$. Then the process $B^* = \{B_s^*\}_{s\geq 0}$ given by*

$$B_s^* = B_s - \log\left(1 - \frac{A_s}{A_\infty}\right), \tag{1.2}$$

is a $\{\widehat{\mathcal{B}}_s\}$-Brownian motion with drift $+\mu$, which is independent of A_∞.

We also put $e_s^* = \exp(B_s^*)$, $Z_s^* = (e_s^*)^{-1}A_s^*$ and $\mathcal{Z}_s^* = \sigma(Z_u^* : u \leq s)$ – the natural filtration of Z^*. We can deduce the following identities given in [17], which will also play important roles in this article. For any $s \geq 0$:

$$Z_s = Z_s^*, \tag{1.3}$$

$$e_s^* = e_s + \frac{Z_s}{A_{s,\infty}}, \tag{1.4}$$

$$\frac{1}{e_s} - \frac{1}{e_s^*} = \frac{Z_s^*}{A_\infty} = \frac{Z_s}{A_\infty}. \tag{1.5}$$

We end this section with recalling two convenient facts regarding conditional independence which will be helpful to better understand some of the arguments in the following sections.

Lemma 1.2. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector, Y be a random variable and $Z_1, \ldots, Z_{n+1}$ be $\mathcal{F}$-measurable random variables. Moreover, assume that $\sigma(Y)$ and $\sigma(X_1, \ldots, X_n) \vee \mathcal{F}$ are independent. Then, for Borel functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ $i = 1, \ldots, n+1$, the random vector $\widetilde{\mathbf{X}} = (f_1(Z_1, X_1), \ldots, f_n(Z_n, X_n))$ and the random variable $f_{n+1}(Z_{n+1}, Y)$ are conditionally independent given $\mathcal{F}$.*

Lemma 1.3. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a random vector and assume that* $\sigma(X_i) \vee \mathcal{F}_i, i = 1, \ldots, n$ *are independent. We also let* $\mathcal{G}$ *be a* σ*-algebra, independent of* $\sigma(\mathbf{X}) \vee \mathcal{F}$*, where* $\mathcal{F} = \mathcal{F}_1 \vee \cdots \vee \mathcal{F}_n$*, and let* $Z_1, \ldots, Z_n$ *be* $\mathcal{F} \vee \mathcal{G}$*-measurable random variables. Then, for any Borel functions* $f_i : \mathbb{R}^2 \to \mathbb{R}$*,* $i = 1, \ldots, n$*, the random variables* $f_1(Z_1, X_1), \ldots, f_n(Z_n, X_n)$ *are conditionally independent given* $\mathcal{F} \vee \mathcal{G}$*.*

2. Bivariate independence property and its interpretation

The original independence property can be equivalently formulated as follows. For details, see [11].

Proposition 2.1. *For a pair* (K_1, K_2) *of positive random variables, the distribution of* $(K_1, K_2 - (K_1)^{-1})$ *is* $\mathrm{GIG}(q; b, a) \otimes \gamma(q, a)$ *for positive parameters* q, a, b *if and only if that of* $(K_1 - (K_2)^{-1}, K_2)$ *is* $\gamma(q, b) \otimes \mathrm{GIG}(q; a, b)$*.*

In this section we adapt the interpretation for the independence property obtained in [17] to the symmetric statement given in this proposition. While, in [17], the result is first mentioned in the case of $a = b$ and then it is extended to a general pair (a, b), the general case is treated directly here by using the scaling property of the GIG and gamma laws. The approach we present below may be treated as a warm-up introducing the ideas which will be exploited, while studying the multivariate case in Sections 3 and 4.

Fix $\mu > 0$ and use the same notations as in the previous section. For $\alpha > 0$, we let (K_1, K_2) be a pair of positive random variables such that

$$\mathcal{L}\left(K_1, K_2 - \frac{1}{K_1}\right) = \mathcal{L}\left(\left(\frac{\alpha}{e_t}, \frac{Z_t}{\alpha A_{t,\infty}}\right) \middle| Z_{[0,t]} = z\right).$$

Since $A_{(t,\infty)}$ is independent of $\sigma\{B_t\} \vee \mathcal{Z}_t$, the two random variables on the right hand side are conditionally independent given $\mathcal{Z}_t$ (see Lemma 1.2). As is mentioned in the previous section, the conditional law $\mathcal{L}(e_t | Z_{[0,t]} = z)$ is $\mathrm{GIG}(-\mu; 1/2z_t, 1/2z_t)$. Hence it is easy to see

$$\mathcal{L}\left(\frac{\alpha}{e_t} \middle| Z_{[0,t]} = z\right) = \mathrm{GIG}\left(\mu; \frac{1}{2\alpha z_t}, \frac{\alpha}{2z_t}\right).$$

Moreover, since $\sigma\{A_{t,\infty}\}$ and $\mathcal{Z}_t$ are independent, the conditional law of $Z_t(\alpha A_{t,\infty})^{-1}$ given $Z_{[0,t]} = z$ is $\gamma(\mu, \alpha/2z_t)$. Hence we have shown that the distribution of $(K_1, K_2 - (K_1)^{-1})$ is $\mathrm{GIG}(\mu; b, a) \otimes \gamma(\mu, a)$, where $b = (2\alpha z_t)^{-1}$ and $a = \alpha/2z_t$.

The pair $(K_1 - (K_2)^{-1}, K_2)$ can be handled in the same way. Using the identities (1.4) and (1.5), we deduce

$$\begin{aligned}\mathcal{L}\left(K_1 - \frac{1}{K_2}, K_2\right) &= \mathcal{L}\left(\left(\frac{\alpha}{e_t} - \frac{\alpha}{e_t + Z_t/A_{t,\infty}}, \frac{1}{\alpha}\left(e_t + \frac{Z_t}{A_{t,\infty}}\right)\right) \middle| Z_{[0,t]} = z\right) \\ &= \mathcal{L}\left(\left(\frac{\alpha Z_t}{A_\infty}, \frac{e_t^*}{\alpha}\right) \middle| Z_{[0,t]} = z\right).\end{aligned}$$

At first we recall that $\sigma\{A_\infty\}$ and $\sigma\{B_t^*\} \vee \mathcal{Z}_t^*$ are independent, which implies that $\sigma\{A_\infty\}$ and $\sigma\{B_t^*\} \vee \mathcal{Z}_t$ are independent, since $Z_s = Z_s^*$, $s \geq 0$. Therefore, by Lemma 1.2, we see that the random variables $\alpha Z_t / A_\infty$ and e_t^* / α are conditionally independent given $\mathcal{Z}_t$.

Hence, noting that $\mathcal{L}(e_t|Z_{[0,t]}=z)$ is $\mathrm{GIG}(\mu;1/2z_t,1/2z_t)$ and by (1.3), we obtain

$$\mathcal{L}\left(\frac{\alpha Z_t}{A_\infty}\,\middle|\,Z_{[0,t]}=z\right)=\mathcal{L}\left(\frac{\alpha Z_t^*}{A_\infty}\,\middle|\,Z^*_{[0,t]}=z\right)=\gamma\left(\mu,\frac{1}{2\alpha z_t}\right)$$

and

$$\mathcal{L}\left(\frac{e_t^*}{\alpha}\,\middle|\,Z_{[0,t]}=z\right)=\mathcal{L}\left(\frac{e_t^*}{\alpha}\,\middle|\,Z^*_{[0,t]}=z\right)=\mathrm{GIG}\left(\mu;\frac{\alpha}{2z_t},\frac{1}{2\alpha z_t}\right).$$

We now can conclude that

$$\left(K_1-\frac{1}{K_2},K_2\right)\sim\gamma(\mu,b)\otimes\mathrm{GIG}(\mu;a,b),$$

which completes the interpretation.

3. Conditional independence properties of the exponential functionals of Brownian motion

For $t_1,\ldots,t_{n-1}>0, n\ge 2$, we define a family of Brownian motions $B^{(1)}=\{B_s^{(1)}\}_{s\ge 0},\ldots,B^{(n-1)}=\{B_s^{(n-1)}\}_{s\ge 0}$ with drift $-\mu$, $\mu\in\mathbb{R}$ by

$$B_s^{(1)}=B_s,$$
$$B_s^{(i+1)}=B^{(i)}_{t_i+s}-B^{(i)}_{t_i},\quad i=1,\ldots,n-2.$$

Setting $\widehat{t_{i-1}}=t_1+\cdots+t_{i-1}$ for $i=2,\ldots,n-1$, we have

$$B_s^{(i)}=B_{\widehat{t_{i-1}}+s}-B_{\widehat{t_{i-1}}}.$$

We also consider, correspondingly, the exponential functional $A^{(i)}_{s,t}$ given by

$$A^{(i)}_{s,t}=\int_s^t\exp\left(2\left(B_u^{(i)}-B_s^{(i)}\right)\right)\mathrm{d}u.$$

Lemma 3.1. *For $\mu>0$ and $i=2,\ldots,n-1$,*

$$A^{(i)}_\infty=A^{(i-1)}_{t_{i-1},\infty}. \tag{3.1}$$

Proof. From the definition of $\{B_s^{(i)}\}$, we have

$$A^{(i)}_\infty=\int_0^\infty\exp\left(2\left(B^{(i-1)}_{t_{i-1}+s}-B^{(i-1)}_{t_{i-1}}\right)\right)\mathrm{d}s$$
$$=\int_{t_{i-1}}^\infty\exp\left(2\left(B^{(i-1)}_u-B^{(i-1)}_{t_{i-1}}\right)\right)\mathrm{d}u=A^{(i-1)}_{t_{i-1},\infty}.\quad\square$$

Lemma 3.2. *For $\mu>0$ and $i=1,\ldots,n-1$, the random variable $A^{(i)}_{t_i,\infty}$ is independent of $\mathcal{B}_{\widehat{t_i}}$.*

Proof. Since $B^{(i)}_{t_i+s}-B^{(i)}_{t_i}=B_{\widehat{t_i}+s}-B_{\widehat{t_i}}$, we have

$$A^{(i)}_{t_i,\infty}=\int_0^\infty\exp\left(2\left(B_{\widehat{t_i}+s}-B_{\widehat{t_i}}\right)\right)\mathrm{d}s.$$

Hence the assertion of the lemma is a simple consequence of the Markov property of Brownian motion. □

We also consider the corresponding exponential functionals $e^{(i)}$, $Z^{(i)}$ given by

$$e_s^{(i)} = \exp\left(-B_s^{(i)}\right), \qquad Z_s^{(i)} = \exp\left(-B_s^{(i)}\right) \int_0^s \exp\left(2B_u^{(i)}\right) \mathrm{d}u = \left(e_s^{(i)}\right)^{-1} A_s^{(i)}.$$

We set $\mathcal{Z}^{(i)} = \sigma\{Z_s^{(i)}; s \le t_i\}$ and

$$\overline{\mathcal{Z}} = \bigvee_{i=1}^{n-1} \mathcal{Z}^{(i)}.$$

Note that $\mathcal{Z}^{(i)}$'s are independent.

Moreover we put

$$\overline{Z} = \left(Z_{[0,t_1]}^{(1)}, \ldots, Z_{[0,t_{n-1}]}^{(n-1)}\right).$$

Then it is clear that $\overline{\mathcal{Z}} = \sigma(\overline{Z})$. For a family of continuous paths $\overline{z} = (z^{(1)}, \ldots, z^{(n-1)})$, $z^{(i)} : [0, t_i] \to \mathbb{R}$, we write $\overline{Z} = \overline{z}$ when $Z_{[0,t_i]}^{(i)} = z^{(i)}$ holds for all $i = 1, \ldots, n-1$.

In view of Proposition 1.1, it is natural to consider a stochastic process $B^{(i)*} = \{B_s^{(i)*}\}_{s \ge 0}$ for $i = 1, \ldots, n-1$ defined by

$$B_s^{(i)*} = B_s^{(i)} - \log\left(1 - \frac{A_s^{(i)}}{A_\infty^{(i)}}\right)$$

only for $\mu > 0$. This process is a Brownian motion with drift $+\mu$ with respect to the initially enlarged filtration $\{\widehat{\mathcal{B}}_s^{(i)}\}$ given by

$$\widehat{\mathcal{B}}_s^{(i)} = \mathcal{B}_s^{(i)} \vee \sigma\{A_\infty^{(i)}\},$$

where $\{\mathcal{B}_s^{(i)}\}$ is the natural filtration of $B^{(i)}$. Of course $B^{(i)*}$ is independent of the random variable $A_\infty^{(i)}$.

For the Brownian motion $B^{(i)*}$, we also associate the exponential functionals

$$e_s^{(i)*} = \exp\left(B_s^{(i)*}\right), \qquad A_s^{(i)*} = \int_0^s \left(e_u^{(i)*}\right)^2 \mathrm{d}u,$$

$$Z_s^{(i)*} = \left(e_s^{(i)*}\right)^{-1} A_s^{(i)*}.$$

In the proof of the main result of this paper we will use the following lemma, which shows the correspondence between the processes $B^{(i)*}$ for $i = 1, \ldots, n-1$.

Lemma 3.3. *For $\mu > 0$ and any $i = 1, \ldots, n-2$,*

$$B_s^{(i+1)*} = B_{t_i+s}^{(i)*} - B_{t_i}^{(i)*}, \quad s \ge 0.$$

Proof. By definition we have

$$B_s^{(i+1)*} = B_s^{(i+1)} - \log\left(1 - \frac{A_s^{(i+1)}}{A_\infty^{(i+1)}}\right)$$

and $B_s^{(i+1)} = B_{t_i+s}^{(i)} - B_{t_i}^{(i)}$. Moreover,

$$A_\infty^{(i+1)} - A_s^{(i+1)} = \int_s^\infty \exp\left(2\left(B_{t_i+u}^{(i)} - B_{t_i}^{(i)}\right)\right) \mathrm{d}u = \left(e_{t_i}^{(i)}\right)^{-2} \int_{t_i+s}^\infty \left(e_u^{(i)}\right)^2 \mathrm{d}u$$
$$= \left(e_{t_i}^{(i)}\right)^{-2} \left(A_\infty^{(i)} - A_{t_i+s}^{(i)}\right)$$

and

$$A_\infty^{(i+1)} = \left(e_{t_i}^{(i)}\right)^{-2} \left(A_\infty^{(i)} - A_{t_i}^{(i),(-\mu)}\right).$$

Combining these identities, we get

$$B_s^{(i+1)*} = B_{t_i+s}^{(i)} - B_{t_i}^{(i)} - \log \frac{A_\infty^{(i)} - A_{t_i+s}^{(i)}}{A_\infty^{(i)} - A_{t_i}^{(i)}} = B_{t_i+s}^{(i)*} - B_{t_i}^{(i)*}. \quad \square$$

Remark 3.1. The above lemma implies that the stochastic processes

$$\{B_s^{(1)*}\}_{0\le s\le t_1}, \ldots, \{B_s^{(n-1)*}\}_{0\le s\le t_{n-1}}$$

are independent.

Remark 3.2. Set $\mathcal{B}_s^{(k)*} = \sigma\{B_u^{(k)*}; u \le s\}$. Then, for any $i = 1, \ldots, n-1$, $\sigma\{A_\infty^{(i)}\}$ and $\bigvee_{k=i}^{n-1} \mathcal{B}_{t_k}^{(k)*}$ are independent. This is because the random variable $A_\infty^{(i)}$ is independent of the Brownian motion $B^{(i)*}$ and, for $k \ge i+1$, the process $B^{(k)*}$ is formed from $B^{(i)*}$.

The conditional independence property we want to discuss can be introduced conveniently through a set of mappings based on an integer valued function c we will describe now. Let $n \ge 2$ be an integer and consider a discrete function c from $\{1, \ldots, n-1\}$ into $\{2, \ldots, n\}$ satisfying

$$i < c(i) < n \quad i = 1, \ldots, n-2, \text{ and } c(n-1) = n. \tag{3.2}$$

Note that, for a given function c and a fixed $r \in \{1, \ldots, n-1\}$, there exists a unique sequence $(i_1, \ldots, i_s)$ such that $i_1 = r$, $i_s = n$ and

$$i_{k+1} = c(i_k) \quad \text{for } k = 1, \ldots, s-1.$$

Hence we may define two subsets $I_r(c)$ and $J_r(c)$ of the set $\{1, \ldots, n\}$ by

$$I_r(c) = \{i_1, \ldots, i_s\} \quad \text{and} \quad J_r(c) = \{1, \ldots, n\} \setminus I_r(c).$$

Note also that $i_1 < \cdots < i_s$ and $i_{s-1} = n-1$. For $r = n$, we simply put $I_n(c) = \{i_1 = n\}$ and $J_n(c) = \{1, \ldots, n-1\}$. Note also that the sets $I_r(c)$ and $J_r(c)$ are uniquely determined by the function c and $r \in \{1, \ldots, n\}$.

For such a function c and r as above, we put, assuming $I_r(c) = \{i_1, \ldots, i_s\}$ and setting $x_{i_0} = x_{i_{s+1,(r)}} = \infty$,

$$x_{i,(r)} = x_i, \quad \text{for } i \in J_r(c),$$
$$x_{i_j,(r)} = x_{i_j} + \frac{1}{x_{i_{j-1}}} - \frac{1}{x_{i_{j+1,(r)}}} \quad \text{for } j = 1, \ldots, s,$$

and define a mapping $\phi_r^{(c)}$ by

$$\phi_r^{(c)}(x_1, \dots, x_n) = (x_{1,(r)}, \dots, x_{n,(r)}). \tag{3.3}$$

For an explicit computation for (3.3), we should start from $x_{i_s,(r)} = x_{n,(r)}$, that is,

$$x_{n,(r)} = x_{i_s,(r)} = x_n + \frac{1}{x_{n-1}},$$
$$x_{n-1,(r)} = x_{i_{s-1},(r)} = x_{n-1} + \frac{1}{x_{i_{s-2}}} - \frac{1}{x_n + 1/x_{n-1}},$$

and so on. Note that $\phi_n^{(c)}$ is an identity mapping. We set $\Phi^{(c)} = \{\phi_r^{(c)} : r = 1, \dots, n\}$.

Remark 3.3. Let $r \in \{1, \dots, n-1\}$. From the definition of $\phi_r^{(c)}$, it follows that the vector $(x_{i,(r)} : i \in I_r(c))$ does not depend on $\{x_i : i \in J_r(c)\}$. Namely, $x_{i,(r)}$ for $i \in I_r(c)$ are functions of $\{x_i : i \in I_r(c)\}$.

Lemma 3.4. *For any $r = 1, \dots, n$, the mapping $\phi_r^{(c)}$ is a bijection of $(0, \infty)^n$ onto itself. In particular the inverse $(\phi_r^{(c)})^{-1}$ is given by*

$$x_i = x_{i,(r)}, \quad \textit{for } i \in J_r(c),$$
$$x_{i_j} = x_{i_j,(r)} + \frac{1}{x_{i_{j+1},(r)}} - \frac{1}{x_{i_{j-1}}} \quad \textit{for } j = 1, \dots, s.$$

Proof. Let $(x_1, \dots, x_n) \in (0, \infty)^n$. It is sufficient to consider $x_{i,(r)}$ only for $i \in I_r(c)$. Let $y_{i_j} = x_{i_j} - \frac{1}{x_{i_{j+1},(r)}}$, throughout the proof. Then $x_{i_j,(r)} = y_{i_j} + \frac{1}{x_{i_{j-1}}}$ and

$$y_{i_s} = x_{i_s}, \qquad y_{i_j} = x_{i_j} - \frac{1}{y_{i_{j+1}} + 1/x_{i_j}}, \quad j = 1, \dots, s-1.$$

Since $x - \frac{1}{c+1/x} > 0$ for any $x, c > 0$, a simple induction shows that $y_{i_j} > 0$ for any $j = 1, \dots, s$ and in consequence $(x_{1,(r)} \dots, x_{n,(r)}) \in (0, \infty)^n$. Moreover, since $(\phi_r^{(c)})^{-1}$ has the analogous structure as $\phi_r^{(c)}$, for a given vector $(x_{1,(r)} \dots, x_{n,(r)}) \in (0, \infty)^n$, there exists $(x_1, \dots, x_n) \in (0, \infty)^n$ such that $\phi_r^{(c)}(x_1, \dots, x_n) = (x_{1,(r)} \dots, x_{n,(r)})$. $\square$

In the following we fix $\mu > 0$. The following theorem, which is the main result of the paper, presents independence properties of functionals of exponential Brownian motion. It appears that these properties are related to independencies based on ordered trees obtained in [11] (see Section 4).

Theorem 3.5. *Let c be a function described above and $\alpha_1(z), \dots, \alpha_n(z)$ be a family of arbitrary measurable positive functionals, where $z = (z_1, \dots, z_n)$ and z_i is an arbitrary positive continuous function on $[0, t_i]$ with $z_i(0) = 0$, satisfying*

$$\alpha_i(z)\alpha_{c(i)}(z) = \frac{z_i(t_i)}{z_{c(i)}(t_{c(i)})}, \quad i = 1, \dots, n-2 \quad \textit{and} \quad \alpha_{n-1}(z)\alpha_n(z) = 1. \tag{3.4}$$

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector whose distribution is given by

$$\mathcal{L}(\mathbf{X}) = \mathcal{L}\left(\left(\frac{\alpha_1(\overline{Z})}{e_{t_1}^{(1)}}, \dots, \frac{\alpha_{n-1}(\overline{Z})}{e_{t_{n-1}}^{(n-1)}}, \frac{\alpha_n(\overline{Z}) Z_{t_{n-1}}^{(n-1)}}{A_{t_{n-1},\infty}^{(n-1)}}\right) \middle| \overline{Z} = \overline{z}\right).$$

Then, for $r = 1, \ldots, n$, the distribution of $\phi_r^{(c)}(\mathbf{X}) = (X_{1,(r)}, \ldots, X_{n,(r)})$ is the product of $n-1$ GIG-distributions and one gamma distribution. More precisely, $X_{i,(r)} \sim \mathrm{GIG}(\mu; a_i, a_{c(i)})$ for $i \in J_r(c)$, $X_{i_1,(r)} \sim \gamma(\mu, a_{i_1})$ and $X_{i_j,(r)} \sim \mathrm{GIG}(\mu; a_{i_j}, a_{i_{j-1}})$ for $j = 2, \ldots, s$, where

$$a_i = \frac{1}{2\alpha_i(\overline{z}) z_{t_i}^{(i)}} \quad \textit{for } i = 1, \ldots, n-1, \quad \textit{and} \quad a_n = \frac{1}{2\alpha_n(\overline{z}) z_{t_{n-1}}^{(n-1)}}.$$

Remark 3.4. The random variables $\alpha_1(\overline{Z}), \ldots, \alpha_n(\overline{Z})$ are not uniquely determined by (3.4) and we fix an arbitrary family satisfying this condition.

Proof. To shorten notation, we set

$$X_i' = \frac{\alpha_i(\overline{Z})}{e_{t_i}^{(i)}}, \quad i = 1, \ldots, n-1 \quad \text{and} \quad X_n' = \frac{\alpha_n(\overline{Z}) Z_{t_{n-1}}^{(n-1)}}{A_{t_{n-1}}^{(n-1)}}. \tag{3.5}$$

We begin with computing the distribution of $\phi_n(\mathbf{X}) = \mathbf{X}$. For this purpose we first note that $\sigma\{B_{t_1}^{(1)}, \ldots, B_{t_{n-1}}^{(n-1)}\} \vee \overline{\mathcal{Z}}$ and $\sigma\{A_{t_{n-1},\infty}^{(n-1)}\}$ are independent. Then we see, by Lemma 1.2, that $(X_1', \ldots, X_{n-1}')$ and X_n' are conditionally independent given $\overline{\mathcal{Z}}$. Moreover, since $\mathcal{Z}^{(i)} \subset \mathcal{B}_{t_i}^{(i)}$, $\sigma\{B_{t_i}^{(i)}\} \vee \mathcal{Z}^{(i)}$, $i = 1, \ldots, n-1$, is a family of independent σ-algebras. Hence, by Lemma 1.3, $X_1', \ldots, X_{n-1}'$ are conditionally independent given $\overline{\mathcal{Z}}$.

Next note that, for any $i = 1, \ldots, n-1$,

$$\overline{\mathcal{Z}} = \mathcal{Z}^{(i)} \vee \left(\bigvee_{k=1, k\neq i}^{n-1} \mathcal{Z}^{(k)} \right)$$

and that $\sigma\{B_{t_i}^{(i)}\} \vee \mathcal{Z}^{(i)}$ and $\bigvee_{k=1,k\neq i}^{n-1} \mathcal{Z}^{(k)}$ are independent. Since $\mathcal{L}((e_{t_i}^{(i)})^{-1} | \overline{Z} = \overline{z}) = \mathrm{GIG}(\mu; 1/2z_{t_i}^{(i)}, 1/2z_{t_i}^{(i)})$, we have

$$\mathcal{L}\left(\frac{\alpha_i(\overline{Z})}{e_{t_i}^{(i)}} \middle| \overline{Z} = \overline{z} \right) = \mathrm{GIG}\left(\mu; \frac{1}{2\alpha_i\,(\overline{z})\, z_{t_i}^{(i)}}, \frac{\alpha_i\,(\overline{z})}{2z_{t_i}^{(i)}} \right).$$

Furthermore, since $\sigma\{A_{t_{n-1},\infty}^{(n-1)}\}$ is independent of $\overline{\mathcal{Z}}$ and the distribution of $A_{t_{n-1},\infty}^{(n-1)}$ is $\gamma(\mu, 1/2)$, we also have

$$\mathcal{L}\left(\frac{\alpha_n(\overline{Z}) Z_{t_{n-1}}^{(n-1)}}{A_{t_{n-1},\infty}^{(n-1)}} \middle| \overline{Z} = \overline{z} \right) = \gamma\left(\mu, \frac{1}{2\alpha_n\,(\overline{z})\, z_{t_{n-1}}^{(n-1)}} \right).$$

Finally, by the definition of α_i's, we obtain

$$\begin{aligned} \phi_n(\mathbf{X}) &\sim \bigotimes_{i=1}^{n-1} \mathrm{GIG}\left(\mu; \frac{1}{2\alpha_i\,(\overline{z})\, z_{t_i}^{(i)}}, \frac{\alpha_i\,(\overline{z})}{2z_{t_i}^{(i)}} \right) \otimes \gamma\left(\mu, \frac{1}{2\alpha_n\,(\overline{z})\, z_{t_{n-1}}^{(n-1)}} \right) \\ &= \bigotimes_{i=1}^{n-1} \mathrm{GIG}\left(\mu; a_i, a_{c(i)}\right) \otimes \gamma(\mu, a_n), \end{aligned}$$

which is the assertion of the theorem when $r = n$.

We now proceed to the general case, where $r = 1, \dots, n-1$, and compute the distribution of $\phi_r(\mathbf{X})$. It is clear that

$$\mathcal{L}(\phi_r(\mathbf{X})) = \mathcal{L}((X'_{1,(r)}, \dots, X'_{n,(r)})|\overline{Z} = \overline{z}),$$

where $(X'_{1,(r)}, \dots, X'_{n,(r)}) = \phi_r(\mathbf{X}')$ and the random vector $\mathbf{X}'$ is given by (3.5). Since $X'_{i,(r)} = X'_i$ for $i \in J_r(c)$, by Remark 3.3, the random vectors $(X'_{i,(r)} : i \in J_r(c))$ and $(X'_{i,(r)} : i \in I_r(c))$ are conditionally independent given $\overline{\mathcal{Z}}$. Moreover, we have

$$\begin{aligned}\mathcal{L}((X'_{i,(r)} : i \in J_r(c))|\overline{Z} = \overline{z}) &= \bigotimes_{i \in J_r(c)} \mathrm{GIG}\left(\mu; \frac{1}{2\alpha_i(\overline{z})\, z^{(i)}_{t_i}}, \frac{\alpha_i(\overline{z})}{2z^{(i)}}\right) \\ &= \bigotimes_{i \in J_r(c)} \mathrm{GIG}(\mu; a_i, a_{c(i)}).\end{aligned}$$

The next lemma allows us to find the regular conditional distribution of $(X'_{i,(r)} : i \in I_r(c))$ given $\overline{Z} = \overline{z}$ and is crucial for our proof of the theorem.

Lemma 3.6. *For any $k = 1, \dots, s-1$, where $s = \#I_r$, it holds that*

$$\mathcal{L}((X'_{i,(r)} : i \in I_r(c))|\overline{Z} = \overline{z}) = \mathcal{L}((Y_1^{(k)}, \dots, Y_s^{(k)})|\overline{Z} = \overline{z}), \tag{3.6}$$

where the random variables $Y_j^{(k)}$, $j = 1, \dots, s$, are given inductively by

$$\begin{aligned} Y_j^{(k)} &= \frac{e^{(i_{j-1})*}_{t_{i_{j-1}}}}{\alpha_{i_{j-1}}} \quad \text{for } j = k+1, \dots, s, \\ Y_j^{(k)} &= \frac{\alpha_{i_j}}{e^{(i_j)}_{t_{i_j}}} + \frac{e^{(i_{j-1})}_{t_{i_{j-1}}}}{\alpha_{i_{j-1}}} - \frac{1}{Y_{j+1}^{(k)}} \quad \text{for } j = 2, \dots, k, \\ Y_1^{(k)} &= \frac{\alpha_{i_1}}{e^{(i_1)}_{t_{i_1}}} - \frac{1}{Y_2^{(k)}}. \end{aligned}$$

We postpone a proof of Lemma 3.6 and, assuming it as proved, we complete our proof of the theorem. Taking $k = 1$ in Lemma 3.6, we obtain

$$\mathcal{L}((X'_{i,(r)} : i \in I_r(c))|\overline{Z} = \overline{z}) = \mathcal{L}\left(\left(\frac{\alpha_{i_1}(\overline{Z})Z^{(i_1)}_{t_{i_1}}}{A^{(i_1)}_\infty}, \frac{e^{(i_1)*}_{t_{i_1}}}{\alpha_{i_1}(\overline{Z})}, \dots, \frac{e^{(i_{s-1})*}_{t_{i_{s-1}}}}{\alpha_{i_{s-1}}(\overline{Z})}\right) \middle| \overline{Z} = \overline{z}\right).$$

As is mentioned in Remark 3.2, we conclude that $\sigma\{A^{(i_1)}_\infty\}$ and $\sigma\{B^{(i_1)*}_{t_{i_1}}, \dots, B^{(i_{s-1})*}_{t_{i_{s-1}}}\}$ are independent, which implies that the random variable $\alpha_{i_1}(\overline{Z})Z^{(i_1)}_{t_{i_1}}/A^{(i_1)}_\infty$ and the random vector

$$\left(\frac{e^{(i_1)*}_{t_{i_1}}}{\alpha_{i_1}(\overline{Z})}, \dots, \frac{e^{(i_{s-1})*}_{t_{i_{s-1}}}}{\alpha_{i_{s-1}}(\overline{Z})}\right) \tag{3.7}$$

are conditionally independent given $\overline{\mathcal{Z}}$, and

$$\mathcal{L}\left(\frac{\alpha_{i_1}(\overline{Z})Z^{(i_1)}_{t_{i_1}}}{A^{(i_1)}_\infty}\,\middle|\,\overline{Z}=\overline{z}\right)=\gamma\left(\mu,\frac{1}{2\alpha_{i_1}(\overline{z})z^{(i_1)}_{t_{i_1}}}\right).$$

Now, it remains to find the regular conditional distribution of the random vector (3.7) given $\overline{Z}=\overline{z}$. First, we show that the random vector $(e^{(i)*}_{t_i}/\alpha_i(\overline{Z}) : i=1,\dots,n-1)$ has conditionally independent components given $\overline{\mathcal{Z}}$. Since

$$\sigma\left(B^{(i)*}_{t_i}\right)\vee\mathcal{Z}^{(i)*},\quad i=1,\dots,n,$$

are a family of independent σ-algebras (see Remark 3.1), we see that

$$\sigma\left(B^{(i)*}_{t_i}\right)\vee\mathcal{Z}^{(i)},\quad i=1,\dots,n,$$

are also a family of independent σ-algebras by (1.3). Applying Lemma 1.3, we conclude that the random vector $(e^{(i)*}_{t_i}/\alpha_i(\overline{Z}) : i=1,\dots,n-1)$ and hence also the random vector given by (3.7) have conditionally independent components given $\overline{\mathcal{Z}}$. Moreover we have

$$\mathcal{L}\left(e^{(i)*}_{t_i}\,\middle|\,\overline{Z}=\overline{z}\right)=\mathrm{GIG}\left(\mu;\frac{1}{2z^{(i)}_{t_i}},\frac{1}{2z^{(i)}_{t_i}}\right),\quad i=1,\dots,n-1,$$

and, hence, by independence arguments similar to these from the beginning of the proof of the theorem and by (3.4), we obtain

$$\mathcal{L}\left(\left(\frac{e^{(i_1)*}_{t_{i_1}}}{\alpha_{i_1}(\overline{Z})},\dots,\frac{e^{(i_{s-1})*}_{t_{i_{s-1}}}}{\alpha_{i_{s-1}}(\overline{Z})}\right)\,\middle|\,\overline{Z}=\overline{z}\right)=\bigotimes_{j=2}^{s}\mathrm{GIG}\left(\mu;\frac{\alpha_{i_{j-1}}(\overline{z})}{2z^{(i_{j-1})}_{t_{i_{j-1}}}},\frac{1}{2\alpha_{i_{j-1}}(\overline{z})\,z^{(i_{j-1})}_{t_{i_{j-1}}}}\right)$$
$$=\bigotimes_{j=2}^{s}\mathrm{GIG}(\mu,a_{i_j},a_{i_{j-1}}).$$

Hence, if we prove Lemma 3.6, our proof of Theorem 3.5 will be completed.

Proof of Lemma 3.6. We prove the lemma by induction on k. Let $k=s-1$. Recalling that $i_s=n$ and $i_{s-1}=n-1$, we obtain the almost sure equality

$$X'_{i_s,(r)}=X'_{i_s}+\frac{1}{X'_{i_{s-1}}}=\frac{\alpha_{i_s}(\overline{Z})Z^{(i_{s-1})}_{t_{i_{s-1}}}}{A^{(i_{s-1})}_{t_{i_{s-1}},\infty}}+\frac{e^{(i_{s-1})}_{t_{i_{s-1}}}}{\alpha_{i_{s-1}}(\overline{Z})}$$
$$=\frac{1}{\alpha_{i_{s-1}}(\overline{Z})}\cdot\left[\frac{Z^{(i_{s-1})}_{t_{i_{s-1}}}}{A^{(i_{s-1})}_{t_{i_{s-1}},\infty}}+e^{(i_{s-1})}_{t_{i_{s-1}}}\right]=\frac{e^{(i_{s-1})*}_{t_{i_{s-1}}}}{\alpha_{i_{s-1}}(\overline{Z})}$$

from the definition of ϕ_r, (3.4) and (1.4). Thus, once again by the definition of ϕ_r, we have $X'_{i_j,(r)}\overset{a.s.}{=}Y^{(s-1)}_j$ for $j=1,\dots,s$ and this completes the proof for $k=s-1$.

Now, assuming (3.6) to hold for $2 \le k \le s-1$, we prove it for $k-1$. By the induction assumption, we have

$$\mathcal{L}((X'_{i,(r)} : i \in I_r(c))|\overline{Z} = \overline{z}) = \mathcal{L}\left(\left(\left(Y_1^{(k)}, \dots, Y_k^{(k)}, \frac{e_{t_{i_k}}^{(i_k)*}}{\alpha_{i_k}(\overline{Z})}, \dots, \frac{e_{t_{i_{s-1}}}^{(i_{s-1})*}}{\alpha_{i_{s-1}}(\overline{Z})}\right)\right| \overline{Z} = \overline{z}\right).$$

Since $c(i_{k-1}) = i_k$, we have from (3.4)

$$Y_k^{(k)} = \frac{1}{\alpha_{i_{k-1}}(\overline{Z})}\left[\frac{Z_{t_{i_{k-1}}}^{(i_{k-1})}}{A_\infty^{(i_k)}} + e_{t_{i_{k-1}}}^{(i_{k-1})}\right].$$

Suppose first that $i_k - i_{k-1} = 1$. Then, by (3.1) and (1.4), we get

$$Y_k^{(k)} = \frac{1}{\alpha_{i_{k-1}}(\overline{Z})} \cdot \left[\frac{Z_{t_{i_{k-1}}}^{(i_{k-1})}}{A_{t_{i_{k-1}},\infty}^{(i_{k-1})}} + e_{t_{i_{k-1}}}^{(i_{k-1})}\right] = \frac{e_{t_{i_{k-1}}}^{(i_{k-1})*}}{\alpha_{i_{k-1}}(\overline{Z})}.$$

It follows that $Y_j^{(k)} \overset{a.s.}{=} Y_j^{(k-1)}$ for $j = 1, \dots, s$ and (3.6) is proved for $k-1$.

On the other hand, let $i_k - i_{k-1} > 1$. In this case we consider a random vector $U = (U_1, \dots, U_s)$ given by

$$U_j = \frac{\alpha_{i_j}(\overline{Z})}{e_{t_{i_j}}^{(i_j)}}, \quad j = 1, \dots, k-1, \qquad U_k = \frac{\widetilde{\alpha}_{i_k}(\overline{Z})}{A_\infty^{(i_k)}},$$

$$U_j = \frac{e_{t_{i_{j-1}}}^{(i_{j-1})*}}{\alpha_{i_{j-1}}(\overline{Z})}, \quad j = k+1, \dots, s,$$

and also its slight modification

$$\widetilde{U} = \left(U_1, \dots, U_{k-1}, \widetilde{U}_k, U_{k+1}, \dots, U_s\right),$$

where $\widetilde{\alpha}_{i_k}(\overline{Z}) = (\alpha_{i_{k-1}}(\overline{Z}))^{-1} Z_{t_{i_{k-1}}}^{(i_{k-1})}$ and

$$\widetilde{U}_k = \frac{\widetilde{\alpha}_{i_k}(\overline{Z})}{A_\infty^{(i_{k-1}+1)}} = \frac{\widetilde{\alpha}_{i_k}(\overline{Z})}{A_{t_{i_{k-1}},\infty}^{(i_{k-1})}}.$$

The random vector $(Y_1^{(k)}, \dots, Y_s^{(k)})$ is a function of U and we write

$$f_k(U) = \left(Y_1^{(k)}, \dots, Y_s^{(k)}\right).$$

Now our goal is to prove that the random vectors U and $\widetilde{U}$ have the same regular conditional distribution given $\overline{\mathcal{Z}}$. If we show it, then we have

$$\begin{aligned}\mathcal{L}((X'_{i,(r)} : i \in I_r(c))|\overline{Z} = \overline{z}) &= \mathcal{L}((Y_1^{(k)}, \dots, Y_s^{(k)})|\overline{Z} = \overline{z}) \\ &= \mathcal{L}(f_k(U)|\overline{Z} = \overline{z}) = \mathcal{L}(f_k(\widetilde{U})|\overline{Z} = \overline{z})\end{aligned}$$

and, by the same arguments as in the case of $i_k - i_{k-1} = 1$, we obtain

$$f_k(\widetilde{U}) \overset{a.s.}{=} \left(Y_1^{(k-1)}, \dots, Y_s^{(k-1)}\right),$$

which will complete the proof.

First, we show the conditional independence between the random variable U_k and the random vector $(U_1, \dots, U_{k-1}, U_{k+1}, \dots, U_s)$ given $\overline{\mathcal{Z}}$. Since all α's and $\widetilde{\alpha}$ are $\overline{\mathcal{Z}}$-measurable random variables, it is sufficient, by Lemma 1.2, to check that $\sigma\{A_\infty^{(i_k)}\}$ and

$$\sigma\left\{B_{t_{i_1}}^{(i_1)}, \dots, B_{t_{i_{k-1}}}^{(i_{k-1})}, B_{t_{i_k}}^{(i_k)*}, \dots, B_{t_{i_{s-1}}}^{(i_{s-1})*}\right\} \vee \overline{\mathcal{Z}}$$

are independent. Since the processes $Z^{(i)}$ and $Z^{(i)*}$ have the same trajectories, we show equivalent independence between $\sigma\{A_\infty^{(i_k)}\}$ and $\sigma\{B_{t_{i_1}}^{(i_1)}, \dots, B_{t_{i_{k-1}}}^{(i_{k-1})}, B_{t_{i_k}}^{(i_k)*}, \dots, B_{t_{i_{s-1}}}^{(i_{s-1})*}\} \vee \widetilde{\mathcal{Z}}$, where

$$\widetilde{\mathcal{Z}} = \left(\bigvee_{i=1}^{i_k-1} \mathcal{Z}^{(i)}\right) \vee \left(\bigvee_{i=i_k}^{n-1} \mathcal{Z}^{(i)*}\right) \tag{3.8}$$

and $\mathcal{Z}^{(i)*} = \sigma\{Z_s^{(i)*}; 0 \le s \le t_i\}$. With this decomposition, recalling that $i_1 < \cdots < i_s$, we get

$$\sigma\left\{B_{t_{i_1}}^{(i_1)}, \dots, B_{t_{i_{k-1}}}^{(i_{k-1})}, B_{t_{i_k}}^{(i_k)*}, \dots, B_{t_{i_{s-1}}}^{(i_{s-1})*}\right\} \vee \widetilde{\mathcal{Z}} \subset \left(\bigvee_{i=1}^{i_k-1} \mathcal{B}^{(i)}\right) \vee \left(\bigvee_{i=i_k}^{n-1} \mathcal{B}^{(i)*}\right), \tag{3.9}$$

where $\mathcal{B}^{(i)} = \sigma\{B_s^{(i)}; s \le t_i\}$ and $\mathcal{B}^{(i)*} = \sigma\{B_s^{(i)*}; s \le t_i\}$. Since $A_\infty^{(i_k)}$ and $B^{(i)*}$, $i = i_k, i_k+1, \dots, n-1$, are formed from $B^{(i_k)}$, the σ-algebras $\bigvee_{i=1}^{i_k-1} \mathcal{B}^{(i)}$ and $\sigma\{A_\infty^{(i_k)}\} \vee (\bigvee_{i=i_k}^{n-1} \mathcal{B}^{(i)*})$ are independent. Moreover, by Remark 3.2, $\sigma\{A_\infty^{(i_k)}\}$ and $\bigvee_{i=i_k}^{n-1} \mathcal{B}^{(i)*}$ are independent. Recall that, in general, for three σ-algebras $\mathcal{F}_i$, $i = 1, 2, 3$, independence of $\mathcal{F}_1$ and $\mathcal{F}_2$ together with independence of $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_3$ implies independence of $\mathcal{F}_1$ and $\mathcal{F}_2 \vee \mathcal{F}_3$. Therefore $\sigma\{A_\infty^{(i_k)}\}$ and $(\bigvee_{i=1}^{i_k-1} \mathcal{B}^{(i)}) \vee (\bigvee_{i=i_k}^{n-1} \mathcal{B}^{(i)*})$ are independent and hence, by inclusion (3.9), we obtain the desired independence.

Furthermore, since $\sigma\{A_\infty^{(i_k)}\}$ and $\overline{\mathcal{Z}}$ are independent, we have

$$\mathcal{L}\left(\frac{\widetilde{\alpha}_{i_k}(\overline{Z})}{A_\infty^{(i_k)}} \,\middle|\, \overline{Z} = \overline{z}\right) = \gamma\left(\mu, \frac{1}{2\widetilde{\alpha}_{i_k}(\overline{z})}\right).$$

Now, in the same manner, except that in (3.8), we take

$$\widetilde{\mathcal{Z}} = \left(\bigvee_{i=1}^{i_{k-1}} \mathcal{Z}^{(i)}\right) \vee \left(\bigvee_{i=i_{k-1}+1}^{n-1} \mathcal{Z}^{(i)*}\right),$$

one can verify that $\widetilde{U}_k$ and $(U_1, \dots, U_{k-1}, U_{k+1}, \dots, U_s)$ are conditionally independent given $\overline{\mathcal{Z}}$ and

$$\mathcal{L}\left(\frac{\widetilde{\alpha}_{i_k}(\overline{Z})}{A_\infty^{(i_{k-1}+1)}} \,\middle|\, \overline{Z} = \overline{z}\right) = \gamma\left(\mu, \frac{1}{2\widetilde{\alpha}_{i_k}(\overline{z})}\right).$$

Thus we have shown that the random vectors U and $\widetilde{U}$ have a common regular conditional distribution given $\overline{\mathcal{Z}}$ and have completed the proof of Lemma 3.6. □

4. The independence property on trees

In this section we give an interpretation of the independence property on trees (originally derived in [11]) in the stochastic processes framework. The approach will be based on the conditional independence of exponential functionals of Brownian motion as obtained in Theorem 3.5.

First, we recall the independence property on trees following Massam and Wesołowski [11].

Let G_n be a tree of size n: $V(G_n) = \{1, \ldots, n\}$ is the set of vertices and $E(G_n)$ is the set of unordered edges $\{u, v\}$. We denote by $L(G_n) \subset V(G_n)$ the set of leaves, i.e. the set of vertices in the undirected tree G_n with only one neighbor. From the undirected tree G_n, we can create a directed tree $G_{n,(r)}$ by choosing a single root $r \in V(G_n)$, where the arrows flow towards the root.

Let (u, v) denote a directed edge going from a vertex u to another one v in the directed tree $G_{n,(r)}$. We then say that the vertex u is a parent of v and that v is a child of u. Each vertex u has at most one child, which is denoted by $c_r(u)$. We write $c_r(r) = \emptyset$. Each vertex v may have several parents. We denote by $p_r(v)$ the set of parents of v in $G_{n,(r)}$. If $v \neq r$ is a leaf, then $p_r(v) = \emptyset$ and $\# c_r(v) = 1$.

Let us assume that a tree G_n of size n is given and that for any $\{i, j\} \in E(G_n)$ a nonzero constant $k_{ij} = k_{ji}$ is also given. For $\{i, j\} \notin E(G_n)$, we put $k_{ij} = k_{ji} = 0$.

For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{R}^n$, we denote by $\bar{\mathbf{k}}$ the $n \times n$ symmetric matrix whose diagonal elements are k_i, $i = 1, \ldots, n$, and whose off-diagonal elements are given by k_{ij}, and set

$$M(G_n) = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{R}^n : \bar{\mathbf{k}} \text{ is positive definite}\}.$$

We also assume that a positive constant q and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ are given and consider a probability distribution on $M(G_n)$ whose density f is given by

$$f(\mathbf{k}) = \frac{1}{C}(\det(\bar{\mathbf{k}}))^{q-1} \exp(-(\mathbf{a}, \mathbf{k})/2) \mathbb{I}_{M(G_n)}(\mathbf{k}), \tag{4.1}$$

where C is the normalizing constant.

Next we fix a root $r \in V(G_n)$ and consider the directed tree $G_{n,(r)}$. Setting

$$k_{i,(r)} = \begin{cases} k_i, & i \in L(G_n) \setminus \{r\}, \\ k_i - \displaystyle\sum_{j \in p_r(i)} \frac{k_{ij}^2}{k_{j,(r)}}, & \text{otherwise}, \end{cases}$$

we attach to $G_{n,(r)}$ the mapping ψ_r defined by

$$\psi_r(k_1, \ldots, k_n) = (k_{1,(r)}, \ldots, k_{n,(r)}).$$

Here we start the definition of $k_{i,(r)}$ from the leaves and move to the root along the directed paths.

For any $r \in V(G_n)$, the mapping ψ_r is a bijection from $M(G_n)$ onto $\mathbb{R}_+^n$ and we have $\det(\bar{\mathbf{k}}) = \prod_{i \in V(G_n)} k_{i,(r)}$.

Now we are in a position to formulate the independence property on trees.

Theorem 4.1 (*see [11]*). *Let G_n be a tree of size $n \geq 2$ and a set of reals $\{k_{ij} = k_{ji} : k_{ij} \neq 0 \text{ iff } \{i, j\} \in E(G_n)\}$ be given. Let $\mathbf{K} = (K_1, \ldots, K_n)$ be a random vector whose density is given by (4.1) with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ and $q > 0$. Define $\mathbf{K}_r = \psi_r(\mathbf{K})$ for $r \in V(G_n)$. Then for all $r \in V(G_n)$ the components of $\mathbf{K}_r = (K_{1,(r)}, \ldots, K_{n,(r)})$ are mutually independent. Moreover, $K_{r,(r)} \sim \gamma(q, a_r)$ and $K_{i,(r)} \sim \mathrm{GIG}(q, a_i, k_{ic_r(i)}^2 a_{c_r(i)})$, $i \in V(G_n) \setminus \{r\}$.*

Remark 4.1. It is clear that since ψ_r's are bijections the distribution of $\mathbf{K}_{r_0}$ for an arbitrary fixed $r_0 \in V(G_n)$, as given above, uniquely determines the distribution of $\mathbf{K}_r$ for any $r \in V(G_n)$

to be also as given above. As a consequence, in Theorem 4.1 we can assume the distribution of $\mathbf{K}_{r_0} = \psi_{r_0}(\mathbf{K})$ to be a product of GIG's and γ instead of assuming the distribution of $\mathbf{K}$, equivalently.

Here we only consider $k_{ij} = 1, \{i, j\} \in E(G_n)$ for simplicity. The extension to the general case, where k_{ij}'s are arbitrary nonzero numbers for $\{i, j\} \in E(G_n)$, is immediate. Under this restriction, the mappings $\psi_r(\mathbf{k}) = (k_{1,(r)}, \dots, k_{n,(r)}), r \in V(G_n)$, are given by

$$k_{i,(r)} = \begin{cases} k_i, & i \in L(G_n) \setminus \{r\} \\ k_i - \sum_{j \in p_r(i)} \frac{1}{k_{j,(r)}}, & \text{otherwise.} \end{cases} \tag{4.2}$$

For any $u, v \in V(G_n)$, there exists a unique path $(i_1, \dots, i_p)$ from vertex u to v, that is, a sequence of the vertices such that, $i_1 = u, i_p = v$ and $\{i_j, i_{j+1}\} \in E(G_n)$ for $j = 1, \dots, p-1$. In this case, the distance $d(u, v)$ between u and v is given by $d(u, v) = p - 1$.

Without loss of generality, we adapt the following numeration to the vertices, which allows us to apply the result in the previous section. We assume that the vertex n is a leaf and that, for $u, v \in V(G_n)$,

$$d(u, n) < d(v, n) \Rightarrow u > v. \tag{4.3}$$

Thus the vertex number decreases along with the distance from the vertex n. This numeration is not unique, but, in general, it holds that $\{v : d(v, n) = 1\} = \{n - 1\}$ and, if the vertex n is additionally a root, $c_n(n-1) = n$, $p_n(n) = \{n-1\}$.

Hence the function $c_n : \{1, \dots, n-1\} \to \{2, \dots, n\}$, where $c_n(v)$ indicates a child of the vertex v in the rooted tree $G_{n,(n)}$ with a root in the vertex n, satisfies the same properties (3.2) as the function c considered in the previous section. From now on, the function c_n will be called the *child*-function.

Lemma 2 in [22] establishes the correspondence between the mappings $\Psi = \{\psi_r : r \in V(G_n)\}$ satisfying (4.2) and the mappings $\Phi^{(c_n)} = \{\phi_r^{(c_n)} : r = 1, \dots, n\}$ considered in the previous section and given by (3.3). Namely, for any $r \in V(G_n) = \{1, \dots, n\}$,

$$\psi_r(k_1, \dots, k_n) = \phi_r^{(c_n)}(\psi_n(k_1, \dots, k_n)) = \phi_r^{(c_n)}(k_{1,(n)}, \dots, k_{n,(n)}). \tag{4.4}$$

Note that it holds only for the tree satisfying $n \in L(G_n)$ and (4.3).

In view of Remark 4.1, we can now state the equivalent formulation of the independence property on trees and prove it by using the results of Section 3.

Theorem 4.2. *Let G_n be a tree of size $n \geq 2$ satisfying $n \in L(G_n)$ and (4.3). Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$, $\mu > 0$ and $\mathbf{K} = (K_1, \dots, K_n)$ be a random vector such that*

$$\psi_n(\mathbf{K}) = (K_{1,(n)}, \dots, K_{n,(n)}) \sim \left[\bigotimes_{i=1}^{n-1} \mathrm{GIG}(\mu, a_i, a_{c_n(i)}) \right] \otimes \gamma\,(\mu, a_n)\,.$$

For each $r \in V(G_n)$ we define $\mathbf{K}_r := \psi_r(\mathbf{K}) = (K_{1,(r)}, \dots, K_{n,(r)})$. Then it holds that

$$\mathbf{K}_r \sim \left[\bigotimes_{i=1}^{r-1} \mathrm{GIG}(\mu, a_i, a_{c_r(i)}) \right] \otimes \gamma(\mu, a_r) \otimes \left[\bigotimes_{i=r+1}^{n} \mathrm{GIG}(\mu, a_i, a_{c_r(i)}) \right].$$

Proof. Let $\mathbf{X}$ be a random vector satisfying the assumptions of Theorem 3.5 such that

$$\mathcal{L}(\mathbf{X}) = \mathcal{L}\left(\left(\frac{\alpha_1(\overline{Z})}{e^{(1)}_{t_1}}, \ldots, \frac{\alpha_{n-1}(\overline{Z})}{e^{(n-1)}_{t_{n-1}}}, \frac{\alpha_n(\overline{Z}) Z^{(n-1)}_{t_{n-1}}}{A^{(n-1)}_{t_{n-1},\infty}}\right) \middle| \overline{Z} = \overline{z}\right),$$

where α's satisfy

$$\alpha_i(\overline{z}) \cdot \alpha_{c_n(i)}(\overline{Z}) = \frac{z^{(i)}_{t_i}}{z^{(c_n(i))}_{t_{c_n(i)}}}, \quad i = 1, \ldots, n-2$$

$$a_i = \frac{1}{2 z^{(i)}_{t_i} \alpha_i(\overline{Z})}, \quad i = 1, \ldots, n-1$$

$\alpha_{n-1}(\overline{z}) \cdot \alpha_{c_n(n-1)}(\overline{z}) = 1$ and $a_n = (2\alpha_n(\overline{z}) z^{(n-1)}_{t_{n-1}})^{-1}$. Solving the above system we obtain

$$z^{(i)}_{t_i} = \frac{1}{2\sqrt{a_i a_{c_n(i)}}}, \qquad \alpha_i(\overline{Z}) = \sqrt{\frac{a_{c_n(i)}}{a_i}} \quad \text{for } i = 1, \ldots, n-1, \qquad \alpha_n(\overline{Z}) = \sqrt{\frac{a_{n-1}}{a_n}}.$$

Moreover, if we take $\alpha_n \equiv \sqrt{\frac{a_{n-1}}{a_n}}$, then $\alpha_1, \ldots, \alpha_{n-1}$, as functionals of $\overline{z}$, are uniquely determined by (3.4).

Then we obtain $\mathbf{X} \stackrel{d}{=} (K_{1,(n)}, \ldots, K_{n,(n)})$ and, by (4.4),

$$\phi^{(c_n)}_r(\mathbf{X}) \stackrel{d}{=} \phi^{(c_n)}_r(K_{1,(n)}, \ldots, K_{n,(n)}) = \psi_r(\mathbf{K}) = (K_{1,(r)}, \ldots, K_{n,(r)}). \tag{4.5}$$

Since $I_r(c_n) = \{i_1, \ldots, i_s\}$, where $(i_1, \ldots, i_s)$ is a path from the vertex r to the vertex n in the directed tree $G_{n,(n)}$, we have for $r \in \{1, \ldots, n-1\}$

$$c_n(i) = c_r(i) \quad \text{for } i \in J_r(c_n)$$

and

$$i_{k-1} = c_r(i_k) \quad \text{for } k = 2, \ldots, s, \quad i_1 = r.$$

Finally, by (4.5) and Theorem 3.5, we see that the distributions of $K_{i,(r)}$'s are $\gamma(\mu, a_r)$ for $i = r$ and $\mathrm{GIG}(\mu, a_i, a_{c_r(i)})$ for $i \in V(G_n) \setminus \{r\}$ and that they are mutually independent. The proof of Theorem 4.2 is now completed. □

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