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ASKEY–WILSON POLYNOMIALS, QUADRATIC HARNESSES AND MARTINGALES

BY WŁODEK BRYC[1] AND JACEK WESOŁOWSKI

University of Cincinnati and Warsaw University of Technology

We use orthogonality measures of Askey–Wilson polynomials to construct Markov processes with linear regressions and quadratic conditional variances. Askey–Wilson polynomials are orthogonal martingale polynomials for these processes.

1. Introduction. Orthogonal martingale polynomials for stochastic processes have been studied by a number of authors (see [5, 14, 19, 23, 25–28]). Orthogonal martingale polynomials play also a prominent role in noncommutative probability [1, 2] and can serve as a connection to the so called "classical versions" of noncommutative processes. On the other hand, classical versions may exist without polynomial martingale structure (see [6]). In [8] we identify intrinsic properties of the first two conditional moments of a stochastic process that guarantee the process has orthogonal martingale polynomials. These properties, linear conditional expectations and quadratic conditional variances, which we call the quadratic harness properties, have already lead to a number of new examples of Markov processes [9–11] with orthogonal martingale polynomials. Random fields with harness properties were introduced by Hammersley [15] and their properties were studied (see, e.g., [20, 34]).

In this paper we use measures of orthogonality of Askey–Wilson polynomials to construct a large class of Markov processes with quadratic harness properties that includes most of the previous examples, either as special cases or as "boundary cases." The main step is the construction of an auxiliary Markov process which has Askey–Wilson polynomials [4] as orthogonal martingale polynomials. The question of probabilistic interpretation of Askey–Wilson polynomials was raised in [12], page 197.

The paper is organized as follows. In the remainder of this section we recall background material on the quadratic harness property and Askey–Wilson polynomials; we also state our two main results. In Section 2 we give an elementary construction that does not cover the entire range of parameters, but it is explicit and does not rely on orthogonal polynomials. The general construction appears in

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Section 3; this proof follows the method from [9] and relies on the martingale property of Askey–Wilson polynomials which extends a projection formula from [21] to a larger range of parameters. Section 4 contains another elementary but computationally more cumbersome construction of a purely discrete quadratic harness which is not covered by Theorem 1.1. Section 5 illustrates how some of our previous constructions, and some new cases, follow from Theorem 1.1 essentially by a calculation. In the Appendix, we discuss two results on orthogonal polynomials in the form we need in this paper: a version of Favard's theorem that does not depend on the support of the orthogonality measure and a version of connection coefficients formula for Askey–Wilson polynomials, [4].

1.1. *Quadratic harnesses.* In [8] the authors consider square-integrable stochastic processes on $(0,\infty)$ such that for all $t,s>0$,

$$\mathbb{E}(X_t)=0, \qquad \mathbb{E}(X_tX_s)=\min\{t,s\}, \tag{1.1}$$

$E(X_t|\mathcal{F}_{s,u})$ is a linear function of X_s, X_u and $\mathrm{Var}[X_t|\mathcal{F}_{s,u}]$ is a quadratic function of X_s, X_u. Here, $\mathcal{F}_{s,u}$ is the two-sided σ-field generated by $\{X_r : r\in(0,s]\cup[u,\infty)\}$. We will also use the one-sided σ-fields $\mathcal{F}_t$ generated by $\{X_r : r\le t\}$.

Then for all $s<t<u$,

$$\mathbb{E}(X_t|\mathcal{F}_{s,u})=\frac{u-t}{u-s}X_s+\frac{t-s}{u-s}X_u \tag{1.2}$$

and under certain technical assumptions, Bryc, Matysiak and Wesolowski [8], Theorem 2.2, assert that there exist numerical constants $\eta,\theta,\sigma,\tau,\gamma$ such that for all $s<t<u$,

$$\begin{aligned}\mathrm{Var}[X_t|\mathcal{F}_{s,u}]&\\ =\frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-\gamma s}\biggl(&1+\sigma\frac{(uX_s-sX_u)^2}{(u-s)^2}+\eta\frac{uX_s-sX_u}{u-s}\\ &+\tau\frac{(X_u-X_s)^2}{(u-s)^2}+\theta\frac{X_u-X_s}{u-s}\\ &-(1-\gamma)\frac{(X_u-X_s)(uX_s-sX_u)}{(u-s)^2}\biggr).\end{aligned} \tag{1.3}$$

We will say that a square-integrable stochastic process $(X_t)_{t\in T}$ is a quadratic harness on T with parameters $(\eta,\theta,\sigma,\tau,\gamma)$ if it satisfies (1.2) and (1.3) on T which may be a proper subset of $(0,\infty)$. In previous papers (see, e.g., [8]) only $T=(0,\infty)$ was considered.

Under the conditions listed in [8], Theorems 2.4 and 4.1, quadratic harnesses on $(0,\infty)$ have orthogonal martingale polynomials. Although several explicit three-step recurrences have been worked out in [8], Section 4, and even though, for some of the recurrences, corresponding quadratic harnesses were constructed in a series of papers [9–11], the general orthogonal martingale polynomials have not been identified, and the question of existence of corresponding quadratic harnesses was left open.

It has been noted that the family of all quadratic harnesses on $(0,\infty)$ that satisfy condition (1.1) is invariant under the action of translations and reflections of $\mathbb{R}$: translation by $a \in \mathbb{R}$ acts as $(X_t) \mapsto e^{-a}X_{e^{2a}t}$ and the reflection at 0 acts as $(X_t) \mapsto (tX_{1/t})$. Since translations and reflection generate also the symmetry group of the Askey–Wilson polynomials [22], it is natural to investigate how to relate the measures of orthogonality of the Askey–Wilson polynomials to quadratic harnesses. The goal of this paper is to explore this idea and significantly enlarge the class of available examples. We show that quadratic harnesses exist and are Markov processes for a wide range of parameters $(\eta,\theta,\sigma,\tau,\gamma)$. The basic Markov process we construct has Askey–Wilson polynomials as orthogonal martingale polynomials. The final quadratic harness is then obtained by appropriate scaling and a deterministic change of time.

THEOREM 1.1. *Fix parameters* $-1 < q < 1$ *and* A, B, C, D *that are either real, or* (A, B) *or* (C, D) *are complex conjugate pairs such that* $ABCD$, $qABCD < 1$. *Assume that*

$$AC, AD, BC, BD, qAC, qAD, qBC, qBD \in \mathbb{C} \setminus [1,\infty). \tag{1.4}$$

Let

$$\eta = -\frac{[(A+B)(1+ABCD) - 2AB(C+D)]\sqrt{1-q}}{\sqrt{(1-AC)(1-BC)(1-AD)(1-BD)(1-qABCD)}}, \tag{1.5}$$

$$\theta = -\frac{[(D+C)(1+ABCD) - 2CD(A+B)]\sqrt{1-q}}{\sqrt{(1-AC)(1-BC)(1-AD)(1-BD)(1-qABCD)}}, \tag{1.6}$$

$$\sigma = \frac{AB(1-q)}{1-qABCD}, \tag{1.7}$$

$$\tau = \frac{CD(1-q)}{1-qABCD}, \tag{1.8}$$

$$\gamma = \frac{q-ABCD}{1-qABCD}. \tag{1.9}$$

With the convention $1/\infty = 0$, *let*

$$T_0 = \max\left\{0, \frac{\gamma-1+\sqrt{(\gamma-1)^2-4\sigma\tau}}{2\sigma}, -\tau\right\}, \tag{1.10}$$

$$\frac{1}{T_1} = \max\left\{0, \frac{\gamma-1+\sqrt{(\gamma-1)^2-4\sigma\tau}}{2\tau}, -\sigma\right\}. \tag{1.11}$$

Then there exists a bounded Markov process $(X_t)_{t\in J}$ *on the nonempty interval* $J=(T_0,T_1)$ *with mean and covariance* (1.1) *such that* (1.2) *holds, and* (1.3) *holds with parameters* $\eta,\theta,\sigma,\tau,\gamma$. *Process* $(X_t)_{t\in J}$ *is unique among the processes with infinitely-supported one-dimensional distributions that have moments of all orders and satisfy* (1.1), (1.2) *and* (1.3) *with the same parameters*, $\eta,\theta,\sigma,\tau,\gamma$.

REMARK 1.1. Formula (2.28) relates process (X_t) to the Markov process (Z_t) from Theorem 1.2.

REMARK 1.2. The assumptions on A, B, C, D are dictated by the desire to limit the number of cases in the proof but do not exhaust all possibilities where the quadratic harness (X_t) with Askey–Wilson transition probabilities exists (see Proposition 4.1).

REMARK 1.3. When $\sigma\tau\geq 0$, Theorem 1.1 can be used to construct quadratic harnesses only for parameters in the range $-1<\gamma<1-2\sqrt{\sigma\tau}$ which is strictly smaller than the admissible range in [8], Theorem 2.2. To see the upper bound, note that

$$1-\gamma=(1-q)(1-ABCD)/(1-qABCD)>0 \tag{1.12}$$

and that $(1-\gamma)^2-4\sigma\tau=(1-ABCD)^2(1-q)^2/(1-qABCD)^2>0$. The lower bound follows from $q>-1$, as (1.9) defines γ as an increasing function of q. In Corollary 5.4 we show that the construction indeed works through the entire range of γ, at least when $\eta=\theta=0$.

From (1.9), the construction will give $\gamma>1$ when $ABCD<-1$. Multiplying (1.7) and (1.8), we see that this may occur only when $\sigma\tau<0$, that is, when the time interval J is a proper subset of $(0,\infty)$ (compare [8], Theorem 2.2).

REMARK 1.4. In terms of the original parameters, the end-points of the interval, are

$$T_0=\max\left\{0,-CD,\frac{-CD(1-q)}{1-qABCD}\right\}, \tag{1.13}$$

$$\frac{1}{T_1}=\max\left\{0,-AB,\frac{-AB(1-q)}{1-qABCD}\right\}. \tag{1.14}$$

This shows that T_0,T_1 are real and $T_0<T_1$. If $CD<0$ or $AB<0$, then the third term under the maximum contributes for $q<0$ only.

REMARK 1.5. As a side property, we also get information about one-sided conditioning: $\mathbb{E}(X_t|\mathcal{F}_s)=X_s$ and $\operatorname{Var}(X_t|\mathcal{F}_s)=\frac{t-s}{1+\sigma s}(1+\eta X_s+\sigma X_s^2)$ for $s<t$. Similarly, $\mathbb{E}(X_t|\mathcal{F}_{\geq u})=tX_u/u$ and $\operatorname{Var}(X_t|\mathcal{F}_{\geq u})=\frac{t(u-t)}{u+\tau}(1+\theta X_u/u+\tau X_u^2/u^2)$ for $t<u$, where $\mathcal{F}_{\geq u}=\sigma(X_r:r\geq u)$.

1.2. *Martingale property of Askey–Wilson polynomials.* For $a, b, c, d \in \mathbb{C}$ such that

$$abcd, qabcd \notin [1, \infty), \tag{1.15}$$

Askey and Wilson [4], (1.24), introduced polynomials defined by recurrence,

$$2x\widetilde{w}_n(x) = \widetilde{A}_n\widetilde{w}_{n+1}(x) + B_n\widetilde{w}_n(x) + \widetilde{C}_n\widetilde{w}_{n-1}(x), \qquad n \geq 0, \tag{1.16}$$

with the initial conditions $\widetilde{w}_{-1} = 0$ and $\widetilde{w}_0 = 1$, and with the coefficients

$$\widetilde{A}_n = \frac{A_n}{(1-abq^n)(1-acq^n)(1-adq^n)},$$
$$B_n = a + 1/a - A_n/a - aC_n,$$
$$\widetilde{C}_n = C_n(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1}),$$

where for future reference we denote

$$A_n = \frac{(1-abcdq^{n-1})(1-abq^n)(1-acq^n)(1-adq^n)}{(1-abcdq^{2n-1})(1-abcdq^{2n})}, \tag{1.17}$$

$$C_n = \frac{(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \tag{1.18}$$

Here we take $A_0 = (1-ab)(1-ac)(1-ad)/(1-abcd)$ and $C_0 = 0$, also if $q = 0$. We remark that B_n coincides with [4], (1.27), so it is symmetric in a, b, c, d and that by taking the limit, B_n is also well defined for $a = 0$. Since trivially $\widetilde{A}_n$ and $\widetilde{C}_n$ are also symmetric in a, b, c, d it follows that polynomials $\{\widetilde{w}_n\}$ do not depend on the order of a, b, c, d.

Except for Section 4, our parameters satisfy a condition stronger than (1.15):

$$abcd, qabcd, ab, qab, ac, qac, ad, qad \in \mathbb{C} \setminus [1, \infty). \tag{1.19}$$

To avoid cumbersome scaling of martingale polynomials later on, when (1.19) holds it is convenient to renormalize the polynomials $\widetilde{w}_n$. Therefore we introduce the following family of polynomials:

$$2x\bar{w}_n(x) = \bar{A}_n\bar{w}_{n+1}(x) + B_n\bar{w}_n(x) + \bar{C}_n\bar{w}_{n-1}(x), \qquad n \geq 0, \tag{1.20}$$

where $\bar{A}_n = (1-abq^n)\widetilde{A}_n$, $\bar{C}_n = \widetilde{C}_n/(1-abq^{n-1})$. The initial conditions are again $\bar{w}_{-1} = 0$ and $\bar{w}_0 = 1$. When we want to indicate the parameters, we will write $\bar{w}_n(x; a, b, c, d)$.

For each n, polynomial $\bar{w}_n$ differs only by a multiplicative constant from $\widetilde{w}_n$ [see (A.10)] so both families have the same orthogonality measure when it exists. For this reason, both families of polynomials are referred to as Askey–Wilson polynomials.

Recall that the polynomials $\{r_n(x; t) : n \in \mathbb{Z}_+, t \in I\}$ are orthogonal martingale polynomials for the process $(Z_t)_{t \in I}$ if:

(i) $\mathbb{E}(r_n(Z_t;t)r_m(Z_t;t)) = 0$ for $m \neq n$ and $t \in I$,
(ii) $\mathbb{E}(r_n(Z_t;t)|\mathcal{F}_s) = r_n(Z_s;s)$ for $s < t$ in I and all $n = 0, 1, 2 \dots$.

The following result shows that Askey–Wilson polynomials define orthogonal martingale polynomials for a family of Markov processes.

THEOREM 1.2. *Suppose that A, B, C, D satisfy the assumptions of Theorem 1.1. Let*

$$I = I(A, B, C, D, q) = \left(\max\{0, CD, qCD\}, \frac{1}{\max\{0, AB, qAB\}}\right) \tag{1.21}$$

with the convention $1/0 = \infty$. (The last terms under the maxima can contribute only when $q < 0$ and CD or AB are negative.) Let

$$r_n(x;t) = t^{n/2}\bar{w}_n\left(\frac{\sqrt{1-q}}{2\sqrt{t}}x; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}\right). \tag{1.22}$$

Then

$$\{r_n(x;t) : n = 0, 1, 2 \dots, t \in I\}$$

are orthogonal martingale polynomials for a Markov process (Z_t) which satisfies (1.2) and (1.3) with $\eta = \theta = \sigma = \tau = 0$ and $\gamma = q$.

2. The case of densities. In this section we give an explicit and elementary construction of a quadratic harness on a possibly restricted time interval and under additional restrictions on parameters A, B, C, D.

PROPOSITION 2.1. *Fix parameters $-1 < q < 1$ and A, B, C, D that are either real or (A, B) or (C, D) are complex conjugate pairs. Without loss of generality, we assume that $|A| \leq |B|$, $|C| \leq |D|$; additionally, we assume that $|BD| < 1$. Then the interval*

$$J = \left(\frac{|D|^2 - CD}{1 - AB|D|^2}, \frac{1 - CD|B|^2}{|B|^2 - AB}\right)$$

has positive length and there exists a unique bounded Markov process $(X_t)_{t\in J}$ with absolutely continuous finite-dimensional distributions which satisfies the conclusion of Theorem 1.1.

REMARK 2.1. Proposition 2.1 is a special case of Theorem 1.1; the latter may allow us to extend the processes constructed here to a wider time-interval,

$$T_0 \leq \frac{|D|^2 - CD}{1 - AB|D|^2} \quad \text{and} \quad \frac{1 - CD|B|^2}{|B|^2 - AB} \leq T_1.$$

The easiest way to see the inequalities is to compare the end points of intervals (2.9) and (1.21) after the Möbius transformation (2.26); for example, $|D|^2 \geq |CD| \geq \max\{0, CD, qCD\}$ where the last term plays a role only when $q < 0$ and $CD < 0$.

The rest of this section contains the construction, ending with the proof of Proposition 2.1.

2.1. *Askey–Wilson densities.* For complex a and $|q| < 1$ we define

$$(a)_n = (a;q)_n = \begin{cases} \prod_{j=0}^{n-1}(1-aq^j), & n = 1, 2, \dots, \\ 1, & n = 0, \end{cases}$$

$$(a)_\infty = (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j),$$

and we denote

$$(a_1, a_2, \dots, a_l)_\infty = (a_1, a_2, \dots, a_l; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_l;q)_\infty,$$

$$(a_1, a_2, \dots, a_l)_n = (a_1, a_2, \dots, a_l; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_l;q)_n.$$

The advantage of this notation over the standard product notation lies both in its conciseness and in mnemonic simplification rules,

$$\frac{(a,b)_n}{(a,c)_n} = \frac{(b)_n}{(c)_n},$$

$$(\alpha)_{M+L} = (q^M\alpha)_L(\alpha)_M \tag{2.1}$$

and

$$(\alpha)_M = (-\alpha)^M q^{M(M-1)/2}\left(\frac{q}{q^M\alpha}\right)_M, \tag{2.2}$$

which often help with calculations. For a reader who is as uncomfortable with this notation, as we were at the beginning of this project, we suggest to re-write the formulas for the case $q = 0$. For example, $(a;0)_n$ is either 1 or $1-a$ as $n = 0$ or $n > 0$, respectively. The construction of Markov process for $q = 0$ in itself is quite interesting as the resulting laws are related to the laws that arise in Voiculescu's free probability; the formulas simplify enough so that the integrals can be computed by elementary means, for example, by residua.

From Askey and Wilson [4], Theorem 2.1, it follows that if a, b, c, d are complex such that $\max\{|a|, |b|, |c|, |d|\} < 1$ and $-1 < q < 1$, then with $\theta = \theta_x$ such that $\cos\theta = x$,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta})_\infty}\, dx = \frac{2\pi(abcd)_\infty}{(q, ab, ac, ad, bc, bd, cd)_\infty}. \tag{2.3}$$

When $-1 < q < 1$ and a, b, c, d are either real or come in complex conjugate pairs and $\max\{|a|, |b|, |c|, |d|\} < 1$, the integrand is real and positive. This allows us to define the Askey–Wilson density,

$$f(x; a, b, c, d) = \frac{K(a, b, c, d)}{\sqrt{1-x^2}} \left| \frac{(e^{2i\theta})_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta})_\infty} \right|^2 I_{(-1,1)}(x), \tag{2.4}$$

where

$$K(a, b, c, d) = \frac{(q, ab, ac, ad, bc, bd, cd)_\infty}{2\pi(abcd)_\infty}. \tag{2.5}$$

The first two moments are easily computed.

PROPOSITION 2.2. *Suppose X has the Askey–Wilson density $f(x; a, b, c, d)$ with parameters a, b, c, d as above. Then the expectation of X is*

$$\mathbb{E}(X) = \frac{a+b+c+d-abc-abd-acd-bcd}{2(1-abcd)} \tag{2.6}$$

and the variance of X is

$$\operatorname{Var}(X) = \frac{(1-ab)(1-ac)(1-ad)(1-bc)(1-bd)(1-cd)(1-q)}{4(1-abcd)^2(1-abcdq)}. \tag{2.7}$$

PROOF. If $a = b = c = d = 0$, $\mathbb{E}(X) = 0$ by symmetry. If one of the parameters, say $a \in \mathbb{C}$, is nonzero, we note that

$$\begin{aligned}(ae^{i\theta}, ae^{-i\theta})_\infty &= (ae^{i\theta}, ae^{-i\theta})_1 (aqe^{i\theta}, aqe^{-i\theta})_\infty \\ &= (1+a^2-2ax)(aqe^{i\theta}, aqe^{-i\theta})_\infty.\end{aligned}$$

Therefore, by (2.3),

$$\mathbb{E}(1+a^2-2aX) = \frac{K(a, b, c, d)}{K(qa, b, c, d)} = \frac{(1-ab)(1-ac)(1-ad)}{1-abcd}.$$

Now (2.6) follows by a simple algebra.

Similarly, for nonzero $a, b \in \mathbb{C}$,

$$\begin{aligned}4ab \operatorname{Var}(X) &= \mathbb{E}[(1+a^2-2aX)(1+b^2-2bX)], \\ &\quad -\mathbb{E}(1+a^2-2aX)\mathbb{E}(1+b^2-2bX) \\ &= \frac{K(a, b, c, d)}{K(qa, qb, c, d)} - \frac{K^2(a, b, c, d)}{K(qa, b, c, d)K(a, qb, c, d)} \\ &= \frac{(1-ab)(1-qab)(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)} \\ &\quad - \frac{(1-ab)^2(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)^2}.\end{aligned}$$

Again after simple transformations we arrive at (2.7).

If only one parameter is nonzero but $q \neq 0$, the calculations are similar, starting with $\mathbb{E}((1 + a^2 - 2aX)(1 + a^2q^2 - 2aqX))$; when $q = 0$ the density is a re-parametrization of Marchenko–Pastur law [16], (3.3.2); we omit the details. If a, b, c, d are zero, $f(x; 0, 0, 0, 0)$ is the orthogonality measure of the continuous q-Hermite polynomials [18], (3.26.3); since $H_2(x) = 2xH_1(x) - (1 - q)H_0 = 4x^2 - (1 - q)$, the second moment is $(1 - q)/4$. □

We need a technical result on Askey–Wilson densities inspired by [21], formula (2.4).

PROPOSITION 2.3. *Let a, b, c, d, q be as above with the additional assumption that the only admissible conjugate pairs are $a = \bar{b}$ or $c = \bar{d}$, and m is real such that $|m| < 1$. Then with $x = \cos\theta_x$,*

$$\begin{aligned}\int_{-1}^{1} f(x; am, bm, c, d) f(y; a, b, me^{i\theta_x}, me^{-i\theta_x})\, dx \\ = f(y; a, b, cm, dm).\end{aligned} \tag{2.8}$$

PROOF. We compute the left-hand side of (2.8) expanding the constants $K(am, bm, c, d)$ and $K(a, b, me^{i\theta_x}, me^{-i\theta_x})$ to better show how some factors cancel out. To avoid case-by-case reasoning when complex conjugate pairs are present, we also expand parts of the density without the use of modulus as in (2.3).

The integrand on the left-hand side of (2.8) is

$$\begin{aligned}&\frac{(q, abm^2, acm, adm, bcm, bdm, cd)_\infty |(e^{2i\theta_x})_\infty|^2}{2\pi(abcdm^2)_\infty (ame^{i\theta_x}, ame^{-i\theta_x}, bme^{i\theta_x}, bme^{-i\theta_x})_\infty |(ce^{i\theta_x}, de^{i\theta_x})_\infty|^2} \\ &\quad\times \frac{(q, ab, ame^{i\theta_x}, ame^{-i\theta_x}, bme^{i\theta_x}, bme^{-i\theta_x}, m^2)_\infty}{2\pi(abm^2)_\infty \sqrt{1 - y^2}} \\ &\quad\times \frac{|(e^{2i\theta_y})_\infty|^2}{|(ae^{i\theta_y}, be^{i\theta_y}, me^{i(\theta_x+\theta_y)}, me^{i(-\theta_x+\theta_y)})_\infty|^2 \sqrt{1 - x^2}}.\end{aligned}$$

Rearranging the terms we rewrite the left-hand side of (2.8) as

$$\begin{aligned}&\frac{(q, abm^2, acm, adm, bcm, bdm, cd, q, ab, m^2)_\infty}{(2\pi)^2 (abm^2, abcdm^2)_\infty \sqrt{1 - y^2}} \\ &\quad\times \frac{|(e^{2i\theta_y})_\infty|^2}{|(ae^{i\theta_y}, be^{i\theta_y})_\infty|^2} \\ &\quad\times \int_{-1}^{1} \frac{|(e^{2i\theta_x})_\infty|^2}{|(me^{i\theta_y}e^{i\theta_x}, me^{-i\theta_y}e^{i\theta_x}, ce^{i\theta_x}, de^{i\theta_x})_\infty|^2} \frac{dx}{\sqrt{1 - x^2}}.\end{aligned}$$

Now we apply formula (2.3) to this integral, so the left-hand side of (2.8) becomes

$$\frac{(q, abm^2, acm, adm, bcm, bdm, cd, q, ab, m^2)_\infty}{(2\pi)^2(abm^2, abcdm^2)_\infty\sqrt{1-y^2}} \times \frac{|(e^{2i\theta_y})_\infty|^2}{|(ae^{i\theta_y}, be^{i\theta_y})_\infty|^2}$$

$$\times \frac{2\pi(cdm^2)_\infty}{(q, m^2, mce^{i\theta_y}, mde^{i\theta_y}, mce^{-i\theta_y}, mde^{-i\theta_y}, cd)_\infty}$$

$$= \frac{(q, ab, acm, adm, bcm, bdm, cdm^2)_\infty|(e^{2i\theta_y})_\infty|^2}{2\pi(abcdm^2)_\infty|(ae^{i\theta_y}, be^{i\theta_y}, mce^{i\theta_y}, mde^{i\theta_y})_\infty|^2\sqrt{1-y^2}},$$

which completes the proof. □

2.2. *Markov processes with Askey–Wilson densities.* We now fix A, B, C, D as in Proposition 2.1. The interval

$$I(A, B, C, D) = \left(|D|^2, \frac{1}{|B|^2}\right) \tag{2.9}$$

is nonempty (here $1/0 = \infty$). For any $t \in I(A, B, C, D)$ and $y \in [-1, 1]$ let

$$p(t, y) = f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right) \tag{2.10}$$

and for any $s < t$ in $I(A, B, C, D)$ and $x, y \in [-1, 1]$ let

$$p(s, x; t, y) = f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right), \qquad x = \cos\theta_x. \tag{2.11}$$

PROPOSITION 2.4. *The family of probability densities* $(p(s, x; t, y), p(t, y))$ *defines a Markov process* $(Y_t)_{t\in I}$ *on the state space* $[-1, 1]$. *That is, for any* $s < t$ *from* $I(A, B, C, D)$ *and* $y \in [-1, 1]$,

$$p(t, y) = \int_{-1}^{1} p(s, x; t, y)p(s, x)\,dx \tag{2.12}$$

and for any $s < t < u$ *from* $I(A, B, C, D)$ *and* $x, z \in [-1, 1]$,

$$p(s, x; u, z) = \int_{-1}^{1} p(t, y; u, z)p(s, x; t, y)\,dy. \tag{2.13}$$

PROOF. To show (2.12) it suffices just to use the identity (2.8) with $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = C/\sqrt{s}$, $d = D/\sqrt{s}$, and $m = \sqrt{s/t} \in (0, 1)$. We note that this substitution preserves the complex conjugate pairs and that, by the definition of $I(A, B, C, D)$, parameters $A\sqrt{t}$, $B\sqrt{t}$, $C/\sqrt{s}$ and $D/\sqrt{s}$ have modulus less than one. So (2.8) applies here and gives the desired formula,

$$\int_{-1}^{1} f\left(x; A\sqrt{s}, B\sqrt{s}, \frac{C}{\sqrt{s}}, \frac{D}{\sqrt{s}}\right) f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right) dx$$
$$= f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right).$$

To get the second formula (2.13) we again use (2.8) this time with $a = A\sqrt{u}$, $b = \sqrt{u}$, $c = \sqrt{s/t}e^{i\theta_x}$, $d = \sqrt{s/t}e^{-i\theta_x}$, and $m = \sqrt{t/u}$. Thus we arrive at

$$\int_{-1}^{1} f\left(z; A\sqrt{u}, B\sqrt{u}, \frac{\sqrt{t}}{\sqrt{u}}e^{i\theta_y}, \frac{\sqrt{t}}{\sqrt{u}}e^{-i\theta_y}\right)$$
$$\times f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right) dy$$
$$= f\left(z; A\sqrt{u}, B\sqrt{u}, \frac{\sqrt{s}}{\sqrt{u}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{u}}e^{-i\theta_x}\right). \qquad \square$$

PROPOSITION 2.5. *Let $(Y_t)_{t\in I(A,B,C,D)}$ be the Markov process from Proposition 2.4, with marginal densities (2.10) and transition densities (2.11). For $t \in I(A, B, C, D)$,*

$$\mathbb{E}(Y_t) = \frac{[A + B - AB(C + D)]t + C + D - CD(A + B)}{2\sqrt{t}(1 - ABCD)}, \tag{2.14}$$

$$\operatorname{Var}(Y_t) = \frac{(1-q)(1 - AC)(1 - AD)(1 - BC)(1 - BD)}{4t(1 - ABCD)^2(1 - qABCD)} \times (t - CD)(1 - ABt) \tag{2.15}$$

and for $s, t \in I(A, B, C, D)$, such that $s < t$,

$$\operatorname{Cov}(Y_s, Y_t) = \frac{(1-q)(1 - AC)(1 - AD)(1 - BC)(1 - BD)}{4\sqrt{st}(1 - ABCD)^2(1 - qABCD)} \times (s - CD)(1 - ABt), \tag{2.16}$$

$$\mathbb{E}(Y_t|\mathcal{F}_s) = \frac{(A + B)(t - s) + 2(1 - ABt)\sqrt{s}Y_s}{2\sqrt{t}(1 - ABs)}, \tag{2.17}$$

$$\operatorname{Var}(Y_t|\mathcal{F}_s) = \frac{(1-q)(t - s)(1 - ABt)}{4t(1 - ABs)^2(1 - qABs)}\left(1 + A^2s - 2A\sqrt{s}Y_s\right) \times \left(1 + B^2s - 2B\sqrt{s}Y_s\right). \tag{2.18}$$

PROOF. Formulas (2.14) and (2.15) follow, respectively, from (2.6) and (2.7) by taking $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = C/\sqrt{t}$ and $d = D/\sqrt{t}$.

Similarly, the formulas (2.17) and (2.18) follow, respectively, from (2.6) and (2.7) by taking $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = \sqrt{\frac{s}{t}}e^{i\theta_x}$ and $d = \sqrt{\frac{s}{t}}e^{-i\theta_x}$.

To obtain the covariance we make use of (2.17) as follows:

$$\begin{aligned}\operatorname{Cov}(Y_s, Y_t) &= \mathbb{E}(Y_s \mathbb{E}(Y_t|\mathcal{F}_s)) - \mathbb{E}(Y_s)\mathbb{E}(Y_t) \\ &= \left(\frac{(A+B)(t-s)}{2\sqrt{t}(1-ABs)} - \mathbb{E}Y_t\right)\mathbb{E}Y_s \\ &\quad + \frac{(1-ABs)\sqrt{s}}{(1-ABt)\sqrt{t}}\left(\operatorname{Var}(Y_s) + [\mathbb{E}Y_s]^2\right).\end{aligned}$$

Now the formula (2.16) follows, after a calculation, from (2.14) and (2.15). □

Next we show that the conditional distribution of Y_t given the past and the future of the process is given by an Askey–Wilson density that does not depend on parameters A, B, C, D.

PROPOSITION 2.6. *Let $(Y_t)_{t\in I(A,B,C,D)}$ be the Markov process with marginal densities* (2.10) *and transition densities* (2.11). *Then for any $s < t < u$ in $I(A, B, C, D)$, the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ has the Askey–Wilson density,*

$$f\left(y; \frac{\sqrt{t}}{\sqrt{u}}\exp(i\theta_z), \frac{\sqrt{t}}{\sqrt{u}}\exp(-i\theta_z), \frac{\sqrt{s}}{\sqrt{t}}\exp(i\theta_x), \frac{\sqrt{s}}{\sqrt{t}}\exp(-i\theta_x)\right). \tag{2.19}$$

(*Here*, $x = \cos\theta_x = Y_s$, $z = \cos\theta_z = Y_u$.) *The first two conditional moments have the form*

$$\mathbb{E}(Y_t|\mathcal{F}_{s,u}) = \frac{(u-t)\sqrt{s}Y_s + (t-s)\sqrt{u}Y_u}{\sqrt{t}(u-s)}, \tag{2.20}$$

$$\begin{aligned}\operatorname{Var}(Y_t|\mathcal{F}_{s,u}) &= \frac{(1-q)(u-t)(t-s)}{t(u-qs)} \\ &\quad \times \left(\frac{1}{4} - \frac{(u\sqrt{s}Y_s - s\sqrt{u}Y_u)(\sqrt{u}Y_u - \sqrt{s}Y_s)}{(u-s)^2}\right).\end{aligned} \tag{2.21}$$

PROOF. By the Markov property it follows that the conditional density is

$$\begin{aligned}&\frac{p(t, y; u, z)p(s, x; t, y)}{p(s, x; u, z)} \\ &\quad = f\left(z; A\sqrt{u}, B\sqrt{u}, \frac{\sqrt{t}}{\sqrt{u}}e^{i\theta_y}, \frac{\sqrt{t}}{\sqrt{u}}e^{-i\theta_y}\right) \\ &\quad\quad \times f\left(y; A\sqrt{t}, B\sqrt{t}, \frac{\sqrt{s}}{\sqrt{t}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{t}}e^{-i\theta_x}\right) \\ &\quad\quad \times \left(f\left(z; A\sqrt{u}, B\sqrt{u}, \frac{\sqrt{s}}{\sqrt{u}}e^{i\theta_x}, \frac{\sqrt{s}}{\sqrt{u}}e^{-i\theta_x}\right)\right)^{-1}.\end{aligned}$$

Now the result follows by plugging in the formula above the definition of the Askey–Wilson density (2.4) with suitably chosen parameters. The mean and variance are calculated from (2.6) and (2.7). □

PROOF OF PROPOSITION 2.1. If we define a new process $(Z_t)_{t\in I(A,B,C,D)}$ through

$$Z_t = \frac{2\sqrt{t}}{\sqrt{1-q}} Y_t, \tag{2.22}$$

then (Z_t) is Markov and, for $s < t$, satisfies

$$\mathbb{E}(Z_t|\mathcal{F}_s) = \frac{(A+B)(t-s)}{\sqrt{1-q}(1-ABs)} + \frac{1-ABt}{1-ABs} Z_s$$

so that

$$\left(\frac{AB\sqrt{1-q}Z_t - (A+B)}{1-ABt}, \mathcal{F}_t\right)$$

is a martingale. Moreover,

$$\begin{aligned}\operatorname{Var}(Z_t|\mathcal{F}_s) = {} & \frac{(t-s)(1-ABt)}{(1-ABs)^2(1-qABs)} \\ & \times \left(1 + A^2 s - A\sqrt{1-q}Z_s\right)\left(1 + B^2 s - B\sqrt{1-q}Z_s\right).\end{aligned}$$

For the double conditioning with respect to the past and future jointly, it follows that (Z_t) satisfies quadratic harness conditions; for $s < t < u$,

$$\mathbb{E}(Z_t|\mathcal{F}_{s,u}) = \frac{u-t}{u-s} Z_s + \frac{t-s}{u-s} Z_u \tag{2.23}$$

and

$$\operatorname{Var}(Z_t|\mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{u-qs}\left(1 - (1-q)\frac{(uZ_s - sZ_u)(Z_u - Z_s)}{(u-s)^2}\right), \tag{2.24}$$

which correspond to the q-Brownian motion (see [10], Theorem 4.1). Here, (Z_t) is defined only on a possibly-bounded time domain $I(A, B, C, D)$, and the covariance is different than in [10]; for $s < t$,

$$\begin{aligned}\operatorname{Cov}(Z_s, Z_t) = {} & \frac{(1-AC)(1-AD)(1-BC)(1-BD)}{(1-ABCD)^2(1-qABCD)} \\ & \times (s - CD)(1 - ABt).\end{aligned} \tag{2.25}$$

(The law of Z_t will differ from the q-Gaussian law if $|A| + |B| + |C| + |D| > 0$.)

The covariance is adjusted by a suitable deterministic time change. Consider a Möbius transformation

$$h(x) = \frac{x - CD}{1 - ABx}, \tag{2.26}$$

which for $ABCD < 1$ is an increasing function with the inverse,

$$T(t) = \frac{t + CD}{1 + ABt}. \tag{2.27}$$

Note that $J = J(A, B, C, D) = h(I(A, B, C, D))$. For $t \in J(A, B, C, D)$, define

$$\begin{aligned} X_t &:= X_{t;A,B,C,D,q} \\ &= \frac{Z_{T(t)} - \mathbb{E}(Z_{T(t)})}{1 - T(t)AB} \times \frac{(1 - ABCD)\sqrt{1 - qABCD}}{\sqrt{(1 - AC)(1 - BC)(1 - AD)(1 - BD)}} \\ &= \frac{\sqrt{1-q}(1 + ABt)Z_{T(t)} - (A + B)t - (C + D)}{\sqrt{(1 - q)(1 - AC)(1 - AD)(1 - BC)(1 - BD)}}\sqrt{1 - qABCD}. \end{aligned} \tag{2.28}$$

A calculation shows that $(X_t)_{t\in J}$ has unconditional and conditional moments as claimed: formula (1.1) is a consequence of (2.25), and (1.2) follows from (2.23). A much longer calculation shows that (2.24) translates into (1.3) with parameters (1.5)–(1.9). □

3. Construction in the general case. Next, we tackle the issue of extending the quadratic harness from Proposition 2.1 to a larger time interval. The main technical difficulty is that such processes may have a discrete component in their distributions. The construction is based on the Askey–Wilson distribution [4], (2.9), with slight correction as in [30], (2.5).

The basic plan of the proof of Theorem 1.1 is the same as that of the proof of Proposition 2.1: we define auxiliary Markov process $(Y_t)_{t\in I}$ through a family of Askey–Wilson distributions that satisfy the Chapman–Kolmogorov equations. Then we use formulas (2.22) and (2.28) to define (X_t). The main difference is that due to an overwhelming number of cases that arise with mixed-type distributions, we use orthogonal polynomials to deduce all properties we need. (A similar approach was used in [9].)

3.1. *The Askey–Wilson law.* The Askey–Wilson distribution $\nu(dx; a, b, c, d)$ is the (probabilistic) orthogonality measure of the Askey–Wilson polynomials $\{\widetilde{w}_n\}$ as defined in (1.16). Therefore it does not depend on the order of parameters a, b, c, d. Since $\{\widetilde{A}_n\}$, $\{\widetilde{B}_n\}$ and $\{\widetilde{C}_n\}$ are bounded sequences, $\nu(dx; a, b, c, d)$ is unique and compactly supported [17], Theorems 2.5.4 and 2.5.5. If $|a|$, $|b|$, $|c|$, $|d| < 1$, this is an absolutely continuous measure with density (2.4). For other values of parameters, $\nu(dx; a, b, c, d)$ may have a discrete component or be purely discrete as in (4.1).

In general, it is quite difficult to give explicit conditions for the existence of the Askey–Wilson distribution $\nu(dx; a, b, c, d)$ in terms of a, b, c, d. To find sufficient conditions, we will be working with sequences $\{A_k\}$, and $\{C_k\}$ defined by (1.17) and (1.18). Since $\widetilde{A}_{k-1}\widetilde{C}_k = A_{k-1}C_k$, by Theorem A.1, measure $\nu(dx; a, b, c, d)$ exists for all a, b, c, d such that sequences $\{A_k\}$, $\{C_k\}$ are real, and (A.1) holds. If

a, b, c, d are either real or come in complex conjugate pairs and (1.19) holds, then $A_k > 0$ and $C_k \in \mathbb{R}$ for all k. So in this case condition, (A.1) becomes

$$\prod_{k=1}^{n} C_k \geq 0 \qquad \text{for all } n \geq 1. \tag{3.1}$$

A simple sufficient condition for (3.1) is that in addition to (1.19) we have

$$bc, qbc, bd, qbd, cd, qcd \in \mathbb{C} \setminus [1, \infty). \tag{3.2}$$

Under this condition, if a, b, c, d are either real or come in complex conjugate pairs, then the corresponding measure of orthogonality $\nu(dx; a, b, c, d)$ exists. Unfortunately, this simple condition is not general enough for our purposes; we need to allow also Askey–Wilson laws with finite support as in [3]. In fact, such laws describe transitions of the Markov process in the atomic part.

We now state conditions that cover all the cases needed in this paper. Let $m_1 = m_1(a, b, c, d)$ denote the number of the products ab, ac, ad, bc, bd, cd that fall into subset $[1, \infty)$ of complex plane, and let $m_2 = m_2(a, b, c, d)$ denote the number of the products qab, qac, qad, qbc, qbd, qcd that fall into $[1, \infty)$. (For $m_1 = 0$, measure ν is described in [29].)

LEMMA 3.1. *Assume that a, b, c, d are either real or come in complex conjugate pairs and that $abcd < 1$, $qabcd < 1$. Then the Askey–Wilson distribution ν exists only in the following cases*:

(i) *If $q \geq 0$ and $m_1 = 0$, then $\nu(dx; a, b, c, d)$ exists and has a continuous component.*

(ii) *If $q < 0$ and $m_1 = m_2 = 0$, then $\nu(dx; a, b, c, d)$ exists and has a continuous component.*

(iii) *If $q \geq 0$ and $m_1 = 2$, then $a, b, c, d \in \mathbb{R}$. In this case, $\nu(dx; a, b, c, d)$ is well defined if either $q = 0$ or the smaller of the two products that fall into $[1, \infty)$ is of the form $1/q^N$, and in this latter case $\nu(dx; a, b, c, d)$ is a purely discrete measure with $N + 1$ atoms.*

(iv) *If $q < 0$ and $m_1 = 2$, $m_2 = 0$ then $a, b, c, d \in \mathbb{R}$. In this case, $\nu(dx; a, b, c, d)$ is well defined if the smaller of the two products in $[1, \infty)$ equals $1/q^N$ with even N. Then $\nu(dx; a, b, c, d)$ is a purely discrete measure with $N + 1$ atoms.*

(v) *If $q < 0$, $m_1 = 0$ and $m_2 = 2$, then $a, b, c, d \in \mathbb{R}$. In this case, $\nu(dx; a, b, c, d)$ is well defined if the smaller of the two products in $[1, \infty)$ equals $1/q^N$ with even N. Then $\nu(dx; a, b, c, d)$ is a purely discrete measure with $N + 2$ atoms.*

PROOF. We first note that in order for ν to exist when $q \geq 0$, we must have either $m_i = 0$ or $m_i = 2$, $i = 1, 2$. This is an elementary observation based on the positivity of A_0C_1 and A_1C_2 [see (1.17), (1.18)].

Similar considerations show that if $q < 0$ and $m_1m_2 > 0$, then (A.1) fails, and $\nu(dx; a, b, c, d)$ does not exist. Furthermore, there are only three possible choices: $(m_1, m_2) = (0, 0), (0, 2), (2, 0)$.

If $m_2 > 0$, then in cases (iii) and (iv), the product $\prod_{k=1}^n A_{k-1}C_k > 0$ for $n < N$ and is zero for $n \geq N+1$. In case (v), $\prod_{k=1}^n A_{k-1}C_k = 0$ for all $n \geq N+2$. $\square$

According to Askey and Wilson [4] the orthogonality law is

$$\nu(dx; a,b,c,d) = f(x; a,b,c,d)1_{|x|\leq 1} + \sum_{x \in F(a,b,c,d)} p(x)\delta_x.$$

Here $F = F(a,b,c,d)$ is a finite or empty set of atoms. The density f is given by (2.4). Note that f is sub-probabilistic for some choices of parameters. The nonobvious fact that the total mass of ν is 1 follows from [4], (2.11), applied to $m = n = 0$.

As pointed out by Stokman [29], condition (A.1) implies that if one of the parameters a, b, c, d has modulus larger than one, then it must be real. When $m_1 = 0$, at most two of the four parameters a, b, c, d have modulus larger than one. If there are two, then one is positive and the other is negative.

Each of the parameters a, b, c, d that has absolute value larger than one gives rise to a set of atoms. For example, if $a \in (-\infty, -1) \cup (1, \infty)$, then the corresponding atoms are at

$$x_j = \frac{aq^j + (aq^j)^{-1}}{2} \tag{3.3}$$

with $j \geq 0$ such that $|q^j a| \geq 1$, and the corresponding probabilities are

$$p(x_0) = \frac{(a^{-2}, bc, bd, cd)_\infty}{(b/a, c/a, d/a, abcd)_\infty}, \tag{3.4}$$

$$p(x_j) = p(x_0)\frac{(a^2, ab, ac, ad)_j(1 - a^2q^{2j})}{(q, qa/b, qa/c, qa/d)_j(1 - a^2)}\left(\frac{q}{abcd}\right)^j, \qquad j \geq 0. \tag{3.5}$$

(This formula needs to be re-written in an equivalent form to cover the cases when $abcd = 0$. It is convenient to count as an "atom" the case $|q^j a| = 1$ even though the corresponding probability is 0. Formula (3.5) incorporates a correction to the typo in [4], (2.10), as in [18], Section 3.1.)

The continuous component is completely absent when $K(a,b,c,d) = 0$ [recall (2.5)]. Although under the assumptions of Theorem 1.1 the univariate distributions are never purely discrete, we still need to consider the case $K(a,b,c,d) = 0$ as we need to allow transition probabilities of Markov processes to be purely discrete.

We remark that if X has distribution $\nu(dx; a,b,c,d)$, then formulas for $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ from Proposition 2.2 hold now for all admissible choices of parameters a, b, c, d, as these expressions can equivalently be derived from the fact that the first two Askey–Wilson polynomials integrate to zero.

3.2. *Construction of Markov process.* Recall $I = I(A, B, C, D; q)$ from (1.21). As in Section 2, we first construct the auxiliary Markov process $(Y_t)_{t\in I}$. We request that the univariate law π_t of Y_t is the Askey–Wilson law

$$\pi_t(dy) = \nu\left(dy; A\sqrt{t}, B\sqrt{t}, \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right). \tag{3.6}$$

In order to ensure that this univariate law exists, we use condition (3.2). This condition is fulfilled when (1.4) holds and the admissible range of values of t is the interval I from (1.21). [The endpoints (1.13) and (1.14) were computed by applying Möbius transformation (2.26) to the endpoints of I.]

For $t \in I$, let U_t be the support of $\pi_t(dy)$. Under the assumption (1.4), this set can be described quite explicitly using the already mentioned results of Askey–Wilson [4]: U_t is the union of $[-1, 1]$ and a finite or empty set F_t of points that are of the form

$$x_j(t) = \frac{1}{2}\left(B\sqrt{t}q^j + \frac{1}{B\sqrt{t}q^j}\right) \quad \text{or} \quad u_j(t) = \frac{1}{2}\left(\frac{Dq^j}{\sqrt{t}} + \frac{\sqrt{t}}{Dq^j}\right) \quad \text{or} \tag{3.7}$$

$$y_j(t) = \frac{1}{2}\left(A\sqrt{t}q^j + \frac{1}{A\sqrt{t}q^j}\right) \quad \text{or} \quad v_j(t) = \frac{1}{2}\left(\frac{Cq^j}{\sqrt{t}} + \frac{\sqrt{t}}{Cq^j}\right). \tag{3.8}$$

There is, at most, a finite number of points of each type. However, not all such atoms can occur simultaneously. All possibilities are listed in the following lemma.

LEMMA 3.2. *Under the assumptions of Theorem 1.1, without loss of generality, assume $|A| \le |B|$ and $|C| \le |D|$. Then the following atoms occur:*

- *Atoms $u_j(t)$ appear for $D, C \in \mathbb{R}$, and $t \in I$ that satisfy $t < D^2$; admissible indexes $j \ge 0$ satisfy $D^2q^{2j} > t$.*
- *Atoms $v_j(t)$ appear for $D, C \in \mathbb{R}$, and $t \in I$ that satisfy $t < C^2$; admissible indexes $j \ge 0$ satisfy $C^2q^{2j} > t$.*
- *Atoms $x_j(t)$ appear for $A, B \in \mathbb{R}$, and $t \in I$ that satisfy $t > 1/B^2$; admissible indexes $j \ge 0$ satisfy $tB^2q^{2j} > 1$.*
- *Atoms $y_j(t)$ appear for $A, B \in \mathbb{R}$, and $t \in I$ that satisfy $t > 1/A^2$; admissible indexes $j \ge 0$ satisfy $tA^2q^{2j} > 1$.*

(The actual number of cases is much larger as in proofs one needs to consider all nine possible choices for the end points of the time interval I.)

Next, we specify the transition probabilities of Y_t.

PROPOSITION 3.3. *For $s < t$, $s, t \in I$ and any real $x \in U_s$ measures*

$$P_{s,t}(x, dy) = \nu\left(dy; A\sqrt{t}, B\sqrt{t}, \sqrt{\frac{s}{t}}\big(x + \sqrt{x^2-1}\big), \sqrt{\frac{s}{t}}\big(x - \sqrt{x^2-1}\big)\right)$$

are well defined. Here, if $|x| \le 1$ we interpret $x \pm \sqrt{x^2-1}$ as $e^{\pm i\theta_x} = e^{\pm i \arccos(x)}$.

PROOF. For $x \in [-1, 1]$, measures $P_{s,t}(x, dy)$ are well defined as conditions (1.19) and (3.2) hold. This covers all possibilities when (A, B) and (C, D) are conjugate pairs or when $|A|\sqrt{s}$, $|B|\sqrt{s}$, $|C|/\sqrt{s}$, $|D|/\sqrt{s} < 1$, as then $U_s = [-1, 1]$.

It remains to consider x in the atomic part of $\pi_s(dx)$. Relabeling the parameters if necessary, we may assume $|A| \le |B|$ and $|C| \le |D|$. For each of the cases listed in Lemma 3.2, we need to show that the choice of parameters $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = \sqrt{\frac{s}{t}}(x + \sqrt{x^2 - 1})$, $d = \sqrt{\frac{s}{t}}(x - \sqrt{x^2 - 1})$ leads to nonnegative products $\prod_{k=0}^n A_k C_{k+1} \ge 0$ [recall (1.17) and (1.18)]. We check this by considering all possible cases for the endpoints of I and all admissible choices of x from the atoms of measure π_s. In the majority of these cases, condition (3.2) holds, so, in fact, $A_k C_{k+1} > 0$ for all k.

Here is one sample case that illustrates what kind of reasoning is involved in the "simpler cases" where (3.2) holds and one example of a more complicated case where (3.2) fails.

- *Case* $CD < 0$, $AB < 0$, $q \ge 0$: in this case, A, B, C, D are real, $I = (0, \infty)$ and assumption $ABCD < 1$ implies $D^2 < 1/B^2$. A number of cases arises from Lemma 3.2, and we present only one of them.
  - *Sub-case* $x = v_j(s)$: then $0 < s < C^2$ and the Askey–Wilson parameters of $P_{s,t}(x; dy)$ are

$$a = A\sqrt{t}, \qquad b = B\sqrt{t}, \qquad c = \frac{q^j C}{\sqrt{t}}, \qquad d = \frac{s}{Cq^j\sqrt{t}}.$$

Thus

$$ab = ABt < 0 < 1, \qquad ac = ACq^j, \qquad ad = \frac{As}{Cq^j}, \qquad bc = q^j BC,$$

$$bd = \frac{Bs}{Cq^j}, \qquad cd = s/t < 1$$

and

$$qab = qABt < 0 < 1, \qquad qac = ACq^{j+1} < ACq^j, \qquad qad = \frac{As}{Cq^{j-1}},$$

$$qbc = q^{j+1}BC, \qquad qbd = \frac{Bs}{Cq^{j-1}}, \qquad qcd = qs/t < 1.$$

Since $s < C^2 q^{2j}$, this implies $|ad| < |A|\sqrt{s} < 1$ as $s < C^2 \le D^2 < 1/B^2 \le 1/A^2$. For the same reason, $|bd| < 1$. Of course, $|qad| < |ad| < 1$, $|qbd| < |bd| < 1$.

Finally, since $AC, qAC, BC, qBC < 1$, we have $ac, qac, bc, qbc < 1$. Thus by Lemma 3.1(i), $P_{s,t}(x, dy)$ is well defined.

[We omit other elementary but lengthy sub-cases that lead to (3.2).]

- *Case* $A, B, C, D \in \mathbb{R}$, $AB > 0$, $CD > 0$: here $I(A, B, C, D) = (CD, 1/(AB))$ is nonempty. Again a number of cases arises from Lemma 3.2, of which we present only one.
  – *Sub-case* $x = x_j(s)$: here $1/B^2 < s < t < 1/(AB)$ and $B\sqrt{s}|q^j| \geq 1$. Then the Askey–Wilson parameters of $P_{s,t}(x; dy)$ are

$$a = A\sqrt{t}, \qquad b = B\sqrt{t}, \qquad c = \frac{sq^j B}{\sqrt{t}}, \qquad d = \frac{1}{q^j B\sqrt{t}}.$$

Here, Lemmas 3.1(ii) or 3.1(iv) applies with $m_1 = 2$ and $m_2 = 0$ when $q \geq 0$ or j is even, and Lemma 3.1(v) applies with $m_1 = 0$, $m_2 = 2$ when $q < 0$ and j is odd. To see this, we look at the two lists of pairwise products.

$$ab = ABt < 1, \qquad ac = ABsq^j < 1, \qquad ad = \frac{A}{Bq^j}, \qquad bc = sq^j B^2,$$

$$bd = 1/q^j, \qquad cd = s/t < 1$$

and

$$qab = qABt < 1, \qquad qac = ABsq^{j+1} < 1, \qquad qad = \frac{A}{Bq^{j-1}},$$

$$qbc = sq^{j+1}B^2, \qquad qbd = 1/q^{j-1}, \qquad qcd = qs/t < 1.$$

Since $|A| \leq |B|$ we see that $1/(AB) < 1/A^2$ so $|A| < 1/\sqrt{s}$ and $|\frac{A}{Bq^j}| < \frac{\sqrt{s}}{|q^j B|} \leq 1$. This shows that $ad < 1$ and $qad < 1$.

It is clear that both $bc, bd > 1$ when $q > 0$ or j is even, and that qbc, $qbd > 1$ when $q < 0$ and j is odd. Thus by Lemma 3.1, $P_{s,t}(x, dy)$ is well defined.

Other cases are handled similarly and are omitted. □

In the continuous case, $p_t(dy) = p(t, y)\,dy$ and $P_{s,t}(x, dy) = p(s, x; t, y)\,dy$ correspond to (2.10) and (2.11), respectively. We now extend Proposition 2.4 to a larger set of measures.

PROPOSITION 3.4. *The family of probability measures $\pi_t(dy)$ together with the family of transition probabilities $P_{s,t}(x, dy)$ defines a Markov process $(Y_t)_{t\in I}$ on $\bigcup_{t\in I} U_t$. That is, $P_{s,t}(x, dy)$ is defined for all $x \in U_s$. For any $s < t$ from $I(A, B, C, D)$ and a Borel set $V \subset \mathbb{R}$,*

$$\pi_t(V) = \int_{\mathbb{R}} P_{s,t}(x, V)\pi_s(dx) \tag{3.9}$$

and for $s < t < u$ from I,

$$P_{s,u}(x, V) = \int_{U_t} P_{t,u}(y, V) P_{s,t}(x, dy) \qquad \text{for all } x \in U_s. \tag{3.10}$$

To prove this result we follow the same plan that we used in [9] and we will use similar notation. We deduce all necessary information from the orthogonal polynomials. Consider two families of polynomials. The first family is

$$p_n(x;t) = t^{n/2}\bar{w}_n(x;a,b,c,d)$$

with $\bar{w}_n$ defined by (1.20) and

$$a = A\sqrt{t}, \qquad b = B\sqrt{t}, \qquad c = C/\sqrt{t}, \qquad d = D/\sqrt{t}. \tag{3.11}$$

The second family is

$$Q_n(y;x,t,s) = t^{n/2}\bar{w}_n(y;a,b,\widetilde{c},\widetilde{d}), \tag{3.12}$$

where a, b are in (3.11), and

$$\widetilde{c} = \sqrt{\frac{s}{t}}\big(x + \sqrt{x^2-1}\big), \qquad \widetilde{d} = \sqrt{\frac{s}{t}}\big(x - \sqrt{x^2-1}\big). \tag{3.13}$$

As real multiples of the corresponding Askey–Wilson polynomials $\widetilde{w}_n$, polynomials $\{p_n\}$ are orthogonal with respect to $\pi_t(dx)$, and polynomials $\{Q_n\}$ are orthogonal with respect to $P_{s,t}(x,dy)$ when $x \in U_s$. It may be interesting to note that if x is in the atomic part of U_s and a, b from (3.11) satisfy $ab < 1 < b$, then the family $\{Q_n\}$ corresponds to a finitely-supported measure. As explained in Theorem A.1 we still have the infinite family of polynomials $\{Q_n\}$ in this case. This is important to us as we use this infinite family to infer the Chapman–Kolmogorov equations and the martingale property of the infinite family $\{p_n\}$.

The following algebraic identity is crucial for our proof.

LEMMA 3.5. *For $n \geq 1$,*

$$Q_n(y;x,t,s) = \sum_{k=1}^{n} b_{n,k}(x,s)\big(p_k(y;t) - p_k(x;s)\big), \tag{3.14}$$

where $b_{n,k}(x,s)$ does not depend on t for $1 \leq k \leq n$, $b_{n,n}(x,s)$ does not depend on x, and $b_{n,n}(x,s) \neq 0$.

PROOF. When $|A| + |B| \neq 0$, due to symmetry, we may assume that $A \neq 0$. From Theorem A.2, with parameters (3.11) and (3.13), we get

$$Q_n(y;x,t,s) = \sum_{k=0}^{n} b_{n,k}\, p_k(y;t), \tag{3.15}$$

where $b_{n,k} = t^{(n-k)/2}\bar{c}_{k,n}$ is given by (A.8). Coefficients $b_{n,k}$ do not depend on t as $t^{(n-k)/2}/a^{n-k} = A^{k-n}$, and t cancels out in all other entries on the right-hand side of (A.7):

$$\begin{aligned} ab\widetilde{c}\widetilde{d} &= ABs, \qquad abcd = ABCD,\\ a\widetilde{c} &= A\sqrt{s}\big(x + \sqrt{x^2-1}\big), \qquad ac = AC,\\ a\widetilde{d} &= A\sqrt{s}\big(x - \sqrt{x^2-1}\big), \qquad ad = AD. \end{aligned}$$

We also see that

$$b_{n,n}(x,s) = (-1)^n q^{n(n+1)/2} \frac{(q^{-n}, q^{n-1}ABs)_n}{(q, q^{n-1}ABCD)_n}$$

does not depend on x. Using (2.2) we get

$$b_{n,n}(x,s) = \frac{(q^{n-1}ABs)_n}{(q^{n-1}ABCD)_n},$$

which is nonzero also when $q = 0$.

The case $A = B = 0$ is handled similarly, based on (A.9). In this case $b_{n,n}(x,s) = 1$.

Next we use (3.12) to show that $Q_n(x; x, s, s) = 0$ for $n \geq 1$. We observe that (1.20) used for parameters (3.11) and (3.13) gives $Q_1(x; x, s, s) = 0$ as $B_0 = a + 1/a - A_0/a - C_0 a = 2x$ when $t = s$, and $Q_2(x; x, s, s) = 0$ as $\bar{C}_1(a, b, \widetilde{c}, \widetilde{d}) = 0$ when $t = s$. So (1.20) implies that $Q_n(x; x, s, s) = 0$ for all $n \geq 1$, and (3.15) implies

$$\sum_{k=0}^{n} b_{n,k}(x,s) p_k(x; s) = 0.$$

Subtracting this identity from (3.15) we get (3.14). □

We also need the following generalization of the projection formula [21].

PROPOSITION 3.6. *Suppose that A, B, C, D satisfy the assumptions in Theorem 1.1. For $x \in U_s$,*

$$\int_{\mathbb{R}} p_n(y; t) P_{s,t}(x, dy) = p_n(x; s). \tag{3.16}$$

PROOF. Since $x \in U_s$, from Proposition 3.3, measures $P_{s,t}(x, dy)$ are well defined. The formula holds true for $n = 0$. Suppose it holds true for some $n \geq 0$. By induction assumption and orthogonality of polynomials $\{Q_n\}$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} Q_{n+1}(y; x, t, s) P_{s,t}(x, dy) \\ &= b_{n+1,n+1}(x,s) \int_{\mathbb{R}} \big(p_{n+1}(y; t) - p_{n+1}(x; s)\big) P_{s,t}(x, dy) \\ &= b_{n+1,n+1}(x,s) \left(\int_{\mathbb{R}} p_{n+1}(y; t) P_{s,t}(x, dy) - p_{n+1}(x; s) \right). \end{aligned}$$ □

PROOF OF PROPOSITION 3.4. This proof follows the scheme of the proof of [9], Proposition 2.5. To prove (3.9), let $\mu(V) = \int_{\mathbb{R}} P_{s,t}(x, V) \pi_s(dx)$, and note that by orthogonality, $\int_{\mathbb{R}} p_n(x;s)\pi_s(dx) = 0$ for all $n \geq 1$. Then from (3.16),

$$\int_{\mathbb{R}} p_n(y;t)\mu(dy) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} p_n(y;t) P_{s,t}(x,dy) \right) \pi_s(dx)$$
$$= \int_{\mathbb{R}} p_n(x;s)\pi_s(dx) = 0.$$

Since $\int_{\mathbb{R}} p_n(y;t)\pi_t(dy) = 0$, this shows that all moments of $\mu(dy)$ and $\pi_t(dy)$ are the same. By the uniqueness of the moment problem for compactly supported measures, $\mu(dy) = \pi_t(dy)$, as claimed.

To prove (3.10), we first note that for $x \in U_s$, $P_{s,t}(x, U_t) = 1$; this can be seen by analyzing the locations of atoms, which arise either from the values of $A\sqrt{t}$ or $B\sqrt{t} > 1$ or from x being one of the atoms of U_s. [Alternatively, use (3.9).]

Fix $x \in U_s$ and let $\mu(V) = \int_{U_t} P_{t,u}(y, V) P_{s,t}(x, dy)$. Then, by (3.15) for $n \geq 1$ and (3.16) used twice,

$$\int_{\mathbb{R}} Q_n(z; x, u, s)\mu(dz)$$
$$= \int_{U_t} \int_{\mathbb{R}} \sum_{k=1}^{n} b_{n,k}(x,s)\big(p_k(z;u) - p_k(x;s)\big) P_{t,u}(y,dz) P_{s,t}(x,dy)$$
$$= \int_{U_t} \sum_{k=1}^{n} b_{n,k}(x,s)\big(p_k(y;t) - p_k(x;s)\big) P_{s,t}(x,dy)$$
$$= \int_{\mathbb{R}} \sum_{k=1}^{n} b_{n,k}(x,s)\big(p_k(y;t) - p_k(x;s)\big) P_{s,t}(x,dy) = 0.$$

Thus the moments of μ and $P_{s,u}(x,dz)$ are equal which, by the method of moments, ends the proof. $\square$

3.3. *Proofs of Theorems 1.1 and 1.2.*

PROOF OF THEOREM 1.2. If A, B, C, D satisfy the assumptions in Theorem 1.1, by Proposition 3.4, there exists a Markov process (Y_t) with orthogonal polynomials $\{p_n(x;t)\}$. From (3.16) we see that $\{p_n(x;t)\}$ are also martingale polynomials for (Y_t). With Z_t defined by (2.22), polynomials $r_n(x;t)$ inherit the martingale property as $r_n(Z_t;t) = p_n(Y_t;t)$. $\square$

PROOF OF THEOREM 1.1. The fact that time interval J is well defined and nondegenerate has been shown in Remark 1.4. From Proposition 3.4, we already have the Markov process (Y_t) with Askey–Wilson transition probabilities. Thus the mean and covariance are (2.14) and (2.16). Formulas (2.22) and (2.28) will therefore give us the process $(X_t)_{t \in J}$ with the correct covariance.

It remains to verify asserted properties of conditional moments. Again transformation (2.28) will imply (1.3), provided (Z_t) satisfies (2.24). For the proof of the latter we use orthogonal martingale polynomials (1.22). Our proof is closely related to [8], Theorem 2.3. We begin by writing the three step recurrence as

$$xr_n(x;t) = (\alpha_n t + \beta_n)r_{n+1}(x;t) + (\gamma_n t + \delta_n)r_n(x;t) + (\varepsilon_n t + \varphi_n)r_{n-1}(x;t),$$

which amounts to decomposing the Jacobi matrix $\mathbf{J}_t$ of $\{r_n(x;t)\}$ as $t\mathbf{x} + \mathbf{y}$. From (1.20) with $a = A\sqrt{t}$, $b = B\sqrt{t}$, $c = C/\sqrt{t}$ and $d = D/\sqrt{t}$, we read out the coefficients:

$$\alpha_n = -ABq^n\beta_n,$$

$$\beta_n = \frac{1 - ABCDq^{n-1}}{\sqrt{1-q}(1 - ABCDq^{2n})(1 - ABCDq^{2n-1})},$$

$$\varepsilon_n = \frac{(1-q^n)(1-ACq^{n-1})(1-ADq^{n-1})(1-BCq^{n-1})(1-BDq^{n-1})}{\sqrt{1-q}(1-ABCDq^{2n-2})(1-ABCDq^{2n-1})},$$

$$\varphi_n = -CDq^{n-1}\varepsilon_n,$$

$$\gamma_n = \frac{A}{\sqrt{1-q}} - \frac{\alpha_n}{A}(1-ACq^n)(1-ADq^n) - \frac{A\varepsilon_n}{(1-ACq^{n-1})(1-ADq^{n-1})},$$

$$\delta_n = \frac{1}{A\sqrt{1-q}} - \frac{\beta_n}{A}(1-ACq^n)(1-ADq^n) - \frac{A\varphi_n}{(1-ACq^{n-1})(1-ADq^{n-1})}.$$

We note that the expressions for γ_n, δ_n after simplification[1] are well defined also for $A = 0$. Moreover, by continuity $\alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0 = 0, \varphi_0 = 0$ are defined also at $q = 0$.

A calculation verifies the q-commutation equation $[\mathbf{x}, \mathbf{y}]_q = \mathbf{I}$ for the two components of the Jacobi matrix. In terms of the coefficients this amounts to verification that the expressions above satisfy for $n \geq 1$:

$$\alpha_n\beta_{n-1} = q\alpha_{n-1}\beta_n, \tag{3.17}$$

$$\beta_n\gamma_{n+1} + \alpha_n\delta_n = q(\beta_n\gamma_n + \alpha_n\delta_{n+1}), \tag{3.18}$$

$$\gamma_n\delta_n + \beta_n\varepsilon_{n+1} + \alpha_{n-1}\varphi_n = q(\gamma_n\delta_n + \beta_{n-1}\varepsilon_n + \alpha_n\varphi_{n+1}) + 1, \tag{3.19}$$

$$\delta_n\varepsilon_n + \gamma_{n-1}\varphi_n = q(\delta_{n-1}\varepsilon_n + \gamma_n\varphi_n), \tag{3.20}$$

$$\varepsilon_n\varphi_{n+1} = q\varepsilon_{n+1}\varphi_n. \tag{3.21}$$

For a similar calculation see [32], Section 4.2. A more general q-commutation equation $[\mathbf{x}, \mathbf{y}]_q = \mathbf{I} + \theta\mathbf{x} + \eta\mathbf{y} + \tau\mathbf{x}^2 + \sigma\mathbf{y}^2$ appears in [8], (2.22)–(2.26).

---

[1] $\gamma_n = q^n \frac{AB(q+1)((A+B)CD+(C+D)q)q^n - ABCD(AB(C+D)+(A+B)q)q^{2n} - (AB(C+D)+(A+B)q)q}{\sqrt{1-q}(q^2-ABCDq^{2n})(ABCDq^{2n}-1)}$.

For compactly supported measures, conditional moments can be now read out from the properties of the Jacobi matrices; formula (1.2) follows from $\mathbf{J}_t = t\mathbf{x}+\mathbf{y}$, and formula (1.3) follows from $[\mathbf{x},\mathbf{y}]_q = \mathbf{I}$. This can be seen from the proof of [8], Lemma 3.4, but for reader's convenience we include some details.

Denote by $\mathbf{r}(x;t) = [r_0(x;t), r_1(x;t), \dots]$. Then the three-step recurrence is $x\mathbf{r}(x;t) = \mathbf{r}(x;t)\mathbf{J}_t$, and the martingale polynomial property from Theorem 1.2 says that $\mathbb{E}(\mathbf{r}(X_u;u)|\mathcal{F}_t) = \mathbf{r}(X_t;t)$. (Here we take all operations componentwise.)

To verify (1.2) for compactly supported measures it suffices to verify that

$$\mathbb{E}(X_t\mathbf{r}(X_u;u)|\mathcal{F}_s) = \frac{u-t}{u-s}\mathbb{E}(X_s\mathbf{r}(X_u;u)|\mathcal{F}_s) + \frac{t-s}{u-s}\mathbb{E}(X_u\mathbf{r}(X_u;u)|\mathcal{F}_s).$$

Using martingale property and the three-step recurrence, this is equivalent to

$$\mathbf{r}(X_s;s)\mathbf{J}_t = \mathbf{r}(X_s;s)\left(\frac{u-t}{u-s}\mathbf{J}_s + \frac{t-s}{u-s}\mathbf{J}_u\right),$$

which holds true as

$$\mathbf{J}_t = \frac{u-t}{u-s}\mathbf{J}_s + \frac{t-s}{u-s}\mathbf{J}_u$$

for linear expressions in t.

To verify (2.24) we write it as

$$\begin{aligned}\mathbb{E}(Z_t^2|\mathcal{F}_{s,u}) &= \frac{(u-t)(u-qt)Z_s^2}{(u-s)(u-qs)} + \frac{(q+1)(t-s)(u-t)Z_uZ_s}{(u-s)(u-qs)} \\ &\quad + \frac{(t-s)(t-qs)Z_u^2}{(u-s)(u-qs)} + \frac{(t-s)(u-t)}{u-qs}.\end{aligned} \tag{3.22}$$

For compactly supported laws, it suffices therefore to verify that

$$\begin{aligned}&\mathbb{E}(Z_t^2\mathbf{r}(X_u;u)|\mathcal{F}_s) \\ &\quad = \frac{(u-t)(u-qt)}{(u-s)(u-qs)}\mathbb{E}(Z_s^2\mathbf{r}(X_u;u)|\mathcal{F}_s) \\ &\qquad + \frac{(q+1)(t-s)(u-t)}{(u-s)(u-qs)}\mathbb{E}(Z_uZ_s\mathbf{r}(X_u;u)|\mathcal{F}_s) \\ &\qquad + \frac{(t-s)(t-qs)}{(u-s)(u-qs)} + \frac{(t-s)(u-t)}{u-qs}\mathbb{E}(Z_u^2\mathbf{r}(X_u;u)|\mathcal{F}_s).\end{aligned} \tag{3.23}$$

Again, we can write this using the Jacobi matrices and martingale property as

$$\begin{aligned}\mathbf{r}(X_s;s)\mathbf{J}_t^2 = \mathbf{r}(X_s;s)\biggl(&\frac{(u-t)(u-qt)}{(u-s)(u-qs)}\mathbf{J}_s^2 \\ &+ \frac{(q+1)(t-s)(u-t)}{(u-s)(u-qs)}\mathbf{J}_s\mathbf{J}_u + \frac{(t-s)(t-qs)}{(u-s)(u-qs)}\mathbf{J}_u^2\biggr) \\ &+ \frac{(t-s)(u-t)}{u-qs}\mathbf{r}(X_s;s).\end{aligned} \tag{3.24}$$

A calculation shows that for $\mathbf{J}_t = t\mathbf{x} + \mathbf{y}$, the q-commutation equation $[\mathbf{x}, \mathbf{y}]_q = \mathbf{I}$ is equivalent to

$$
\begin{aligned}
\mathbf{J}_t^2 = {} & \frac{(u-t)(u-qt)}{(u-s)(u-qs)}\mathbf{J}_s^2 + \frac{(q+1)(t-s)(u-t)}{(u-s)(u-qs)}\mathbf{J}_s\mathbf{J}_u \\
& + \frac{(t-s)(t-qs)}{(u-s)(u-qs)}\mathbf{J}_u^2 + \frac{(t-s)(u-t)}{u-qs}\mathbf{I},
\end{aligned} \tag{3.25}
$$

so (3.24) holds.

Uniqueness of (X_t) follows from the fact that by [8], Theorem 4.1, each such process has orthogonal martingale polynomials; from martingale property (3.16) all joint moments are determined uniquely and correspond to finite-dimensional distributions with compactly supported marginals. □

We remark that the following version of Propositions 2.6 and 4.5 would shorten the proof of Theorem 1.1.

CONJECTURE 3.1. Let $(Y_t)_{t\in I}$ be the Markov process from Proposition 3.4. Then for any $s < t < u$ from $I(A, B, C, D)$, the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ is

$$
\nu\left(y; \frac{z\sqrt{t}}{\sqrt{u}}, \frac{\sqrt{t}}{z\sqrt{u}}, \frac{x\sqrt{s}}{\sqrt{t}}, \frac{\sqrt{s}}{x\sqrt{t}}\right). \tag{3.26}
$$

(Here, $x = Y_s + \sqrt{Y_s^2 - 1}$, $z = Y_u + \sqrt{Y_u^2 - 1}$.)

4. Purely discrete case. Assumption (1.4) arises from the positivity condition (A.1) for the Askey–Wilson recurrence for which it is difficult to give general explicit conditions. The following result exhibits additional quadratic harnesses when condition (1.4) is not satisfied.

PROPOSITION 4.1. *Suppose $q, A, B, C, D > 0$ and $ABCD < 1$. Suppose that there are exactly two numbers among the four products AC, AD, BC, BD that are larger than one, and that the smaller of the two, say AD, is of the form $1/q^N$ for some integer $N \ge 0$. If $Aq^N > 1$, then there exists a Markov process $(X_t)_{t\in(0,\infty)}$ with discrete univariate distributions supported on $N+1$ points such that* (1.1) (1.2) *hold, and* (1.3) *holds with parameters $\eta, \theta, \sigma, \tau, \gamma$ given by* (1.5) *through* (1.9).

After re-labeling the parameters, without loss of generality for the reminder of this section, we will assume that $0 < A < B$, $0 < C < D$, $AC < 1$, $BC < 1$, $AD = 1/q^N$, so that $BD > 1/q^N$.

4.1. *Discrete Askey–Wilson distribution.* The discrete Askey–Wilson distribution $\nu(dx; a, b, c, d)$ arises in several situations, including the case described in Lemma 3.1(iii). This distribution was studied in detail by Askey and Wilson [3] and was summarized in [4].

Here we consider parameters $a, b, c, d > 0$ and $0 < q < 1$ such that $ad = 1/q^N$ and

$$q^N a > 1, \qquad q^N/(bc) > 1, \qquad ac < 1,$$
$$q^N a/b > 1, \qquad q^N a/c > 1, \qquad q^N ab > 1.$$

Note that this implies $abcd < 1$ and

$$ad = 1/q^N > 1, \qquad ac < 1, \qquad bc < 1, \qquad bd < 1, \qquad cd < 1,$$
$$ab > 1/q^N,$$

so from Lemma 3.1(iii), the Askey–Wilson law $\nu(dx; a, b, c, d) = \nu(dx; a, b, c, 1/(aq^N))$ is well defined and depends on parameters a, b, c, q, N only and is supported on $N + 1$ points,

$$\{x_k = (q^k a + q^{-k} a^{-1})/2 : k = 0, \dots, N\}.$$

According to [3], the Askey–Wilson law assigns to x_k the probability $p_{k,N}(a, b, c) = p_k(a, b, c, 1/(q^N a))$ [recall (3.5)]. The formula simplifies to

$$\begin{aligned} &p_{k,N}(a, b, c) \\ &\quad = \begin{bmatrix} N \\ k \end{bmatrix} \frac{(q^{k+1}a/b, q^{k+1}a/c)_{N-k}(ab, ac)_k(1 - q^{2k}a^2)q^{k(k+1)/2}}{(q^k a^2)_{N+1}(q/(bc))_N(-bc)^k}, \\ &\qquad\qquad k = 0, \dots, N. \end{aligned} \tag{4.1}$$

Here $\begin{bmatrix} N \\ k \end{bmatrix} = \frac{(q)_N}{(q)_k(q)_{N-k}}$ denotes the q-binomial coefficient.

We remark that if X is a random variable distributed according to $\nu(dx; a, b, c, 1/(aq^N))$, then $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ are given by formulas (2.6) and (2.7) with $d = 1/(q^N a)$, respectively. This can be seen by a discrete version of the calculations from the proof of Proposition 2.2; alternatively, one can use the fact that the first two Askey–Wilson polynomials, $\bar{w}_1(X)$ and $\bar{w}_2(X)$, integrate to zero.

The discrete version of Proposition 2.3 says that with $d = 1/(maq^N)$,

$$\begin{aligned} &\nu(U; a, b, cm, md) \\ &\quad = \int \nu\bigl(U; a, b, m\bigl(x + \sqrt{x^2 - 1}\bigr), m\bigl(x - \sqrt{x^2 - 1}\bigr)\bigr) \\ &\qquad\quad \times \nu(dx; ma, mb, c, d) \end{aligned} \tag{4.2}$$

and takes the following form.

LEMMA 4.2. *For any* $m \in (0,1)$ *and any* $j = 0, 1, \dots, N$,

$$p_{j,N}(a,b,mc) = \sum_{k=j}^{N} p_{j,k}(a,b,q^k m^2 a) p_{k,N}(ma,mb,c). \tag{4.3}$$

PROOF. Expanding the right-hand side of (4.3) we have

$$\sum_{k=j}^{N} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q^{j+1}a/b, q/(q^{k-j}m^2))_{k-j}}{(q^j a^2)_{k+1}} \frac{(ab, q^k m^2 a^2)_j}{(q/(q^k m^2 ab))_k} \frac{(1-q^{2j}a^2)q^{j(j+1)/2}}{(-q^k m^2 ab)^j}$$

$$\times \begin{bmatrix} N \\ k \end{bmatrix} \frac{(q^{k+1}a/b, q^{k+1}ma/c)_{N-k}}{(q^k m^2 a^2)_{N+1}} \frac{(m^2 ab, mac)_k}{(q/(mbc))_N} \frac{(1-q^{2k}m^2a^2)q^{k(k+1)/2}}{(-mbc)^k}$$

$$= \begin{bmatrix} N \\ j \end{bmatrix} \frac{(q^{j+1}a/b, q^{j+1}a/(mc))_{N-j}(ab, mac)_j(1-a^2q^{2j})q^{j(j+1)/2}}{(q^j a^2)_{N+1}(q/(mbc))_N(-mbc)^j}$$

$$\times \sum_{k=j}^{N} \begin{bmatrix} N-j \\ k-j \end{bmatrix} \Bigg(\left(\frac{q^{k+1}ma}{c}, q^{k+1}a^2 \right)_{N-k} (m^2)_{k-j}$$

$$\times (mq^j ac)_{k-j}(1-q^{2k}m^2a^2)q^{(k-j)(k-j+1)/2} \Bigg)$$

$$\times \left((q^{k+j}m^2a^2)_{N-j+1} \left(\frac{q^{j+1}a}{mc} \right)_{N-j} \left(-\frac{mc}{q^j a} \right)^{k-j} \right)^{-1}. \tag{4.4}$$

Here we used identities (2.1) and (2.2). The first one for: (i) $\alpha = q^{j+1}a/b$, $M = k - j$ and $L = N - k$; (ii) $\alpha = q^k m^2 a^2$, $M = j$, $L = N - j + 1$; (iii) $\alpha = q^j a^2$, $M = k + 1$, $L = N - k$. The second one for: (i) $\alpha = m^2 ab$ and $M = k$; (ii) $\alpha = m^2$ and $M = k - j$.

We transform the sum in (4.4) introducing $K = k - j$, $L = N - j$, $\alpha = mq^j a$, $\beta = \frac{m}{q^j a}$ and $\gamma = c$. Then by (4.1) we get

$$\sum_{K=0}^{L} \begin{bmatrix} L \\ K \end{bmatrix} \frac{(q^{K+1}\alpha/\gamma, q^{K+1}\alpha/\beta)_{L-K}(\alpha\beta)_K(\alpha\gamma)_K}{(q^K\alpha^2)_{L+1}(q/(\beta\gamma))_L} \frac{(1-q^{2K}\alpha^2)q^{K(K+1)/2}}{(-\beta\gamma)^K} = 1.$$

Now the result follows since the first part of the expression at the right-hand side of (4.4) is the desired probability mass function. □

4.2. *Markov processes with discrete Askey–Wilson laws.* We now choose the parameters as in Proposition 4.1: $0 < A < B$, $0 < C < D = 1/(Aq^N)$, $ABCD < 1$, $BC < 1$, and choose the time interval $I = (C(q^N A)^{-1}, (AB)^{-1})$ from (1.21). For any $t \in I$, define the discrete distribution $\pi_t(dx) = \sum_{k=0}^{N} \pi_t(y_k(t))\delta_{y_k(t)}(dx)$ by choosing the support from (3.8) with weights

$$\pi_t(y_k(t)) = p_{k,N}(At^{1/2}, Bt^{1/2}, Ct^{-1/2}). \tag{4.5}$$

Clearly, the support of π_s is $U_s = \{y_0(s), y_1(s), \dots, y_N(s)\}$.

Also for any $s, t \in I$, $s < t$ and for any $k \in \{0, 1, \dots, N\}$, define the discrete Askey–Wilson distribution $P_{s,t}(y_k(s), dy) = \sum_{j=0}^{k} P_{s,t,y_k(s)} \delta_{y_j(t)}(dy)$ by

$$P_{s,t,y_k(s)}(y_j(t)) = p_{j,k}(At^{1/2}, Bt^{1/2}, q^k Ast^{-1/2}). \tag{4.6}$$

Thus $P_{s,t}(x, dy)$ is defined only for x from the support of π_s. Next, we give the discrete version of Proposition 2.4.

PROPOSITION 4.3. *The family of distributions* $(\pi_t, P_{s,t}(x, dy))$, $s, t \in I$, $s < t$, $k \in \{0, 1, \dots, N\}$ *defines a Markov process* $(Y_t)_{t \in I}$ *with trajectories contained in the set of functions* $\{(y_k(t))_{t \in I}, k = 0, 1, \dots, N\}$.

PROOF. We need to check the Chapman–Kolmogorov conditions. Note that for any $s < t$ and any k the support of the measure $P_{s,t,y_k(s)}$ is a subset of the support U_t of the measures π_t. First we check

$$\pi_t(y_j(t)) = \sum_{k=j}^{N} P_{s,t,k}(y_j(t)) \pi_s(y_k(s)), \tag{4.7}$$

which can be written as

$$\begin{aligned} &p_{j,N}(At^{1/2}, Bt^{1/2}, Ct^{-1/2}) \\ &\quad = \sum_{k=j}^{N} p_{j,k}(At^{1/2}, Bt^{1/2}, q^k Ast^{-1/2}) p_{k,N}(As^{1/2}, Bs^{1/2}, Cs^{-1/2}). \end{aligned}$$

Now (4.7) follows from (4.3) with $a = At^{1/2}$, $b = Bt^{1/2}$, $c = Cs^{-1/2}$ and $m = (s/t)^{1/2}$.

Similarly, the condition

$$P_{s,u,u_k(s)}(u_i(u)) = \sum_{j=k}^{i} P_{t,u,u_j(t)}(u_i(u)) P_{s,t,u_k(s)}(u_j(t)), \tag{4.8}$$

assumes the form

$$\begin{aligned} &p_{i,k}(Au^{1/2}, Bu^{1/2}, q^k Asu^{-1/2}) \\ &\quad = \sum_{j=k}^{i} p_{i,j}(Au^{1/2}, Bu^{1/2}, q^j Atu^{-1/2}) p_{j,k}(At^{1/2}, Bt^{1/2}, q^k Ast^{-1/2}). \end{aligned}$$

Therefore (4.8) follows from (4.3) with $(j, k, N) \to (i, j, k)$, $a = Au^{1/2}$, $b = Bu^{1/2}$, $c = q^k Ast^{-1/2}$ and $m = (t/u)^{1/2}$. $\square$

Let $(Y_t)_{t \in I}$ be a Markov process defined by the above Markov family $(\pi_t, P_{s,t,y_k(s)})$.

Note that at the end-points of I, $Y_{C/(q^N A)}$ is degenerate at $\frac{1+ACq^N}{2(q^N AC)^{1/2}}$, and $Y_{1/(AB)}$ is degenerate at $\frac{A+B}{2(AB)^{1/2}}$ [compare Proposition 6.1(i)].

Expressions for conditional expectations and conditional variances are exactly the same as in the absolutely continuous case with $D = q^{-N}A^{-1}$.

PROPOSITION 4.4. *For the process $(Y_t)_{t\in I}$ defined above,*

$$\mathbb{E}(Y_t) = \frac{(1-q^N)A(Bt+C) - (1-q^N A^2 t)(1-BC)}{2At^{1/2}(BC-q^N)},$$

$$\begin{aligned}\operatorname{Cov}(Y_s, Y_t) &= (1-q)(1-q^N)(1-AC)(1-BC)(q^N A - B)(q^N As - C)\\ &\quad \times (1-ABt)\bigl(4q(st)^{1/2}(BC-q^N)^2(BC-q^{N-1})\bigr)^{-1},\end{aligned}$$

$$\operatorname{Var}(Y_t) = \frac{(1-AC)(1-BC)(1-q^N)(1-q)(1-ABt)(C-q^N At)}{4qAt(BC-q^N)^2(BC-q^{N-1})}.$$

PROOF. The result follows from the fact that the marginal and conditional distributions of the process $(Y_t)_{t\in I}$ are finite Askey–Wilson. Therefore one can apply formulas (2.6) and (2.7). The covariance is derived through conditioning $\mathbb{E}(Y_s Y_t) = \mathbb{E}(Y_s \mathbb{E}(Y_t|\mathcal{F}_s))$. □

Since we are interested in the harness properties, we want to find the conditional distributions of the process with conditioning with respect to the past and the future, jointly. The following result says that the conditional distribution of Y_t given the (admissible) values $Y_s = x$, $Y_u = z$ is the discrete Askey–Wilson distribution $\nu(dy; a, b, c, d)$ with parameters

$$a = \sqrt{\frac{t}{u}}\bigl(z + \sqrt{z^2-1}\bigr), \qquad b = \sqrt{\frac{t}{u}}\bigl(z - \sqrt{z^2-1}\bigr),$$

$$c = \sqrt{\frac{s}{t}}\bigl(x + \sqrt{x^2-1}\bigr), \qquad d = \sqrt{\frac{s}{t}}\bigl(x - \sqrt{x^2-1}\bigr)$$

(compare Proposition 2.6). Using notation (4.1), this formula takes the following, more concise, form.

PROPOSITION 4.5. *Let $(Y_t)_{t\in I}$ be the Markov process defined by $(\pi_t, P_{s,t,y_k(s)})$ given by (4.5) and (4.6) with parameters A, B, C, q, N. Then for any $s, t, u \in I$ such that $s < t < u$, the conditional distribution of Y_t given $\mathcal{F}_{s,u}$ is defined by the discrete Askey–Wilson distribution*

$$P\bigl(Y_t = y_j(t) | Y_s = y_k(s), Y_u = y_i(u)\bigr) = p_{j-i,k-i}\biggl(q^i At^{1/2}, \frac{t^{1/2}}{q^i Au}, \frac{q^k As}{t^{1/2}}\biggr).$$

The expressions for the first two conditional moments are the same as in the absolutely continuous case with $D = q^{-N}A^{-1}$.

PROOF. Due to the Markov property of the process $(Y_t)_{t\in I}$, to determine the conditional distribution of Y_t given $\mathcal{F}_{s,u}$, it suffices to find $P(Y_t = y_j(t)|Y_s = y_k(s), Y_u = y_i(u))$ for any $i, j, k \in \{0, 1, \dots, N\}$ such that $i \le j \le k$. Also the Markov property implies that this probability can be expressed in terms of conditional probabilities with respect to the past as

$$\begin{aligned} p(j|k,i) &= P\big(Y_t = y_j(t)|Y_s = y_k(s), Y_u = y_i(u)\big) \\ &= \frac{P(Y_u = y_i(u)|Y_t = y_j(t))P(Y_t = y_j(t)|Y_s = y_k(s))}{P(Y_u = y_i(u)|Y_s = y_k(s))} \\ &= \frac{p_{i,j}(Au^{1/2}, Bu^{1/2}, q^j Atu^{-1/2})p_{j,k}(At^{1/2}, Bt^{1/2}, q^k Ast^{-1/2})}{p_{i,k}(Au^{1/2}, Bu^{1/2}, q^k Asu^{-1/2})}. \end{aligned}$$

Expanding the expression for the probability mass functions according to (4.1) we get

$$\begin{aligned} p(j|k,i) = {} & \begin{bmatrix} j \\ i \end{bmatrix} \frac{(q^{i+1}A/B, qu/(tq^{j-i})_{j-i}}{(q^i A^2 u)_{j+1}} \frac{(ABu, q^j A^2 t)_i}{(q/(q^j ABt))_j} \\ & \times \frac{(1-q^{2i}A^2u)q^{i(i+1)/2}}{(-q^j ABt)^i} \\ & \times \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q^{j+1}A/B, qt/(sq^{k-j})_{k-j}}{(q^j A^2 t)_{k+1}} \frac{(ABt, q^k A^2 s)_j}{(q/(q^k ABs))_k} \\ & \times \frac{(1-q^{2j}A^2t)q^{j(j+1)/2}}{(-q^k ABs)^j} \\ & \times \Bigg(\begin{bmatrix} k \\ i \end{bmatrix} \frac{(q^{i+1}A/B, qu/(sq^{k-i})_{k-i}}{(q^i A^2 u)_{k+1}} \\ & \quad \times \frac{(ABu, q^k A^2 s)_i}{(q/(q^k ABs))_k} \frac{(1-q^{2i}A^2u)q^{i(i+1)/2}}{(-q^k ABs)^i}\Bigg)^{-1}. \end{aligned}$$

This can be reduced in several steps. The q-binomial symbols reduce, as in the classical ($q = 1$) case to $\begin{bmatrix} k-i \\ j-i \end{bmatrix}$. Then we apply (2.1) in the following situations: (i) $\alpha = \frac{q^{i+1}A}{B}$, $M = j - i$, $L = k - j$; (ii) $\alpha = q^i A^2 u$, $M = j + 1$, $L = k - j$; (iii) $\alpha = q^j A^2 t$, $M = i$, $L = k - i + 1$; (iv) $\alpha = q^k A^2 s$, $M = i$, $L = j - i$. Also we apply (2.2) for (i) $\alpha = \frac{u}{t}$, $M = j - i$; (ii) $\alpha = ABt$, $M = j$. Thus

$$\begin{aligned} p(j|k,i) = {} & \begin{bmatrix} k-i \\ j-i \end{bmatrix} \left(\frac{qt/s}{q^{k-j}}, q^{i+j+1}A^2u\right)_{k-j} \\ & \times \left(\frac{t}{u}, q^{k+i}A^2 s\right)_{j-i} (1-q^{2j}A^2t)q^{(j-i)(j-i+1)/2} \\ & \times \left((q^{i+j}A^2t)_{k-i+1}\left(\frac{q}{q^{k-i}u/s}\right)_{k-i}\left(-q^{k-i}\frac{s}{u}\right)^{j-i}\right)^{-1}, \end{aligned}$$

which, through comparison with the definition (4.1), is easily identified as the distribution we sought. □

PROOF OF PROPOSITION 4.1. Since formulas (2.6) and (2.7) hold for all Askey–Wilson distributions, from Proposition 4.5 we see that the conditional moments and variances in the discrete case are also given by formulas from Proposition 2.5. Therefore the transformed process,

$$X_t = \frac{2(1+ABt)T(t)^{1/2}Y_{T(t)} - (A+B)t - (C+1/(q^N A))}{\sqrt{(1-q)(1-AC)(1-q^{-N})(1-BC)(1-B/(q^N A))}}\sqrt{1-q^{-N+1}BC},$$

$t \in J$, is a quadratic harness on J with $\theta, \eta, \tau, \sigma, \gamma$ defined as in the general case with $D = q^{-N}A^{-N}$. [Recall that $T(t)$ is the Möbius transformation (2.27).] □

5. Some worked out examples. This section shows how Theorem 1.1 is related to some previous constructions and how it yields new examples. From examples that have been previously worked out in detail one can see that the boundary of the range of parameters is not covered by Theorem 1.1; in particular it does not cover at all the family of five Meixner Lévy processes characterized by the quadratic harness property in [33]. On the other hand, sometimes new examples arise when processes run only on a subinterval of $(0, \infty)$.

Theorem 1.1 gives L_2-continuous processes on an open interval, so in applications we extend them to the closure of the time domain.

5.1. *q-Meixner processes.* Theorem 1.1 allows us to extend [10], Theorem 3.5, to negative τ. (The cases $\gamma = \pm 1$ which are included in [10] are not covered by Theorem 1.1.)

COROLLARY 5.1. *Fix $\tau, \theta \in \mathbb{R}$ and $-1 < \gamma < 1$, and let*

$$T_0 = \begin{cases} 0, & \text{if } \tau \geq 0, \\ -\tau/(1-\gamma), & \text{if } \tau < 0, \gamma \geq 0, \\ -\tau, & \text{if } \tau < 0, \gamma < 0. \end{cases}$$

Then there exists a Markov process (X_t) on $[T_0, \infty)$ such that (1.1), (1.2) hold, and (1.3) holds with parameters $\eta = 0, \sigma = 0$.

PROOF. Let $q = \gamma$, $A = 0$, $B = 0$, and

$$C = \begin{cases} \dfrac{-\theta + \sqrt{\theta^2 - 4\tau}}{2\sqrt{1-q}}, & \theta^2 \geq 4\tau, \\[2ex] \dfrac{-\theta + i\sqrt{4\tau - \theta^2}}{2\sqrt{1-q}}, & \theta^2 < 4\tau, \end{cases}$$

$$D = \begin{cases} \dfrac{-\theta - \sqrt{\theta^2 - 4\tau}}{2\sqrt{1-q}}, & \theta^2 \geq 4\tau, \\ \dfrac{-\theta - i\sqrt{4\tau - \theta^2}}{2\sqrt{1-q}}, & \theta^2 < 4\tau. \end{cases}$$

Then (1.4) holds trivially, so by Theorem 1.1 and L_2-continuity, (X_t) is well defined on $\overline{J} = [T_0, \infty)$. Straightforward calculation of the parameters from (1.6), (1.8) and (1.9) ends the proof. □

When $\tau < 0$, the univariate laws of X_t form the "sixth" family to be added to the five cases from [10], Theorem 3.5. The orthogonal polynomials, with respect to the law of X_t, satisfy the recurrence

$$xp_n(x;t) = p_{n+1}(x;t) + \theta[n]_q p_n(x;t) + (t + \tau[n-1]_q)[n]_q p_{n-1}(x;t),$$

where $[n]_q = (1-q^n)/(1-q)$. So the polynomials with respect to the standardized law of $X_t/\sqrt{t}$ are

$$x\widetilde{p}_n(x;t) = \widetilde{p}_{n+1}(x;t) + \frac{\theta}{\sqrt{t}}[n]_q \widetilde{p}_n(x;t) + \left(1 + \frac{\tau}{t}[n-1]_q\right)[n]_q \widetilde{p}_{n-1}(x;t). \tag{5.1}$$

The same law appears under the name q-Binomial law in [24] for parameters $n = -t/\tau \in \mathbb{N}$, $\tau = -p(1-p) \in [-1/4, 0)$. When $q \leq 0$ and $t = |\tau|$, this law is a discrete law supported on two roots of $\widetilde{p}_2$ (see Theorem A.1).

A justification of relating this law to the Binomial can be given for $q = 0$. In this case, recurrence (5.1) appears in [7], (3), with their $a = \frac{\theta}{\sqrt{t}}$ and their $b = \frac{\tau}{t}$. By [7], Proposition 2.1, the law ν_t of $\frac{1}{\sqrt{\tau}} X_t$ is a free convolution $\frac{t}{|\tau|}$-fold power of the two-point discrete law that corresponds to $t = -\tau$. That is, $\nu_t = \nu_{-\tau}^{t/|\tau|\boxplus}$; in particular, at $t = -n\tau$, $X_t/\sqrt{\tau}$ has the law that is the n-fold free additive convolution of a centered and standardized two-point law.

5.2. *Bi-Poisson processes.* Next we deduce a version of [9], Theorem 1.2. Here we again have to exclude the boundary cases $\gamma = \pm 1$ as well as the case $1 + \eta\theta = \max\{\gamma, 0\}$.

COROLLARY 5.2. *For $-1 < \gamma < 1$, and $1 + \eta\theta > \max\{\gamma, 0\}$ there exists a Markov process $(X_t)_{t \in [0,\infty)}$ such that* (1.1), (1.2) *hold, and* (1.3) *holds with $\sigma = \tau = 0$.*

PROOF. Let $A = 0$, $B = -\frac{\eta}{\sqrt{\eta\theta + 1 - q}}$, $C = 0$, $D = -\frac{\theta}{\sqrt{\eta\theta + 1 - q}}$. Then $BD = \frac{\eta\theta}{\eta\theta + 1 - q} < 1$. The condition $qBD < 1$ is also satisfied as we assume $\eta\theta + 1 > 0$

when $q < 0$. Thus (1.4) holds and we can apply Theorem 1.1. From formulas (1.5) through (1.9); the quadratic harness has parameters $\eta, \theta, \sigma = 0, \tau = 0, \gamma$, as claimed. $\square$

5.3. *Free harness*. Next we indicate the range of parameters that guarantee existence of the processes described in [8], Proposition 4.3. Let

$$\alpha = \frac{\eta + \theta\sigma}{1 - \sigma\tau}, \qquad \beta = \frac{\eta\tau + \theta}{1 - \sigma\tau}. \tag{5.2}$$

COROLLARY 5.3. *For* $0 \le \sigma\tau < 1$, $\gamma = -\sigma\tau$, *and* η, θ *with* $2 + \eta\theta + 2\sigma\tau \ge 0$ *and* $1 + \alpha\beta > 0$, *there exists a Markov process* $(X_t)_{t\in[0,\infty)}$ *such that* (1.1), (1.2) *and* (1.3) *hold*.

REMARK 5.1. When $2 + \eta\theta + 2\sigma\tau < 0$, two of the products in (1.4) are in the "forbidden region" $[1, \infty)$, so Theorem 1.1 does not apply. However, the univariate Askey–Wilson distributions are still well defined.

PROOF OF COROLLARY 5.3. Take $q = 0$, and let

$$A = -\frac{\alpha + \beta\sigma - \sqrt{-4\sigma + (\alpha - \beta\sigma)^2}}{2\sqrt{1 + \alpha\beta}},$$

$$B = -\frac{\alpha + \beta\sigma + \sqrt{-4\sigma + (\alpha - \beta\sigma)^2}}{2\sqrt{1 + \alpha\beta}},$$

$$C = -\frac{\beta + \alpha\tau - \sqrt{-4\tau + (\beta - \alpha\tau)^2}}{2\sqrt{1 + \alpha\beta}},$$

$$D = -\frac{\beta + \alpha\tau + \sqrt{-4\tau + (\beta - \alpha\tau)^2}}{2\sqrt{1 + \alpha\beta}}.$$

To verify that $AC \notin [1, \infty)$ we proceed as follows. Note that

$$A + B = -\frac{\alpha + \sigma\beta}{\sqrt{1 + \alpha\beta}}, \tag{5.3}$$

$$C + D = -\frac{\alpha\tau + \beta}{\sqrt{1 + \alpha\beta}}, \tag{5.4}$$

$$A - B = \frac{\sqrt{(\alpha - \sigma\beta)^2 - 4\sigma}}{\sqrt{1 + \alpha\beta}}, \tag{5.5}$$

$$C - D = \frac{\sqrt{(\beta - \tau\alpha)^2 - 4\tau}}{\sqrt{1 + \alpha\beta}}. \tag{5.6}$$

Multiplying $(A+B)(C+D)$ and $(A-B)(C-D)$ and using $ABCD=\sigma\tau$, we get

$$AC+\frac{\sigma\tau}{AC}-BC-\frac{\sigma\tau}{BC}=\frac{\sqrt{(\alpha-\sigma\beta)^2-4\sigma}\sqrt{(\beta-\tau\alpha)^2-4\tau}}{1+\alpha\beta}$$

and

$$AC+\frac{\sigma\tau}{AC}+BC+\frac{\sigma\tau}{BC}=\frac{(\alpha+\sigma\beta)(\alpha\tau+\beta)}{1+\alpha\beta}.$$

This gives the following quadratic equation for AC:

$$AC+\frac{\sigma\tau}{AC}=\frac{(\alpha+\sigma\beta)(\alpha\tau+\beta)+\sqrt{(\alpha-\sigma\beta)^2-4\sigma}\sqrt{(\beta-\tau\alpha)^2-4\tau}}{2(1+\alpha\beta)}. \tag{5.7}$$

We now note that a quadratic equation $x+a/x=b$ with $0<a<1$ and complex b can have a root in $[1,\infty)$ only when b is real and $b\ge 1+a$; this follows from the fact that $x+a/x$ is increasing for $x>a$, so $x+a/x\ge 1+a$ for $x\ge 1$.

Suppose, therefore, that the right-hand side of (5.7) is real and larger than $1+\sigma\tau$. Then calculations lead to $\sqrt{\eta^2-4\sigma}\sqrt{\theta^2-4\tau}\ge 2+\eta\theta+2\sigma\tau$. The right-hand side is nonnegative by assumption, so squaring the inequality we get $(1+\alpha\beta)(1-\sigma\tau)^2\le 0$ which contradicts the assumption.

Other cases with AD, BC, BD are handled similarly. Since $ABCD=\sigma\tau<1$ by assumption, by Theorem 1.1 the quadratic harness exists.

It remains to calculate the parameters. From $AB=\sigma$, $CD=\tau$ we see that (1.7) and (1.8) give the correct values, and $\gamma=-\sigma\tau$ from (1.9). To compute the remaining parameters, we re-write the expression under the square root in the denominator of (1.5) as

$$\begin{aligned}&(1-AC)(1-BC)(1-AD)(1-BD)\\&\quad=\left(1+\sigma\tau-\left(AC+\frac{\sigma\tau}{AC}\right)\right)\left(1+\sigma\tau-\left(BC+\frac{\sigma\tau}{BC}\right)\right).\end{aligned}$$

This is the product of two conjugate expressions [see (5.7), and its derivation]. A calculation now simplifies the denominator of (1.5) to $(1-\sigma\tau)/\sqrt{1+\alpha\beta}$. Inserting (5.3) and (5.4), the numerator of (1.5) simplifies to $(\alpha-\beta\sigma)(1-\sigma\tau)/\sqrt{1+\alpha\beta}$. The quotient of these two expressions is $\alpha-\beta\sigma=\eta$. Similar calculation verifies (1.6). □

5.4. *Purely quadratic harness.* The quadratic harness with parameters $\eta=\theta=0$ and $\sigma\tau>0$ has not been previously constructed.

COROLLARY 5.4. *For $\sigma,\tau>0$ with $\sigma\tau<1$ and $-1<\gamma<1-2\sqrt{\sigma\tau}$ there exists a Markov process $(X_t)_{t\in[0,\infty)}$ such that* (1.1), (1.2) *hold, and* (1.3) *holds with $\eta=\theta=0$.*

PROOF. Let

$$q = \frac{4(\gamma + \sigma\tau)}{(1 + \gamma + \sqrt{(1-\gamma)^2 - 4\sigma\tau})^2}.$$

To see that $-1 < q < 1$, note that for $\gamma + \sigma\tau \neq 0$,

$$q = \frac{1 + \gamma^2 - 2\sigma\tau - (1+\gamma)\sqrt{(1-\gamma)^2 - 4\sigma\tau}}{2(\gamma + \sigma\tau)},$$

which gives

$$q - 1 = \frac{-2\sqrt{(1-\gamma)^2 - 4\sigma\tau}}{1 + \gamma + \sqrt{(1-\gamma)^2 - 4\sigma\tau}} < 0 \tag{5.8}$$

and

$$q + 1 = \frac{2(1+\gamma)}{1 + \gamma + \sqrt{(1-\gamma)^2 - 4\sigma\tau}} > 0. \tag{5.9}$$

Noting that $(1-q)^2 + 4q\sigma\tau \geq 4\sigma\tau(1 - \sigma\tau) > 0$, let

$$A = -B = \frac{i\sqrt{2\sigma}}{\sqrt{(1-q) + \sqrt{(1-q)^2 + 4q\sigma\tau}}}$$

and

$$C = -D = \frac{i\sqrt{2\tau}}{\sqrt{(1-q) + \sqrt{(1-q)^2 + 4q\sigma\tau}}}.$$

Since A, B, C, D are purely imaginary, we only need to verify condition $BC < 1$ which reads

$$q + 2\sqrt{\sigma\tau} - 1 < \sqrt{(1-q)^2 + 4q\sigma\tau}. \tag{5.10}$$

This is trivially true when $q + 2\sqrt{\sigma\tau} - 1 < 0$. If $q + 2\sqrt{\sigma\tau} - 1 \geq 0$, squaring both sides we get $4(1-q)\sqrt{\sigma\tau} > 4(1-q)\sigma\tau$, which holds true as $q < 1$ and $0 < \sigma\tau < 1$.

Thus quadratic harness (X_t) exists by Theorem 1.1, and it remains to verify that its parameters are as claimed. A straightforward calculation shows that (1.7) and (1.8) give the correct values of parameters. It remains to verify that formula (1.9) indeed gives the correct value of parameter γ. Since this calculation is lengthy, we indicate major steps: we write (1.9) as (1.12), and evaluate the right-hand side. Substituting values of A, B, C, D we get

$$\frac{(q-1)(1 + ABCD)}{1 - qABCD} = \frac{(1-q)^2 + (1+q)\sqrt{(1-q)^2 + 4q\sigma\tau}}{2q}.$$

Then we use formulas (5.8) and (5.9) to replace $1-q$ and $1+q$ and note that since $\gamma < 1-2\sqrt{\sigma\tau}$ we have $\gamma < 1-2\sigma\tau$ and

$$\sqrt{(1-q)^2+4q\sigma\tau} = \frac{2(1-\gamma-2\sigma\tau)}{\gamma+\sqrt{(1-\gamma)^2-4\sigma\tau+1}}.$$

This eventually simplifies the right-hand side of (1.12) to $\gamma - 1$, so both uses of parameter γ are consistent, as claimed. □

6. Concluding observations. This section contains additional observations that may merit further study.

6.1. *Bridge property.* The following proposition lists combinations of parameters that create a "quadratic harness bridge" between either two-point masses, or degenerated laws.

PROPOSITION 6.1. *Let $(Z_t)_{t\in I}$ be the Markov process from Theorem 1.2. Assume that $AB \neq 0$ so that (1.21) defines a bounded interval $I = (S_1, S_2)$ and extend Z_t to the end-points of I by L_2-continuity.*

(i) *If $AB > 0$, then $Z_{S_2} = (1/A + 1/B)/\sqrt{1-q}$ is deterministic; similarly, if $CD \geq 0$, then $Z_{S_1} = (C+D)/\sqrt{1-q}$.*

(ii) *If $q \leq 0$ and $CD < 0$, then Z_{S_1} takes only two-values. Similarly, if $q \leq 0$ and $AB < 0$, then Z_{S_2} is a two-valued random variable.*

(iii) *If $CD < 0$ and $q > 0$, then Z_0 is purely discrete with the following law:*

$$\Pr\left(Z_0 = \frac{q^k C}{\sqrt{1-q}}\right) = \frac{(AD, BD)_\infty (AC, BC)_k}{(D/C, ABCD)_\infty (q, qC/D)_k} q^k, \qquad k \geq 0, \tag{6.1}$$

$$\Pr\left(Z_0 = \frac{q^k D}{\sqrt{1-q}}\right) = \frac{(AC, BC)_\infty (AD, BD)_k}{(C/D, ABCD)_\infty (q, qD/C)_k} q^k, \qquad k \geq 0. \tag{6.2}$$

PROOF. We can derive the first two statements from moments which are easier to compute for (Y_t) instead of (Z_t). In the first case, $\mathrm{Var}(Y_t) = 0$ at the endpoints [see (2.15)]; in the second case $E(\bar{w}_2^2(Y_t)) = 0$ at the end-points. Alternatively, one can compute the limit of the Askey–Wilson law as in the proof of part (iii).

For part (iii), without loss of generality, assume $|A| \leq |B|$ and $|C| \leq |D|$. Then the discrete part of Z_s has atoms at

$$\left\{\frac{1}{\sqrt{1-q}}\left(q^j C + \frac{s}{Cq^j}\right) : j \geq 0, q^{2j}C^2 > s\right\}$$

and

$$\left\{\frac{1}{\sqrt{1-q}}\left(q^j D + \frac{s}{Dq^j}\right) : j \geq 0, q^{2j}D^2 > s\right\}.$$

The probabilities can be computed from (3.5) with $c = A\sqrt{s}, d = B\sqrt{s}$ and either $a = C/\sqrt{s}, b = D/\sqrt{s}$ for (6.1) or $a = D/\sqrt{s}, b = C/\sqrt{s}$ for (6.2) and converge to (6.1) and (6.2), respectively. To see that the limit distribution is indeed discrete, we note that

$$\sum_{k=0}^{\infty} \Pr\left(Z_0 = \frac{q^k C}{\sqrt{1-q}}\right) + \Pr\left(Z_0 = \frac{q^k C}{\sqrt{1-q}}\right)$$
$$= \frac{(AD, BD)_\infty}{(D/C, ABCD)_\infty} {}_2\varphi_1\left(\begin{matrix} AC, BC \\ qC/D \end{matrix}; q\right)$$
$$+ \frac{(AC, BC)_\infty}{(C/D, ABCD)_\infty} {}_2\varphi_1\left(\begin{matrix} AD, BD \\ qD/C \end{matrix}; q\right) = 1.$$

Here we use hypergeometric function notation

$$(6.3) \qquad {}_{r+1}\varphi_r\left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_k}{(q, b_1, \dots, b_r)_k} z^k.$$

The identity that gives the final equality is [17], (12.2.21), used with $a = AC, b = BC, c = qC/D$. □

6.2. *Transformations that preserve quadratic harness property.* The basic idea behind the transformation (2.28) is that if a covariance

$$(6.4) \qquad \mathbb{E}(Z_t Z_s) = c_0 + c_1 \min\{t, s\} + c_2 \max\{t, s\} + c_3 ts,$$

factors as $(s - \alpha)(1 - t\beta)$ for $s < t$ with $\alpha\beta < 1$, then it can be transformed into $\min\{t, s\}$ by a deterministic time change and scaling.

This transformation is based on the following group action: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$ is invertible, then A acts on stochastic processes $\mathbf{X} = (X_t)$ by $A(\mathbf{X}) := \mathbf{Y}$ with $Y_t = (ct + d)X_{T_A(t)}$ where $T_A(t) = (at + b)/(ct + d)$ is the associated Möbius transformation. It is easy to check that this is a (right) group action: $A(B(\mathbf{X})) = (B \times A)(\mathbf{X})$.

If $\mathbb{E}(Z_t Z_s) = (s - \alpha)(1 - t\beta)$ for $s < t$ and $\alpha\beta < 1$, then $\mathbf{X} = A^{-1}(\mathbf{Z})$ with $A = \begin{bmatrix} 1 & -\alpha \\ -\beta & 1 \end{bmatrix}$ has $\mathbb{E}(X_t X_s) = \min\{s, t\}$. The easiest way to see this is to note that T_A is increasing for $\alpha\beta < 1$, and by group property $\mathbf{Z} = A(\mathbf{X})$. So with $s < t$, $\mathbb{E}(Z_s Z_t) = (1 - s\beta)(1 - t\beta)\mathbb{E}(X_{T_A(s)} X_{T_A(t)}) = (1 - s\beta)(1 - t\beta)T_A(s) = (s - \alpha)(1 - t\beta)$.

It is clear that, at least locally, this group action preserves properties of linearity of regression and of quadratic conditional variance. In fact, one can verify that the general form of the covariance $\mathbb{E}(X_t, X_s) = c_0 + c_1 \min\{t, s\} + c_2 \max\{t, s\} + c_3 ts$ is also preserved, and since this covariance corresponds to (1.2), the latter is also preserved by the group action.

APPENDIX: SUPPLEMENT ON ORTHOGONAL POLYNOMIALS

A.1. General theory. A standard simplifying condition in the general theory of orthogonal polynomials is that the orthogonality measure has infinite support. This condition may fail for the transition probabilities of the Markov process in Theorem 1.14. Since we did not find a suitable reference, for the reader's convenience we state the general result in the form we need and indicate how to modify known proofs to cover the case of discrete orthogonality measure. According to [3], page 1012, related results are implicit in some of Chebyshev's work on continued fractions.

THEOREM A.1. *Let A_n, B_n, C_n be real, $n \geq 0$ and such that*

$$\prod_{k=0}^{n} A_k C_{k+1} \geq 0 \qquad \text{for all } n \geq 0. \tag{A.1}$$

Consider two families of polynomials defined by the recurrences

$$x\overline{p}_n(x) = A_n \overline{p}_{n+1}(x) + B_n \overline{p}_n(x) + C_n \overline{p}_{n-1}(x), \qquad n \geq 0, \tag{A.2}$$

$$x p_n(x) = p_{n+1}(x) + B_n p_n(x) + A_{n-1} C_n p_{n-1}(x), \qquad n \geq 0, \tag{A.3}$$

with the initial conditions $p_0 = \overline{p}_0 = 1$, $p_{-1} = \overline{p}_{-1} = 0$. Then:

(i) *Polynomials $\{\overline{p}_n\}$ are well defined for all $n \geq 0$ such that $\prod_{k=0}^{n-1} A_k \neq 0$. (Here and below, the product for $n = 0$ is taken as 1.)*

(ii) *Monic polynomials $\{p_n\}$ are defined for all $n \geq 0$. For n such that $\prod_{k=0}^{n-1} A_k \neq 0$, the polynomials differ only by normalization*

$$p_n(x) = \overline{p}_n(x) \prod_{k=0}^{n-1} A_k. \tag{A.4}$$

(iii) *There exists a probability measure ν such that both families $\{\overline{p}_n\}$ and $\{p_n\}$ are orthogonal with respect to ν. In particular for all $m, n \geq 0$,*

$$\int p_n(x) p_m(x) \nu(dx) = \delta_{m,n} \prod_{k=0}^{n-1} A_k C_{k+1}. \tag{A.5}$$

Furthermore, if N is the first positive integer such that $A_{N-1}C_N = 0$, then $\nu(dx)$ is a discrete probability measure supported on the finite set of $N \geq 1$ real and distinct zeros of the polynomial p_N.

PROOF. It is clear that recurrence (A.2) can be solved (uniquely) for $\overline{p}_{n+1}$ as long as $A_0, \dots, A_n \neq 0$ while recurrence (A.3) has a unique solution for all n. It is also clear that transformation (A.4) maps the solutions of recurrence (A.2) to the solutions of (A.3).

If $\prod_{k=0}^{n} A_k C_{k+1} > 0$ for all n, then each factor $A_{n-1}C_n$ must be positive, so measure $\nu(dx)$ exists and (A.5) holds for all m, n by Favard's theorem as stated, for example, in [17], Theorem 2.5.2. If the product (A.1) is zero starting from some n, and N is the first positive integer such that $A_{N-1}C_N = 0$, then $N \ge 1$, and (A.3) implies that for $n > N$, polynomial p_n is divisible by p_N. So if $\nu(dx)$ is a discrete measure supported on the finite set of N zeros of the polynomial p_N, then once we show that the zeros are real, (A.5) holds trivially if either $n \ge N$ or $m \ge N$. To see that (A.5) holds when $0 \le m, n \le N-1$, and to see that all zeros of p_N are distinct and real, we apply known arguments. First, the proof of [31], (3.2.4) (or recursion) implies that

$$\text{(A.6)} \quad p_n'(x)p_{n-1}(x) - p_{n-1}'(x)p_n(x) > 0 \qquad \text{for all } x \in \mathbb{R} \text{ and all } 1 \le n \le N,$$

so the proof of [31], Theorem 3.3.2, establishes recurrently that each of the polynomials $p_1, \dots, p_N$ has real and distinct zeros. Now let $\lambda_0, \dots, \lambda_{N-1}$ be the zeros of p_N. The remainder of the proof is an adaptation of the proof of Theorem 1.3.12 in [13]. (Unfortunately, we cannot apply [13], Theorem 1.3.12, directly since the Nth polynomial is undefined there.) Let $J = [J_{i,j}]$ be the $N \times N$ Jacobi matrix whose nonzero entries are $J_{n,n} = B_n$, $n = 0, 1, \dots, N-1$ and $J_{n,n+1} = 1$, $J_{n+1,n} = A_n C_{n+1}$, $n = 0, 1, \dots, N-2$. Then (A.3) says that vector $\vec{v}_j = [p_0(\lambda_j), \dots, p_{N-1}(\lambda_j)]^T$ is the eigenvector of J with eigenvalue λ_j.

Let D be the diagonal matrix with diagonal entries

$$d_j = \left(\prod_{k=0}^{j-1} A_k C_{k+1}\right)^{-1/2} > 0, \qquad 0 \le j \le N-1.$$

Thus $d_0 = 1$ and $d_{N-1} = (A_0 \cdots A_{N-2} C_1 \cdots C_{N-1})^{-1/2}$. Then DJD^{-1} is a symmetric matrix with the eigenvectors $D\vec{v}_0, \dots, D\vec{v}_{N-1}$ which correspond to the distinct eigenvalues $\lambda_0, \dots, \lambda_{N-1}$. So the matrix

$$\left[\frac{1}{\|D\vec{v}_0\|} D\vec{v}_0, \frac{1}{\|D\vec{v}_1\|} D\vec{v}_1, \dots, \frac{1}{\|D\vec{v}_{N-1}\|} D\vec{v}_{N-1}\right]$$

has orthonormal columns, and hence also orthonormal rows. The latter gives (A.5) with $\nu(dx) = \sum_{j=0}^{N-1} \gamma_j \delta_{\lambda_j}$ where $\gamma_j = (\sum_{k=0}^{N-1} p_k(\lambda_j)^2 d_k^2)^{-1} > 0$ (recall that $p_0 = 1$). Note that since $d_0 = 1$, from (A.5) applied to $m = n = 0$ we see that $\sum \gamma_j = 1$, so ν is a probability measure. □

As an illustration, for the degenerate measure $\mu = \delta_a$, one has $A_n = 0$, $B_n = a$, $C_n = 0$. Here, $N = 1$, so the family $\{\bar{p}_n(x)\} = \{1\}$ consists of just one polynomial, while the monic family is infinite, $\{p_n(x) : n \ge 0\} = \{(x-a)^n : n \ge 0\}$, and ν is concentrated on the set of zeros of $p_1 = x - a$.

A.2. Connection coefficients of Askey–Wilson polynomials. This section contains a re-statement of the special case of [4], formula (6.1), which we need in this paper.

THEOREM A.2. *Let* $\{\bar{w}_n\}$ *be defined by* (1.20). *If* $a \neq 0$ *then*

$$\bar{w}_n(x; a, b, \widetilde{c}, \widetilde{d}) = \sum_{k=0}^{n} \bar{c}_{k,n} \bar{w}_k(x; a, b, c, d), \tag{A.7}$$

where

$$\begin{aligned}\bar{c}_{k,n} &= (-1)^k q^{k(k+1)/2} \\ &\quad \times \frac{(q^{-n}, q^{n-1}ab\widetilde{c}\widetilde{d})_k (a\widetilde{c}, a\widetilde{d})_n}{a^{n-k}(q, q^{k-1}abcd, a\widetilde{c}, a\widetilde{d})_k} \\ &\quad \times {}_4\varphi_3\left(\begin{matrix} q^{k-n}, ab\widetilde{c}\widetilde{d}q^{n+k-1}, acq^k, adq^k \\ abcdq^{2k}, a\widetilde{c}q^k, a\widetilde{d}q^k \end{matrix}; q\right).\end{aligned} \tag{A.8}$$

[*Recall the hypergeometric function* (6.3).]

If $a = b = 0$ *and* $cd\widetilde{d} \neq 0$, *then*

$$\begin{aligned}\bar{c}_{k,n} &= (-1)^k q^{k(2n+1-k)/2} \frac{(q^{-n})_k d^{n-k} (\widetilde{d}/d)_{n-k}}{(q)_k} \\ &\quad \times {}_2\varphi_1\left(\begin{matrix} q^{-n}, \widetilde{c}/c \\ q^{k+1-n} d/\widetilde{d} \end{matrix}; qc/\widetilde{d}\right).\end{aligned} \tag{A.9}$$

Since $d^m(\widetilde{d}/d)_m = \prod_{j=0}^{m-1}(d - q^j \widetilde{d})$, *expression* (A.9) *is also well defined when* $d = 0$. *Similarly, it is well defined for* $c = 0$, $\widetilde{d} = 0$ [*see* (6.3)].

PROOF. The monic form of the Askey–Wilson polynomials $\{\widetilde{w}_n\}$ and $\{\bar{w}_n\}$ is the same. Applying (A.4) twice we see that

$$\widetilde{w}_n(x; a, b, c, d) = (ab)_n \bar{w}_n(x; a, b, c, d). \tag{A.10}$$

Since our $\widetilde{w}_n$ is denoted by p_n in [4], formula (A.7) is recalculated from [4], (6.1), with swapped parameters a, d and with $\beta = b$, $\gamma = \widetilde{c}$, $\alpha = \widetilde{d}$.

To prove the second part, first take $b = 0$ and all other parameters nonzero to write

$$\bar{c}_{k,n} = (-1)^k q^{k(k+1)/2} \frac{(a\widetilde{c}, a\widetilde{d})_n (q^{-n})_k}{a^{n-k}} {}_3\varphi_2\left(\begin{matrix} q^{k-n}, acq^k, adq^k \\ a\widetilde{c}q^k, a\widetilde{d}q^k \end{matrix}; q\right). \tag{A.11}$$

Then we apply the limiting case of Sears transformation [17], Theorem 12.4.2, to rewrite

$${}_3\varphi_2\left(\begin{matrix} q^{k-n}, acq^k, adq^k \\ a\widetilde{c}q^k, a\widetilde{d}q^k \end{matrix}; q\right) = \frac{(adq^k)^{n-k}(\widetilde{d}/d)_{n-k}}{(a\widetilde{d}q^k)_{n-k}} {}_3\varphi_2\left(\begin{matrix} q^{k-n}, adq^k, \widetilde{c}/c \\ a\widetilde{c}q^k, q^{k+1-n}d/\widetilde{d} \end{matrix}; q\right).$$

This allows us to take the limit $a \to 0$ in (A.11), proving (A.9). □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CINCINNATI
PO BOX 210025
CINCINNATI, OHIO 45221–0025
USA
E-MAIL: Wlodzimierz.Bryc@UC.edu
URL: http://math.uc.edu/~brycw

FACULTY OF MATHEMATICS
AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
PL. POLITECHNIKI 1
00-661 WARSZAWA
POLAND
E-MAIL: wesolo@mini.pw.edu.pl