STATISTICS IN TRANSITION new series, October 2010
Vol.11, No. 2, pp. 267-285

RECURSIVE OPTIMAL ESTIMATION IN SZARKOWSKI ROTATION SCHEME

Jacek Wesołowski[1]

ABSTRACT

In late 90ties Szarkowski observed that under the rotation pattern typical for the Labour Force Survey the recursion for the optimal estimator of the mean on a given occasion has to use estimators and observations only from three last occasions. Since the fundamental work of Patterson (1950) it had been known that for rotation patterns with "holes" it is a difficult problem to determine the depth of such recursion formulas. Under special assumptions the problem has been settled only recently in Kowalski and Wesołowski (2010). In the present paper it is shown that these assumptions are always satisfied in the case of the Szarkowski rotation pattern 110011. Moreover, explicit formulas for the coefficients of recursion are derived.

1. Introduction

Andrzej Szarkowski passed away in June 2003. He was a creative and passionate statistician with considerable mathematical background. He devoted his talents to the Labour Force Survey (LFS) conducted by the Central Statistical Office in Poland taking care about mathematical methodology of this survey for more than 10 years, just from its beginning in 1992. For details on development of methodology of this survey in Poland, see Szarkowski and Witkowski (1994) and Popiński (2006). In particular, in 1993 Szarkowski introduced in the LFS a rotation pattern 110011. One of the issues related to this approach he was very concerned about was the recurrence form of optimal linear estimators of mean on every occasion under this pattern. In late nineties he studied Patterson (1955) paper, where the rotation pattern with no holes had been thoroughly treated. However, it appeared to be of not much help since a real challenge is posed by the HOLES! On the basis of intensive numerical experiments Szarkowski conjectured that the pattern

[1] Główny Urząd Statystyczny and Politechnika Warszawska, Warszawa, POLAND
e-mail: wesolo@mini.pw.edu.pl

110011 forces the recurrence to move THREE STEPS back when the correlation is exponential in the occasions span and SEVEN STEPS back when there are no restrictions on correlations. For a while we had sought together a mathematical explanation of this phenomena with no luck.

It took more than ten years to answer in affirmative Szarkowski's THREE STEPS conjecture. The explanation is given in the present paper. It is based on a general approach described in Kowalski and Wesołowski (2010) (KW in the sequel). Earlier the problem for rotation schemes with singleton holes was solved in Kowalski (2009) (particular cases of 1011 and 1101 rotation patterns were covered even earlier, in Ciepiela (2004)). In Section 2 the general approach from KW, which is based on TWO ASSUMPTIONS, is adjusted to a setup with a single hole of any size h, that is for the rotation pattern 11...110...011...11. In Section 3 we prove that for the Szarkowski scheme these TWO ASSUMPTIONS **are necessarily satisfied** and thus the general procedure works. Moreover, explicit formulas for the coefficients of the recursion are derived. In Section 4 we give proofs of lemmas which are used in Section 3 to derive the main result.

Szarkowski's SEVEN STEPS conjecture remains open. Even in the case of one singleton hole it is not known how far back one has to go in the recursion formula.

2. General method

Consider a doubly-infinite matrix of random variables (X_{ij}), $i, j \in \mathbb{Z}$. Index i identifies a unit and index j is an occasion number (time). We assume that for any $j \in \mathbb{Z}$ we have

$$\mathbb{E}\, X_{i,j} = \mu_j, \quad \text{for all } i \in \mathbb{Z},$$

and, without loss of generality we assume that $\mathbb{V}\mathrm{ar}(X_{i,j}) = 1$ for all $i, j \in \mathbb{Z}$. The correlation structure is described as follows

$$\mathbb{C}\mathrm{orr}(X_{i,j}, X_{k,l}) = I(k = i)\rho^{|j-l|}.$$

Fix natural numbers n and h and consider a sequence of random vectors $\underline{X}_j = (X_{j,j}, \ldots, X_{j,j+n+h-1})$, $j \in \mathbb{Z}$. Note that $C = \mathbb{C}\mathrm{ov}\, \underline{X}_j$ is an $(n + h) \times (n + h)$ matrix with all entries equal zero except the entries just above the diagonal which are all equal ρ. Moreover

$$\mathbb{C}\mathrm{ov}(\underline{X}_j, \underline{X}_k) = C^{|k-j|}$$

and C^j is a matrix with all entries equal zero except the jth over diagonal with all entries equal ρ^j when $j \leq n + h - 1$ and it is a zero matrix when $j > n + h - 1$.

A rotation pattern is any vector $(\epsilon_1, \ldots, \epsilon_{n+h})$ with 0-1 entries such that $\epsilon_1 = \epsilon_{n+h} = 1$ and there are exactly h zeros among the entries. Let $p-1$ denotes the dimension of the largest zero subvector of subsequent entries in the rotation pattern.

We modify vectors $\underline{X}_j$ into

$$\underline{Y}_j = (X_{j,k}\epsilon_{k-j+1},\, k = j, \ldots, j+n+h-1), \quad j \in \mathbb{Z}.$$

For a given $j \in \mathbb{Z}$ let $\widehat{\mu}_j$ denote the BLUE of μ_j based on $\underline{Y}_l$, $l \leq j$.

We study the recurrence formula for the BLUE estimators of the following form

$$\widehat{\mu}_t = a_1\,\widehat{\mu}_{t-1} + \ldots + a_p\,\widehat{\mu}_{t-p} + \langle \underline{r}_0, \underline{Y}_t\rangle + \langle \underline{r}_1, \underline{Y}_{t-1}\rangle + \ldots + \langle \underline{r}_p, \underline{Y}_{t-p}\rangle,$$

for any $t \in \mathbb{Z}$, where the parameters $a_1, \ldots, a_p \in \mathbb{R}$ and $\underline{r}_0, \underline{r}_1, \ldots, \underline{r}_p \in \mathbb{R}^{n+h}$ are to be identified. Here we use the symbol $\langle \underline{a}, \underline{b}\rangle$ to denote the scalar product of vectors $\underline{a} = (a_1, \ldots, a_d)$ and $\underline{b} = (b_1, \ldots, b_d)$, that is $\langle \underline{a}, \underline{b}\rangle = \sum_{i=1}^{d} a_i b_i$. Note that the parameters are assumed to be constant, i.e. they do not depend on t.

Note that, alternatively, $\widehat{\mu}_t$ can be defined as optimal unbiased linear estimator $\sum_{s\leq t} \langle \underline{w}_s, \underline{X}_s\rangle$, with additional constraints

$$\underline{w}_{s,j}(1-\epsilon_j) = 0, \quad j = 1, \ldots, n+h, \quad s \leq t, \tag{1}$$

imposed by the holes in the rotation pattern. Therefore the above recursion can be written in the form

$$\widehat{\mu}_t = a_1\,\widehat{\mu}_{t-1} + \ldots a_p\,\widehat{\mu}_{t-p} + \langle \underline{r}_0, \underline{X}_t\rangle + \langle \underline{r}_1, \underline{X}_{t-1}\rangle + \ldots + \langle \underline{r}_p, \underline{X}_{t-p}\rangle. \tag{2}$$

Note that (1) forces respective entries of vectors $\underline{r}_j$, $j = 0, \ldots, h+1$, to be equal zero.

Under certain assumptions (see ASSUMPTION 1 and 2, below) there exists a general algorithm, described in KW (see also Kowalski (2010)), which completely solves the problem. It is rather complicated. Here we describe it in the case of a single hole of any size h in the rotation pattern (thus $p = h+1$). More precisely we assume that the rotation patterns have a form $[1, 1, \ldots, 1, 0, 0, \ldots, 0, 0, 1, 1, \ldots, 1]$ where the zeros occur at places $s+1, s+2, \ldots, s+h$, for arbitrary s satisfying $1 \leq s < n$.

Recall that the Chebyshev polynomials of the first kind (T_n) *are defined through a three step recurrence*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \ldots$$

and $T_0 \equiv 1$, $T_1(x) = x$.

Consider a polynomial P *of degree* p *defined by*

$$P(x) = 1 - \rho^2 + (n + h - 1)(1 + \rho^2 - 2\rho x) + \\ -(1 + \rho^2 - 2\rho x)^2 \ \mathrm{tr}\left(\mathbf{T}_p(x)\mathbf{R}_p^{-1}(\rho)\right), \tag{3}$$

where $\mathbf{T}_p(x)$ *is a* $h \times h$ *symmetric matrix polynomial*

$$\mathbf{T}_p(x) = \begin{bmatrix} T_0(x) & T_1(x) & T_2(x) & \dots & T_{p-3}(x) & T_{p-2}(x) \\ T_1(x) & T_0(x) & T_1(x) & \dots & T_{p-4}(x) & T_{p-3}(x) \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ T_{p-3}(x) & T_{p-4}(x) & T_{p-5}(x) & \dots & T_0(x) & T_1(x) \\ T_{p-2}(x) & T_{p-3}(x) & T_{p-4}(x) & \dots & T_1(x) & T_0(x) \end{bmatrix} \tag{4}$$

and $\mathbf{R}_p$ *is a* $h \times h$ *invertible constant tridiagonal matrix*

$$\mathbf{R}_p = \begin{bmatrix} 1+\rho^2 & \rho & 0 & \dots & 0 & 0 \\ \rho & 1+\rho^2 & \rho & \dots & 0 & 0 \\ 0 & \rho & 1+\rho^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 1+\rho^2 & \rho \\ 0 & 0 & 0 & \dots & \rho & 1+\rho^2 \end{bmatrix} \tag{5}$$

ASSUMPTION 1: All the roots $x_1, \dots, x_p$ **(real or complex) of the polynomial** P **defined through** (3), (4), (5) **are distinct and do not belong to the interval** $[-1, 1]$**.**

Under ASSUMPTION 1 *equation*

$$d + \frac{1}{d} = 2x_i,$$

has exactly one solution d_i *such that* $|d_i| < 1$, $i = 1, \dots, p$.

Let $\underline{d} = [d_1, \dots, d_p]^T$*. Consider linear system*

$$S(\underline{d})\underline{c} = (1 - \rho^2)\underline{e}, \tag{6}$$

where $S(\underline{d})$ *is a* $(p+1)p \times p^2$ *matrix of the form*

$$S(\underline{d}) = \begin{bmatrix} G(d_1) & G(d_2) & G(d_3) & \dots & G(d_p) \\ H(d_1) & 0 & 0 & \dots & 0 \\ 0 & H(d_2) & 0 & \dots & 0 \\ 0 & 0 & H(d_3) & \dots & 0 \\ 0 & 0 & 0 & \dots & H(d_p) \end{bmatrix}$$

with $p \times p$ *blocks* $G(d_i)$, $H(d_i)$, $i = 1, \dots, p$ *defined as*

$$G(v) = \begin{bmatrix} g_0(v) & g_1(v) & g_1(v) & g_1(v) & \dots & g_1(v) & g_1(v) \\ g_1(v) & 1 & -v\rho & 0 & \dots & 0 & 0 \\ g_1(v) & 0 & 1 & -v\rho & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ g_1(v) & 0 & 0 & 0 & \dots & 1 & -v\rho \\ g_1(v) & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

$$H(v) = \begin{bmatrix} h_0(v) & h_1(v) & h_1(v) & \dots & h_1(v) & h_1(v) & h_1(v) \\ h_1(v) & v(1+\rho^2) & -v^2\rho & \dots & 0 & 0 & 0 \\ h_1(v) & -\rho & v(1+\rho^2) & \dots & 0 & 0 & 0 \\ h_1(v) & 0 & -\rho & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ h_1(v) & 0 & 0 & \dots & v(1+\rho^2) & -v^2\rho & 0 \\ h_1(v) & 0 & 0 & \dots & -\rho & v(1+\rho^2) & -v^2\rho \\ h_1(v) & 0 & 0 & \dots & 0 & -\rho & v(1+\rho^2) \end{bmatrix},$$

and

$$g_1(v) = 1 - v\rho, \quad g_0(v) = 1 - \rho^2 + (n+h-1)g_1(v),$$

$$h_1(v) = (1-v\rho)(v-\rho), \quad h_0(v) = v(1-\rho^2) + (n+h-1)h_1(v).$$

The unknown vector $\underline{c}$ *has the following structure*

$$\underline{c} = \left[\underline{c}_1,\ \underline{c}_2,\ \dots,\ \underline{c}_p\right]^T,$$

where $\underline{c}_i = [c_{0,i}, c_{1,i}, \dots, c_{h,i}]^T$, $i = 0, 1, \dots$. *Finally,* $\underline{e}$ *is the* $(p+1)p$-*dimensional unit vector* $\underline{e} = [1, 0, \dots, 0]^T$.

ASSUMPTION 2. Linear system (6) **has a unique solution.**

Under ASSUMPTION*s* 1 *and* 2 *the recurrence* (2) *holds with parameters* $a_1, \dots, a_p$ *and* $\underline{r}_0, \dots, \underline{r}_p$ *defined as follows:*

- *The linear system*

$$x_1 d_i^{p-1} + x_2 d_i^{p-2} + \dots + x_{p-1} d_i + x_p = d_i^p, \quad i = 1, \dots, p$$

has a unique solution, which equals $\underline{a} = [a_1, \dots, a_p]^T$, *that is*

$$\underline{a} = \begin{bmatrix} d_1^{p-1} & d_1^{p-2} & \dots & d_1 & 1 \\ d_2^{p-1} & d_2^{p-2} & \dots & d_2 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ d_p^{p-1} & d_p^{p-2} & \dots & d_p & 1 \end{bmatrix}^{-1} \begin{bmatrix} d_1^p \\ d_2^p \\ \vdots \\ d_p^p \end{bmatrix}.$$

- *For any $i = 1, \ldots, p$ let D_i be an $(n+h) \times (p+1)p$ matrix defined as*

$$D_i = \begin{bmatrix} d_1^i & 0 & 0 & \ldots & 0 & d_2^i & 0 & 0 & \ldots & 0 & \ldots & d_p^i & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ d_1^i & 0 & 0 & \ldots & 0 & d_2^i & 0 & 0 & \ldots & 0 & \ldots & d_p^i & 0 & 0 & \ldots & 0 \\ d_1^i & d_1^i & 0 & \ldots & 0 & d_2^i & d_2^i & 0 & \ldots & 0 & \ldots & d_p^i & d_p^i & 0 & \ldots & 0 \\ d_1^i & 0 & d_1^i & \ldots & 0 & d_2^i & 0 & d_2^i & \ldots & 0 & \ldots & d_p^i & 0 & d_p^i & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ d_1^i & 0 & 0 & \ldots & d_1^i & d_2^i & 0 & 0 & \ldots & d_2^i & \ldots & d_p^i & 0 & 0 & \ldots & d_p^i \\ d_1^i & 0 & 0 & \ldots & 0 & d_2^i & 0 & 0 & \ldots & 0 & \ldots & d_p^i & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ d_1^i & 0 & 0 & \ldots & 0 & d_2^i & 0 & 0 & \ldots & 0 & \ldots & d_p^i & 0 & 0 & \ldots & 0 \end{bmatrix}$$

where the first s rows are identical, then the rows with numbers $s+1$, $s+2$, $\ldots$, $s+h$, associated to the hole in the rotation pattern, are perturbed in a regular manner, and the last rows with numbers $s+h+1$, $s+h+2$, $\ldots$, $n+h$ are again identical and the same as the first s rows.

Let Δ be a $(n+h) \times (n+h)$ diagonal matrix defined as

$$\Delta = (\mathrm{Id} - CC^T)^{-1}.$$

All the elements of the diagonal of Δ are equal $(1-\rho^2)^{-1}$ except the last one which equals 1.

Let

$$V_i = \Delta(D_i - CD_{i+1}), \quad i = 0, 1, \ldots$$

Let $\underline{c} = \underline{c}(\underline{d})$ be the unique solution of (6), which by ASSUMPTION 2 is guaranteed to exist. Then, denoting additionally $a_0 = -1$, we have

$$\underline{r}_0 = V_0\, \underline{c}(\underline{d}), \quad \underline{r}_j = \left(V_j + \sum_{i=0}^{j-1} (a_i C^T - a_{i+1}\mathrm{Id}) V_{j-1-i} \right) \underline{c}(\underline{d}) \tag{7}$$

for $j = 1, \ldots, p$.

Thus the problem has a solution provided the ASSUMPTIONs 1 and 2 are satisfied. It was proved in KW that the ASSUMPTIONs are always satisfied when $p = 0$ (no holes) or $p = 1$ (singleton hole; actually, any number of singleton holes has been allowed). Intensive numerical experiments provided strong motivation to conjecture that ASSUMPTIONs 1 and 2 are satisfied for any $p \geq 0$. However, proving this conjecture seems to be rather difficult even in the case of a single hole of any size.

3. Szarkowski scheme

Here we concentrate on a special rotation pattern 110011 with a single hole of size 2, called the Szarkowski scheme. As already mentioned, this scheme has been adopted for the Labuor Force Survey in the CSO in Poland. We will prove that under this rotation pattern ASSUMPTIONS 1 and 2 are satisfied. Moreover, we will derive explicit analytic formulas for the parameters of the recurrence a_i, $i = 1, 2, 3$, and $\underline{r}_j$, $j = 0, 1, 2, 3$.

Note that under this pattern $n = 4$, $h = 2$, $p = 3$ and the missing elements are at positions defined by the vectors $\underline{e}_3$ and $\underline{e}_4$ in six-dimensional space $\mathbb{R}^6$. We seek a representation

$$\begin{aligned} \widehat{\mu}_t = a_1\, \widehat{\mu}_{t-1} + a_2\, \widehat{\mu}_{t-2} + a_3\, \widehat{\mu}_{t-3} \\ +\langle \underline{r}_0, \underline{X}_t\rangle + \langle \underline{r}_1, \underline{X}_{t-1}\rangle + \langle \underline{r}_2, \underline{X}_{t-2}\rangle + \langle \underline{r}_3, X_{t-3}\rangle \end{aligned} \tag{8}$$

for the BLUE of the mean μ_t on the t-th occasion.

The main result will be preceded by three auxiliary lemmas, proofs of which are postponed to Section 4.

Lemma 1. *Let $\rho \in (-1, 1) \setminus \{0\}$. Then the polynomial*

$$W_3(x) = x^3 + (2 - \rho^2 + 2\rho^4)x + 2(2 + 2\rho^2 + 2\rho^4 + \rho^6)$$

has one real root $x_1 < -2|\rho|$ and two conjugate complex roots x_2 and x_3.

Lemma 2. *For any $\rho \in (-1, 1) \setminus \{0\}$ let*

$$Q(d) = \begin{bmatrix} 5(1-d\rho)(d-\rho) + d(1-\rho^2) & (1-d\rho)(d-\rho) & (1-d\rho)(d-\rho) \\ (1-d\rho)(d-\rho) & d(1+\rho^2) & -d^2\rho \\ (1-d\rho)(d-\rho) & -\rho & d(1+\rho^2) \end{bmatrix}.$$

The equation

$$\det\, Q(d) = 0 \tag{9}$$

has exactly three distinct roots $d_1 = d_1(\rho)$, $d_2 = d_2(\rho)$ and $d_3 = d_3(\rho)$ such that $|d_i| < 1$, $i = 1, 2, 3$. The number d_i is the unique solution of equation

$$-\rho\left(d + \frac{1}{d}\right) = x_i(\rho),$$

where $x_i(\rho)$ is the root of the polynomial W_3, satisfying $|d_i| < 1$, $i = 1, 2, 3$.

The root d_1 is real and the roots d_2, and d_3 are conjugate complex. Moreover,

$$d_1 d_2 d_3 (d_1 + d_2 + d_3) = -d_1 d_2 - d_2 d_3 - d_3 d_1. \tag{10}$$

Lemma 3. *Let* $\rho \in (-1,1)\setminus\{0\}$*. Let* $\alpha = d_1$*,* $\beta = d_2$ *and* $\gamma = d_3$ *be the roots of* (9) *defined in Lemma* 2*. Let* $\underline{e}_1 = (1,0,\dots,0) \in \mathbb{R}^{12}$*. Consider the following* 12×9 *system of linear equations in* $\underline{c}_j = [c_{0,j},\, c_{1,j},\, c_{2,j}]^T$*,* $j = 1,2,3$*,*

$$\mathbb{Q}\,\underline{c} = \begin{bmatrix} Q_0(\alpha) & Q_0(\beta) & Q_0(\gamma) \\ Q(\alpha) & 0 & 0 \\ 0 & Q(\beta) & 0 \\ 0 & 0 & Q(\gamma) \end{bmatrix} \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \end{bmatrix} = (1-\rho^2)\underline{e}_1, \tag{11}$$

where $Q(d)$ *is defined in Lemma* 2 *and*

$$Q_0(d) = \begin{bmatrix} 5(1-d\rho)+1-\rho^2 & 1-d\rho & 1-d\rho \\ 1-d\rho & 1 & -d\rho \\ 1-d\rho & 0 & 1 \end{bmatrix}.$$

The linear system (11) *has the unique solution*

$$\begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \end{bmatrix} = (1-\rho^2)\widetilde{\mathbb{Q}}^{-1}\underline{\widetilde{e}}_1,$$

where $\underline{\widetilde{e}}_1 = (1,0,\dots,0) \in \mathbb{R}^9$ *and* $\widetilde{\mathbb{Q}}$ *is* 9×9 *invertible matrix defined as*

$$\widetilde{\mathbb{Q}} = \begin{bmatrix} Q_0(\alpha) & Q_0(\beta) & Q_0(\gamma) \\ \widetilde{Q}(\alpha) & 0 & 0 \\ 0 & \widetilde{Q}(\beta) & 0 \\ 0 & 0 & \widetilde{Q}(\gamma) \end{bmatrix} \tag{12}$$

with

$$\widetilde{Q} = \begin{bmatrix} (1-d\rho)(d-\rho) & d(1+\rho^2) & -d^2\rho \\ (1-d\rho)(d-\rho) & -\rho & d(1+\rho^2) \end{bmatrix}. \tag{13}$$

Now we are ready to formulate and prove the main result of the paper, which completely covers the problem of recursive optimal estimation under the pattern 110011.

Theorem 1. *Let* $\rho \in (-1,1)\setminus\{0\}$*. Let* $\alpha = d_1$*,* $\beta = d_2$ *and* $\gamma = d_3$ *be the roots of* (9) *defined in Lemma* 2*. Let* $\underline{c} = \underline{c}(d)$ *be defined as in Lemma* 3*.*

Under the Szarkowski rotation pattern recurrence (8) *always holds with*

$$a_1 = \alpha+\beta+\gamma, \quad a_2 = -(\alpha\beta+\beta\gamma+\gamma\alpha), \quad a_3 = \alpha\beta\gamma. \tag{14}$$

Denote

$$V = \begin{bmatrix} 1-\rho\alpha & 0 & 0 & 1-\rho\beta & 0 & 0 & 1-\rho\gamma & 0 & 0 \\ 1-\rho\alpha & 1 & 1 & 1-\rho\beta & 1 & 1 & 1-\rho\gamma & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-\rho\alpha & 0 & 0 & 1-\rho\beta & 0 & 0 & 1-\rho\gamma & 0 & 0 \\ 1-\rho^2 & 0 & 1 & 1-\rho^2 & 0 & 1 & 1-\rho^2 & 0 & 1 \end{bmatrix}$$

Then

$$\underline{r}_i = \frac{1}{1-\rho^2} V\, D_i\, \underline{c}, \quad i = 0, 1, 2, \tag{15}$$

with

$$D_0 = \mathrm{Diag}[1, -\alpha\rho, 0, 1, -\beta\rho, 0, 1, -\gamma\rho, 0],$$

$$D_1 = \mathrm{Diag}[\beta+\gamma, -\alpha(\beta+\gamma)\rho, -\rho, \gamma+\alpha, -\beta(\gamma+\alpha)\rho, -\rho, \alpha+\beta, -\gamma(\alpha+\beta)\rho, -\rho],$$

$$D_2 = \mathrm{Diag}[\beta\gamma, -\alpha\beta\gamma\rho, -(\beta+\gamma)\rho, \gamma\alpha, -\alpha\beta\gamma\rho, -(\gamma+\alpha)\rho, \alpha\beta, -\alpha\beta\gamma\rho, -(\alpha+\beta)\rho].$$

and

$$\underline{r}_3 = \frac{1}{1-\rho^2} \widetilde{I}\, V\, D_3\, \underline{c} \tag{16}$$

with

$$D_3 = -\rho\mathrm{Diag}[\beta\gamma, 0, \beta\gamma, \gamma\alpha, 0, \gamma\alpha, \alpha\beta, 0, \alpha\beta].$$

and

$$\widetilde{I} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof of Theorem 1. First, note that $p = 3$ and the matrices $\mathbf{T}_3(x)$ and $\mathbf{R}_3(\rho)$ have the forms

$$\mathbf{T}_3(x) = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{R}_3(\rho) = \begin{bmatrix} 1+\rho^2 & -\rho \\ -\rho & 1+\rho^2 \end{bmatrix}.$$

Therefore det $\mathbf{R}_3(\rho) = 1 + \rho^2 + \rho^4$,

$$\mathbf{R}_3^{-1}(\rho) = \frac{1}{1+\rho^2+\rho^4} \begin{bmatrix} 1+\rho^2 & \rho \\ \rho & 1+\rho^2 \end{bmatrix}$$

and

$$\mathrm{tr}\ (\mathbf{T}_3(x)\,\mathbf{R}_3(\rho)) = \frac{2(1+\rho^2+x\rho)}{1+\rho^2+\rho^4}.$$

Consequently the polynomial P has the form

$$P(x) = \frac{-8x^3\rho^3 - 2x(2\rho+\rho^3-2\rho^5) + 2(2+2\rho^2+2\rho^4+\rho^6)}{1+\rho^2+\rho^4}$$

Observe that

$$P\left(-\frac{x}{2\rho}\right) = \frac{W_3(x)}{1+\rho^2+\rho^4},$$

where polynomial W_3 is defined in Lemma 1. By Lemma 1 polynomial W_3 has one real root less than -2ρ and two complex roots. Therefore polynomial P has one real root outside interval $[-1,1]$ and two complex roots. Hence ASSUMPTION 1 is satisfied.

To show that ASSUMPTION 2 also holds we note that the matrix $S = S(d)$ in (6) has dimensions 12×9. Moreover, (6) is identical to (11). Now, from Lemmas 2 and 3 it follows that ASSUMPTION 2 is also satisfied.

The coefficients a_1, a_2 and a_3 solve the Vandermonde linear system

$$a_1 d_i^2 + a_2 d_i + a_3 = d_i^3, \quad i = 1,2,3.$$

Therefore

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} d_1^2 & d_1 & 1 \\ d_2^2 & d_2 & 1 \\ d_3^2 & d_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} d_1^3 \\ d_2^3 \\ d_3^3 \end{bmatrix}$$

$$= -\frac{\begin{bmatrix} d_2-d_3 & d_3-d_1 & d_1-d_2 \\ d_3^2-d_2^2 & d_1^2-d_3^2 & d_2^2-d_1^2 \\ d_2(d_2-d_3)d_3 & d_1(d_3-d_1)d_3 & d_1(d_1-d_2)d_2 \end{bmatrix} \begin{bmatrix} d_1^3 \\ d_2^3 \\ d_3^3 \end{bmatrix}}{(d_1-d_2)(d_2-d_3)(d_3-d_1)}.$$

Thus

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} d_1+d_2+d_3 \\ -d_1d_2-d_2d_3-d_3d_1 \\ d_1d_2d_3 \end{bmatrix}.$$

Denote

$$\underline{u}_1 = (1-d_1\rho,\ 1,\ -d_1\rho,\ 1-d_2\rho,\ 1,\ -d_2\rho,\ 1-d_3\rho,\ 1,\ -d_3\rho)^T,$$

$$\underline{u}_2 = (1-d_1\rho,\ 0,\ 1,\ 1-d_2\rho,\ 0,\ 1-d_3\rho,\ 0,\ 1)^T$$

and

$$\underline{v} = [\underline{v}_1^T, \underline{v}_2^T, v_3^T], \qquad \underline{w} = [\underline{w}_1^T, \underline{w}_2^T, \underline{w}_3^T],$$

where for $i = 1, 2, 3$

$$\underline{v}_i = ((1 - d_i\rho)(d_i - \rho),\, d_i(1 + \rho^2),\, -d_i^2\rho)^T,$$

$$\underline{w}_i = ((1 - d_i\rho)(d_i - \rho),\, -\rho,\, d_i(1 + \rho^2))^T.$$

Note that from (11) (its second and third row) it follows that

$$\underline{u}_i^T \underline{c} = 0, \quad i = 1, 2. \tag{17}$$

Similarly, the fourth, sixth and eighth row of (11) imply

$$\underline{v}_i^T \underline{c}_i = 0, \quad i = 1, 2, 3 \tag{18}$$

and the fifth, seventh and ninth row of (11) imply

$$\underline{w}_i^T \underline{c}_i = 0, \quad i = 1, 2, 3. \tag{19}$$

Let $\alpha = d_1$, $\beta = d_2$ and $\gamma = d_3$. Denote also $\widetilde{\alpha} = 1 - \alpha\rho$, $\widetilde{\beta} = 1 - \beta\rho$ and $\widetilde{\gamma} = 1 - \gamma\rho$. From (7) we get

$$(1 - \rho^2)V_0 = (1 - \rho^2)\Delta(D_0 - CD_1)$$

$$= \begin{bmatrix} \widetilde{\alpha} & 0 & 0 & \widetilde{\beta} & 0 & 0 & \widetilde{\gamma} & 0 & 0 \\ \widetilde{\alpha} & -\alpha\rho & 0 & \widetilde{\beta} & -\beta\rho & 0 & \widetilde{\gamma} & -\gamma\rho & 0 \\ \widetilde{\alpha} & 1 & -\alpha\rho & \widetilde{\beta} & 1 & -\beta\rho & \widetilde{\gamma} & 1 & -\gamma\rho \\ \widetilde{\alpha} & 0 & 1 & \widetilde{\beta} & 0 & 1 & \widetilde{\gamma} & 0 & 1 \\ \widetilde{\alpha} & 0 & 0 & \widetilde{\beta} & 0 & 0 & \widetilde{\gamma} & 0 & 0 \\ 1-\rho^2 & 0 & 0 & 1-\rho^2 & 0 & 0 & 1-\rho^2 & 0 & 0 \end{bmatrix}.$$

Note that the second and third rows of the above matrix are equal $\underline{u}_1^T$ and $\underline{u}_2^T$, respectively. Thus (17) implies

$$\underline{r}_0 = V_0\,\underline{c} = \frac{1}{1-\rho^2} \begin{bmatrix} \widetilde{\alpha} & 0 & 0 & \widetilde{\beta} & 0 & 0 & \widetilde{\gamma} & 0 & 0 \\ \widetilde{\alpha} & -\alpha\rho & 0 & \widetilde{\beta} & -\beta\rho & 0 & \widetilde{\gamma} & -\gamma\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\alpha} & 0 & 0 & \widetilde{\beta} & 0 & 0 & \widetilde{\gamma} & 0 & 0 \\ 1-\rho^2 & 0 & 0 & 1-\rho^2 & 0 & 0 & 1-\rho^2 & 0 & 0 \end{bmatrix} \underline{c}$$

and thus (15) for $j = 0$ holds.

To simplify formulas, denote $\widehat{\alpha} = \beta + \gamma$, $\widehat{\beta} = \alpha + \gamma$ and $\widehat{\gamma} = \alpha + \beta$. From (7) we obtain

$$(1 - \rho^2)[V_1 - (C^T + a_1 \mathrm{Id})V_0]$$

$$= - \left[\begin{array}{cccccc|} \widetilde{\alpha}\,\widehat{\alpha} & 0 & 0 & \widetilde{\beta}\,\widehat{\beta} & 0 & 0 \\ \widetilde{\alpha}(\widehat{\alpha} + \rho) & -\alpha\,\widehat{\alpha}\rho & 0 & \widetilde{\beta}(\widehat{\beta} + \rho) & -\beta\,\widehat{\beta}\rho & 0 \\ \widetilde{\alpha}(\widehat{\alpha} + \rho) & \widehat{\alpha} - \alpha\rho^2 & -\alpha\,\widehat{\alpha}\rho & \widetilde{\beta}(\widehat{\beta} + \rho) & \widehat{\beta} - \beta\rho^2 & -\beta\,\widehat{\beta}\rho \\ \widetilde{\alpha}(\widehat{\alpha} + \rho) & \rho & \widehat{\alpha} - \alpha\rho^2 & \widetilde{\beta}(\widehat{\beta} + \rho) & \rho & \widehat{\beta} - \beta\rho^2 \\ \widetilde{\alpha}(\widehat{\alpha} + \rho) & 0 & \rho & \widetilde{\beta}(\widehat{\beta} + \rho) & 0 & \rho \\ \widetilde{\alpha}\rho + \widehat{\alpha}(1 - \rho^2) & 0 & 0 & \widetilde{\beta}\rho + \widehat{\beta}(1 - \rho^2) & 0 & 0 \end{array} \right.$$

$$\left. \begin{array}{|ccc} \widetilde{\gamma}\,\widehat{\gamma} & 0 & 0 \\ \widetilde{\gamma}(\widehat{\gamma} + \rho) & -\gamma\,\widehat{\gamma}\rho & 0 \\ \widetilde{\gamma}(\widehat{\gamma} + \rho) & \widehat{\gamma} - \gamma\rho^2 & -\gamma\,\widehat{\gamma}\rho \\ \widetilde{\gamma}(\widehat{\gamma} + \rho) & \rho & \widehat{\gamma} - \gamma\rho^2 \\ \widetilde{\gamma}(\widehat{\gamma} + \rho) & 0 & \rho \\ \widetilde{\gamma}\rho + \widehat{\gamma}(1 - \rho^2) & 0 & 0 \end{array} \right]$$

Note that the second row of this matrix can be written in the form

$$(\widetilde{\alpha}\,\widehat{\alpha},\ -\alpha\rho\,\widehat{\alpha},\ -\rho,\ \widetilde{\beta}\,\widehat{\beta},\ -\beta\rho\,\widehat{\beta},\ -\rho,\ \widetilde{\gamma}\,\widehat{\gamma},\ -\gamma\rho\,\widehat{\gamma},\ -\rho) + \rho\underline{u}_2.$$

The third and fourth rows, respectively, can be written as

$$a_1\underline{u}_1 - \underline{v} \quad \text{and} \quad a_1\underline{u}_2 + \underline{w}.$$

The fifth and sixth rows, respectively, can be written as

$$(\widetilde{\alpha}\,\widehat{\alpha},\ 0,\ 0,\ \widetilde{\beta}\,\widehat{\beta},\ 0,\ 0,\ \widetilde{\gamma}\,\widehat{\gamma},\ 0,\ 0) + \rho\underline{u}_2$$

and

$$((1 - \rho^2)\widehat{\alpha},\ 0,\ -\rho,\ (1 - \rho^2)\widehat{\beta},\ 0,\ -\rho,\ (1 - \rho^2)\widehat{\gamma},\ 0,\ -\rho) + \rho\underline{u}_2.$$

Therefore, from (17), (18) and (19) we get

$$-(1 - \rho^2)\underline{r}_1$$

$$= \begin{bmatrix} \widetilde{\alpha}\,\widehat{\alpha} & 0 & 0 & \widetilde{\beta}\,\widehat{\beta} & 0 & 0 & \widetilde{\gamma}\,\widehat{\gamma} & 0 & 0 \\ \widetilde{\alpha}\,\widehat{\alpha} & -\rho\alpha\,\widehat{\alpha} & -\rho & \widetilde{\beta}\,\widehat{\beta} & -\rho\beta\,\widehat{\beta} & -\rho & \widetilde{\gamma}\,\widehat{\gamma} & -\rho\gamma\,\widehat{\gamma} & -\rho \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\alpha}\,\widehat{\alpha} & 0 & 0 & \widetilde{\beta}\,\widehat{\beta} & 0 & 0 & \widetilde{\gamma}\,\widehat{\gamma} & 0 & 0 \\ (1 - \rho^2)\widehat{\alpha} & 0 & -\rho & (1 - \rho^2)\widehat{\beta} & 0 & -\rho & (1 - \rho^2)\widehat{\gamma} & 0 & -\rho \end{bmatrix} \underline{c}.$$

Consequently, (15) holds for $j = 1$.

Now we consider the case $j = 2$. Denote additionally $\overline{\alpha} = \beta\gamma$, $\overline{\beta} = \gamma\alpha$, $\overline{\gamma} = \alpha\beta$. From (7) we get

$$(1-\rho^2)\left[V_2 - (C^T + a_1\mathrm{Id})V_1 + (a_1 C^T - a_2\mathrm{Id})V_0\right]$$

$$= \left[\begin{array}{cccccc|} \widetilde{\alpha}\,\overline{\alpha} & 0 & 0 & \widetilde{\beta}\,\overline{\beta} & 0 & 0 \\ \widetilde{\alpha}(\widehat{\alpha}\rho + \overline{\alpha}) & -a_3\rho & 0 & \widetilde{\beta}(\widehat{\beta}\rho + \overline{\beta}) & -a_3\rho & 0 \\ \widetilde{\alpha}(\widehat{\alpha}\rho + \overline{\alpha}) & \overline{\alpha} - \widehat{\alpha}\alpha\rho^2 & -a_3\rho & \widetilde{\beta}(\widehat{\beta}\rho + \overline{\beta}) & \overline{\beta} - \widehat{\beta}\beta\rho^2 & -a_3\rho \\ \widetilde{\alpha}(\widehat{\alpha}\rho + \overline{\alpha}) & \widehat{\alpha}\rho & \overline{\alpha} - \widehat{\alpha}\alpha\rho^2 & \widetilde{\beta}(\widehat{\beta}\rho + \overline{\beta}) & \widehat{\beta}\rho & \overline{\beta} - \widehat{\beta}\beta\rho^2 \\ \widetilde{\alpha}(\widehat{\alpha}\rho + \overline{\alpha}) & 0 & \widehat{\alpha}\rho & \widetilde{\beta}(\widehat{\beta}\rho + \overline{\beta}) & 0 & \widehat{\beta}\rho \\ \widetilde{\alpha}\,\widehat{\alpha}\rho + \overline{\alpha}(1-\rho^2) & 0 & 0 & \widetilde{\beta}\,\widehat{\beta}\rho + \overline{\beta}(1-\rho^2) & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{|ccc} \widetilde{\gamma}\,\overline{\gamma} & 0 & 0 \\ \widetilde{\gamma}(\widehat{\gamma}\rho + \overline{\gamma}) & -a_3\rho & 0 \\ \widetilde{\gamma}(\widehat{\gamma}\rho + \overline{\gamma}) & \overline{\gamma} - \widehat{\gamma}\gamma\rho^2 & -a_3\rho \\ \widetilde{\gamma}(\widehat{\gamma}\rho + \overline{\gamma}) & \widehat{\gamma}\rho & \overline{\gamma} - \widehat{\gamma}\gamma\rho^2 \\ \widetilde{\gamma}(\widehat{\gamma}\rho + \overline{\gamma}) & 0 & \widehat{\gamma}\rho \\ \widetilde{\gamma}\,\widehat{\gamma}\rho + \overline{\gamma}(1-\rho^2) & 0 & 0 \end{array}\right]$$

Note that the second row of this matrix can be written as

$$[\widetilde{\alpha}\,\overline{\alpha},\ -\rho a_3,\ -\rho\,\widehat{\alpha},\ \widetilde{\beta}\,\overline{\beta},\ -\rho a_3,\ -\rho\,\widehat{\beta},\ \widetilde{\gamma}\,\overline{\gamma}, -\rho a_3,\ -\rho\,\widehat{\gamma}] + \rho a_1 \underline{u}_2 - \rho\underline{w} - \rho^2\underline{u}_1$$

while the third and the fourth rows, respectively, are

$$-a_2\underline{u}_1 - [\widehat{\alpha}\underline{v}_1^T,\ \widehat{\beta}\underline{v}_2^T,\ \widehat{\gamma}\underline{v}_3^T] \quad \text{and} \quad -a_2\underline{u}_2 - [\widehat{\alpha}\underline{w}_1^T,,\ \widehat{\beta}\underline{w}_2^T,\ \widehat{\gamma}\underline{w}_3^T]$$

and the fifth and sixth row, respectively, are

$$[\widetilde{\alpha}\,\overline{\alpha},\ 0,\ 0, \widetilde{\beta}\,\overline{\beta},\ 0,\ 0, \widetilde{\gamma}\,\overline{\gamma}] + a_1\rho\underline{u}_2 - \rho\underline{w} - \rho^2\underline{u}_1$$

and

$$[(1-\rho^2)\overline{\alpha},\ 0,\ -\rho\,\widehat{\alpha},\ (1-\rho^2)\overline{\beta},\ 0,\ -\rho\,\widehat{\beta},\ (1-\rho^2)\overline{\gamma},\ 0,\ -\rho\,\widehat{\gamma}] + a_1\rho\underline{u}_2 - \rho\underline{w} - \rho^2\underline{u}_1.$$

Therefore, from (17), (18) and (19) we get

$$-(1-\rho^2)\underline{r}_2$$

$$= \begin{bmatrix} \widetilde{\alpha}\,\overline{\alpha} & 0 & 0 & \widetilde{\beta}\,\overline{\beta} & 0 & 0 & \widetilde{\gamma}\,\overline{\gamma} & 0 & 0 \\ \widetilde{\alpha}\,\overline{\alpha} & -\rho a_3 & -\rho\,\widehat{\alpha} & \widetilde{\beta}\,\overline{\beta} & -\rho a_3 & -\rho\,\widehat{\beta} & \widetilde{\gamma}\,\overline{\gamma} & -\rho a_3 & -\rho\,\widehat{\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\alpha}\,\overline{\alpha} & 0 & 0 & \widetilde{\beta}\,\overline{\beta} & 0 & 0 & \widetilde{\gamma}\,\overline{\gamma} & 0 & 0 \\ (1-\rho^2)\overline{\alpha} & 0 & -\rho\,\widehat{\alpha} & (1-\rho^2)\overline{\beta} & 0 & -\rho\,\widehat{\beta} & (1-\rho^2)\overline{\gamma} & 0 & -\rho\,\widehat{\gamma} \end{bmatrix} \underline{c}.$$

Consequently, (15) holds for $j = 2$.

Now we consider the case $j = 3$. From (7) we get

$$(1-\rho^2)\left[V_3 - (C^T + a_1\mathrm{Id})V_2 + (a_1C^T - a_2\mathrm{Id})V_1 + (a_2C^T - a_3\mathrm{Id})V_0\right]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\widetilde{\alpha}\,\overline{\alpha}\rho & 0 & 0 & -\widetilde{\beta}\,\overline{\beta}\rho & 0 & 0 & -\widetilde{\gamma}\,\overline{\gamma}\rho & 0 & 0 \\ -\widetilde{\alpha}\,\overline{\alpha}\rho & a_3\rho^2 & 0 & -\widetilde{\beta}\,\overline{\beta}\rho & a_3\rho^2 & 0 & -\widetilde{\gamma}\,\overline{\gamma}\rho & a_3\rho^2 & 0 \\ -\widetilde{\alpha}\,\overline{\alpha}\rho & -\overline{\alpha}\rho & a_3\rho^2 & -\widetilde{\beta}\,\overline{\beta}\rho & -\overline{\beta}\rho & a_3\rho^2 & -\widetilde{\gamma}\,\overline{\gamma}\rho & -\overline{\gamma}\rho & a_3\rho^2 \\ -\widetilde{\alpha}\,\overline{\alpha}\rho & 0 & -\rho\,\overline{\alpha} & -\widetilde{\beta}\,\overline{\beta}\rho & 0 & -\rho\,\overline{\beta} & -\widetilde{\gamma}\,\overline{\gamma}\rho & 0 & -\rho\,\overline{\gamma} \\ -\widetilde{\alpha}\,\overline{\alpha}\rho & 0 & 0 & -\widetilde{\beta}\,\overline{\beta}\rho & 0 & 0 & -\widetilde{\gamma}\,\overline{\gamma}\rho & 0 & 0 \end{bmatrix}$$

Note that the third and fourth row of this matrix, respectively, are

$$a_3\underline{u}_1 - [\overline{\alpha}\underline{v}_1^T,\, \overline{\beta}\underline{v}_2^T,\, \overline{\gamma}\underline{v}_3^T] \quad \text{and} \quad -\, a_3\underline{u}_2 + [\overline{\alpha}\underline{w}_1^T,\, \overline{\beta}\underline{w}_2^T,\, \overline{\gamma}\underline{w}_3^T]$$

Again, using (17), (18) and (19) we conclude that

$$\underline{r}_3 = -\frac{\rho}{1-\rho^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\alpha}\,\overline{\alpha} & 0 & 0 & \widetilde{\beta}\,\overline{\beta} & 0 & 0 & \widetilde{\gamma}\,\overline{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\alpha}\,\overline{\alpha} & 0 & \overline{\alpha} & \widetilde{\beta}\,\overline{\beta} & 0 & \overline{\beta} & \widetilde{\gamma}\,\overline{\gamma} & 0 & \overline{\gamma} \\ \widetilde{\alpha}\,\overline{\alpha} & 0 & 0 & \widetilde{\beta}\,\overline{\beta} & 0 & 0 & \widetilde{\gamma}\,\overline{\gamma} & 0 & 0 \end{bmatrix}\underline{c}.$$

Consequently, (16) holds. □

4. Proofs of lemmas

Proof of Lemma 1. The coefficients of the polynomial W_3 are positive. Therefore it is strictly increasing. Thus there is exactly one real root $x_1 = x_1(\rho) < 0$ and two complex conjugate roots $x_2 = x_2(\rho)$ and $x_3 = x_3(\rho)$, $x_2 = \overline{x}_3$. We need to show that the real root x_1 is outside of the interval $[-2|\rho|, 2|\rho|]$. Since W_3 is strictly increasing to show that $x_1 < -2|\rho|$ it suffices to prove that $W_3(-2|\rho|)$ is positive. We note that

$$W_3(-2|\rho|) = -8|\rho|^3 - (2-\rho^2+2\rho^4)|\rho| + 2(2+2\rho^2+2\rho^4+\rho^6)$$

$$= 4 - 2|\rho| + 4\rho^2 - 7|\rho|^3 + 4\rho^4 - 2|\rho|^5 + 2\rho^6$$

$$= (1-|\rho|)^2 + 2(1-|\rho|^3)^2 + (1-|\rho|^5)^2 + 3\rho^2(1-|\rho|) + \rho^4(1-\rho^6) + 3\rho^4 > 0$$

□

Proof of Lemma 2. Expanding the determinant in the equation (9) and introducing the variable

$$x = -\rho \left(d + \frac{1}{d} \right)$$

we arrive at the equivalent equation

$$W_3(x) = 0.$$

By Lemma 1 we conclude that

$$-\frac{x_i}{2\rho} = \frac{1}{2} \left(d + \frac{1}{d} \right) \notin [-1, 1], \quad i = 1, 2, 3. \tag{20}$$

That is exactly one of the two solutions of the above equation (for the variable d) is inside open unit disc and exactly one outside. In particular,

$$d_1 = \frac{-x_1 - \sqrt{x_1^2 - 4\rho^2}}{2\rho} \in \mathbb{R}. \tag{21}$$

Since x_2 and x_3 are complex conjugate, then d_2 and d_3 are also complex conjugate, since they both are in open unit disc.

By the Viete formulas for W_3 we have

$$x_1 + x_2 + x_3 = 0. \tag{22}$$

Note that if d_i is the solution we seek, then the remaining solution of (20) is $1/d_i$. Therefore (22) is equivalent to

$$d_1 + \frac{1}{d_1} + d_2 + \frac{1}{d_2} + d_3 + \frac{1}{d_3} = 0.$$

Multiplying the above identity by $d_1 d_2 d_3$ we arrive at (10). □

Proof of Lemma 3. Due to Lemma 2 the linear system

$$\widetilde{\mathbb{Q}}\, \underline{c} = (1 - \rho^2)\widetilde{e}_1 \tag{23}$$

is equivalent to (11).

We will prove that the matrix $\widetilde{\mathbb{Q}}$ is invertible by showing that its determinant is non-zero. Equivalently we consider determinant of the matrix

$$\widetilde{\mathbb{Q}}_1 =$$

$$\begin{bmatrix} i_\alpha & i_\beta & i_\gamma & j_\alpha & j_\alpha & j_\beta & j_\beta & j_\gamma & j_\gamma \\ j_\alpha & j_\beta & j_\gamma & 1 & -\rho\alpha & 1 & -\rho\beta & 1 & -\rho\gamma \\ j_\alpha & j_\beta & j_\gamma & 0 & 1 & 0 & 1 & 0 & 1 \\ k_\alpha & 0 & 0 & (1+\rho^2)\alpha & -\rho\alpha^2 & 0 & 0 & 0 & 0 \\ k_\alpha & 0 & 0 & -\rho & (1+\rho^2)\alpha & 0 & 0 & 0 & 0 \\ 0 & k_\beta & 0 & 0 & 0 & (1+\rho^2)\beta & -\rho\beta^2 & 0 & 0 \\ 0 & k_\beta & 0 & 0 & 0 & -\rho & (1+\rho^2)\beta & 0 & 0 \\ 0 & 0 & k_\gamma & 0 & 0 & 0 & 0 & (1+\rho^2)\gamma & -\rho\gamma^2 \\ 0 & 0 & k_\gamma & 0 & 0 & 0 & 0 & -\rho & (1+\rho^2)\gamma \end{bmatrix},$$

where

$$i_x = 5(1-\rho x) + 1 - \rho^2, \quad j_x = 1 - \rho x \quad \text{and} \quad k_x = j_x(x - \rho),$$

for $x = \alpha,\ \beta,\ \gamma$.

Let

$$A(d) = \frac{(1+\rho^2 - \rho(d+1/d))(1+\rho^2+\rho d)}{1+\rho^2+\rho^4} \quad \text{and} \quad B(d) = \alpha(1/d).$$

We add 4th column multiplied by $A(\alpha)$ and 5th column multiplied by $B(\beta)$ and subtract the result from the 1st column. Then we add 6th column multiplied by $A(\beta)$ and 7th column multiplied by $B(\beta)$ and subtract the result from the 2nd column. Finally, we add 8th column multiplied by $A(\gamma)$ and 9th column multiplied by $B(\gamma)$ and subtract the result from the 3rd column. All these operations do not change the absolute value of the determinant of $\widetilde{\mathbb{Q}}$ and the resulting matrix is block diagonal with the following blocks on the diagonal

$$B_0 = \begin{bmatrix} i_\alpha - j_\alpha(A(\alpha)+B(\alpha)) & i_\beta - j_\beta(A(\beta)+B(\beta)) & i_\gamma - j_\gamma(A(\gamma)+B(\gamma)) \\ j_\alpha - A(\alpha) + \rho\alpha B(\alpha) & j_\beta - A(\beta) + \rho\beta B(\beta) & j_\gamma - A(\gamma) + \rho\gamma B(\gamma) \\ j_\alpha - B(\alpha) & j_\beta - B(\beta) & j_\gamma - B(\gamma) \end{bmatrix}$$

and

$$B_i = \begin{bmatrix} (1+\rho^2)d & -\rho d^2 \\ -\rho & (1+\rho^2)d \end{bmatrix}, \quad i = 1, 2, 3.$$

It suffices to prove that det $B_i \neq 0$, $i = 0, 1, 2, 3$.

Note that

$$\det B_i = d^2[(1+\rho^2)^2 + \rho^2] > 0 \quad i = 1, 2, 3.$$

Now we consider $G(\rho) := \det B_0$. Expanding the determinant of B_0 we arrive at a "polynomial of twelfth degree" in ρ

$$G(\rho) = -4 + 2s_1\rho - 2\rho^2 + (2s_1 + s_1^3 - 6s_1s_2 + 9s_3)\rho^3 - 2(1 - s_2^2 + s_1s_3)\rho^4$$
$$+2(s_1 - s_1s_2 + 2s_3)\rho^5 + (1 + 2s_2^2 - 2s_1s_3 - 6s_3^2 - s_2^3 + 3s_1s_2s_3)\rho^6 - 2(s_1s_2 - 2s_3)\rho^7$$
$$+2(s_2^2 - s_1s_3 - s_3^2)\rho^8 + (s_1s_2 - 2s_3 + s_3^3)\rho^9 - 2s_3^2\rho^{10} + s_3^2\rho^{12}.$$

where

$$s_1 = d_1 + d_2 + d_3, \quad s_2 = d_1d_2 + d_2d_3 + d_3d_1, \quad s_3 = d_1d_2d_3. \tag{24}$$

By (10) we get

$$G(\rho) = -4 + 2s_1\rho - 2\rho^2 + (2s_1 + s_1^3 + 6s_1^2s_3 + 9s_3)\rho^3 - 2(1 - s_1^2s_3^2 + s_1s_3)\rho^4$$
$$+2(s_1 + s_1^2s_3 + 2s_3)\rho^5 + (1 - s_1^2s_3^2 - 2s_1s_3 - 6s_3^2 - s_1^3s_3^3)\rho^6 + 2(s_1^2s_3 + 2s_3)\rho^7$$
$$+2(s_1^2s_3^2 - s_1s_3 - s_3^2)\rho^8 + (-s_1^2s_3 - 2s_3 + s_3^3)\rho^9 - 2s_3^2\rho^{10} + s_3^2\rho^{12};$$

Thet Viete formulas for W_3 give

$$t_2 = x_1x_2 + x_2x_3 + x_3x_1 = 2 - \rho^2 + 2\rho^4 > 0.$$

On the other hand

$$t_2 = \rho^2\left[\left(d_1 + \frac{1}{d_1}\right)\left(d_2 + \frac{1}{d_2}\right) + \left(d_2 + \frac{1}{d_2}\right)\left(d_3 + \frac{1}{d_3}\right) + \left(d_3 + \frac{1}{d_3}\right)\left(d_1 + \frac{1}{d_1}\right)\right].$$

By (24) and (10)

$$t_2 = \rho^2\left(\frac{s_2s_3 + s_1 + s_1s_2}{s_3} - 3\right) = \rho^2\left(\frac{s_1 - s_1s_3(s_1 + s_3)}{s_3} - 3\right).$$

Thus

$$s_1 = s_3\frac{3 + s_1^2 + t_2/\rho^2}{1 - s_3^2} \tag{25}$$

and consequently

$$s_1s_3 = s_3^2\,\frac{3 + s_1^2 + t_2/\rho^2}{1 - s_3^2} > 0$$

since $t_2 > 0$ and $|s_3| = |d_1d_2d_3| < 1$.

Since the coefficients of the polynomial W_3 depend only on ρ^2 then x_1 is a function of $|\rho|$. Since $x_1 < 0$ (see Lemma 1) it follows from (21) that

$$\rho d_1(\rho) < 0$$

and thus

$$\rho s_3 = \rho d_1(\rho)d_2(\rho)d_3(\rho) = \rho d_1(\rho)|d_2(\rho)|^2 < 0.$$

Moreover

$$\rho s_1 = \frac{\rho^2 s_1 s_3}{\rho s_3} < 0.$$

Transform (25) into

$$s_1^2 s_3 - (1 - s_3^2)s_1 + s_3(t_2/\rho^2 + 3) = 0$$

leading to

$$s_1 s_3(1 - s_1 s_3) = s_1 s_3^3 + s_3^2(t_2/\rho^2 + 3) > 0.$$

That is $0 < s_1 s_3 < 1$.

Now we will use the inequalities we have just derived

$$\rho s_1 < 0, \quad \rho s_3 < 0, \quad \text{and} \quad 0 < s_1 s_3 < 1.$$

to show that $G(\rho) < 0$ for any $\rho \in (-1, 1)$. To this end we split $G(\rho)$ into several terms and show that each of these terms is negative:

$$s_1\rho < 0,$$

$$(2s_1 + s_1^3 + 6s_1^2 s_3 + 9s_3)\rho^3 < 0,$$

$$-2(1 - s_1^2 s_3^2 + s_1 s_3)\rho^4 + (1 - s_1^2 s_3^2 - 2s_1 s_3 - 6s_3^2 - s_1^3 s_3^3)\rho^6$$

$$= -\rho^4(2 - \rho^2) - 2s_1 s_3(1 - s_1 s_3)\rho^4 - (s_1^2 s_3^2 + 2s_1 s_3 + 6s_3^2 + s_1^3 s_3^3)\rho^6 < 0,$$

$$2(s_1 + s_1^2 s_3 + 2s_3)\rho^5 + (-s_1^2 s_3 - 2s_3 + s_3^3)\rho^9$$

$$= 2s_1\rho^5 + s_1^2 s_3 \rho^5(2 - \rho^4) + 2s_3\rho^5(1 - \rho^4) + s_3^3\rho^9 < 0,$$

$$2(s_1^2 s_3 + 2s_3)\rho^7 < 0,$$

$$2(s_1^2 s_3^2 - s_1 s_3 - s_3^2)\rho^8 = -2[s_1 s_3(1 - s_1 s_3) + s_3^2]\rho^8 < 0,$$

$$-2s_3^2\rho^{10} + s_3^2\rho^{12} = -s_3^2\rho^{10}(2 - \rho^2) < 0.$$

□

REFERENCES

CIEPIELA, P. (2004). Estimation of the mean in rotation schemes. MSc Thesis, Warsaw Univ. Techn., Fac. Math. Inform. Sci., Warsaw (in Polish)

KOWALSKI, J. (2009). Optimal estimation in rotation patterns. J. Statist. Plann. Infer. 139(4), 2429-2436.

KOWALSKI, J. (2010). Optimal recurrence estimation of the mean in rotation schemes. PhD Thesis, Warsaw Univ. Techn., Fac. Math. Inform. Sci., Warsaw (in Polish).

KOWALSKI, J. and WESOŁOWSKI, J. (2010). Recurrence optimal estimators for rotation cascade patterns with holes - unpublished manuscript

PATTERSON, H.D. (1955). Sampling on successive occasions. J. Roy. Statist. Soc. B 12, 241-255.

POPIŃSKI, W. (2006). Development of the Polish Labour Force Survey. Statist. Trans. 7(5), 1009-1030.

SZARKOWSKI, A. and WITKOWSKI, J. (1994). The Polish Labour Force Survey. Statist. Trans. 1(4), 467-483.