

# Renorming divergent perpetuities

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We consider a sequence of random variables  $(R_n)$  defined by the recurrence  $R_n = Q_n + M_n R_{n-1}$ ,  $n \geq 1$ , where  $R_0$  is arbitrary and  $(Q_n, M_n)$ ,  $n \geq 1$ , are i.i.d. copies of a two-dimensional random vector  $(Q, M)$ , and  $(Q_n, M_n)$  is independent of  $R_{n-1}$ . It is well known that if  $E \ln |M| < 0$  and  $E \ln^+ |Q| < \infty$ , then the sequence  $(R_n)$  converges in distribution to a random variable  $R$  given by  $R \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{j=1}^{k-1} M_j$ , and usually referred to as perpetuity. In this paper we consider a situation in which the sequence  $(R_n)$  itself does not converge. We assume that  $E \ln |M|$  exists but that it is non-negative and we ask if in this situation the sequence  $(R_n)$ , after suitable normalization, converges in distribution to a non-degenerate limit.

*Keywords:* convergence in distribution; perpetuity; stochastic difference equation

## 1. Introduction

We consider the following iterative scheme

$$R_n = Q_n + M_n R_{n-1}, \quad n \geq 1, \quad (1.1)$$

where  $R_0$  is arbitrary and  $(Q_n, M_n)$ ,  $n \geq 1$ , are i.i.d. copies of a two-dimensional random vector  $(Q, M)$ , and  $(Q_n, M_n)$  is independent of  $R_{n-1}$ . Writing out the above recurrence we see that

$$R_n = Q_n + M_n Q_{n-1} + M_n M_{n-1} Q_{n-2} + \cdots + M_n \cdots M_2 Q_1 + M_n \cdots M_1 R_0. \quad (1.2)$$

Note that although  $(R_n)$  is not a sequence of partial sums, after renumbering  $(Q_n, M_n)$  in the opposite direction, we can write

$$R_n \stackrel{d}{=} \prod_{j=1}^n M_j R_0 + \sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j, \quad (1.3)$$

where “ $\stackrel{d}{=}$ ” denotes the equality in distribution and we adopt the convention that the product or sum over the empty range is 1 or 0, respectively. Much of the impetus for studying such equations stems from numerous applications of schemes like (1.1) in mathematics and other disciplines of science. We refrain here from giving a long list of fields in which equation (1.1) appeared, referring instead to some of the references we give (most notably to [6,22] for the status up to the early nineties of the past century) for more detailed information. Examples of more recent

applications closer to statistics are given in [2], an application related to neuronal modeling may be found in [13] and, for an application in the analysis of algorithms, see [11] and references therein.

Most of the up-to-date research focused on the situation when the sequence  $(R_n)$  converges in distribution and on analyzing properties of its limit. It has been known for a long time (see [22], Lemma 1.7, or [15]) that if

$$E \ln |M| < 0 \quad \text{and} \quad E \ln^+ |Q| < \infty,$$

then the sequence of partial sums in (1.3) converges almost surely and the product in the first term asymptotically vanishes. Thus,  $(R_n)$  converges in distribution to a random variable  $R$  given by

$$R \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{j=1}^{k-1} M_j.$$

This  $R$ , referred to as perpetuity, is of central interest. In particular, its tail behavior has been thoroughly investigated (see, e.g., [7–9,14,15,22] and references therein for more information).

In this paper we focus on a situation in which the sequence  $(R_n)$  does not converge. We will refer in such situations to the whole sequence  $(R_n)$  as a divergent perpetuity. We will assume throughout that  $E \ln |M|$  exists but that

$$\mu := E \ln |M| \geq 0. \tag{1.4}$$

Under the above assumption we ask if  $(R_n)$  can be renormed to converge in distribution to a non-degenerate limit. As is the case when  $(R_n)$  converges, the roles of  $R_0$  and  $Q$  seem to be of much less importance than the role of  $M$ . In fact, since  $R_0$  plays no significant role from now on we assume that  $R_0 = 0$ . Furthermore, assuming that  $M$  is non-random would lead to  $(R_n)$  being a sequence of sums of independent random variables. Since this situation has been extensively studied, we will exclude it from our considerations by assuming from now on that  $M$  is non-constant. On the other hand, we freely impose moment conditions on  $Q$  when necessary, or sometimes even assume that it is of a special form. For example, if  $Q = 1$ , then (1.3) is just a partial sum of successive partial products of i.i.d. copies of  $M$ . While there is very little known technical connection, it is perhaps worth mentioning that an analogous problem of investigating the asymptotic properties of the consecutive products of partial sums

$$\prod_{k=1}^n \sum_{j=1}^k M_j \quad \text{vs.} \quad \sum_{k=1}^n \prod_{j=1}^k M_j,$$

has been intensely investigated in the past several years; see, for example, [19,21,23].

As  $x \rightarrow \ln x$  is concave, assumption (1.4) implies that necessarily  $E |M| \geq 1$ . (We do not assume, however, that  $E |M|$  is finite.) Thus there are at least four situations to consider, namely,

- (i)  $E \ln |M| > 0, E |M| > 1, |M|$  non-random,
- (ii)  $E \ln |M| > 0, E |M| > 1, |M|$  random,

- (iii)  $E \ln |M| = 0$  and  $E |M| > 1$ ,
- (iv)  $E \ln |M| = 0$  and  $E |M| = 1$ .

These cases are discussed in detail below and we will show that in each of these situations  $(R_n)$  can be renormed (each time differently) so that it converges in distribution to a non-degenerate limit. As was brought to our attention by A. Iksanov, some particular cases of (ii) and (iii) were studied by Rachev and Samorodnitsky [20] who, for non-negative  $Q$  and  $M$  and under the assumption that  $\log M_n$  belongs to the  $\alpha$ -stable domain of attraction, obtained the log-stable limit law for suitably normalized  $(R_n)$  – see also comments on this connection in Sections 3 and 4. Furthermore, weak convergence in the situation complementary to (1.4) is considered in [17] where, for non-negative  $Q$  and  $M$ , it is assumed that  $-\infty < \ln M < 0$ , but the perpetuity is divergent because  $E \ln^+ Q = \infty$ .

## 2. The case $E \ln |M| > 0, E |M| > 1$ and non-random $|M|$

In this section we consider the following situation: for a fixed  $\rho > 1$  let

$$M \stackrel{d}{=} \rho \varepsilon, \quad \text{where } P(\varepsilon = 1) = p, \quad P(\varepsilon = -1) = 1 - p = q.$$

**Theorem 1.** *Let  $M$  be as above. Assume that  $Q \stackrel{d}{=} \varepsilon$  and that it is independent of  $M$ .*

- (i) *Symmetric case: Let  $p = 1/2$ . Then*

$$\frac{R_n}{\rho^{n-1}} \xrightarrow{d} \sum_{k=1}^{\infty} \lambda^{k-1} \varepsilon_k,$$

where  $\lambda = \rho^{-1} < 1$  and  $(\varepsilon_k)$  is the sequence of i.i.d. copies of  $\varepsilon$ .

- (ii) *Asymmetric case: Suppose  $p \neq 1/2$ . Then we have*

$$\frac{R_n}{\rho^{n-1}} \xrightarrow{d} r X, \tag{2.1}$$

where  $r$  is a symmetric Bernoulli random variable (i.e.,  $P(r = 1) = P(r = -1) = 1/2$ ),  $X \stackrel{d}{=} \sum_{k=0}^{\infty} \lambda^k \prod_{j=1}^k \varepsilon_j$  and  $r$  and  $X$  are independent.

**Remark.** *The sums of the form  $\sum_{k=1}^{\infty} \lambda^{k-1} \varepsilon_k$  considered in part (i) are well-known objects, usually referred to as “symmetric Bernoulli convolutions”. Their properties have been extensively studied since mid-1930s. In particular, it is known that the limiting distribution is uniform on  $[-2, 2]$  if  $\lambda = 1/2$ , singular if  $0 < \lambda < 1/2$  and absolutely continuous for almost all (but not all)  $\lambda \in (1/2, 1)$ . A good description of the current state of knowledge can be found, for example, in [18].*

**Proof of Theorem 1.** We have

$$R_n = \varepsilon_n + \rho\varepsilon_n\varepsilon_{n-1} + \dots + \rho^{n-1} \prod_{j=1}^n \varepsilon_j \stackrel{d}{=} \sum_{k=1}^n \rho^{k-1} \prod_{j=1}^k \varepsilon_j. \tag{2.2}$$

If  $p = 1/2$  then the sequences

$$(\varepsilon_1, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_2\varepsilon_3, \dots) \quad \text{and} \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) \tag{2.3}$$

are identically distributed. Thus, if we normalize  $R_n$  by  $\rho^{n-1}$ , we get

$$\frac{R_n}{\rho^{n-1}} \stackrel{d}{=} \sum_{k=1}^n \rho^{k-n} \varepsilon_k \stackrel{d}{=} \sum_{k=1}^n \lambda^{k-1} \varepsilon_k.$$

Since  $0 < \lambda < 1$  the series of partial sums converges almost surely and thus  $R_n/\rho^{n-1}$  converges in distribution to the given limit.

If  $p \neq 1/2$ , then the distributional equality in (2.3) is no longer valid. However, we can write the right-hand side of (2.2) as

$$R_n \stackrel{d}{=} \sum_{k=1}^n \rho^{k-1} \prod_{j=1}^k \varepsilon_j = \rho^{n-1} \prod_{j=1}^n \varepsilon_j \left( \sum_{k=1}^n \frac{\lambda^{k-1}}{\prod_{j=1}^{k-1} \varepsilon_j} \right).$$

Since  $\varepsilon_k = 1/\varepsilon_k$  we get

$$\frac{R_n}{\rho^{n-1}} \stackrel{d}{=} T_n \prod_{j=1}^n \varepsilon_j, \tag{2.4}$$

where

$$T_\ell = \sum_{k=1}^{\ell} \lambda^{k-1} \prod_{j=1}^{k-1} \varepsilon_j.$$

Fix arbitrary  $m$ . Then, for any  $n > m$ , write the term on the right-hand side of (2.4) as

$$T_m \prod_{j=1}^m \varepsilon_j + \prod_{j=1}^m \varepsilon_j \left( \sum_{k=m+1}^n \lambda^{k-1} \prod_{j=1}^{k-1} \varepsilon_j \right).$$

The second summand is bounded in absolute value by  $\lambda^m/(1 - \lambda)$  and thus it can be made arbitrarily small by choosing  $m$  sufficiently large. Consequently, we consider only the first part of the above expression. Note that  $T_m$  depends on  $(\varepsilon_1, \dots, \varepsilon_m)$  only. Therefore upon conditioning on  $(\varepsilon_1, \dots, \varepsilon_m)$  we obtain

$$E(e^{itT_m} \prod_{j=1}^n \varepsilon_j) = E \left[ \phi_{n-m} \left( t T_m \prod_{j=1}^m \varepsilon_j \right) \right],$$

where  $\phi_k$  is the characteristic function of the product  $\prod_{j=1}^k \varepsilon_j$ . Since  $\prod_{j=1}^n \varepsilon_j = 1$  if and only if the number of  $j$ 's such that  $\varepsilon_j = -1$  is even, we have

$$\begin{aligned} P\left(\prod_{j=1}^n \varepsilon_j = 1\right) - P\left(\prod_{j=1}^n \varepsilon_j = -1\right) &= \sum_{\substack{k=0 \\ k\text{-even}}}^n \binom{n}{k} q^k p^{n-k} - \sum_{\substack{k=0 \\ k\text{-odd}}}^n \binom{n}{k} q^k p^{n-k} \\ &= (p - q)^n, \end{aligned} \tag{2.5}$$

which vanishes as  $n \rightarrow \infty$ . Thus the product  $\prod_{j=1}^n \varepsilon_j$  converges in distribution to  $r$  (recall that its characteristic function is  $\cos(t)$ ). Therefore for arbitrary fixed  $m$

$$E(e^{itT_m \prod_{j=1}^n \varepsilon_j}) \xrightarrow{n \rightarrow \infty} E \cos\left(tT_m \prod_{j=1}^m \varepsilon_j\right) = E \cos(tT_m).$$

As  $m \rightarrow \infty$  the sum  $T_m$  converges almost surely to  $X$ . By the Lebesgue dominated convergence it follows therefore that

$$E \cos(tT_m) \xrightarrow{m \rightarrow \infty} E \cos(tX),$$

the latter being the characteristic function of  $rX$ , with  $r$  and  $X$  independent. This proves (2.1). □

### 3. The case $E \ln|M| > 0$ , $E|M| > 1$ and random $|M|$

In this section we assume that  $\mu = E \ln|M| > 0$  and that  $|M|$  is random. This forces  $E|M| > 1$ , but it may be infinite. Set  $v^2 := \text{var}(\ln|M|)$ . Then we have

**Theorem 2.** *Let  $(R_n)$  be given by (1.3) with the pair  $(Q, M)$  satisfying  $\mu > 0$ ,  $0 < v^2 < \infty$ , and  $E \ln^+|Q| < \infty$ .*

(i) *As  $n \rightarrow \infty$ ,*

$$\frac{|R_n|^{1/(v\sqrt{n})}}{\exp(\mu\sqrt{n}/v)} \xrightarrow{d} e^{\mathcal{N}},$$

*where  $\mathcal{N}$  is a standard normal random variable.*

(ii) *Assume  $P(M > 0) > 0$  and  $P(M < 0) > 0$  and define for real  $t$  and  $x$ ,  $x^t := \text{sgn}(x)|x|^t$ . Then we have*

$$\frac{R_n^{1/(v\sqrt{n})}}{\exp(\mu\sqrt{n}/v)} \xrightarrow{d} r e^{\mathcal{N}},$$

*where  $r$  is a symmetric Bernoulli random variable independent of  $\mathcal{N}$ .*

**Remark.** The first part of this theorem overlaps with Theorem 2.1(a) in [20] where, for non-negative  $Q$  and  $M$  and under the assumption that  $\ln M$  is in the domain of attraction of an  $\alpha$ -stable law,  $1 < \alpha \leq 2$ , the authors obtained (i) with  $\mathcal{N}$  replaced by an  $\alpha$ -stable random variable and suitable normalization of  $(R_n)$  on the left-hand side (the normalizing constants are the constants implied by the definition of a domain of attraction).

**Proof of Theorem 2.** Consider  $R_n$  given in (1.2) and factor the product of  $M_j$ 's to write it as

$$R_n = \left( \prod_{j=1}^n M_j \right) \sum_{k=1}^n Q_k \prod_{j=1}^k \frac{1}{M_j}. \tag{3.1}$$

Consider the first factor. By the classical CLT

$$\frac{|\prod_{j=1}^n M_j|^{1/(v\sqrt{n})}}{\exp((\mu/v)\sqrt{n})} = \left( \frac{\prod_{j=1}^n |M_j|}{e^{\mu n}} \right)^{1/(v\sqrt{n})} = \exp \left\{ \frac{\sum_{j=1}^n \ln |M_j| - n\mu}{v\sqrt{n}} \right\} \xrightarrow{d} e^{\mathcal{N}}.$$

To finish the proof by Slutsky's theorem it suffices to show the second factor on the right-hand side of (3.1) converges to 1 in distribution. To this end note that

$$S_n := \sum_{k=1}^n Q_k \prod_{j=1}^k \frac{1}{M_j} = \sum_{k=1}^n \frac{Q_k}{M_k} \prod_{j=1}^{k-1} \frac{1}{M_j} \tag{3.2}$$

is a perpetuity generated by  $(Q/M, 1/M)$ . Since we are working under the condition  $E \ln |M| > 0$ , we have  $E \ln |1/M| = -E \ln |M| < 0$ . Furthermore, by our assumption on  $Q$ ,  $E \ln^+ |Q/M| < \infty$  and thus  $(S_n)$  converges in distribution to, say  $S$  (see [22], Theorem 1.6(b)). Moreover,  $P(1/M = 0) = 0$  and hence by Theorem 1.3 of [1] (see also [10] and [3], Lemma 2.1) it follows that the distribution of  $S$  is continuous. In particular,  $|S|^{1/v}$  does not have an atom at zero, which is all that is important for our purposes. Denote by  $\nu_n$  and  $\nu$  the distributions of  $|S_n|^{1/v}$  and  $|S|^{1/v}$ , respectively. We want to show that  $|S_n|^{1/(v\sqrt{n})} \xrightarrow{d} 1$ . Consider first an arbitrary  $x \in (0, 1)$ , fix an arbitrary  $m$  and take any  $n > m$ . Then

$$P(|S_n|^{1/(v\sqrt{n})} \leq x) = \nu_n([0, x\sqrt{n}]) \leq \nu_n([0, x\sqrt{m}]) \xrightarrow{n \rightarrow \infty} \nu([0, x\sqrt{m}]).$$

Letting now  $m \rightarrow \infty$  we conclude that  $P(|S_n|^{1/(v\sqrt{n})} \leq x) \rightarrow 0$  for  $x \in (0, 1)$ .

Now we take an arbitrary  $x \geq 1$ . Then again we fix some  $m$ . For  $n > m$  we obtain

$$P(|S_n|^{1/(v\sqrt{n})} \leq x) = \nu_n([0, x\sqrt{n}]) \geq \nu_n([0, x\sqrt{m}]) \xrightarrow{n \rightarrow \infty} \nu([0, x\sqrt{m}]) \xrightarrow{m \rightarrow \infty} 1$$

so that  $P(|S_n|^{1/(v\sqrt{n})} \leq x) \rightarrow 1$  for  $x \geq 1$  and it follows that  $|S_n|^{1/(v\sqrt{n})} \xrightarrow{d} 1$ , which completes the proof of part (i).

To prove part (ii) let  $\varepsilon_j = \text{sgn}(M_j)$  and consider again

$$R_n = \left( \prod_{j=1}^n M_j \right) S_n,$$

where  $(S_n)$  is defined by (3.2). By definition of  $x^t$ ,

$$Z_n := \frac{R_n^{1/(v\sqrt{n})}}{\exp(\mu\sqrt{n}/v)} = \text{sgn}(S_n)|S_n|^{1/(v\sqrt{n})} \left( \prod_{j=1}^n \varepsilon_j \right) \exp\left( \sum_{j=1}^n \frac{\ln|M_j| - \mu}{v\sqrt{n}} \right). \tag{3.3}$$

For any  $m < n$  write the sum in the exponent of (3.3) as

$$\sum_{j=1}^m \frac{\ln|M_j| - \mu}{v\sqrt{m}} + \sum_{j=m+1}^n \frac{\ln|M_j| - \mu}{v\sqrt{n}} + \left( \sqrt{\frac{m}{n}} - 1 \right) \sum_{j=1}^m \frac{\ln|M_j| - \mu}{v\sqrt{m}}.$$

Splitting the product of signs on the right-hand side of (3.3) in two factors, and using the above equation, we see that (3.3) can be written as

$$Z_n = \left( \prod_{j=m+1}^n \varepsilon_j \right) Z_m V_{n,m}, \tag{3.4}$$

where

$$V_{n,m} := \frac{\text{sgn}(S_n) |S_n|^{1/(v\sqrt{n})}}{\text{sgn}(S_m) |S_m|^{1/(v\sqrt{m})}} \exp\left( \sum_{j=m+1}^n \frac{\ln|M_j| - \mu}{v\sqrt{n}} + \left( \sqrt{\frac{m}{n}} - 1 \right) \sum_{j=1}^m \frac{\ln|M_j| - \mu}{v\sqrt{m}} \right).$$

We claim that  $(V_{n,m})$  converges in probability to 1 as long as  $n, m \rightarrow \infty$  in such a way that  $n - m = o(n)$ . To this end, consider the first sum in the exponent above. Its variance is  $(n - m)/n$ , and thus as long as  $n - m = o(n)$  it goes to 0 in probability by Chebyshev’s inequality. Furthermore, under the same condition on  $n - m$ ,  $\sqrt{\frac{m}{n}} - 1 = \sqrt{1 - \frac{o(n)}{n}} - 1 = o(1)$  and thus the second term in the exponent above goes to 0 in probability as well. (Note that the sum of  $\frac{\ln|M_j| - \mu}{v\sqrt{m}}$  converges in distribution to  $\mathcal{N}$  by the classical CLT.)

As for the other factors in  $V_{n,m}$ , just as in part (i),  $|S_k|^{1/(v\sqrt{k})} \xrightarrow{P} 1$  as  $k \rightarrow \infty$ , where “ $\xrightarrow{P}$ ” denotes the convergence in probability. It remains to show that  $\text{sgn}(S_n)/\text{sgn}(S_m) \xrightarrow{P} 1$ . To this end, note that the equation (3.2) defining  $(S_n)$  holds a.s. and not only in distribution and thus the sequence  $(S_n)$  is a sequence of partial sums. Hence, by the basic convergence result for perpetuities (see, e.g., [22], Lemma 1.7), the conditions  $E\ln|1/M| < 0$  and  $E\ln^+|Q/M| < \infty$  suffice for the a.s. convergence of the series

$$\sum_{k=1}^{\infty} \frac{Q_k}{M_k} \prod_{j=1}^{k-1} \frac{1}{M_j}.$$

This, together with the fact that the limit of  $(S_m)$  has a continuous distribution function (see an argument following (3.2)), clearly implies that  $\text{sgn}(S_n)/\text{sgn}(S_m) \xrightarrow{P} 1$  (and a.s.) as  $m, n \rightarrow \infty$ . This proves our claim about  $(V_{n,m})$ .

We now go back to (3.4) and note that the first two random variables on the right-hand side are independent. Furthermore, the product  $\prod_{j=m+1}^n \varepsilon_j$  converges in distribution to a symmetric Bernoulli random variable  $r$  (see (2.5) above) as long as  $n - m \rightarrow \infty$ . Hence, if the sequence (3.3) converges in distribution, then its limit, say  $Z$ , has to satisfy the distributional identity  $Z \stackrel{d}{=} rZ$ , with  $r, Z$  independent on the right-hand side. This would complete the proof since any such  $Z$  is symmetric and we would have

$$Z \stackrel{d}{=} r|Z| \stackrel{d}{=} re^{\mathcal{N}}, \tag{3.5}$$

where the second equality follows by part (i).

Thus, it remains only to prove that the sequence given in (3.3) does, in fact, converge in distribution. We note that this sequence is tight because the sequence of absolute values

$$|Z_n| = \frac{\prod_{j=1}^n |M_j|}{\exp(\mu\sqrt{n}/v)} |S_n|^{1/(v\sqrt{n})}$$

converges in distribution to  $e^{\mathcal{N}}$  by part (i). We will complete the proof by showing that every converging subsequence of (3.3) converges to the same limit, namely  $re^{\mathcal{N}}$ . Let  $(k_n)$  be any subsequence for which  $(Z_{k_n})$  converges in distribution. Define

$$\ell_n = \max\{m : k_m \leq \sqrt{k_n}\} \quad \text{and set} \quad m_n = k_n - k_{\ell_n}.$$

By tightness we can choose a subsequence  $(m_{n_j})$  of  $(m_n)$  for which  $(Z_{m_{n_j}})$  converges in distribution. Since  $m_{n_j} = k_{n_j} - k_{\ell_{n_j}}$  we have

$$Z_{k_{n_j}} = \left( \prod_{i=m_{n_j}+1}^{k_{n_j}} \varepsilon_i \right) Z_{m_{n_j}} V_{k_{n_j}, m_{n_j}}.$$

It is readily seen from the construction that  $k_{n_j} - m_{n_j} = o(k_{n_j})$  and that it converges to infinity. Therefore,  $V_{k_{n_j}, m_{n_j}} \xrightarrow{P} 1$  as  $j \rightarrow \infty$ . Thus, if  $Z$  and  $\bar{Z}$  are the limits in distribution of  $(Z_{k_n})$  and  $(Z_{m_{n_j}})$ , respectively, then the above identity implies that  $Z \stackrel{d}{=} r\bar{Z}$ , where  $r$  and  $\bar{Z}$  are independent. It follows that  $Z$  is symmetric and thus must satisfy (3.5). Since  $Z$  was a limit along the arbitrary converging subsequence of  $(Z_n)$ , the proof is complete.  $\square$

### 4. The case $E\ln|M| = 0$ and $E|M| > 1$

To see the difference between the current situation and the preceding one, note that because  $E\ln|M| = 0$  the perpetuity defined by (3.2) in the course of the proof of Theorem 2 is not guaranteed to converge. Specifically, consider the following example: let  $X$  be a non-degenerate, integrable, symmetric random variable and set  $M = e^X$ . Then,

$$E\ln|M| = EX = 0 \quad \text{and} \quad E|M| > e^{EX} = 1,$$



where the strict inequality follows from the non-degeneracy of  $X$ . By the symmetry of  $X$ ,  $M$  and  $1/M$  have *exactly* the same distribution. Hence, if we take  $Q = 1$  in (1.3) and (3.2) we find that  $R_n \stackrel{d}{=} S_n$  for  $n \geq 1$ . Therefore,  $(R_n)$  and  $(S_n)$  converge or diverge in distribution simultaneously. If  $M$  is non-negative this difficulty can be handled by factoring the largest product in (1.3) rather than the last one; however, the limiting distribution will no longer be lognormal. It seems very likely that the case of general  $M$  is similar, however it remains open.

Before stating our result in the non-negative case, recall that a function  $h : \mathbf{R} \rightarrow \mathbf{R}$  is regularly varying at infinity with index  $\rho$  if for every  $x > 0$

$$\lim_{t \rightarrow \infty} \frac{h(xt)}{h(t)} = x^\rho.$$

Recall also that any such function can be written as  $h(t) = t^\rho \ell(t)$ , where  $\ell(t)$  is slowly varying at infinity; that is, for every  $x > 0$

$$\lim_{t \rightarrow \infty} \frac{\ell(xt)}{\ell(t)} = 1.$$

It follows from that definition in particular that for any  $\delta > 0$ ,  $t^\delta \ell(t) \rightarrow \infty$  as  $t \rightarrow \infty$  (see [5] for these and much more on regularly varying functions). The following holds true:

**Theorem 3.** *Let  $(Q, M)$  be given by  $(e^Y, e^X)$ , where  $EX = 0$  and  $v^2 := \text{var}(X)$  satisfies  $0 < v^2 < \infty$  (so that  $E \ln M = 0$ ;  $EM > 1$ , but may be infinite; and  $v^2 = \text{var}(\ln M) < \infty$ ).*

(i) *If  $EY^2 < \infty$ , then, as  $n \rightarrow \infty$ ,*

$$R_n^{1/(v\sqrt{n})} \xrightarrow{d} e^{|\mathcal{N}|}, \tag{4.1}$$

where  $|\mathcal{N}|$ , the absolute value of the standard normal random variable, has distribution given by

$$P(|\mathcal{N}| \leq x) = 2\Phi(x) - 1 = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} dt.$$

(ii) *Let  $h(t) := P(Y > t)$  be a tail function of a random variable  $Y$  and define a sequence  $(\gamma_n)$  by*

$$\gamma_n := \inf \left\{ t : h(t) \leq \frac{1}{n} \right\}, \quad n \geq 1.$$

*If  $h(t)$  is regularly varying at infinity with index  $\alpha$  for some  $-2 < \alpha < 0$  then, as  $n \rightarrow \infty$*

$$R_n^{1/\gamma_n} \xrightarrow{d} e^{V_\alpha}, \tag{4.2}$$

where  $V_\alpha$  has Fréchet distribution  $\Phi_\alpha$  given by

$$P(V_\alpha \leq x) = \Phi_\alpha(x) = \begin{cases} 0, & x \leq 0; \\ \exp(-x^\alpha), & x > 0. \end{cases}$$

- (iii) Assume that  $h(t)$  is regularly varying at infinity with index  $\alpha = -2$ , that is, that  $h(t) = t^{-2}\ell(t)$ , where  $\ell(t)$  is slowly varying at infinity. If  $\lim_{t \rightarrow \infty} \ell(t) = \infty$ , then (4.2) holds, while if  $\lim_{t \rightarrow \infty} \ell(t) = 0$ , then (4.1) holds.

**Remarks.** (1) The only case not covered by the above theorem is  $\alpha = -2$  and  $\ell(t) \sim \text{const}$ . We suspect that in that case, at least when  $X$  and  $Y$  are independent, we have

$$\frac{\max_{1 \leq k \leq n} \{Y_k + S_{k-1}\}}{\sqrt{n}} \xrightarrow{d} V_\alpha + \sigma \mathcal{N},$$

where  $V_\alpha$  is a Fréchet random variable with parameter  $\alpha = -2$ ,  $\sigma$  is the variance of  $X$  and  $V_\alpha$  and  $\mathcal{N}$  are independent, but we have not managed to prove it.

(2) Part (i) of Theorem 3 overlaps with Theorem 2.1(c) of [20], where the authors worked under the assumption that  $X$  is in the domain of attraction of an  $\alpha$ -stable law,  $1 < \alpha \leq 2$ , but assumed additionally that  $X$  and  $Y$  are independent. Under a suitable normalization that agrees with ours for  $\alpha = 2$  and results in a slightly weaker condition on  $Y$  than  $EY^2 < \infty$ , they obtained as a limit law the distribution of a supremum of the Lévy  $\alpha$ -stable motion on  $[0, 1]$ . A similar comment applies to the second part of (iii); we assume the regular variation of the tail of  $Y$  but not independence of  $X$  and  $Y$  as was assumed in [20], Theorem 2.1(c). When  $\alpha = 2$ , the assumptions about the decay of the tail of  $Y$  in both papers are identical.

**Proof of Theorem 3.** Set

$$W_n := \max_{1 \leq k \leq n} Q_k \prod_{j=1}^{k-1} M_j,$$

and write

$$R_n^{1/(v\sqrt{n})} = W_n^{1/(v\sqrt{n})} \left( \sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{1/(v\sqrt{n})}. \tag{4.3}$$

Since  $x \rightarrow \ln x$  is increasing,

$$W_n^{1/(v\sqrt{n})} = \exp \left\{ \frac{1}{v\sqrt{n}} \ln \left( \max_{1 \leq k \leq n} Q_k \prod_{j=1}^{k-1} M_j \right) \right\} = \exp \left\{ \frac{\max_{1 \leq k \leq n} \{ \ln Q_k + \sum_{j=1}^{k-1} \ln M_j \}}{v\sqrt{n}} \right\}.$$

We let  $Y_k = \ln Q_k$ ,  $X_k = \ln M_k$ ,  $S_k = \sum_{j=1}^k X_j$  and, for any sequence of random variables  $(Z_k)$ , we will write  $Z_m^* = \max_{1 \leq k \leq m} Z_k$ .

Subadditivity of the maxima implies that for any numerical sequences  $(u_k)$ ,  $(w_k)$ ,

$$\max_{1 \leq k \leq n} \{u_k\} - \max_{1 \leq k \leq n} \{|w_k|\} \leq \max_{1 \leq k \leq n} \{u_k + w_k\} \leq \max_{1 \leq k \leq n} \{u_k\} + \max_{1 \leq k \leq n} \{w_k\}. \tag{4.4}$$

To prove part (i) we note that

$$\frac{|Y_n|^*}{\sqrt{n}} \xrightarrow{P} 0.$$

Indeed, for  $\varepsilon > 0$  we have

$$P\left(\frac{|Y_n|^*}{\sqrt{n}} > \varepsilon\right) = 1 - (1 - P(|Y| > \varepsilon\sqrt{n}))^n \leq nP(Y^2 > \varepsilon^2 n) \rightarrow 0, \quad n \rightarrow \infty,$$

where the last assertion follows from  $EY^2 < \infty$ . Using this and (4.4) with  $u_k = S_{k-1}$  and  $w_k = Y_k$  we obtain that

$$\frac{\max_{1 \leq k \leq n} \{Y_k + S_{k-1}\}}{v\sqrt{n}} = \frac{S_{n-1}^*}{v\sqrt{n}} + o_P(1),$$

where  $o_P(1)$  denotes a quantity that goes to zero in probability. Furthermore, our assumptions on  $M_j$ 's imply that  $(\sum_{j=1}^k X_j)$  is a sequence of partial sums of random walk whose increments  $X_j$  have mean zero and a finite variance  $v^2$ . Thus, the Erdős–Kac theorem for the maxima of random walks (see, e.g., [12], Theorem 12.2) implies that

$$W_n^{1/(v\sqrt{nv})} \xrightarrow{d} e^{|\mathcal{N}|}.$$

Just as in the proof of Theorem 2, to complete the proof we need to show that the second factor in (4.3) converges to 1 in distribution. But that is clear since, on the one hand,

$$\left(\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n}\right)^{1/(v\sqrt{n})} \leq n^{1/(v\sqrt{n})} = \exp\left(\frac{\ln n}{v\sqrt{n}}\right) \rightarrow 1$$

and, on the other hand, we clearly have

$$\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \geq 1.$$

This implies that

$$\sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \xrightarrow{P} 1,$$

and proves part (i).

The argument for the second part is parallel to the one just given with the following adjustment: In the first part, the assumption  $EY^2 < \infty$  ensures that the maximum of the random walk with increments  $X_j$ ,  $j < n$ , dominates the maximum of  $\{Y_k, k \leq n\}$ . If this assumption is weakened this may no longer be true, and, in fact, the maximum of  $Y_k$ 's may dominate. In that case, we can just use the basics of extreme value theory (Chapter 1 of [16] being more than enough) instead of the Erdős–Kac theorem to complete the argument. This time, using (4.4) we write

$$Y_n^* - |S_{n-1}|^* \leq \max_{1 \leq k \leq n} \left\{ Y_k + \sum_{j=1}^{k-1} X_j \right\} \leq Y_n^* + S_{n-1}^*.$$

By the characterization theorem in the extreme value theory (see, e.g., [16], Theorem 1.6.2), our assumption on  $Y$  is a necessary and sufficient condition for the existence of constants  $(a_n), (b_n)$  for which

$$a_n(Y_n^* - b_n) \xrightarrow{d} V_\alpha,$$

where  $V_\alpha$  has the Fréchet distribution (also referred to as a type-II extreme value distribution) described above. Furthermore (see [16], Corollary 1.6.3), we may take  $a_n = 1/\gamma_n$  and  $b_n = 0$ ,  $n \geq 1$ , and, if we do, we obtain that

$$\frac{Y_n^*}{\gamma_n} \xrightarrow{d} V_\alpha, \quad n \rightarrow \infty.$$

We now observe that  $\gamma_n/\sqrt{n} \rightarrow \infty$ . In fact, there exists  $\beta > 1/2$  such that  $\gamma_n \geq n^\beta$  for all sufficiently large  $n$ . Indeed, since  $h$  is decreasing it is enough to see that  $h(n^\beta) > 1/n$ . But as  $h$  is regularly varying, we have

$$h(n^\beta) = n^{\beta\alpha} \ell(n^\beta) = \frac{n^{1+\beta\alpha} \ell(n^\beta)}{n} = \frac{(n^\beta)^{1/\beta+\alpha} \ell(n^\beta)}{n} = \frac{(n^\beta)^\delta \ell(n^\beta)}{n}.$$

If we now take  $1/2 < \beta < -1/\alpha$  (which is possible since  $-2 < \alpha < 0$ ) then  $\delta > 0$  and the numerator on the right-hand side goes to infinity with  $n$ , proving the claim that  $h(n^\beta) > 1/n$  for large  $n$ . We now get

$$\frac{|S_{n-1}|^*}{\gamma_n} = \frac{|S_{n-1}|^* \sqrt{n}}{\sqrt{n} \gamma_n} \xrightarrow{P} 0,$$

and also, since  $\gamma_n/\sqrt{n} \rightarrow \infty$ ,

$$1 \leq \left( \sum_{k=1}^n \frac{Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{1/\gamma_n} \leq n^{1/\gamma_n} \rightarrow 1,$$

which proves the second part.

Finally, the last part follows by essentially the same reasoning. Assume  $\ell(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . By the just-given argument, to establish (4.2), it suffices to verify  $\gamma_n/\sqrt{n} \rightarrow \infty$  as  $n \rightarrow \infty$ . Assume this is not the case. Then there exist  $C < \infty$  and an infinite subsequence  $(n_k)$  such that  $\gamma_{n_k}/\sqrt{n_k} \leq C$  for all  $k \geq 1$ . By the definition of  $(\gamma_n)$  this means that  $P(Y \geq C\sqrt{n_k}) \leq 1/n_k$ , that is, that  $\ell(C\sqrt{n_k}) \leq C^2$ . But that contradicts the assumption that  $\ell(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . If, on the other hand,  $\ell(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then  $\gamma_n/\sqrt{n} \rightarrow 0$  (for otherwise there would exist  $c > 0$  and a subsequence  $(n_k)$  such that  $\gamma_{n_k}/\sqrt{n_k} \geq c$ ,  $k \geq 1$ , implying that  $\ell(c\sqrt{n_k}) \geq c^2$  and contradicting  $\ell(t) \rightarrow 0$  as  $t \rightarrow \infty$ ). Now, it follows from the proof of the first part that  $\gamma_n/\sqrt{n} \rightarrow 0$  is enough to conclude (4.1). The proof of part (iii) is completed.  $\square$

### 5. The case $E \ln |M| = 0$ and $E |M| = 1$

Under this assumption we have that  $|M| \equiv 1$ , that is,  $M$  takes on values  $\pm 1$ . Consequently, because of a non-degeneracy assumption on  $M$ , its expected value must satisfy  $-1 < EM < 1$ . We have

**Theorem 4.** *Suppose that  $EQ^2 < \infty$ . Then, as  $n \rightarrow \infty$ ,*

$$\frac{R_n}{\sqrt{n}} \xrightarrow{d} \beta \mathcal{N}, \tag{5.1}$$

where  $\beta^2 = EQ^2 + 2 \frac{EQ}{1-EM} E(QM)$  and  $\mathcal{N}$  is the standard normal random variable.

**Remark.** *Since  $-1 < EM < 1$ , by straightforward calculation we see that  $ER_n = O(1)$  and  $\text{var}(R_n) = \beta^2 n + O(1)$ . Thus, (5.1) is equivalent to*

$$\frac{R_n - ER_n}{\sqrt{\text{var}(R_n)}} \xrightarrow{d} \mathcal{N}.$$

**Proof.** Let  $\alpha := EM$  and  $q := EQ$  so that

$$\beta^2 = EQ^2 + 2 \frac{q}{1-\alpha} EQM = \text{var} \left( Q + \frac{q}{1-\alpha} M \right).$$

To prove (5.1) we write

$$R_n = \sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j = \sum_{k=1}^n (Q_k - q) \prod_{j=1}^{k-1} M_j + q \sum_{k=1}^n \prod_{j=1}^{k-1} M_j. \tag{5.2}$$

Furthermore, as

$$\sum_{k=1}^n (M_k - \alpha) \prod_{j=1}^{k-1} M_j = -\alpha + \prod_{j=1}^n M_j + (1-\alpha) \sum_{k=2}^n \prod_{j=1}^{k-1} M_j = -1 + \prod_{j=1}^n M_j + (1-\alpha) \sum_{k=1}^n \prod_{j=1}^{k-1} M_j,$$

the second term on the right-hand side of (5.2) is

$$\frac{q}{1-\alpha} \sum_{k=1}^n (M_k - \alpha) \prod_{j=1}^{k-1} M_j - \frac{q}{1-\alpha} \prod_{j=1}^n M_j + \frac{q}{1-\alpha}.$$

Set

$$d_k := \left( Q_k - q + \frac{q}{1-\alpha} (M_k - \alpha) \right) \prod_{j=1}^{k-1} M_j, \quad k = 1, \dots, n.$$

Then, by independence of  $(Q_j, M_j)$ ,  $j \geq 1$ ,  $(d_k)$  is a martingale difference sequence with respect to  $(\mathcal{F}_n)$  where  $\mathcal{F}_k = \sigma(M_1, Q_1, \dots, M_k, Q_k)$ ,  $k \geq 1$ . Let  $E_k$  denote the conditional expectation given  $\mathcal{F}_k$ . Since  $M_j^2 = 1$  we have

$$E_{k-1}d_k^2 = E\left(Q - q + \frac{q}{1-\alpha}(M - \alpha)\right)^2 = \beta^2,$$

so that, trivially,

$$\frac{\sum_{k=1}^n E_{k-1}d_k^2}{n} \xrightarrow{P} \beta^2.$$

Moreover, since the  $M_k$ 's are uniformly bounded, for a given  $\varepsilon > 0$  and  $n$  sufficiently large

$$E_{k-1}d_k^2 I_{\{|d_k| > \varepsilon\sqrt{n}\}} \leq 4EQ^2 I_{\{|Q| > \varepsilon\sqrt{n}/2\}}.$$

Since  $EQ^2 < \infty$ , the last quantity converges to zero as  $n \rightarrow \infty$  by the dominated convergence theorem. This verifies the conditional version of Lindeberg's condition:

$$\forall \varepsilon > 0 \quad \frac{\sum_{k=1}^n E_{k-1}d_k^2 I_{\{|d_k| > \varepsilon\sqrt{n}\}}}{n} \xrightarrow{P} 0.$$

It follows by the martingale version of the CLT (see, e.g., [4], Theorem 35.12) that

$$\frac{\sum_{k=1}^n d_k}{\sqrt{n}} \xrightarrow{d} \beta\mathcal{N}.$$

Now, by the above manipulations we have

$$\frac{R_n}{\sqrt{n}} = \frac{\sum_{k=1}^n d_k}{\sqrt{n}} + \frac{q}{(1-\alpha)\sqrt{n}} - \frac{q}{(1-\alpha)\sqrt{n}} \prod_{j=1}^n M_j.$$

Since each of the last two terms goes to 0 (deterministically and in probability, respectively), Theorem 4 follows. □

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