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Infinitesimal generators of q-Meixner processes

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Abstract

We show that the weak infinitesimal generator of a class of Markov processes acts on bounded continuous functions with bounded continuous second derivative as a singular integral with respect to the orthogonality measure of the explicit family of polynomials.

Keywords: Infinitesimal generators; Quadratic conditional variances; Harnesses; Polynomial processes

1. Introduction

We are interested in the class of non-homogeneous Markov processes that were introduced in [6] under the name q-Meixner processes. The transition probabilities $\{P_{s,t}(x, dy) : s < t,\ x \in \mathbb{R}\}$ of a q-Meixner process with parameters $\tau \geq 0, \theta \in \mathbb{R}$, and $q \in [-1, 1]$ are defined as the unique orthogonality measures of the polynomials $Q_n(y|x, s, t)$ in variable y which solve the three step recurrence

$$\begin{aligned} yQ_n(y|x,s,t) = {} & Q_{n+1}(y|x,s,t) + (\theta[n]_q + xq^n)Q_n(y|x,s,t) \\ & + (t - sq^{n-1} + \tau[n-1]_q)[n]_q Q_{n-1}(y|x,s,t), \end{aligned} \tag{1}$$

where $n \geq 1$, and $Q_{-1}(y|x, s, t) = 0$, $Q_0(y|x, s, t) = 1$, so $Q_1(y|x, s, t) = y - x$.

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(Recall the q-notation: $[n]_q = \sum_{j=0}^{n-1} q^j$. The Chapman–Kolmogorov equations hold by [6, Proposition 3.2].)

For $-1 < q < 1$ recurrence (1) can be reparametrized into a recurrence for the so called Al-Salam–Chihara polynomials, so the explicit formula for $P_{s,t}(x, dy)$ can be read out from known results, see e.g. [9,10]. However, the explicit form of the transition probabilities does not play a role in our proofs, and the expressions for the transition probabilities are rather complicated, as they may have both the discrete and the absolutely continuous parts.

Cases $q = -1$ and $q = 1$ are included in (1). In the first case the recursion degenerates with $P_{s,t}(x, dy)$ supported on two points. In the second case polynomials $\{Q_n(y|x, s, t) : n = 0, 1, \dots\}$ are the reparametrization of the Meixner polynomials, and we get Lévy processes in the Meixner class [11]. Since the infinitesimal generators of Lévy processes are well understood, in this paper we concentrate on the case $-1 < q < 1$, see Remark 1.2.

Several other special cases have appeared in the literature and have been studied by other authors. If $q = 0$, then the corresponding q-Meixner Markov processes arise as the so called classical versions of the free-Meixner Lévy processes; that is, the time ordered moments of Borel functions coincide, see [4, Definition 4.1]. For more details, including connections with [2] see [5, Appendix, Notes 2 and 3]. If $\theta = \tau = 0$ then the corresponding q-Meixner Markov processes arise as the classical version of the noncommutative q-Brownian motion; this can be seen by comparing the transition probabilities in [4, Theorem 4.6] and in [6, Section 4.1].

Finally we note that from [6, Proposition 3.3] it follows that q-Meixner processes are examples of (nonhomogeneous) "polynomial processes" studied in [7,14] with explicit "time-space harmonic polynomials" [13].

1.1. *Infinitesimal generators*

Inhomogeneous Markov processes with state space $\mathbb{R}$ are often turned into the homogeneous Markov processes with state space $\mathbb{R}\times[0, \infty)$ by considering $\widetilde{X}_t = (X_t, t)$, see for example [15]. We will work in the non-homogeneous setting as we use one-variable polynomials in some of the proofs.

From (1) it is clear that $Q_n(y|x, s, t)$ is a polynomial in y, x, s, t with the leading term y^n for every n. It follows that the moments $(x, s, t) \mapsto \int y^n P_{s,t}(x, dy)$ are polynomials in variables x, s, t of degree at most n.

We will be interested only in the case $-1 < q < 1$, in which case probability measures $\{P_{s,t}(x, dy) : s < t, x \in \mathbb{R}\}$ are compactly supported, see Proposition 1.8(v). Since for compactly supported measures, convergence of moments implies weak convergence, the fact that conditional moments are polynomials in variables x, s, t implies that the transition probabilities $\{P_{s,t}(x, dy) : x \in \mathbb{R}, 0 \le s < t\}$ define a Feller process. That is, if f is a bounded continuous function on $\mathbb{R}$ then $x \mapsto \int f(y)P_{s,t}(x, dy)$ is a bounded continuous function.

We will denote by $\mathsf{P}_{s,t}$ the linear operators $f \mapsto \int f(y)P_{s,t}(x, dy)$. We will consider $\mathsf{P}_{s,t}$ as a contraction on various subspaces of Banach space $C_b(\mathbb{R})$ of bounded continuous functions with norm $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. We will use the same symbol $\mathsf{P}_{s,t}$ for the linear mappings on the vector space of all polynomials in variable y, defined on monomials by $y^n \mapsto \int z^n P_{s,t}(y, dz)$.

We will work with several notions of the infinitesimal generator.

The *weak left infinitesimal generator* of a non-homogeneous Markov process with transition operators $\mathsf{P}_{s,t}$ is defined for $t > 0$ by

$$\mathbb{A}_t^- f = \lim_{h\to 0^+} \frac{1}{h}(\mathsf{P}_{t-h,t} f - f). \tag{2}$$

The domain $\mathcal{D}_t^-$ of the weak left generator is the set of all $f \in C_b(\mathbb{R})$ where the convergence is pointwise and the expression

$$\left\| \frac{1}{h}(\mathsf{P}_{t-h,t} f - f) \right\|_\infty$$

under the limit is bounded, compare [8, Chapter 1, Section 6] for the homogeneous case.

The *weak right infinitesimal generator* of a non-homogeneous Markov process with transition operators $\mathsf{P}_{s,t}$ is defined for $t \geq 0$ by the right-generator

$$\mathbb{A}_t^+ f = \lim_{h \to 0^+} \frac{1}{h}(\mathsf{P}_{t,t+h} f - f). \tag{3}$$

The domain $\mathcal{D}_t^+$ of the weak right generator is the set of all $f \in C_b(\mathbb{R})$ where the convergence is pointwise and the expression under the limit $\left\| \frac{1}{h}(\mathsf{P}_{t,t+h} f - f) \right\|_\infty$ is bounded.

We will also consider (2) and (3) with pointwise convergence on polynomials. Since $\mathsf{P}_{s,t}$ preserves the degree of a polynomial, it is clear that the pointwise limits (2) and (3) exist for any polynomial f, and that both limits are polynomials of degree at most $\deg(f)$ in variable x. Thus when f is a polynomial, the limits (2) and (3) define two linear operators $\widetilde{\mathbb{A}}_t^\pm$ that map polynomials to polynomials without increasing their degrees.

Our goal is to derive the common integral representation for these infinitesimal generators. We will write the generators as singular integrals with respect to an appropriate probability measure $\nu_{x,t}(dy)$ which we determine as an orthogonality measure of appropriate orthogonal polynomials.

For $x \in \mathbb{R}$ and $t > 0$, let $\nu_{x,t}(dy)$ be the orthogonality measure of the following monic polynomials in real variable y. With $W_{-1}(y; x, t) = 0$, $W_0(y; x, t) = 1$, for $n \geq 0$ consider polynomials

$$\begin{aligned} yW_n(y; x, t) &= W_{n+1}(y; x, t) + (\theta[n+1]_q + xq^{n+1})W_n(y; x, t) \\ &\quad + ((1-q)t + \tau)[n]_q[n+1]_q W_{n-1}(y; x, t). \end{aligned} \tag{4}$$

By Favard's theorem, these polynomials are orthogonal, and since for $-1 \leq q < 1$ the coefficients of the recurrence are bounded, their orthogonality measure is compactly supported. In fact, the three step recursion (4) can be reparametrized into a recursion for the so called Al-Salam–Chihara polynomials [9,10]. (The dependence of measure $\nu_{x,t}(dy)$ on parameters θ, τ, q is suppressed in our notation.)

Our main result is the following "singular integral" expression for the generator of the q-Meixner processes with $|q| < 1$.

Theorem 1.1. *Fix $\theta \in \mathbb{R}$, $\tau \geq 0$ and $q \in (-1, 1)$.*

(i) *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function with bounded continuous second derivative. Then $f \in \mathcal{D}_t^- \cap \mathcal{D}_t^+$, both weak infinitesimal generators coincide on f and are given by*

$$\mathbb{A}_t^\pm(f)(x) = \frac{1}{2} f''(x)\nu_{x,t}(\{x\}) + \int_{\mathbb{R}\setminus\{x\}} \left(\frac{\partial}{\partial x} \frac{f(y) - f(x)}{y - x} \right) \nu_{x,t}(dy). \tag{5}$$

(ii) *If f is a polynomial then both left and right infinitesimal generators $\widetilde{\mathbb{A}}_t^\pm$ coincide on f, and are given by the right hand side of (5). On polynomials, the latter expression takes a slightly* *simpler form*

$$\widetilde{\mathbb{A}}_t^{\pm}(f)(x) = \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} \frac{f(y) - f(x)}{y - x} \right) \nu_{x,t}(dy). \tag{6}$$

Remark 1.2. With minimal changes in our proofs, Theorem 1.1 holds also for $q = 1$. However, for $q = 1$ the q-Meixner processes are Lévy processes with finite moments, and the infinitesimal generators for centered Lévy processes with finite second moments have been studied in more detail, see e.g. [1]. In this case, the restriction of the infinitesimal generator of a square-integrable Lévy process to our class of functions is given by (5) with $\nu_{x,t}(dy) = (\delta_x * K)(dy)$ where $K(dy)$ is the measure from the Kolmogorov representation for the characteristic function of the infinitely divisible measure $P_{0,1}(0, dy)$. Kolmogorov measures $K(dy)$ for centered Lévy processes in the Meixner class are known explicitly, see [12]. They can also be read out as the orthogonality measures of polynomials (4) for $x = 0, q = 1$.

Remark 1.3. Theorem 1.1 holds also for $q = -1$ with $\nu_{x,t}(dy) = \delta_{\theta - x}$. Our proof could be modified to account for the possibility that $[n]_q = 0$ when n is even. However, in this case the transition probabilities are supported on two points:

$$\begin{aligned} P_{s,t}(x, dy) = &\left(\frac{1}{2} + \frac{\theta - 2x}{2\sqrt{(\theta - 2x)^2 + 4(t - s)}} \right) \delta_{\frac{1}{2}\left(\theta - \sqrt{(\theta - 2x)^2 + 4(t-s)}\right)} \\ &+ \left(\frac{1}{2} - \frac{\theta - 2x}{2\sqrt{(\theta - 2x)^2 + 4(t - s)}} \right) \delta_{\frac{1}{2}\left(\theta + \sqrt{(\theta - 2x)^2 + 4(t-s)}\right)}. \end{aligned}$$

So the fact that the weak infinitesimal generator on twice differentiable functions is given by (5) with the degenerate $\nu_{x,t}(dy) = \delta_{\theta - x}$ is just an exercise, and we omit proof for this case.

The following technical result is an intermediate step in the proof of Theorem 1.1.

Theorem 1.4. *Measures*

$$\frac{(y - x)^2}{t - s} P_{s,t}(x, dy) \tag{7}$$

are probability measures and converge weakly as $s \to t^-$ *to* $\nu_{x,t}(dy)$. *Similarly, probability measures* (7) *converge weakly as* $t \to s^+$ *to* $\nu_{x,s}(dy)$.

The proofs are in Sections 2 and 3. The plan of proof is as follows. We first prove Theorem 1.1(ii). We then derive Theorem 1.4 from Theorem 1.1(ii). Finally, we show that Theorem 1.4 implies Theorem 1.1(i).

We end this section with a short list of more explicit examples.

1.2. Some special cases

Measures $\nu_{x,t}(dy)$ take more explicit form in some special cases. In the corollaries, $\mathbb{A}_t$ denotes either the left or the right weak infinitesimal generator if it is applied to bounded continuous functions, or one of the generators $\tilde{\mathbb{A}}_t^{\pm}$ if it is acting on polynomials.

The generator of the q-Brownian process was determined [1, Section 5, Theorem 23]; this result inspired our study of generators for more general q-Meixner processes.

Corollary 1.5. *The infinitesimal generator of the q-Wiener process acts on a polynomial f or on a bounded continuous function f with bounded continuous second derivative as follows:*

$$(\mathbb{A}_t f)(x) = \int \left(\frac{\partial}{\partial x} \frac{f(y) - f(x)}{y - x} \right) P_{q^2 t, t}(qx, dy). \tag{8}$$

Here $P_{s,t}(x, dy)$ denotes the transition probability measure of the q-Wiener process.

Proof. From (1) with $\theta = \tau = 0$, we read out that the transition probabilities $P_{s,t}(x, dy)$ of the q-Brownian motion are the orthogonality measures of the monic polynomials $\{Q_n(y|x, s, t) n \geq 0\}$ in variable y which are given by the three step recurrence

$$\begin{aligned} y Q_n(y|x, s, t) = {} & Q_{n+1}(y|x, s, t) + x q^n Q_n(y|x, s, t) \\ & + (t - s q^{n-1})[n]_q Q_{n-1}(y|x, s, t), \quad n \geq 1, \end{aligned} \tag{9}$$

with $Q_{-1}(y|x, s, t) = 0$, $Q_0(y|x, s, t) = 1$. Comparing (4) with $\theta = \tau = 0$ and (9) with $s = q^2 t$ and x replaced by qx we see that $\nu_{x,t}(dy) = P_{q^2 t, t}(qx, dy)$. So (8) follows from (5). $\square$

The free Brownian motion corresponds to $q = 0$ and has been studied in [3, p. 392]. The domain of the closely related generator for the free Ornstein–Uhlenbeck process is described in more detail in [4, p. 150].

Corollary 1.6. *For $q = 0$, the infinitesimal generator of the q-Wiener process acts on a polynomial f or on a bounded continuous function f with bounded continuous second derivative as follows:*

$$(\mathbb{A}_t f)(x) = \int_{(-2,2)} \left(\frac{\partial}{\partial x} \frac{f\left(\sqrt{t} y\right) - f(x)}{\sqrt{t} y - x} \right) \sqrt{4 - y^2} dy / \pi. \tag{10}$$

Proof. With $q = 0$, this follows formula (8): $P_{0,t}(0, dy)$ is the univariate law of the free Brownian motion X_t started at $X_0 = 0$, which is known to be the semicircle law of mean 0 and variance t. Then $X_t / \sqrt{t}$ has the semicircle law of variance 1, so by a change of variable

$$\int f(y) P_{0,t}(0, dy) = \int f\left(\sqrt{t} y\right) \sqrt{4 - y^2} dy / \pi$$

for any (say polynomial) f. $\square$

The q-Meixner processes with $q = 0$ arise as classical versions of certain free Lévy processes. The generator of such processes was studied in [5]. Anshelevich [1, Theorem 15] determined the strong infinitesimal generators for the more general class of Markov processes that arise from arbitrary free Lévy processes, and identified a large subset of their domain.

Corollary 1.7. *The generator of the q-Meixner process for $q = 0$ acts on a polynomial f or on a bounded continuous function f with bounded continuous second derivative as follows:*

$$(\mathbb{A}_t f)(x) = \int_{(\theta - 2\sqrt{t+\tau}, \theta + 2\sqrt{t+\tau})} \left(\frac{\partial}{\partial x} \frac{f(y) - f(x)}{y - x} \right) w_{\theta, t+\tau}(dy), \tag{11}$$

where $w_{m,\sigma^2}(y) \sim \sqrt{4\sigma^2 - (y - m)^2}$ is the semicircle density of mean m and variance σ.

Proof. For $q = 0$, recurrence (4) becomes

$$W_1(y; x; t) = y - \theta,$$
$$(y - \theta)W_n(y; x, t) = W_{n+1}(y; x; t) + (t + \tau)W_{n-1}(y; x, t), \quad n \geq 1.$$

The corresponding probability measure ν is the semicircle law of mean θ and variance $t + \tau$. $\square$

1.3. Some additional observations

Here we collect some simple "regularity" properties of the transition operators for the q-Meixner Markov processes.

Proposition 1.8. *Suppose (X_t) is a q-Meixner Markov process with transition probabilities $P_{s,t}(x, dy)$ that are orthogonality measures of polynomials (1). Then the following properties hold.*

(i) *Process (X_t) is a martingale:*

$$\int y P_{s,t}(x, dy) = x.$$

(ii) *More generally, $M_n(y; t) := Q_n(y|0, 0, t)$ are martingale polynomials:*

$$\int M_n(y; t) P_{s,t}(x, dy) = M_n(x; s).$$

(iii) *For $s < t$, and fixed $x \in \mathbb{R}$, the positive measure (7) is a probability measure.*

(iv) *For $s < t$, fixed $x \in \mathbb{R}$ and $U = (x - \delta, x + \delta)$ with $\delta > 0$, we have*

$$P_{s,t}(x, U) \geq 1 - (t - s)/\delta^2.$$

(v) *For fixed $s < t, x \in \mathbb{R}$ and $-1 \leq q < 1$, probability measure $P_{s,t}(x, dy)$ has compact support.*

(vi) *Transition probabilities $P_{s,t}(x, dy)$ have the Feller property: if $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and $s < t$ then $g(x) := \int f(y) P_{s,t}(x, dy)$ defines a bounded continuous function $g : \mathbb{R} \to \mathbb{R}$. Furthermore, if $\lim_{x\to\pm\infty} f(x) = 0$ then $\lim_{x\to\pm\infty} g(x) = 0$.*

Proof. (i) The martingale property follows from the fact that $Q_1(y|x, s, t) = y - x$ is orthogonal to $Q_0 = 1$ with respect to the measure $P_{s,t}(x, dy)$.

(ii) The more general martingale property is [6, Proposition 3.3] (the polynomials $M_n(y; t)$ are not orthogonal unless $X_0 = 0$).

(iii) Clearly, this is a positive measure so we only need to verify that it integrates to 1. Since

$$Q_2(y|x, s, t) = (y - x)^2 - (y - x)(\theta + (q - 1)x) - (t - s)$$

is orthogonal to $Q_0 = 1$, we get constant conditional variance

$$\int (y - x)^2 P_{s,t}(x, dy) = t - s.$$

This shows that (7) is a probability measure.

(iv) By Chebyshev's inequality,

$$P_{s,t}(x, U') \leq \frac{1}{\delta^2} \int (y - x)^2 P_{s,t}(x, dy) = \frac{(t - s)}{\delta^2}.$$

(v) Compact support follows from the fact that for $-1 \leq q < 1$ the coefficients of recurrence (1) are bounded, see e.g. [9, Theorem 2.5.4].

(vi) Clearly, for any bounded measurable f we have $\|g\|_\infty \leq \|f\|_\infty$. As explained in the introduction, the fact that $g(x)$ is continuous follows from the fact that conditional moments $\int y^n P_{s,t}(x, dy)$ are polynomials in variable x. To show that the transition operators preserve vanishing at infinity property, given $\varepsilon > 0$ choose $A > 0$ such that $\|f\|_\infty (t-s)/A^2 < \varepsilon$ and $\sup_{y>A} |f(y)| < \varepsilon$. Then for $x > 2A$ we have

$$\int f(y) P_{s,t}(x, dy) = \int_{y>A} f(y) P_{s,t}(x, dy) + \int_{y \leq A} f(y) P_{s,t}(x, dy)$$

and the first term is bounded by

$$\left| \int_{y>A} f(y) P_{s,t}(x, dy) \right| \leq \sup_{y>A} |f(y)| < \varepsilon.$$

Since $\{y : y \leq A\} \subset \{y : |y - x| \geq A\}$ for $x > 2A$, the second term is bounded by Chebyshev's inequality that was already used in the proof of (iv):

$$\begin{aligned} \left| \int_{y \leq A} f(y) P_{s,t}(x, dy) \right| &\leq \|f\|_\infty P_{s,t}(x, \{y : |y - x| \geq A\}) \\ &\leq \|f\|_\infty (t-s)/A^2 < \varepsilon. \end{aligned}$$

The proof for the case $x \to -\infty$ is similar. □

2. Proof of Theorem 1.1(ii)

We begin by checking that both left and right infinitesimal generators $\widetilde{\mathbb{A}}_t^{\pm}$ coincide on the polynomials.

Lemma 2.1. *Suppose $M_n(y; t) := Q_n(y|0, 0, t)$ are the martingale polynomials from Proposition 1.8(ii). Then*

$$\widetilde{\mathbb{A}}_t^{\pm}(M_n(\cdot; t))(x) = -\frac{\partial}{\partial t} M_n(x; t). \tag{12}$$

In particular, for any polynomial p the left and right infinitesimal generators $\widetilde{\mathbb{A}}_t^{\pm}(p)$ exist and are equal.

Proof. Since $M_n(y; t)$ is given by a special case of recurrence (1), it is clear that $M_n(y; t)$ is a polynomial in t and hence it is differentiable with respect to t.

By the martingale property,

$$\int M_n(y, t) P_{s,t}(x, dy) - M_n(x; t) = M_n(x; s) - M_n(x; t)$$

for $s < t$, so

$$\begin{aligned} \widetilde{\mathbb{A}}_t^{-}(M_n(\cdot; t))(x) &= \lim_{s \to t^-} \frac{1}{t-s} \big(\mathsf{P}_{s,t}(M_n(\cdot; t))(x) - M_n(x; t) \big) \\ &= \lim_{s \to t^-} \frac{1}{t-s} (M_n(x; s)(x) - M_n(x; t)) = -\frac{\partial}{\partial t} M_n(x; t). \end{aligned}$$

We now consider the right generator. Writing

$$M_n(y;t)=\sum_{k=0}^{n}a_k(t)y^k$$

we have

$$\frac{1}{h}\left(\int M_n(y,t)P_{t,t+h}(x,dy)-M_n(x;t)\right)=\int\frac{M_n(y;t)-M_n(y;t+h)}{h}P_{t,t+h}(x,dy)$$
$$=\sum_{k=0}^{n}\frac{a_k(t)-a_k(t+h)}{h}\int y^kP_{t,t+h}(x,dy).$$

Since

$$\int y^kP_{t,t+h}(x,dy)\to x^k\quad\text{as }h\to 0$$

and

$$\frac{a_k(t)-a_k(t+h)}{h}\to -a_k'(t)\quad\text{as }h\to 0,$$

the formula follows.

To prove the second part, we write $p(y)$ as a (finite) linear combination

$$a_0(t)M_0(y;t)+\cdots+a_{n-1}(t)M_{n-1}(y;t)+a_n(t)M_n(y;t).$$

Then by linearity

$$\mathsf{P}_{s,u}(p)(x)=\sum_{k=0}^{n}a_k(t)\mathsf{P}_{s,u}(M_k(\cdot;t))(x),$$

so $\tilde{\mathbb{A}}_t^{\pm}(p)(x)=-\sum_{k=0}^n a_k(t)\frac{\partial}{\partial t}M_k(x;t)$. $\square$

Since both left and right generators coincide on polynomials, from now on write $\widetilde{\mathbb{A}}_t(p)$ for their common value.

Next, consider an auxiliary operator H_t acting on a polynomial p as the difference of $\widetilde{\mathbb{A}}_t$ applied to polynomial $yp(y)$ and $x\tilde{\mathbb{A}}_t(p)(x)$. Define,

$$\mathsf{H}_t(p)(x):=\tilde{\mathbb{A}}_t(yp(y))(x)-x\widetilde{\mathbb{A}}_t(p(y))(x). \tag{13}$$

Lemma 2.2.

$$\mathsf{H}_t(M_k(\cdot;t))(x)=[k]_qM_{k-1}(x;t). \tag{14}$$

Proof. From (1) we get the recurrence

$$yM_k(y;t)=M_{k+1}(y;t)+\theta[k]_qM_k(y;t)+(t+\tau[k-1])[k]_qM_{k-1}(y;t).$$

Therefore, by linearity and Lemma 2.1 we have

$$\tilde{\mathbb{A}}_t(yM_k(y;t))(x)=-\frac{\partial}{\partial t}M_{k+1}(x;t)-\theta[k]_q\frac{\partial}{\partial t}M_k(x;t)$$
$$-(t+\tau[k-1])[k]_q\frac{\partial}{\partial t}M_{k-1}(x;t).$$

On the other hand,

$$x\widetilde{\mathbb{A}}_t(M_k(y;t))(x) = -x\frac{\partial}{\partial t}M_k(x;t) = -\frac{\partial}{\partial t}(xM_k(x;t))$$
$$= -\frac{\partial}{\partial t}\left(M_{k+1}(x;t) + \theta[k]_q M_k(x;t) + (t+\tau[k-1])[k]_q M_{k-1}(x;t)\right).$$

Subtracting the two expressions yields the answer. □

Lemma 2.3. *With $\nu_{x,t}(dy)$ as in Theorem 1.1, we have*

$$\mathsf{H}_t(p)(x) = \int \frac{p(y)-p(x)}{y-x}\nu_{x,t}(dy). \tag{15}$$

Proof. In view of (14), we only need to show that for all $n = 0, 1, \ldots$ we have

$$\int \frac{M_n(y;t) - M_n(x;t)}{y-x}\nu_{x,t}(dy) = [n]_q M_{n-1}(x;t). \tag{16}$$

We prove (16) by induction. Since $M_0 = 1$ and $M_1(y;t) = y$, it is clear that (16) holds for $n = 0, 1$.

Suppose now that (16) holds for some $n \geq 1$ and for all the previous non-negative integers.

The induction step relies on the following algebraic identities from [6, Lemma 3.1]:

$$0 = Q_n(x|x,t,t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q Q_{n-k}(0|x,t,0)M_k(x,t) \tag{17}$$

and

$$Q_{n+1}(y|x,t,t) = \sum_{k=1}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q Q_{n+1-k}(0|x,t,0)(M_k(y;t) - M_k(x;t)). \tag{18}$$

Here we use the q-notation $[n]_q! = \prod_{k=1}^{n}[k]_q$ with $[0]_q! = 1$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

(The latter is well defined as we only consider $q > -1$.)

From (1) applied to $s = t$ we see that for $n \geq 0$ we have

$$Q_{n+1}(y|x,t,t) = (y-x)W_n(y;x,t),$$

where polynomials $\{W_n(y;x,t)\}$ satisfy recurrence (4). Identity (18) gives

$$W_n(y;x,t) = \sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q Q_{n+1-k}(0|x,t,0)\,\tfrac{M_k(y;t)-M_k(x;t)}{y-x}.$$

Since $n \geq 1$, polynomial W_n is orthogonal to $W_0 = 1$. Integrating the above equality with respect to the measure $\nu_{x,t}(dy)$ we get

$$0 = \sum_{k=1}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q Q_{n+1-k}(0|x,t,0) \int \tfrac{M_k(y,t)-M_k(x,t)}{y-x}\,\nu_{x,t}(dy). \tag{19}$$

We now observe that identity (17) gives

$$\sum_{k=1}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q Q_{n+1-k}(0|x,t,0)[k]_q M_{k-1}(x,t)$$
$$= [n+1]_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q Q_{n-k}(0|x,t,0) M_k(x,t) = 0.$$

Subtracting the left hand side of the above from (19) we see that all but the last terms cancel by the inductive assumption (16). The remaining term is

$$\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_q Q_0(0|x,s,0) \left(\int \tfrac{M_{n+1}(y,t)-M_{n+1}(x,t)}{y-x} \nu_{x,t}(dy) - [n+1]_q M_n(x,t) \right)$$
$$= 0.$$

Since $Q_0(0|x,s,0) = 1$ and $q \neq -1$, this proves that (16) holds for $n+1$, hence for all n, and (15) follows. □

Lemma 2.4. *If H_t, as defined on polynomials by (13), is given by an integral (15), then $\widetilde{\mathbb{A}}_t$ acts on polynomials by (6).*

Proof. By linearity it is enough to prove (6) for $f(x) = x^n$. We proceed by induction. Since $\widetilde{\mathbb{A}}_t(1) = 0$, the formula holds true for $n = 0$. Since

$$\frac{\partial}{\partial x} \frac{f(y) - f(x)}{y - x} = \frac{f(y) - f(x)}{(y-x)^2} - \frac{f'(x)}{y-x}, \tag{20}$$

assuming (6) holds for x^n, we have

$$\begin{aligned} \tilde{\mathbb{A}}_t(x^{n+1}) &= \mathsf{H}_t(x^n) + x\widetilde{\mathbb{A}}_t(x^n) \\ &= \int \frac{y^n - x^n}{y - x} \nu_{x,t}(dy) + \int \left(x \frac{y^n - x^n}{(y-x)^2} - \frac{nx^n}{y-x} \right) \nu_{x,t}(dy) \\ &= \int \left(\frac{y^n(y-x)}{(y-x)^2} + x \frac{y^n - x^n}{(y-x)^2} - \frac{(n+1)x^n}{y-x} \right) \nu_{x,t}(dy) \\ &= \int \left(\frac{y^{n+1} - x^{n+1}}{(y-x)^2} - \frac{(n+1)x^n}{y-x} \right) \nu_{x,t}(dy) \\ &= \int \left(\frac{\partial}{\partial x} \frac{y^{n+1} - x^{n+1}}{y-x} \right) \nu_{x,t}(dy). \quad \square \end{aligned} \tag{21}$$

Proof of Theorem 1.1(ii). By Lemma 2.1 both left and right generators coincide on polynomials. The integral representation follows by combining Lemma 2.3 with Lemma 2.4. □

3. Proof of Theorems 1.4 and 1.1 (i)

We will deduce Theorem 1.4 from Theorem 1.1(ii), which we have already proved. Consider operator H_t defined in (13), and let C_t be defined by the similar expression:

$$\mathsf{C}_t(p)(x) = \mathsf{H}_t(yp(y))(x) - x\mathsf{H}_t(p(y))(x).$$

Lemma 3.1. *If H_t is given by (15), then $\mathsf{C}_t(p)(x) = \int p(y)\nu_{x,t}(dy)$.*

Proof. Recall that if p is a polynomial then $(p(y)-p(x))/(y-x)$ is a polynomial in variable y. By simple algebraic manipulations we get

$$\begin{aligned}\mathsf{H}_t(yp(y))(x) - x\mathsf{H}_t(p(y))(x) &= \int \frac{yp(y)-xp(x)}{y-x}\nu_{x,t}(dy) - x\frac{p(y)-p(x)}{y-x}\nu_{x,t}(dy)\\ &= \int \frac{(y-x)p(y)}{y-x}\nu_{x,t}(dy). \quad \square\end{aligned}$$

Proof of Theorem 1.4. We give the proof for the first part only. By Theorem 1.1(ii), for a polynomial f we have

$$\begin{aligned}\mathsf{C}_t(f)(x) &= \mathsf{H}_t(yf(y))(x) - x\mathsf{H}_t(f(y))(x)\\ &= \widetilde{\mathbb{A}}_t(y^2f(y))(x) - 2x\widetilde{\mathbb{A}}_t(yf(y))(x) + x^2\widetilde{\mathbb{A}}f(y)(x)\\ &= \lim_{h\to0^+}\int \frac{y^2f(y)-x^2f(x)}{h}P_{t,t+h}(x,dy)\\ &\quad -2x\lim_{h\to0^+}\int \frac{yf(y)-xf(x)}{h}P_{t,t+h}(x,dy)\\ &\quad +x^2\lim_{h\to0^+}\int \frac{f(y)-f(x)}{h}P_{t,t+h}(x,dy) = \lim_{h\to0^+}\int \frac{(y-x)^2f(y)}{h}P_{t,t+h}(x,dy).\end{aligned}$$

In view of Lemma 3.1, this shows that as $h\to 0$, all moments of probability measure

$$\frac{(y-x)^2}{h}P_{t,t+h}(x,dy)$$

converge to the moments of measure $\nu_{x,t}(dy)$. Probability measure $\nu_{x,t}(dy)$ is compactly supported (recall that $|q|<1$), so it is uniquely determined by moments, and weak convergence follows. $\square$

Proof of Theorem 1.1(i). If f is bounded and has bounded and continuous second derivative then for a fixed x

$$\varphi(y) = \begin{cases}\dfrac{\partial}{\partial x}\dfrac{f(y)-f(x)}{y-x} & \text{if } y\neq x\\ \dfrac{1}{2}f''(x) & \text{if } y=x\end{cases}$$

is a bounded continuous function. Indeed, by Taylor's theorem

$$\left|\frac{\partial}{\partial x}\frac{f(y)-f(x)}{y-x}\right| = \frac{1}{(y-x)^2}\left|\int_x^y f''(z)(z-x)dz\right| \le \frac{1}{2}\sup_{z\in\mathbb{R}}|f''(z)|.$$

Next, we observe that since $\int yP_{s,t}(x,dy)=x$, from (20) we get

$$\begin{aligned}\frac{1}{t-s}\int (f(y)-f(x))\,P_{s,t}(x,dy) &= \frac{1}{t-s}\int_{\mathbb{R}-\{x\}}(f(y)-f(x))\,P_{s,t}(x,dy)\\ &= \int_{\mathbb{R}-\{x\}}\left(\frac{\partial}{\partial x}\frac{f(y)-f(x)}{y-x}\right)\frac{(y-x)^2}{t-s}P_{s,t}(x,dy)\\ &= \int_{\mathbb{R}}\varphi(y)\frac{(y-x)^2}{t-s}P_{s,t}(x,dy).\end{aligned}$$

Therefore, by Theorem 1.4,

$$\lim_{s\to t^-}\frac{1}{t-s}\int_{\mathbb{R}}(f(y)-f(x))\,P_{s,t}(x,dy)=\lim_{s\to t^-}\int_{\mathbb{R}}\varphi(y)\frac{(y-x)^2}{t-s}P_{s,t}(x,dy)$$
$$=\int_{\mathbb{R}}\varphi(y)\nu_{x,t}(dy)$$
$$=\frac{1}{2}f''(x)\nu_{x,t}(\{x\})+\int_{\mathbb{R}\setminus\{x\}}\left(\frac{\partial}{\partial x}\frac{f(y)-f(x)}{y-x}\right)\nu_{x,t}(dy).$$

Note that by Proposition 1.8(iii),

$$\sup_{0\le s<t}\sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}\varphi(y)\frac{(y-x)^2}{t-s}P_{s,t}(x,dy)\right|\le\|\varphi\|_\infty,$$

so f is indeed in the domain of the weak generator. This proves (5) for the left generator. A similar argument applies to the right generator. □

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