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On the Matsumoto–Yor type regression characterization of the gamma and Kummer distributions

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ABSTRACT

In this paper we study a Matsumoto–Yor type property for the gamma and Kummer independent variables discovered by Koudou and Vallois (2012). We prove that constancy of regressions of $U = (1 + (X + Y)^{-1})/(1 + X^{-1})$ given $V = X + Y$ and of U^{-1} given V, where X and Y are independent and positive random variables, characterizes the gamma and Kummer distributions. This result completes characterizations by independence of U and V obtained, under smoothness assumptions for densities, in Koudou and Vallois (2011, 2012). Since we work with differential equations for the Laplace transforms, no density assumptions are needed.

1. Introduction

Let X and Y be independent random variables. There are several well known settings in which $U = \psi(X, Y)$ and $V = X + Y$ are also independent. Related characterizations of distributions of X and Y by properties of independence of X and Y and independence of U and V have been also studied. The most prominent seem to be:

- Bernstein's characterization of the normal law by independence of $U = X - Y$ and V (Bernstein, 1941),
- Lukacs's characterization of the gamma law by independence of $U = X/Y$ and V (Lukacs, 1955).

In the end of 1990s a new independence phenomenon of this kind, called Matsumoto–Yor property, see e.g. Stirzaker (2005, p. 43), was discovered. It says that for X with a GIG (generalized inverse Gaussian) law and independent Y with a gamma law (both distributions with suitably adjusted parameters), random variables $U = 1/X - 1/(X + Y)$ and V are independent. This elementary property was identified while the authors analyzed structure of functionals of Brownian motion—see Matsumoto and Yor (2001, 2003). A related characterization of the GIG and gamma laws by independence of X and Y and of U and V was obtained in Letac and Wesołowski (2000). Both these results: the Matsumoto–Yor property and the characterization were generalized in several directions. Matrix variate analogues were studied e.g. in Letac and Wesołowski (2000), Wesołowski (2002) and Massam and Wesołowski (2006)—the last one including a relation with conditional structure of Wishart matrices. Recently it has been extended to symmetric cones setting in Kołodziejek (2014). Multivariate versions related to specific transformations governed by a tree were considered in Barndorff-Nielsen and Koudou (1998), Massam and Wesołowski (2004), Koudou (2006) and very recently in Bobecka (2015). Further connections with (exponential) Brownian motion were investigated in Witkowski and Wesołowski (2007) and Matsumoto et al. (2009). There are also regression versions of Matsumoto–Yor type characterizations, as given in Seshadri and Wesołowski (2001), Wesołowski (2002) and

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Chou and Huang (2004). A survey of these results together with other characterizations of the GIG law can be found in a recent paper (Koudou and Ley, 2014).

In 2009 Koudou and Vallois tried to generalize Matsumoto–Yor property by a search of distributions of independent X and Y and functions f such that $V = f(X+Y)$ and $U = f(X) - f(X+Y)$ are independent. Their research lead to a discovery of another pair $U = \psi(X, Y)$ and $V = X + Y$ with independence property: Assume that X and Y are independent random variables, X has the Kummer distribution $\mathrm{K}(a, b, c)$ with the density

$$f_X(x) \propto \tfrac{x^{a-1}e^{-cx}}{(1+x)^{a+b}}\, I_{(0,\infty)}(x), \quad a, b, c > 0,$$

and Y has the gamma distribution $\mathrm{G}(b, c)$ with the density

$$f_Y(y) \propto y^{b-1}e^{-cy} I_{(0,\infty)}(y).$$

Then, see Koudou and Vallois (2012), random variables

$$U = \frac{1+\frac{1}{X+Y}}{1+\frac{1}{X}} \quad \text{and} \quad V = X + Y \tag{1}$$

are independent, U has the beta first kind distribution $\mathrm{B}_I(a, b)$ with the density

$$f_U(u) \propto u^{a-1}(1-u)^{b-1} I_{(0,1)}(u)$$

and V has the Kummer distribution, $\mathrm{K}(a + b, -b, c)$. (Note that the Kummer distribution $\mathrm{K}(\alpha, \beta, \gamma)$ is well-defined iff $\alpha, \gamma > 0$ and $\beta \in \mathbb{R}$.)

It is an interesting question if a theory, similar to the one for the original Matsumoto–Yor property described in the literature recalled above, can be developed for this new independence property. There have already been some successful efforts in this direction. The property was extended to matrix variate distributions in Koudou (2012). It is also known that, under appropriate smoothness assumptions on densities, a characterization counterpart of the property holds: if X and Y are independent positive random variables, and U and V, given by (1), are also independent then $X \sim \mathrm{K}(a, b, c)$ and $Y \sim \mathrm{G}(b, c)$ for some positive constants a, b, c. Originally this result was proved in Koudou and Vallois (2012) under requirements that the densities of X and Y are strictly positive and twice differentiable on $(0, \infty)$. Then, in Koudou and Vallois (2011) it was proved under strict positivity of densities and local integrability of their logarithms. Letac (2009) conjectured that such a characterization is possibly true with no assumptions on densities. In this note we contribute further to this development but instead assuming independence of X and Y and independence of U and V we assume constancy of regressions of U and U^{-1} with respect to V, while the assumption of independence X and Y is kept. Obviously, up to necessary moment assumption, this is weaker than independence of U and V. In the proof we use Laplace transform and therefore no assumptions on densities are needed.

2. Regression characterization

Our main result is a characterization of the Kummer and gamma laws by constancy of regressions of U and U^{-1} given V in the setting described in (1). Since $U \in (0, 1)$ $\mathbb{P}$-a.s. $\mathbb{E}\, U < \infty$, and one can consider conditional moment $\mathbb{E}(U|V)$ without any additional restrictions. This is not the case of $\mathbb{E}(U^{-1}|V)$ since, a priori, the moment $\mathbb{E}\, U^{-1}$ may not be finite. Since

$$U^{-1} = \tfrac{1+X}{X}\, \tfrac{X+Y}{1+X+Y} \le 1 + \tfrac{1}{X}$$

we have $\mathbb{E}\, U^{-1} \le 1 + \mathbb{E}\, X^{-1}$. So, under the assumption $\mathbb{E}\, X^{-1} < \infty$ the conditional moment $\mathbb{E}(U^{-1}|V)$ is well defined.

Now we are ready to state the main result of this note.

Theorem 2.1. *Let X and Y be independent positive non-degenerate random variables and $\mathbb{E}\, X^{-1} < \infty$. Define U and V through (1). If*

$$\mathbb{E}(U|V) = \alpha \quad \textit{and} \quad \mathbb{E}(U^{-1}|V) = \beta \tag{2}$$

for real constants α and β then there exists a constant $c > 0$ such that

$$X \sim \mathrm{K}\left(\tfrac{\alpha(\beta-1)}{\alpha\beta-1}, \tfrac{(1-\alpha)(\beta-1)}{\alpha\beta-1}, c\right) \quad \textit{and} \quad Y \sim \mathrm{G}\left(\tfrac{(1-\alpha)(\beta-1)}{\alpha\beta-1}, c\right).$$

Proof. First, rewrite Eqs. (2) as

$$\mathbb{E}\left(\tfrac{X}{1+X}\middle| X+Y\right) = \alpha\tfrac{X+Y}{1+X+Y} \quad \text{and} \quad \mathbb{E}\left(\tfrac{1+X}{X}\middle| X+Y\right) = \beta\tfrac{1+X+Y}{X+Y}.$$

Equivalently, we have

$$\mathbb{E}\left(\tfrac{1}{1+X}\middle| X+Y\right) = 1 - \alpha + \tfrac{\alpha}{1+X+Y} \tag{3}$$

and

$$\mathbb{E}\left(\tfrac{1}{X}\Big|X+Y\right)=\beta-1+\tfrac{\beta}{X+Y}. \tag{4}$$

Eq. (3) implies

$$\mathbb{E}\,\frac{e^{s(1+X+Y)}}{1+X}=(1-\alpha)\mathbb{E}\,e^{s(1+X+Y)}+\alpha\mathbb{E}\,\frac{e^{s(1+X+Y)}}{1+X+Y} \tag{5}$$

at least for $s\le 0$.

Similarly, from (4) we get the equation

$$\mathbb{E}\,\frac{e^{s(X+Y)}}{X}=(\beta-1)\mathbb{E}\,e^{s(X+Y)}+\beta\mathbb{E}\,\frac{e^{s(X+Y)}}{X+Y},\quad s\le 0. \tag{6}$$

Differentiating (5) with respect to s (it is possible at least for $s<0$) we obtain

$$\mathbb{E}\,e^{s(1+X+Y)}+\mathbb{E}\,\frac{Y}{1+X}\,e^{s(X+Y+1)}=(1-\alpha)\mathbb{E}\,(1+X+Y)e^{s(1+X+Y)}+\alpha\,\mathbb{E}\,e^{s(X+Y+1)}.$$

After dividing by e^s both sides of the above equation and canceling the term $\mathbb{E}\,e^{s(X+Y)}$ we arrive at

$$\mathbb{E}\,\frac{e^{sX}}{1+X}\,\mathbb{E}\,Ye^{sY}=(1-\alpha)\left(\mathbb{E}\,Xe^{sX}\,\mathbb{E}\,e^{sY}+\mathbb{E}\,e^{sX}\,\mathbb{E}\,Ye^{sY}\right).$$

Below we use the notation $L_Z(s)=\mathbb{E}\,e^{sZ}$ and $H_Z(s)=\mathbb{E}\,\frac{e^{sZ}}{Z}$ for any positive random variable Z, such that $\mathbb{E}\,Z^{-1}<\infty$ and for $s\le 0$. Thus the above equation can be written as

$$e^{-s}H_{1+X}\,L_Y'=(1-\alpha)(L_X\,L_Y)'. \tag{7}$$

On the other hand differentiating (6) we get

$$\mathbb{E}\,e^{s(X+Y)}+\mathbb{E}\,\frac{Y}{X}\,e^{s(X+Y)}=(\beta-1)\mathbb{E}\,(X+Y)e^{s(X+Y)}+\beta\,\mathbb{E}\,e^{s(X+Y)}.$$

Consequently,

$$\mathbb{E}\,\frac{e^{sX}}{X}\,\mathbb{E}\,Ye^{sY}=(\beta-1)\left(\mathbb{E}\,Xe^{sX}\,\mathbb{E}\,e^{sY}+\mathbb{E}\,e^{sX}\,\mathbb{E}\,Ye^{sY}+\mathbb{E}\,e^{sX}\,\mathbb{E}\,e^{sY}\right).$$

Therefore

$$H_X\,L_Y'=(\beta-1)((L_X\,L_Y)'+L_X\,L_Y). \tag{8}$$

By deriving the formula for $(L_XL_Y)'$ from (7) and (8) we get

$$ae^{-s}\,H_{1+X}\,L_Y'=bH_X\,L_Y'-L_X\,L_Y, \tag{9}$$

with $a=(1-\alpha)^{-1}$ and $b=(\beta-1)^{-1}$. The numbers a and b are well defined since $\alpha=\mathbb{E}\,U<1$ and $\beta=\mathbb{E}\,U^{-1}>1$. Differentiate (9) to get

$$-ae^{-s}H_{1+X}L_Y'+ae^{-s}H_{1+X}'L_Y'+ae^{-s}H_{1+X}L_Y''=bH_X'L_Y'+bH_XL_Y''-(L_XL_Y)'.$$

Note that $H_X'=L_X=e^{-s}H_{1+X}'$. Therefore the above equation together with (7) and (8), after multiplying both sides by L_Y' implies

$$-(L_XL_Y)'L_Y'+aL_XL_Y'^2+(L_XL_Y)'L_Y''=bL_XL_Y'^2+((L_XL_Y)'+L_XL_Y)L_Y''-(L_XL_Y)'L_Y'$$

which after cancelations (which are allowed since $L_X\neq 0$) gives

$$L_YL_Y''=(a-b)L_Y'^2. \tag{10}$$

Moreover,

$$a-b=\frac{1}{1-\alpha}+\frac{1}{1-\beta}=\frac{2-\alpha-\beta}{(1-\alpha)(1-\beta)}=1+\frac{\alpha\beta-1}{(1-\alpha)(\beta-1)}=:1+\frac{1}{p}.$$

Since, as it has already been observed, $\alpha<1$ and $\beta>1$, and, due to the Schwartz inequality, $\alpha\beta=\mathbb{E}\,U\,\mathbb{E}\,U^{-1}>1$, we conclude that $p>0$. Therefore, by a standard calculation, see e.g. (3) in Wesołowski (1990), the only probabilistic solution of (10) has the form $L_Y(s)=\frac{c^p}{(c-s)^p}$, where c is a positive constant. Consequently, Y has the gamma distribution $\mathrm{G}(p,\,c)$.

Now we differentiate Eq. (8) for $s<0$ getting

$$bH_X'L_Y'+bH_XL_Y''=(L_XL_Y)''+(L_XL_Y)'.$$

Multiplying both sides by L_Y' and using again (8) we arrive at

$$bL_XL_Y'^2+((L_XL_Y)'+L_XL_Y)L_Y''=(L_XL_Y)''L_Y'+(L_XL_Y)'L_Y'$$

which yields

$$L_X''\tfrac{L_Y'}{L_Y}+L_X'\left[2\left(\tfrac{L_Y'}{L_Y}\right)^2+\tfrac{L_Y'}{L_Y}-\tfrac{L_Y''}{L_Y}\right]-L_X\left[\tfrac{L_Y''}{L_Y}-(1-b)\left(\tfrac{L_Y'}{L_Y}\right)^2\right]=0.$$

After inserting known values for L_Y, L_Y' and L_Y'' the above equation transforms into

$$(c-s)L_X''(s)+(p-1+c-s)L_X'(s)-(1+bp)L_X(s)=0,\quad s<0.$$

Change the variable $t:=c-s$ and define $N(t)=L_X(c-t)$. It follows that

$$tN''(t)+(1-p-t)N'(t)-(1+bp)N(t)=0\quad t>c.$$

We read two linearly independent solutions, M and U, of this equation from Abramovitz and Stegun (1964, Ch. 13). One of these solutions is the generalized hypergeometric function

$$N(t)=M(1+bp,1-p,t)={}_1F_1(1+bp,1-p,t)$$

which is of the order $e^t t^{(1+b)p}$ for $t\to\infty$, see (13.1.4) in Abramovitz and Stegun (1964). Consequently, $N(t)\to\infty$ as $t\to\infty$ yielding $L_X(s)\to\infty$ as $s\to-\infty$, the latter being impossible since the Laplace transform of a negative argument of a positive probability measure has to be bounded. The second solution

$$N(t)=U(1+bp,1-p,t)=C\int_0^\infty e^{-tx}\tfrac{x^{bp}}{(1+x)^{p(b+1)+1}}\,dx,$$

yields

$$L_X(s)=C\int_0^\infty e^{sx}\tfrac{x^{bp}}{(1+x)^{p(b+1)+1}}e^{-cx}\,dx,$$

which is the Laplace transform of the Kummer $\mathrm{K}(1+bp,p,c)$ distribution. $\square$

Remark 2.1. Note that, since the first parameter of the Kummer distribution of X is $1+bp>1$, it follows that $\mathbb{E}X^{-1}$ is finite. Similarly, $U\sim\mathrm{Beta}_I(1+bp,p)$ and thus $\mathbb{E}\,U^{-1}<\infty$, as expected.

Remark 2.2. It is still not clear if independence of U and V for independent, positive and non-degenerate X and Y without any additional assumptions characterizes the gamma and Kummer laws. Theorem 2.1 answers the question under additional restriction that $\mathbb{E}X^{-1}<\infty$.

Remark 2.3. Since U as defined in (1) is (0, 1) valued random variable, without any additional moment assumptions we can write regression conditions of the form

$$\mathbb{E}\left((1-U)^k\middle|\,V\right)=\mathbb{E}(1-U)^k=:\alpha_k \tag{11}$$

for any positive k. Obviously, such conditions are weaker than independence. A little of algebra allows to see that (11) is equivalent to

$$\mathbb{E}\left(\tfrac{Y^k}{(1+X)^k}\middle|\,X+Y\right)=\alpha_k(X+Y)^k.$$

However, we failed to prove characterization assuming (11) for, say, $k=1,2$.

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