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Singularities and Symplectic Geometry

Part VIII

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Approximation of sets defined by polynomials with holomorphic coefficients

Marcin Bilski [1] [2]

Abstract

Let X be an analytic set defined by polynomials whose coefficients $a_1, \ldots, a_s$ are holomorphic functions. We formulate conditions such that for all sequences $\{a_{1,\nu}\}, \ldots, \{a_{s,\nu}\}$ of holomorphic functions converging locally uniformly to $a_1, \ldots, a_s$ respectively the following holds true. If $a_{1,\nu}, \ldots, a_{s,\nu}$ satisfy the conditions then the sequence of the sets $\{X_\nu\}$ obtained by replacing a_j's by $a_{j,\nu}$'s in the polynomials, converge to X.

Keywords: Analytic set, Nash set, approximation
MSC (2000): 32C25

1 Introduction and main results

The problem of approximating analytic objects by simpler algebraic ones with similar properties appears in many contexts of complex geometry and hasattracted the attention of several mathematicians (see [2], [3], [10], [11], [14], [15], [16], [17], [19], [24], [25], [26]). In the present paper we concern the problem in the case where the approximated objects are complex analytic sets whereas the approximating ones are complex Nash sets (see Section 2.1). The approximation is expressed by means of the convergence of holomorphic chains (for the definition see Section 2.2).

For sets with proper projection the existence of such approximation was discussed in [5], [6]. In a subsequent paper [7] it was proved that the order of tangency of a limit set and the approximating sets can be arbitrarily high. The first results on approximation of analytic sets by higher order tangent

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algebraic varieties are due to R. W. Braun, R. Meise and B. A. Taylor [11] with applications in [12].

Both in [6] and in [7] analytic sets are represented as mappings defined on an open subset of $\mathbf{C}^n$ with values in an appropriate symmetric power of $\mathbf{C}^m$. However, in many cases such sets are defined by systems of equations which in general carry more information than the sets themselves. Therefore it is natural to look for approximations of the functions appearing in the equations. Throughout this paper we restrict our attention to the case where the description is given by a system of polynomials with holomorphic coefficients whereas the approximated set is with proper projection onto an appropriate affine space. Our aim is to show how to approximate the coefficients of the polynomials to obtain Nash approximations of the set.

If the number of the functions describing the analytic set X is equal to the codimension of X then it is sufficient to take generic approximations of the coefficients in order to get local uniform approximation of X. Such approach clearly does not work in the case of a non-complete intersection as it leads to sets of dimensions strictly smaller than the dimension of X. Yet, it is natural to expect that there are algebraic relations satisfied by the coefficients such that if the approximating coefficients also satisfy the relations then the original polynomials with these new coefficients define appropriate approximations.

Before stating the main result let us recall that for any analytic set Y by $Y_{(n)}$ we denote the union of all n-dimensional irreducible components of Y.

Let $U \subset \mathbf{C}^n$ be a domain. Abbreviate $v = (v_1, \dots, v_p)$, $z = (z_1, \dots, z_m)$. Assuming the notation of Section 2 and treating analytic sets as holomorphic chains with components of multiplicity one we prove

Theorem 1.1. *Let $q_1, \dots, q_s \in \mathbf{C}[v, z]$, for some $s \in \mathbf{N}$, and let $H : U \to \mathbf{C}^p$ be a holomorphic mapping. Assume that*

$$X = \{(x, z) \in U \times \mathbf{C}^m : q_i(H(x), z) = 0, i = 1, \dots, s\}$$

is an analytic set of pure dimension n with proper projection onto U. Then there is an algebraic subvariety F of $\mathbf{C}^p$ with $H(U) \subset F$ such that for every sequence $\{H_\nu : U \to F\}$ of holomorphic mappings converging locally uniformly to H the following holds. The sequence $\{X_\nu\}$, where

$$X_\nu = \{(x, z) \in U \times \mathbf{C}^m : q_i(H_\nu(x), z) = 0, i = 1, \dots, s\},$$

converges to X locally uniformly and the sequence $\{(X_\nu)_{(n)}\}$ converges to X in the sense of holomorphic chains.

The following example shows that the sets from $\{X_\nu\}$ are in general not purely dimensional:

Example 1.2. *Define $X = \{(x,z) \in \mathbf{C}^2 : zxe^x = 0, z^2 - zx = 0\}$. Then $X = \{(x,z) \in \mathbf{C}^2 : z = 0\}$, therefore it is purely 1-dimensional. On the other hand, $\mathbf{C}^2 \times \{1\}$ is the smallest algebraic set in $\mathbf{C}^3$ containing the image of the mapping $x \mapsto (-x, xe^x, 1)$. By approximating this mapping by $x \mapsto (-x, (x - \frac{1}{\nu})e^x, 1)$ one obtains $X_\nu = \{(x,z) \in \mathbf{C}^2 : z(x - \frac{1}{\nu})e^x = 0, z^2 - zx = 0\}$ containing an isolated point $(\frac{1}{\nu}, \frac{1}{\nu})$.*

Let U be a connected Runge domain in $\mathbf{C}^n$, let X be a purely n-dimensional analytic subset of $U \times \mathbf{C}^m$ with proper projection onto U and let $Q_1, \ldots, Q_s \in \mathcal{O}(U)[z]$, for some $s \in \mathbf{N}$, satisfy

$$X = \{(x,z) \in U \times \mathbf{C}^m : Q_1(x,z) = \ldots = Q_s(x,z) = 0\}.$$

(An example of such $Q_1, \ldots, Q_s$ are the canonical defining functions for X (see [29], [13]).)

We check that combining Theorem 1.1 with one of results of L. Lempert (Theorem 3.2 from [19], see Theorem 2.3 below) one obtains Nash approximations of X by approximating its holomorphic description by a Nash description. (Let us mention that the proof of Theorem 2.3 is based on the affirmative solution to the Artin's conjecture first presented in [21], [22], see also [1], [20], [23].)

Let $H = (H_1, \ldots, H_s)$ denote the holomorphic mapping defined on U where, for every $j \in \{1, \ldots, s\}$, H_j is the mapping whose components are all the non-zero coefficients of the polynomial Q_j; by n_j denote the number of these coefficients. More precisely, the components of H_j are indexed by m-tuples from some finite set $S_j \subset \mathbf{N}^m$ in such a way that the component indexed by a fixed $(\alpha_1, \ldots, \alpha_m)$ is the coefficient standing at the monomial $z_1^{\alpha_1} \cdot \ldots \cdot z_m^{\alpha_m}$ in Q_j.

Let F be the intersection of all algebraic subvarieties of $\mathbf{C}^{(\sum_j n_j)}$ containing $H(U)$ and let $\tilde{U}$ be any open relatively compact subset of U. Then $\tilde{U}$ is contained in a polynomially convex compact subset of U hence by Theorem 2.3 there exists a sequence $\{H_\nu : \tilde{U} \to F\}$ of Nash mappings,

$H_\nu = (H_{1,\nu}, \ldots, H_{s,\nu})$, such that $\{H_{j,\nu}\}$ converges uniformly to $H_j|_{\tilde{U}}$, for every $j = 1, \ldots, s$. Now let

$$X_\nu = \{(x, z) \in \tilde{U} \times \mathbf{C}^m : Q_{1,\nu}(x, z) = \ldots = Q_{s,\nu}(x, z) = 0\},$$

where $Q_{j,\nu} \in \mathcal{O}(\tilde{U})[z]$, for $j = 1, \ldots, s$, is defined as follows. The coefficient of $Q_{j,\nu}$ standing at the monomial $z_1^{\alpha_1} \cdot \ldots \cdot z_m^{\alpha_m}$ is the component of $H_{j,\nu}$ indexed by $(\alpha_1, \ldots, \alpha_m)$ (if $(\alpha_1, \ldots, \alpha_m) \notin S_j$ then the coefficient equals zero).

Finally, let $q_1, \ldots, q_s$ be the polynomials obtained from $Q_1, \ldots, Q_s$ by replacing the holomorphic coefficients of the latter polynomials by independent new variables. It is easy to see that $q_1, \ldots, q_s$ together with the mapping H satisfy the hypotheses of Theorem 1.1. Hence the sequence of Nash sets $\{(X_\nu)_{(n)}\}$, where X_ν defined in the previous paragraph, converges to $X \cap (\tilde{U} \times \mathbf{C}^m)$ in the sense of holomorphic chains. Thus we recover the main result of [6]:

Corollary 1.3. *Let X be a purely n-dimensional analytic subset of $U \times \mathbf{C}^m$ with proper projection onto U. Then for every open set $\tilde{U} \subset\subset U$ there is a sequence $\{X_\nu\}$ of purely n-dimensional Nash subsets of $\tilde{U} \times \mathbf{C}^m$ converging to $X \cap (\tilde{U} \times \mathbf{C}^m)$ in the sense of chains.*

Every purely n-dimensional analytic set is locally with proper projection onto an open subset of an n-dimensional affine space. Hence, by Corollary 1.3 every analytic set can be locally approximated by Nash ones. Let us mention that to obtain this result, one does not need to use the advanced methods of commutative algebra; see [8] for a purely geometrical approach to the problem. As for the local version of Theorem 2.3, it can be derived by combining the ideas of [2] and [15] or [8] (see [9]).

Note that the convergence of positive chains appearing in this paper is equivalent to the convergence of currents of integration over the considered sets (see [18], [13]). The organization of this paper is as follows. In Section 2 preliminary material is presented whereas Section 3 contains the proof of Theorem 1.1.

2 Preliminaries

2.1 Nash sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. A holomorphic mapping defined on Ω with values in $\mathbf{C}^N$ is said to be a Nash mapping if each of its components is a Nash function.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions $f_1, \ldots, f_s$ on U such that

$$Y \cap U = \{x \in U : f_1(x) = \ldots = f_s(x) = 0\}.$$

The fact from [27] stated below explains the relation between Nash and algebraic sets.

Theorem 2.1. *Let X be an irreducible Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then there exists an algebraic subset Y of $\mathbf{C}^n$ such that X is an analytic irreducible component of $Y \cap \Omega$. Conversely, every analytic irreducible component of $Y \cap \Omega$ is an irreducible Nash subset of Ω.*

2.2 Convergence of closed sets and holomorphic chains

Let U be an open subset in $\mathbf{C}^m$. By a holomorphic chain in U we mean the formal sum $A = \sum_{j\in J} \alpha_j C_j$, where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{C_j\}_{j\in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of U (see [28], cf. also [4], [13]). The set $\bigcup_{j\in J} C_j$ is called the support of A and is denoted by $|A|$ whereas the sets C_j are called the components of A with multiplicities α_j. The chain A is called positive if $\alpha_j > 0$ for all $j \in J$. If all the components of A have the same dimension n then A will be called an $n-$chain.

Below we introduce the convergence of holomorphic chains in U. To do this we first need the notion of the local uniform convergence of closed sets.

Let Y, Y_ν be closed subsets of U for $\nu \in \mathbf{N}$. We say that $\{Y_\nu\}$ converges to Y locally uniformly if:

(1l) for every $a \in Y$ there exists a sequence $\{a_\nu\}$ such that $a_\nu \in Y_\nu$ and $a_\nu \to a$ in the standard topology of $\mathbf{C}^m$,
(2l) for every compact subset K of U such that $K \cap Y = \emptyset$, $K \cap Y_\nu = \emptyset$ holds for almost all ν.

Then we write $Y_\nu \to Y$. For details concerning the topology of local uniform convergence see [28].

We say that a sequence $\{Z_\nu\}$ of positive n-chains converges to a positive n-chain Z if:

(1c) $|Z_\nu| \to |Z|$,
(2c) for each regular point a of $|Z|$ and each submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact and $|Z| \cap \overline{T} = \{a\}$, we have $deg(Z_\nu \cdot T) = deg(Z \cdot T)$ for almost all ν.

Then we write $Z_\nu \mapsto Z$. (By $Z \cdot T$ we denote the intersection product of Z and T (cf. [28]). Observe that the chains $Z_\nu \cdot T$ and $Z \cdot T$ for sufficiently large ν have finite supports and the degrees are well defined. Recall that for a chain $A = \sum_{j=1}^{d} \alpha_j \{a_j\}$, $deg(A) = \sum_{j=1}^{d} \alpha_j$).

The following lemma from [28] will be useful to us.

Lemma 2.2. *Let $n \in \mathbf{N}$ and Z, Z_ν, for $\nu \in \mathbf{N}$, be positive n-chains. If $|Z_\nu| \to |Z|$ then the following conditions are equivalent:*
(1) $Z_\nu \mapsto Z$,
(2) for each point a from a given dense subset of $Reg(|Z|)$ there exists a submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact, $|Z| \cap \overline{T} = \{a\}$ and $deg(Z_\nu \cdot T) = deg(Z \cdot T)$ for almost all ν.

2.3 Approximation of holomorphic mappings

In the proof of Corollary 1.3 we use the following theorem which is due to L. Lempert (see [19], Theorem 3.2).

Theorem 2.3. *Let K be a holomorphically convex compact subset of $\mathbf{C}^n$ and $f : K \to \mathbf{C}^k$ a holomorphic mapping that satisfies a system of equations $Q(z, f(z)) = 0$ for $z \in K$. Here Q is a Nash mapping from a neighborhood*

$U \subset \mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Then f can be uniformly approximated by a Nash mapping $F : K \to \mathbf{C}^k$ satisfying $Q(z, F(z)) = 0$.

3 Proof of Theorem 1.1

Denote $B_m(r) = \{z \in \mathbf{C}^m : ||z||_{\mathbf{C}^m} < r\}$ and recall $v = (v_1, \ldots, v_p)$. Let U be a domain in $\mathbf{C}^n$. We prove the following

Proposition 3.1. *Let $q_1, \ldots, q_s \in \mathbf{C}[v, z]$, for some $s \in \mathbf{N}$, and let $H : U \to \mathbf{C}^p$ be a holomorphic mapping. Assume that*

$$X = \{(x, z) \in U \times \mathbf{C}^m : q_i(H(x), z) = 0, i = 1, \ldots, s\}$$

is an analytic set of pure dimension n with proper projection onto U. Then there is an algebraic subvariety F of $\mathbf{C}^p$ with $H(U) \subset F$ such that for every domain $\tilde{U} \subset\subset U$ and every sequence $\{H_\nu : \tilde{U} \to F\}$ of holomorphic mappings converging uniformly to H on $\tilde{U}$ the following holds. There is $r_0 > 0$ such that for every $r > r_0$ the sequence $\{X_\nu\}$, where

$$X_\nu = \{(x, z) \in \tilde{U} \times B_m(r) : q_i(H_\nu(x), z) = 0, i = 1, \ldots, s\},$$

satisfies:

(1) X_ν is n-dimensional with proper projection onto $\tilde{U}$ for almost all ν,
(2) $\max\{\sharp(X \cap (\{x\} \times \mathbf{C}^m)) : x \in U\} =$
$\max\{\sharp((X_\nu)_{(n)} \cap (\{x\} \times \mathbf{C}^m)) : x \in \tilde{U}\}$ *for almost all ν,*
(3) $\{X_\nu\}$, $\{(X_\nu)_{(n)}\}$ converge to $X \cap (\tilde{U} \times \mathbf{C}^m)$ locally uniformly.

Proof of Proposition 3.1. Define the algebraic set

$$V = \{(v, z) \in \mathbf{C}^p \times \mathbf{C}^m : q_i(v, z) = 0, i = 1, \ldots, s\}.$$

Next, by F denote the intersection of all algebraic subsets of $\mathbf{C}^p$ containing the image of H. Clearly, F is irreducible (because U is connected) hence of pure dimension, say $\bar{n}$. Fix an open connected subset $\tilde{U} \subset\subset U$. In the following lemma F is endowed with the topology induced by the standard topology of $\mathbf{C}^p$.

Lemma 3.2. *Let $r > 0$ be such that $(\tilde{U} \times B_m(r)) \cap X \neq \emptyset$ and $(\overline{\tilde{U}} \times \partial B_m(r)) \cap X = \emptyset$. Then there is an open neighborhood C of $\overline{H(\tilde{U})}$ in F such that $(C \times B_m(r)) \cap V$ is $\bar{n}$-dimensional with proper projection onto C. Moreover, for every $(a, z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$ it holds $dim_{(a,z)}((C \times B_m(r)) \cap V) = \bar{n}$.*

Proof of Lemma 3.2. First we check that there is an open neighborhood C of $\overline{H(\tilde{U})}$ in F such that $(\overline{C} \times \partial B_m(r)) \cap V = \emptyset$, which implies the properness of the projection of $(C \times B_m(r)) \cap V$ onto C.

It is sufficient to show that for every $a \in \overline{H(\tilde{U})}$ there is an open neighborhood C_a in F such that $(C_a \times \partial B_m(r)) \cap V = \emptyset$. Fix $a \in \overline{H(\tilde{U})}$. Now, if for every open neighborhood C_a of a we had $(C_a \times \partial B_m(r)) \cap V \neq \emptyset$ then there would be $(\{a\} \times \partial B_m(r)) \cap V \neq \emptyset$. But then $(\overline{\tilde{U}} \times \partial B_m(r)) \cap X \neq \emptyset$ as $a \in \overline{H(\tilde{U})} \subset H(\overline{\tilde{U}})$, a contradiction.

Let us show that $dim_{(a,z)}((C \times B_m(r)) \cap V) = \bar{n}$ for every $(a, z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$. First observe that $dim((C \times B_m(r)) \cap V)$ cannot exceed the dimension of C because $(C \times B_m(r)) \cap V$ is with proper projection onto C. Next suppose that there is $(a, z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$ such that $dim_{(a,z)}((C \times B_m(r)) \cap V) < \bar{n}$. Let V_1 be the union of the irreducible analytic components of $(C \times B_m(r)) \cap V$ containing (a, z) and let $\pi : \mathbf{C}^p \times \mathbf{C}^m \to \mathbf{C}^p$ denote the natural projection. It is easy to see that $H^{-1}(\pi(V_1))$ is a non-empty nowhere dense analytic subset of $H^{-1}(C)$ (nowhere-density because otherwise $H(U)$ would be contained in an algebraic set of dimension smaller than $\bar{n}$). Let P be a neighborhood of (a, z) in $C \times B_m(r)$ such that $P \cap V = P \cap V_1 \neq \emptyset$. Now consider the set

$$E = \{(w, y) \in (U \times B_m(r)) \cap X : (H(w), y) \in P \cap V\}.$$

One observes that $E \neq \emptyset$, because $H^{-1}(\{a\}) \times \{z\} \subset E$, and that E has a non-empty interior in X, and moreover, the projection of E onto U is contained in $H^{-1}(\pi(V_1))$. This contradicts the fact that X is purely n-dimensional.

Since $(\tilde{U} \times B_m(r)) \cap X \neq \emptyset$ then $(H(\tilde{U}) \times B_m(r)) \cap V \neq \emptyset$ so by what we have proved so far $(C \times B_m(r)) \cap V$ is $\bar{n}$-dimensional. □

Proof of Proposition 3.1 (continuation). Let $r_0 > 0$ be such that $(\tilde{U} \times B_m(r_0)) \cap X = (\tilde{U} \times \mathbf{C}^m) \cap X$ and let $r > r_0$. Then $(\overline{\tilde{U}} \times \partial B_m(r)) \cap X = \emptyset$

and by Lemma 3.2, there is a neighborhood C of $\overline{H(\tilde{U})}$ in F such that $(C \times B_m(r)) \cap V$ is $\bar{n}$-dimensional with proper projection onto C. Moreover, for every $(a,z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$ it holds $dim_{(a,z)}((C \times B_m(r)) \cap V) = \bar{n}$. Let $\{H_\nu : \tilde{U} \to F\}$ be a sequence of holomorphic mappings converging uniformly to H on $\tilde{U}$. Define the sequence $\{X_\nu\}$ as in the statement of Proposition 3.1.

First we show (1): X_ν is n-dimensional and with proper projection onto $\tilde{U}$ for almost all ν. To do this observe that for sufficiently large ν it holds $H_\nu(\tilde{U}) \subset C$ and then

$$X_\nu = \{(x,z) \in \tilde{U} \times B_m(r) : (H_\nu(x), z) \in (C \times B_m(r)) \cap V\}.$$

Thus the properness of the projection of X_ν onto $\tilde{U}$ is obvious by the choice of C in Lemma 3.2.

Now we check the following claim: for sufficiently large ν every fiber in X_ν over $\tilde{U}$ is not empty. Indeed, let C_0 denote the irreducible Nash component of C containing $H(\tilde{U})$. Then the projection of $(C_0 \times B_m(r)) \cap V$ onto C_0 is surjective which follows by Lemma 3.2. On the other hand, for sufficiently large ν, $H_\nu(\tilde{U}) \subset C_0$ which clearly implies the claim. Consequently, X_ν is n-dimensional for almost all ν.

Let us turn to (2). Since C_0 is an irreducible Nash set then $Reg(C_0)$ is connected. There is a nowhere dense Nash subset C' of C_0 such that the function $\rho : Reg(C_0) \setminus C' \to \mathbf{N}$ given by

$$\rho(v) = \sharp((\{v\} \times B_m(r)) \cap V)$$

is constant. By $\tilde{m}$ we denote the only value of ρ.

Neither $H(\tilde{U})$ nor $H_\nu(\tilde{U})$ (for large ν) can be contained in $Sing(C_0) \cup C'$ so $(H^{-1}(Sing(C_0) \cup C') \cup H_\nu^{-1}(Sing(C_0) \cup C')) \cap \tilde{U}$ is a nowhere dense analytic subset of $\tilde{U}$. This means that for the generic $x \in \tilde{U}$ the fibers in X and in X_ν over x have $\tilde{m}$ elements which completes the proof of (2).

Finally, let us prove (3). To check the condition (2l) of the definition of local uniform convergence it is sufficient to show that for every $(x_0, z_0) \in (\tilde{U} \times \mathbf{C}^m) \setminus X$ there is a neighborhood D of (x_0, z_0) in $\tilde{U} \times \mathbf{C}^m$ such that $D \cap X_\nu = \emptyset$ for almost all ν. This is obvious as there is $i \in \{1, \ldots, s\}$ such that $q_i(H(x_0), z_0) \neq 0$. Then $q_i(H_\nu(x_0), z_0) \neq 0$ for almost all ν in some neighborhood of (x_0, z_0).

As for the condition (1l), it suffices to show that for a fixed $x_0 \in \tilde{U} \setminus H^{-1}(Sing(C))$ the sequence $\{(\{x_0\} \times \mathbf{C}^m) \cap (X_\nu)_{(n)}\}$ converges to $(\{x_0\} \times \mathbf{C}^m) \cap X$ locally uniformly. Take $(x_0, z_0) \in X \cap (\tilde{U} \times \mathbf{C}^m) = X \cap (\tilde{U} \times B_m(r))$. Then by Lemma 3.2 it holds $dim_{(H(x_0),z_0)}(C \times B_m(r)) \cap V = dim(C)$. Consequently, (since $H(x_0) \in Reg(C)$ and $(C \times B_m(r)) \cap V$ is with proper projection onto C) there is a sequence $\{z_\nu\}$ converging to z_0 such that $dim_{(H_\nu(x_0),z_\nu)}(C \times B_m(r)) \cap V = dim(C)$ for almost all ν. This implies that for sufficiently large ν, the image of the projection of every open neighborhood of (x_0, z_ν) in X_ν onto $\tilde{U}$ contains a neighborhood of x_0 in $\tilde{U}$. Thus $(x_0, z_\nu) \in (X_\nu)_{(n)}$ for almost all ν and the proof is complete. □

Proof of Theorem 1.1 (end). Let F denote the intersection of all algebraic subvarieties of $\mathbf{C}^p$ containing $H(U)$ and let $\{H_\nu : U \to F\}$ be a sequence of holomorphic mappings converging locally uniformly to H. Define X_ν as in the statement of Theorem 1.1.

It is sufficient to show that for every relatively compact subset $\tilde{U}$ of U the sequences $\{X_\nu \cap (\tilde{U} \times \mathbf{C}^m)\}$ and $\{(X_\nu)_{(n)} \cap (\tilde{U} \times \mathbf{C}^m)\}$ converge to $X \cap (\tilde{U} \times \mathbf{C}^m)$ locally uniformly and in the sense of holomorphic chains respectively. Fix $\tilde{U} \subset\subset U$. Then by Proposition 3.1 there is r_0 such that for every $r > r_0$ the following hold. $\{X_\nu \cap (\tilde{U} \times B_m(r))\}$ and $\{(X_\nu)_{(n)} \cap (\tilde{U} \times B_m(r))\}$ converge to $X \cap (\tilde{U} \times \mathbf{C}^m)$ locally uniformly. Moreover, for almost all ν, $X_\nu \cap (\tilde{U} \times B_m(r))$ is n-dimensional with proper projection onto $\tilde{U}$ and $\max\{\sharp(X \cap (\{x\} \times \mathbf{C}^m)) : x \in \tilde{U}\} = \max\{\sharp((X_\nu)_{(n)} \cap (\{x\} \times B_m(r))) : x \in \tilde{U}\}$. Thus by Lemma 2.2 we have: $\{(X_\nu)_{(n)} \cap (\tilde{U} \times B_m(r))\}$ converges to $X \cap (\tilde{U} \times \mathbf{C}^m)$ in the sense of holomorphic chains. Since r can be taken arbitrarily large we get our claim. □

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On smooth real-compactness of countably generated differential spaces

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Abstract

We will show that if a differential structure of a differential space in the Sikorski [3, 4] sense has countable number of generators then all its real homomorphisms are evaluations.

1 Introduction

When all the real homomorphisms defined on an algebra of functions of some kind of space are evaluations then we say that such a space is smoothly real-compact. There are many articles stating about this property of some spaces. In the articles [6],[7] it is shown that the spaces of real continuous functions on $\mathbf{R}$ and $\mathbf{R}^n$ are smoothly real-compact. In [10] this property has been shown for the spaces of the functions of class C^k $(k = 1, \ldots, \infty)$ on separable Banach spaces. Many discussions on this topic can be found in [9]. The most important from the point of view of Sikorski spaces is the article [1] since it discusses smooth real-compactness of smooth spaces, which are a wider category than the Sikorski spaces. Many conditions for this spaces to be smoothly real-compact are given there. In our article we give other conditions using techniques proper for Sikorski spaces. The concept of generators of the structure is crucial. Some results were already obtained before but we have given other proofs because of usage of this concept. We have obtained that if there exists at most countable set of generators then the differential space is smoothly real-compact.

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2 Basic concepts and definitions

Let M be a nonempty set and $\mathcal{C}$ a set of real functions on M. We introduce on M topology $\tau_{\mathcal{C}}$ - the weakest topology in which the functions from $\mathcal{C}$ are continuous. We say that the set $\mathcal{C}$ is closed with respect to superposition if all functions of the form $\omega\circ(f_1,\ldots,f_n)$ where $f_1,\ldots,f_n\in\mathcal{C}, \quad \omega\in C^\infty(\mathbf{R}^n)$ are in C. Adding to $\mathcal{C}$ all the functions of this form we obtain its superposition closure; we denote it $sc\mathcal{C}$. For any $A\subseteq M$ the symbol $\mathcal{C}_A$ will denote the set of all functions f on A such that for any $p\in A$ there exists an open neighbourhood $U\in\tau_{\mathcal{C}}$ of p and a function $g\in\mathcal{C}$ such that $f|_{U\cap A}=g|_{U\cap A}$. If $\mathcal{C}=\mathcal{C}_M$ then we say that $\mathcal{C}$ is closed with respect to localization. We call the set of real functions $\mathcal{C}$ on a nonempty set M a differential structure if it is:

1) Closed with respect to superposition $\mathcal{C}=sc\mathcal{C}$.

2) Closed with respect to localization $\mathcal{C}=\mathcal{C}_M$.

A differential structure is always an algebra with unity and with all constant functions.

Definition 2.1. *A pair $(M,\mathcal{C})$ is a differential space if M is a nonempty set and $\mathcal{C}$ a differential structure on it.*

We call a differential subspace of the differential space $(M,\mathcal{C})$ any pair $(A,\mathcal{C}_A)$ where $A\subseteq M$.

Definition 2.2. *The differential structure $(M,\mathcal{C})$ is generated by the set of functions $\mathcal{C}_0$ if $\mathcal{C}$ is the smallest differential structure that contains $\mathcal{C}_0$. Then we write $\mathcal{C}=Gen\mathcal{C}_0$.*

If $\mathcal{C}=gen\mathcal{C}_0$ then $\mathcal{C}=(sc\mathcal{C}_0)_M$, and for any $f\in\mathcal{C}$ and any point $p\in M$ there exists an open neighbourhood $U\in\tau_{\mathcal{C}}$ of p and there exist functions $f_1,\ldots,f_n\in\mathcal{C}_0, \quad \omega\in C^\infty(\mathbf{R}^n)$ such that $f|_U=\omega\circ(f_1,\ldots,f_n)|_U$. We say that the differential space $(M,\mathcal{C})$ is finitely generated if there exists a finite set of real functions on M that generates the differential structure $\mathcal{C}$. A differential space is countably generated if there exists a countable set of real functions on M that generates the differential structure $\mathcal{C}$ and the structure $\mathcal{C}$ is not finitely generated.

By $(\mathbf{R}^I,\varepsilon_I)$ we denote the differential space with the structure ε_I generated by the set of projections $\mathcal{C}_0=\{\pi_i : i\in I\}$, where

$\pi_i : \mathbf{R}^I \to \mathbf{R}$ is defined by: $\pi_i(x) = x_i$ for $x = \{(x_i) : i \in I\}$. This is a generalization of the Euclidean space $(\mathbf{R}^n, \varepsilon_n)$ where $\varepsilon_n = C^\infty(\mathbf{R}^n)$.

The spectrum of an algebra $\mathcal{C}$ is the set
$Spec\mathcal{C} = \{\chi : \mathcal{C} \to \mathbf{R} : \chi$ is homomorphism that preserves unity$\}$.
Let $(M, \mathcal{C})$ be a differential space. Evaluation of the algebra $\mathcal{C}$ at the point $p \in M$ is the homomorphism $\chi \in Spec\mathcal{C}$ of the following form:

$$\chi(f) = f(p) \quad \forall f \in \mathcal{C}; \tag{1}$$

we will denote this homomorphism by ev_p. We will define the mapping $ev : M \to Spec\mathcal{C}$ by the formula:

$$ev(p) = ev_p \tag{2}$$

Definition 2.3. *We say that a differential space $(M, \mathcal{C})$ is smoothly real-compact iff any $\chi \in Spec\mathcal{C}$ is an evaluation at some point $p \in M$.*

From this definition it follows that the space $(M, \mathcal{C})$ is smoothly real-compact when the mapping ev is onto. For any $f \in \mathcal{C}$ we define the function $\hat{f} : Spec\mathcal{C} \to \mathbf{R}$ by the formula:

$$\hat{f}(\chi) = \chi(f) \quad \forall \chi \in Spec\mathcal{C} \tag{3}$$

The set of all functions of the form $\hat{f}$ will be denoted by $\hat{C}$. By $\tau : \mathcal{C} \to \hat{\mathcal{C}}$ we will denote the mapping defined as follows:

$$\tau(f) = \hat{f} \quad \forall f \in \mathcal{C} \tag{4}$$

The mapping τ is an isomorphism between the algebra $\mathcal{C}$ and the algebra $\hat{C}$.

3 Main results

Lemma 3.1. *The differential space $(\mathbf{R}^n, \varepsilon_n)$ is smoothly real-compact.*

Proof. Let $\chi \in Spec\varepsilon_I$. We define a point $p \in \mathbf{R}^n$ by the equations $p_i := \chi(\pi_i)$ for $i = 1, \dots, n$. We will show that $\chi = ev_p$. We know from [4] that any $f \in \varepsilon_n$ can be presented in the form:

$$f = f(p) + \sum_{i=1}^{n} g_i(\pi_i - p_i), \quad for \quad g_1, \dots, g_n \in \varepsilon_n. \tag{5}$$

Then $\chi(f) = \chi(f(p)) + \sum_{i=1}^{n} \chi(g_i)(\chi(\pi_i) - \chi(p_i)) = f(p)+$
$\sum_{i=1}^{n} \chi(g_i)(p_i - p_i) = f(p)$. Therefore $\chi(f) = f(p)$ for all $f \in \varepsilon_n$. $\square$

Lemma 3.2. *A differential subspace of the differential space* $(\mathbf{R}^n, \varepsilon_n)$ *is smoothly real-compact.*

Proof. Let $(M, \mathcal{C})$ be a differential subspace of $(\mathbf{R}^n, \varepsilon_n)$. The inclusion mapping $\iota_M : M \to \mathbf{R}^n$ is smooth and therefore
$\iota_M^* : \varepsilon_n \to M$ is a homomorphism. From the definition we know that $\iota_M(f) = f|_M \quad \forall f \in \varepsilon_n$. For any $\chi \in Spec\mathcal{C}$ the mapping
$\chi \circ \iota_M^* \in Spec\varepsilon_n$. From Lemma 3.1 we know that $\exists p \in \mathbf{R}^n$ such that $\chi \circ \iota_M^*(f) = \chi(\iota_M^*(f)) = \chi(f|_M) = f(p) \quad \forall f \in \varepsilon_n$. Let us suppose that $p \notin M$. There exists the function $\omega \in \varepsilon_n$ defined by the formula:

$$\omega(x_1, \ldots, x_n) = (x_1 - p_1)^2 + \cdots + (x_n - p_n)^2 \qquad (6)$$

The function $\omega|_M > 0$ so $\frac{1}{\omega|_M} \in \mathcal{C}$. We also know that $\chi((\omega|_M)(\frac{1}{\omega|_M})) = \chi(1) = 1$, and $(\chi \circ \iota_M^*)(\omega) = \chi(\omega|M) = \omega(p) = 0$. So it is a contradiction.

We will show that $\chi = ev_p$. Let $f \in \mathcal{C}$. There exists an open neighbourhood $U \in \tau_{\varepsilon_n}$ of the point p and a function $\kappa \in \varepsilon_n$ such that $f|_{U\cap M} = \kappa|_{U\cap M}$. From [4] we know that there exists a bump function $\phi \in \varepsilon_n$ and $\phi(p) = 1$, $\phi|_{M\cap U} > 0$ such that $\phi|_{(\mathbf{R}^n - (M\cap U))} = 0$. From these properties it follows that $(f - \kappa|_M)\phi|_M = 0$. Then $\chi((f - \kappa|_M)\phi|_M) = (\chi(f) - \chi(\kappa|_M))\chi(\phi|_M) = 0$. But
$\chi(\phi|_M) = (\iota_M \circ \chi)(\phi) = \phi(p) = 1$ so $\chi(f) = \chi(\kappa|_M) = \kappa(p) = f(p)$. We have shown that $\chi(f) = f(p) \quad \forall f \in \mathcal{C}$.

When the differential structure $\mathcal{C}$ of the differential space $(M, \mathcal{C})$ is generated by the set of functions $\mathcal{C}_0$ then we can define the mapping $\phi : M \to \mathbf{R}^{\mathcal{C}_0}$ by the following formula:

$$\phi(p)(f) = f(p) \quad f \in \mathcal{C}_0 \qquad (7)$$

We will call this mapping generatory embedding. We can prove the following lemma:

Lemma 3.3. *A differential space* $(M, \mathcal{C})$ *with* $\mathcal{C} = Gen\mathcal{C}_0$ *is smoothly real-compact iff the differential space* $(\phi(M), (\varepsilon_I)_{\phi(M)})$ *for* $I = |\mathcal{C}_0|$ *is smoothly real-compact.*

Proof. If $\mathcal{C}_0$ separates points of M then ϕ is a diffeomorphism on its image and all is clear. So let us assume that $\mathcal{C}_0$ does not separate points. Then $\bar{\phi} : M \to \phi(M)$ where $\bar{\phi}(p) = \phi(p)$ is surjective but not injective. Let us denote $F := \bar{\phi}$. We know that $F^* : (\varepsilon_I)_{\phi(M)} \to \mathcal{C}$ is an isomorphism of algebras. If $(M, \mathcal{C})$ has the spectral property then for any $\nu \in Spec(\varepsilon_I)_{\phi(M)}$ there exists $\mu \in Spec\mathcal{C}$ such that $\mu = \nu \circ (F^*)^{-1}$. Then for any $g \in (\varepsilon_I)_{\phi(M)}$ $\nu(g) = \mu(F^*(g)) = \mu(g \circ F) = g(F(p))$. So if $\mu = ev_p$ then $\nu = ev_{F(p)}$.

If $(\phi(M), (\varepsilon_I)_{\phi(M)})$ is smoothly real-compact then for any $\mu \in Spec\mathcal{C}$ there exists $\nu \in Spec(\varepsilon_I)_{\phi(M)}$ defined by $\nu = \mu \circ F^*$; then $\mu = \nu \circ (F^*)^{-1}$. Therefore for any $f \in \mathcal{C}$ we have $\mu(f) = (\nu \circ (\phi^*)^{-1})(f) = \nu((\phi^*)^{-1}(f)) = ((\phi^*)^{-1}(f))(q) = f(p)$ for any $p \in F^{-1}(q)$. So if $\nu = ev_q$ then $\mu = ev_p \quad \forall p \in F^{-1}(q)$. □

From last lemma we know that it is sufficient to work on subspaces of Euclidean spaces.

Corollary 3.4. *Let $(M, \mathcal{C})$ be a differential space with $\mathcal{C} = Gen\mathcal{C}_0$ for some finite $\mathcal{C}_0$. Then $(M, \mathcal{C})$ is smoothly real-compact.*

Proof. By using the generators $\mathcal{C}_0$ we can embed $(M, \mathcal{C})$ into $(\mathbf{R}^{\mathcal{C}_0}, (\varepsilon_{\mathcal{C}_0})_{\phi(M)})$ and then from Lemmas 3.2,3.3 we derive that $(M, \mathcal{C})$ is smoothly real-compact. □

Lemma 3.5. *Let $(M, \mathcal{C})$ be a differential space. Any $\chi \in Spec\mathcal{C}$ satisfies the following condition:*

$$\chi(\omega \circ (f_1, \ldots, f_n)) = \omega(\chi(f_1), \ldots, \chi(f_n)) \tag{8}$$

for all $\omega \in \varepsilon_n$ and $f_1, \ldots, f_n \in \mathcal{C}$.

Proof. Let $\beta_1, \ldots, \beta_n \in \mathcal{C}$ be arbitrary functions. We can define the mapping $F : (M, \mathcal{C}) \to (\mathbf{R}^n, \varepsilon_n)$ by the formula:

$$F(p) = (\beta_1(p), \ldots, \beta_n(p)) \quad p \in M$$

This mapping is smooth and it is onto its image. Therefore the mapping $F^* : (\varepsilon_n)_{F(M)} \to \mathcal{C}$ is a homomorphism. For any $\chi \in Spec\mathcal{C}$ the composition $\chi \circ F^* \in Spec((\varepsilon_n)_{F(M)})$. From Corollary 3.4 we know that $\exists q \in F(M)$ s.t. $\chi \circ F^* = ev_q$ for some $q \in F(M)$. Also $\exists p \in M$ s.t.

$$(\chi \circ F^*)(\omega|_{F(M)}) = ev_{F(p)}(\omega|_{F(M)}) \quad \forall \omega \in \varepsilon_n$$

We can rewrite it in the form:

$$\chi(\omega \circ F) = \omega(F(p)) = \omega(\beta_1(p), \ldots, \beta_n(p)) \quad \forall \omega \in \varepsilon_n$$

By setting $\omega = \pi_i$, $i = 1, \ldots, n$ we obtain: $\chi(\beta_i) = \chi(\pi_i \circ F) = \pi_i(F(p)) = \beta_i(p)$ and finally: $\chi(\omega \circ (\beta_1, \ldots, \beta_n)) = \omega(\chi(\beta_1), \ldots, \chi(\beta_n))$ $\forall \omega \in \varepsilon_n$. □

Lemma 3.6. *Let $(M, \mathcal{C})$ be a differential space such that $\mathcal{C} = Gen\mathcal{C}_0$. If some $\chi \in Spec\mathcal{C}$ satisfies the condition $\chi|_{\mathcal{C}_0} = ev_p|_{\mathcal{C}_0}$ then $\chi = ev_p$.*

Proof. First we will show that if $f \in sc\mathcal{C}_0$ then $\chi(f) = f(p)$. From Lemma 3.5 we know that $\chi(\omega \circ (\beta_1, \ldots, \beta_n)) = \omega(\chi(\beta_1), \ldots, \chi(\beta_n))$ for $\omega \in \varepsilon_n$ and $\beta_1, \ldots, \beta_n \in \mathcal{C}_0$. We also know that $\chi(\beta_i) = ev_p(\beta_i) = \beta_i(p)$. We can write $\chi(\omega \circ (\beta_1, \ldots, \beta_n) = \omega(\beta_1(p), \ldots, \beta_n(p)) = \omega \circ (\beta_1, \ldots, \beta_n)(p) = ev_p(\omega \circ (\beta_1, \ldots, \beta_n))$. So we see that $\chi|_{sc\mathcal{C}_0} = ev_p|_{sc\mathcal{C}_0}$.

Now let $f \in \mathcal{C}$ be an arbitrary function. We know that $\forall p \in M$ there exists an open neighbourhood $U \in \tau_{\mathcal{C}}$, functions $\beta_1, \ldots, \beta_n \in \mathcal{C}_0$ and a function $\omega \in \varepsilon_n$ s.t. $f|_U = \omega \circ (\beta_1, \ldots, \beta_n)|_U$. There also exists a bump function ψ which separates the point p in the set U. This function is constructed from composition of some function from ε_n with some generators from $\mathcal{C}_0$. We know that the homomorphism χ is the evaluation at the point p on this function, so $\chi(\phi) = \phi(p) = 1$. Now the following equality holds: $\phi \cdot (f - \omega \circ (\beta_1, \ldots, \beta_n)) = 0$. By applying homomorphism χ to this equality we will obtain: $\chi(\phi) \cdot \chi(f - \omega \circ (\beta_1, \ldots, \beta_n)) = \chi(f) - \chi(\omega \circ (\beta_1, \ldots, \beta_n)) = 0$ so $\chi(f) = \chi(\omega \circ (\beta_1, \ldots, \beta_n)) = ev_p(\omega \circ (\beta_1, \ldots, \beta_n)) = f(p) = ev_p(f)$. We see that $\chi(f) = f(p) \quad \forall f \in \mathcal{C}$. □

As an obvious corollary from this lemma we get:

Corollary 3.7. *The differential space $(\mathbf{R}^I, \varepsilon_I)$ is smoothly real-compact.*

Proof. Let $\chi \in Spec\varepsilon_I$ be any homomorphism. We can define the point $p \in \mathbf{R}^I$ by the equations $\pi_i = \chi(\pi_i)$ for $i \in I$. Then $\chi(\pi_i) = \pi_i(p)$ so $\chi(\pi_i) = ev_p(\pi_i)$. Since the structure ε_I is generated by the set $\{\pi_i : i \in I\}$ we see that χ is the evaluation at the point p on the generators. From the last lemma we derive that χ is an evaluation on whole ε_I. □

By using whole $\mathcal{C}$ as the set of generators we can embed M in $\mathbf{R}^{\mathcal{C}}$. We denote this embedding by ι, so $\iota : M \to \mathbf{R}^{\mathcal{C}}$, $\iota(p)_f = f(p)$. This is a special case of generatory embedding. We can also map $Spec\mathcal{C}$ into $\mathbf{R}^{\mathcal{C}}$ using the mapping $\kappa : Spec\mathcal{C} \to \mathbf{R}^{C}$ defined by: $\kappa(\chi)_f = \hat{f}(\chi) = \chi(f)$. It is obvious that $\iota = \kappa \circ ev$. In [1] Kriegl, Michor and Schachermayer have shown that $\iota(M)$ is dense in $\kappa(Spec\mathcal{C})$ in the Tichonov topology in $\mathbf{R}^{\mathcal{C}}$. Since the mapping κ is a homeomorphism we derive that:

Corollary 3.8. *$ev(M)$ is dense in $Spec\mathcal{C}$ in the topology $\tau_{\hat{\mathcal{C}}}$.*

This property will allow us to prove an interesting fact about the space $(Spec\mathcal{C}, \hat{\mathcal{C}})$.

Lemma 3.9. *If $(M, \mathcal{C})$ is a differential space then $(Spec\mathcal{C}, \hat{\mathcal{C}})$ is a differential space.*

Proof. To prove that $(Spec\mathcal{C}, \hat{\mathcal{C}})$ is a differential space we will have to show that the set $\hat{\mathcal{C}}$ is closed with respect to superposition with smooth functions from ε_n and closed with respect to localization.

Let us define the function $g = \omega \circ (\hat{f}_1, \ldots, \hat{f}_n)$ for some $\omega \in \varepsilon_n$ and $\hat{f}_1, \ldots, \hat{f}_n \in \hat{\mathcal{C}}$. From Lemma 3.5 we know that $g(\chi) = \omega \circ (\hat{f}_1, \ldots, \hat{f}_n)(\chi) = \tau(\omega \circ (f_1, \ldots, f_n))(\chi) \quad \forall \chi \in Spec\mathcal{C}$. We have shown that $g \in \hat{C}$, so $\hat{\mathcal{C}}$ is closed with respect to superposition.

Let a function $f : Spec\mathcal{C} \to \mathbf{R}$ satisfy the localization condition in the space $(Spec\mathcal{C}, \hat{C})$. For any open subset $\hat{U} \in Spec\mathcal{C} \quad \exists \hat{g} \in \hat{\mathcal{C}}$ s. t. $f|_{\hat{U}} = \hat{g}|_{\hat{U}}$. We can uniquely define the function $h : M \to \mathbf{R}$ satisfying the condition $h(p) = f(ev_p) \quad \forall p \in M$. For any open set $\hat{U} \in Spec\mathcal{C}$ there exists the open set $U \in M$ defined by $U = \{p \in M : ev_p \in \hat{U}\}$. From the definitions of the function h and the set U we know that $h|_U = g|_U$. Because $g \in \mathcal{C}$ it follows that $h \in \mathcal{C}$. We also know that $\hat{h}|_{evM} = f|_{evM}$. From Corollary 3.8 we derive that $f = \hat{h}$. This means that $f \in \hat{\mathcal{C}}$, so $\hat{\mathcal{C}}$ is closed with respect to localization. □

Lemma 3.10. *If $(M, \mathcal{C})$ is a differential space with the structure $\mathcal{C}$ generated by $\mathcal{C}_0$ then the differential structure $\hat{\mathcal{C}}$ of the differential space $(Spec\mathcal{C}, \hat{\mathcal{C}})$ is generated by $\hat{\mathcal{C}}_0$.*

Proof. Let us assume that $\mathcal{C}_0 = \{f_i : i \in I\}$. We know that for any $f \in \mathcal{C}$ there exists such an open covering of M that on each set U of this covering the function f can be expressed in the form $\omega \circ (f_1, \ldots, f_n)$ where $f_1, \ldots, f_n \in \mathcal{C}$ and $\omega \in \varepsilon_n$. For each open set U of this covering we can define the set $\hat{U} = \{ev_p \in Spec\mathcal{C} : p \in U\}$. On the set $\hat{U}$ the function $\hat{f} = \tau(\omega \circ (f_1, \ldots, f_n))$. The sets of form $\hat{U}$ might not be a covering of $Spec\mathcal{C}$ but the sum of them is dense in $Spec\mathcal{C}$. Therefore we can prolong uniquely this representation of $\hat{f}$ on whole $Spec\mathcal{C}$. We have shown that $\hat{\mathcal{C}} = gen\hat{\mathcal{C}}_0$. □

Lemma 3.11. *For any differential space* $(M, \mathcal{C})$ *the differential space* $(Spec\mathcal{C}, \hat{C})$ *is smoothly real-compact.*

Proof. We need to show that for every homomorphism $\hat{\chi} \in Spec\hat{\mathcal{C}}$ there exists a homomorphism $\psi \in Spec\mathcal{C}$ s.t. $\hat{\chi} = ev_\psi$. Since the algebras $\mathcal{C}$ and $\hat{\mathcal{C}}$ are isomorphic we can define uniquely $\chi \in Spec\mathcal{C}$ by the formula $\chi(f) = \hat{\chi}(\hat{f})$. We will show that $\hat{\chi} = ev_\chi$. Let us compute $ev_\chi(\hat{f}) = \hat{f}(\chi) = \chi(f) = \hat{\chi}(\hat{f})$, so by setting $\psi = \chi$ we obtain that $\hat{\chi} = ev_\psi$. □

Lemma 3.12. *Let* $(M, \mathcal{C})$ *be a differential space and* $\mathcal{C} = Gen\mathcal{C}_0$. *If* $\chi_1, \chi_2 \in Spec\mathcal{C}$ *are equal on the generators* $\chi_1|_{\mathcal{C}_0} = \chi_2|_{\mathcal{C}_0}$ *then they are equal* $\chi_1 = \chi_2$.

Proof. Let us assume that $\chi_1|_{\mathcal{C}_0} = \chi_2|_{\mathcal{C}_0}$ and $\chi_1 \neq \chi_2$. From the last lemma we know that the differential structure $\hat{C}$ of the differential space $(Spec\mathcal{C}, \hat{C})$ is generated by $\hat{\mathcal{C}}_0$. From the condition $\chi_1|_{\mathcal{C}_0} = \chi_2|_{\mathcal{C}_0}$ we derive that $\forall \hat{f} \in \hat{\mathcal{C}} \quad \hat{f}(\chi_1) = \hat{f}(\chi_2)$. But we know that if the generators do not separate points then all the functions do not separate points, so $\forall \hat{f} \in \hat{\mathcal{C}} \hat{f}(\chi_1) = \hat{f}(\chi_2)$ and it follows that $\forall f \in \mathcal{C} \quad \chi_1(f) = \chi_2(f)$. This means that $\chi_1 = \chi_2$. □

Lemma 3.13. *If* $(M, \mathcal{C})$ *is a differential subspace of the space* $(\mathbf{R}^I, \varepsilon_I)$ *then any function* $f \in \mathcal{C}$ *is uniquely continuously prolongable to* $\tilde{f} : \tilde{M} \to \mathbf{R}$, *where* $\tilde{M} = \{p \in \mathbf{R}^I : \exists \chi \in Spec\mathcal{C} \text{ s.t. } p_i = \chi(\pi_i|_M) \quad \forall i \in I\}$.

Proof. We will define the function $\tilde{f}$ by the formula $\tilde{f}(p) = \hat{f}(\chi)$, where $\chi \in Spec\mathcal{C}$ is s.t. $\chi(\pi_i) = p_i \quad \forall i \in I$. Since a homomorphism is uniquely defined by its values on the generators (Lemma 3.12) this definition works well. We see that if $p \in M$ then $\chi = ev_p$ and $\tilde{f}(p) = \hat{f}(ev_p) = f(p)$ so this is indeed a prolongation. This prolongation is continuous since the function $\tilde{f}$ is the realization of the function $\hat{f}$ in the set $\tilde{M}$ which is the image of the set $Spec\mathcal{C}$ by the generatory embedding using the generators $\tau(\pi_i|_M) : i \in I$. Uniqueness follows from the fact that the set M is dense in the set $\tilde{M}$ in the topology of $\mathbf{R}^I$. □

Corollary 3.14. *When $(M, \mathcal{C})$ is a differential subspace of $(\mathbf{R}^I, \varepsilon_I)$ generated by $\mathcal{C}_0 = \{\pi_i|_M : \quad i \in I\}$ then the mapping $\chi : \mathcal{C}_0 \to \mathbf{R}$ defined on generators as $\chi(\pi_i|_M) = p_i$ for some $p \in \tilde{M} - M$ can be prolonged to homomorphism on whole $\mathcal{C}$ iff all the functions from $\mathcal{C}$ are prolongable to p.*

Let $M = \mathbf{R}^{\mathbf{N}} - \{0\}$, where by 0 we denote the zero sequence and $\mathcal{C}_M = (\varepsilon_{\mathbf{N}})_M$. Then $(M, \mathcal{C}_M)$ is a differential subspace of the differential space $(\mathbf{R}^{\mathbf{N}}, \varepsilon_{\mathbf{N}})$. We will show that this space is smoothly real-compact.

Lemma 3.15. *There exists a function $\xi \in \mathcal{C}_M$ which is non-prolongable to any continuous function on $\mathbf{R}^{\mathbf{N}}$.*

Proof. We know that there exists a function $\phi \in C^\infty(\mathbf{R})$ satisfying the following properties:
1. $\forall x \in \mathbf{R} \quad \phi(x) \in < 0, 1 >$
2. $supp(\phi) \subseteq (-\infty, 1 >$
3. $\phi|_{<0,\frac{1}{2}>} = 1$

For any $k \in \mathbf{N}$ we will define the function $\tilde{\rho}_k : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ by the formula:

$$\tilde{\rho}_k((x_n)) = \sum_{i=1}^{k} x_i^2,$$

for $(x_n) \in \mathbf{R}^{\mathbf{N}}$. The function $\tilde{\rho}_k \in \mathcal{C}^\infty(\mathbf{R}^{\mathbf{N}})$, and the function $\rho_k = \tilde{\rho}_k|_M$ is in $\mathcal{C}_M$. We will define the function $\xi : M \to \mathbf{R}$ by the following formula:

$$\xi((x_n)) = \sum_{k=1}^{\infty} \phi(k^2 \rho_k(x_n)). \tag{9}$$

We will show that this function belongs to the structure $\mathcal{C}_M$. For any $k \in \mathbf{N}$ we can define the closed subset $A_k = \{(x_n) \in M : k^2 \rho_k((x_n)) \leq 1\} = \{(x_n) \in M : \rho_k(x_n) \leq \frac{1}{k^2}\}$. We see that $supp(\phi \circ (k^2 \rho_k)) \subseteq A_k$. For any $(x_n) \in M$ the sequence $\rho_k((x_n))$ is non-decreasing with respect to k and there exists $k_0 \in \mathbf{N}$ for which $\frac{1}{k^2} < \rho_{k_0}(x_n)$. This means that $(x_n) \notin A_k$. Therefore $\bigcap_{k \in \mathbf{N}} A_k = \emptyset$. We also know that $A_{k+1} \subseteq A_k$. Let us define the family of open subsets $U_k = M - A_k$. Of course $\bigcup_{k \in \mathbf{N}} U_k = M$. If $(x_n) \in U_k$ then $\phi(k^2 \rho_k((x_n)) = 0$. Then $\forall m > k \quad x_n \in U_m$ so $\phi(m^2 \rho_m(x_n)) = 0$. This means that only a finite number of elements are non-zero in the sum (9) and therefore:

$$\xi(x_n) = \sum_{j=1}^{k-1} \phi(j^2 \rho_j(x_n)),$$

$\forall k \in \mathbf{N}$ the function $\xi|_{U_k} \in \mathcal{C}_{U_k} = (\mathcal{C}_M)_{U_k}$. From the localization closeness of the differential structure we derive that $\xi \in \mathcal{C}_M$. Now we will define a sequence in M convergent to 0 on which the function ξ will diverge. Let us define $z_k = (x_{n,k})$ where

$$x_{n,k} = \begin{cases} \frac{1}{k\sqrt{2}} & \text{for} \quad n = k \\ 0 & \text{for} \quad n \neq k \end{cases}$$

We can see that $\lim_{k\to\infty} z_k = 0 \in \mathbf{R}^{\mathbf{N}}$ and

$$\rho_j(z_k) = \begin{cases} \frac{1}{2k^2} & \text{for} \quad j \geq k \\ 0 & \text{for} \quad j < k \end{cases}$$

For $j \leq k$ we obtain $\phi(j^2 \rho_j(x_k)) = 1$ and therefore

$$\xi(x_k) = \sum_{j=1}^{\infty} \phi(j^2 \rho_j(x_k)) \geq \sum_{j=1}^{k} 1 = k$$

This means that $\lim_{k\to\infty} \xi(x_k) = +\infty$. The function ξ is non-prolongable to any continuous function in $\mathbf{R}^{\mathbf{N}}$. □

Lemma 3.16. *The differential space $(M, \mathcal{C}_M)$ is smoothly real-compact.*

Proof. From Lemma 3.12 we know that the set $Spec\mathcal{C}_M$ may contain only one homomorphism χ_0 which is not an evaluation. This homomorphism would be defined on the generators by the formula $\chi_0(\pi_i|_M) = 0 \quad \forall i \in I$. So there would be only one point $0 \in \tilde{M} - M$. But it cannot be so since from Corollary 3.14 we know that all the functions from $\mathcal{C}_M$ should be prolongable to the point 0. From the last lemma we know that there exists a function $\xi \in \mathcal{C}_M$ which is not prolongable. □

Corollary 3.17. *Differential space $(\mathbf{R}^{\mathbf{N}} - \{p\}, (\varepsilon_{\mathbf{N}})_{\mathbf{R}^{\mathbf{N}} - \{p\}})$ where $p \in \mathbf{R}^{\mathbf{N}}$ is arbitrary is smoothly real-compact.*

Proof. This space is diffeomorphic to the space $(M, \mathcal{C}_M)$ so it must be smoothly real-compact. $\square$

Definition 3.18. *The disjoint union of differential spaces* $(M, \mathcal{C})$ *and* $(N, \mathcal{D})$ *where* $M \cap N = \emptyset$ *is the differential space* $(M \cup N, \mathcal{C} \oplus \mathcal{D})$*. The structure* $\mathcal{C} \oplus \mathcal{D}$ *is defined by the property* $f \in \mathcal{C} \oplus \mathcal{D} \iff f|_M \in \mathcal{C}$ *and* $f|_N \in \mathcal{D}$*.*

Lemma 3.19. *If the differential spaces* $(M, \mathcal{C})$ *and* $(N, \mathcal{D})$ *are smoothly real-compact then the differential space* $(M \cup N, \mathcal{C} \oplus \mathcal{D})$ *is smoothly real-compact.*

Proof. Elements of the algebra $\mathcal{C} \oplus \mathcal{D}$ are pairs (f, g) where $f \in \mathcal{C}$ and $g \in \mathcal{D}$. Let $\chi \in Spec(\mathcal{C} \oplus \mathcal{D})$. We shall show that it is an evaluation at some point $p \in M \cup N$. From the equations $(0,1) + (1,0) = (1,1)$ and $(0,1)(1,0) = (0,0)$ we get that we have two cases:
1) $\chi((1,0)) = 1$ and $\chi((0,1)) = 0$
2) $\chi((1,0)) = 0$ and $\chi((0,1)) = 1$.
Since every function from $\mathcal{C} \oplus \mathcal{D}$ can be uniquely decomposed as $(f, g) = (f, 0)(1, 0) + (0, g)(0, 1)$ the homomorphism χ acts as follows:
$\chi((f,g)) = \chi((f,0))\chi((1,0)) + \chi((0,g))\chi((0,1))$. In the case 1) we will get: $\chi((f,g)) = \chi((f,0))$ and in the case 2) $\chi((f,g)) = \chi((0,f))$.
The algebra of functions of the form $((f,0)) \in \mathcal{C} \oplus \mathcal{D}$ is isomorphic to $\mathcal{C}$. Therefore homomorphisms from $\psi \in Spec\mathcal{C}$ can be extended to homomorphisms from $\mathcal{C} \oplus \mathcal{D}$ by the formula $\bar{\psi}((f,g)) = \psi(f)$. All the homomorphisms in case 1) are of this form. Therefore in case 1) the homomorphism $\chi((f,g)) = \psi(f)$ where $\psi \in Spec\mathcal{C}$ is such that $\bar{\psi} = \chi$. But since the space $(M, \mathcal{C})$ is smoothly real-compact there exists a point $p \in M$ s.t. $\psi = ev_p$. Then we can write $\chi((f,g)) = ev_p((f,g)) = (f,g)(p) = f(p) + g(p)$ for $p \in M \cup N$. We have shown that in the case 1) homomorphism χ is an evaluation. For the case 2) the proof is analogous. $\square$

Definition 3.20. *By* $\tilde{\varepsilon}$ *we denote the differential structure on* $\mathbf{R}^{\mathbf{N}}$ *generated by the set* $\mathcal{C}_0 = \{\pi_i : i \in \mathbf{N}\} \cup \{\theta_p\}$*, where* θ_p *is the characteristic function of the point* $p \in M$*.*

Lemma 3.21. *The differential space* $(\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$ *is smoothly real-compact.*

Proof. We can decompose the space $(\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$ into the direct sum of the spaces $(\mathbf{R}^{\mathbf{N}} - \{p\}, (\varepsilon_{\mathbf{N}})_{\mathbf{R}^{\mathbf{N}} - \{p\}})$ and $(\{p\}, F(p))$ where $F(p)$ is the algebra of all possible functions on one point. From the definition of the space $(\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$

it is obvious that $\mathbf{R}^{\mathbf{N}} = \{p\} \cup (\mathbf{R}^{\mathbf{N}} - \{p\})$ and $\tilde{\varepsilon} = (\varepsilon_{\mathbf{N}})_{\mathbf{R}^{\mathbf{N}} - \{p\}} \oplus F(p)$. Both the spaces in the direct sum are smoothly real-compact, so the space $(\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$ is smoothly real-compact. □

Theorem 3.22. *Any differential subspace of the differential space* $(\mathbf{R}^{\mathbf{N}}, \varepsilon_{\mathbf{N}})$ *is smoothly real-compact.*

Proof. Let $\iota_M : (M, \mathcal{C}) \to (\mathbf{R}^{\mathbf{N}}, \varepsilon_{\mathbf{N}})$ be the inclusion mapping. For any $\chi \in Spec\mathcal{C}$ the composition $\chi \circ \iota_M^* \in Spec(\varepsilon_{\mathbf{N}})$. The space $(\mathbf{R}^{\mathbf{N}}, \varepsilon_{\mathbf{N}})$ is smoothly real-compact so $\exists p \in \mathbf{R}^{\mathbf{N}}$ such that $\chi \circ \iota_M^* = ev_p|_{\varepsilon_{\mathbf{N}}}$.

We need to show that $p \in M$. Let us assume that $p \notin M$. We can treat the space $(M, \mathcal{C})$ as a differential subspace of $(\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$. Let us denote this inclusion by $\nu_M : (M, \mathcal{C}) \to (\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$. The composition $\chi \circ \nu_M^* \in Spec\tilde{\varepsilon}$. Because the space $(\mathbf{R}^{\mathbf{N}}, \tilde{\varepsilon})$ is smoothly real-compact there exists a point $q \in \mathbf{R}^{\mathbf{N}}$ such that $\chi \circ \nu_M^* = ev_q|_{\tilde{\varepsilon}}$. We know that on common generators π_i the equalities $\chi(\pi_i|_M) = ev_p(\pi_i) = p_i$ and $\chi(\pi_i|_M) = ev_q(\pi_i) = q_i$ hold $\forall i \in \mathbf{N}$. This specifies all the coordinates, so $p = q$. Therefore we can write $\chi \circ \nu_M^* = ev_p|_{\tilde{\varepsilon}}$. So $(\chi \circ \nu_M^*)(\theta_p) = ev_p(\theta_p) = 1$. We have a contradiction with the fact that $(\chi \circ \nu_M^*)(\theta_p) = \chi(\theta_p|_M) = \chi(0) = 0$. We see that $p \in M$ and $\chi \circ \iota_M^* = ev_p|_{\varepsilon_{\mathbf{N}}}$. So $\chi(\pi_i|_M) = ev_p(\pi_i|M) \quad \forall i \in \mathbf{N}$. The set $\{\pi_i : i \in \mathbf{N}\}$ is the set of generators of the differential space $(M, \mathcal{C})$. We derive that $\chi = ev_p$. □

Corollary 3.23. *Any countably generated differential space is smoothly real-compact.*

Proof. A countably generated differential space can be treated as a subspace of the space $(\mathbf{R}^{\mathbf{N}}, \varepsilon_{\mathbf{N}})$. From Theorem 3.22 we know that all subspaces of this space are smoothly real-compact. □

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Symplectic T_7 singularities and Lagrangian tangency orders

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Abstract

We study the local symplectic algebra of curves. We use the method of algebraic restrictions to classify symplectic T_7 singularities. We define discrete symplectic invariants - the Lagrangian tangency orders. We use these invariants to distinguish symplectic singularities of classical $A-D-E$ singularities of planar curves, S_5 singularity and T_7 singularity. We also give the geometric description of these symplectic singularities.

1 Introduction

In this paper we study the symplectic classification of singular curves under the following equivalence:

Definition 1.1. *Let N_1, N_2 be germs of subsets of symplectic space $(\mathbb{R}^{2n}, \omega)$. N_1, N_2 are* ***symplectically equivalent*** *if there exists a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$ such that $\Phi(N_1) = N_2$.*

We recall that ω is a symlectic form if ω is a smooth nondegenerate closed 2-form, and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if Φ is diffeomorphism and $\Phi^*\omega = \omega$.

Symplectic classification of curves were first studied by V. I. Arnold. In [A1] V. I. Arnold discovered new symplectic invariants of singular curves. He proved that the A_{2k} singularity of a planar curve (the orbit with respect to standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by

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different orders of tangency of the parameterized curve to the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these invariants in terms of the local algebra's interaction with the symplectic structure and he proposed to call this interaction the local symplectic algebra.

In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. The orbit of action of all diffeomorphism-germs agrees with volume-preserving orbit or splits into two volume-preserving orbits (in the case $\mathbb{K} = \mathbb{R}$) for germs which satisfy a special weak form of quasi-homogeneity e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights $w_i \geq 0$ and $\sum_i w_i > 0$.

Symplectic singularity is stably simple if it is simple and remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space. In [K] P. A. Kolgushkin classified the stably simple symplectic singularities of parameterized curves (in the $\mathbb{C}$-analytic category). All stably simple symplectic singularities of curves are quasi-homogeneous too.

In [DJZ2] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential k-forms:

Differential k-forms ω_1 and ω_2 have the same **algebraic restriction** to a subset N if $\omega_1 - \omega_2 = \alpha + d\beta$, where α is a k-form vanishing on N and β is a $(k-1)$-form vanishing on N.

In [DJZ2] the generalization of Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space was obtained. This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant except the algebraic restriction ([DJZ2], [DJZ1]). The dimension of the space of algebraic restrictions of closed 2-

forms to a 1-dimensional quasi-homogeneous isolated complete intersection singularity C is equal to the multiplicity of C ([DJZ2]). In [D] it was proved that the space of algebraic restrictions of closed 2-forms to a 1-dimensional (singular) analytic variety is finite-dimensional. In [DJZ2] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular the complete symplectic classification of classical $A-D-E$ singularities of planar curves and S_5 singularity were obtained. Most of different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.

In this paper following ideas from [A1] and [D] we define new discrete symplectic invariants - the Lagrangian tangency orders (section 3.1). These invariants let us distinguish all symplectic $A-D-E$ singularities of planar curves including E_6^3, E_6^4 and E_8^5, E_8^6 singularities which were not distinguished by the index of isotropy and the symplectic multiplicity (Tables 4 and 6). Using Lagrangian tangency orders we are able to give more detailed classification of S_5 singularity (Theorem 5.5) and to present an alternative geometric description of its symplectic orbits (Theorem 5.3).

We also obtain the complete symplectic classification of the classical isolated complete intersection singularity T_7 using the method of algebraic restrictions (Theorem 6.1). We calculate discrete symplectic invariants for this classification (Theorems 6.7 and 6.4) and we present geometric descriptions of symplectic orbits (Theorem 6.10).

The paper is organized as follows. In section 2 we recall the method of algebraic restrictions. In section 3 we present known discrete symplectic invariants and introduce Lagrangian tangency orders. Lagrangian tangency orders of symplectic $A-D-E$ singularities of planar curves are studied in section 4. In section 5 we obtain more detailed symplectic classification of S_5 using Lagrangian tangency orders and an alternative geometric description of symplectic singularities. Symplectic classification of T_7 singularity is studied in section 6.

2 The method of algebraic restrictions

In this section we present basic facts on the method of algebraic restrictions, which is a very powerful tool for the symplectic classification. The proofs of all results of this section can be found in [DJZ2].

Given a germ of a non-singular manifold M denote by $\Lambda^p(M)$ the space of all germs at 0 of differential p-forms on M. Given a subset $N \subset M$ introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M) : \quad \omega(x) = 0 \text{ for any } x \in N\};$$

$$\mathcal{A}^p_0(N, M) = \{\alpha + d\beta : \quad \alpha \in \Lambda^p_N(M), \ \beta \in \Lambda^{p-1}_N(M).\}$$

The relation $\omega(x) = 0$ means that the p-form ω annihilates any p-tuple of vectors in T_xM, i.e. all coefficients of ω in some (and then any) local coordinate system vanish at the point x.

Definition 2.1. *Let N be the germ of a subset of M and let $\omega \in \Lambda^p(M)$. The* **algebraic restriction** *of ω to N is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.*

Notation. The algebraic restriction of the germ of a p-form ω on M to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. Writing $[\omega]_N = 0$ (or saying that ω has zero algebraic restriction to N) we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in A^p_0(N, M)$.

Let M and $\widetilde{M}$ be non-singular equal-dimensional manifolds and let $\Phi : \widetilde{M} \to M$ be a local diffeomorphism. Let N be a subset of M. It is clear that $\Phi^*\mathcal{A}^p_0(N, M) = \mathcal{A}^p_0(\Phi^{-1}(N), \widetilde{M})$. Therefore the action of the group of diffeomorphisms can be defined as follows: $\Phi^*([\omega]_N) = [\Phi^*\omega]_{\Phi^{-1}(N)}$, where ω is an arbitrary p-form on M.

Definition 2.2. *Two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_{\widetilde{N}}$ are called* **diffeomorphic** *if there exists the germ of a diffeomorphism $\Phi : \widetilde{M} \to M$ such that $\Phi(\widetilde{N}) = N$ and $\Phi^*([\omega]_N) = [\widetilde{\omega}]_{\widetilde{N}}$.*

Remark 2.3. *The above definition does not depend on the choice of ω and $\widetilde{\omega}$ since a local diffeomorphism maps forms with zero algebraic restriction to N to forms with zero algebraic restrictions to $\tilde{N}$. If $M = \widetilde{M}$ and $N = \widetilde{N}$ then the definition of diffeomorphic algebraic restrictions reduces to the following one: two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_N$ are diffeomorphic if there exists a local symmetry Φ of N (i.e. a local diffeomorphism preserving N) such that $[\Phi^*\omega]_N = [\widetilde{\omega}]_N$.*

Definition 2.4. *A subset N of $\mathbb{R}^m$ is quasi-homogeneous if there exists a coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{R}^m$ and positive numbers $\lambda_1, \cdots, \lambda_n$ such that for any point $(y_1, \cdots, y_m) \in \mathbb{R}^m$ and any $t \in \mathbb{R}$ if $(y_1, \cdots, y_m)$ belongs to N then a point $(t^{\lambda_1} y_1, \cdots, t^{\lambda_m} y_m)$ belongs to N.*

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.

Theorem 2.5 (Theorem A in [DJZ2]). *Let N be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

Two germs of quasi-homogeneous subsets N_1, N_2 of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectically equivalent if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.

Theorem 2.5 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of zero algebraic restriction is explained by the following theorem.

Theorem 2.6 (Theorem **B** in [DJZ2]). *The germ of a quasi-homogeneous set N of a symplectic space (R^{2n}, ω) is contained in a non-singular Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N.*

Proposition 2.7 (Lemma 2.20 in [DJZ2]). *Let $N \subset \mathbb{R}^m$. Let $W \subseteq T_0 R^m$ be the tangent space to some (and then any) non-singular submanifold containing N of minimal dimension within such submanifolds. If ω is the germ of a p-form with zero algebraic restriction to N then $\omega|_W = 0$.*

The following result shows that the method of algebraic restrictions is very powerful tool in symplectic classification of singular curves.

Theorem 2.8 (Theorem 2 in [D]). *Let C be the germ of a $\mathbb{K}$-analytic curve (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Then the space of algebraic restrictions of germs of closed 2-forms to C is a finite dimensional vector space.*

By a $\mathbb{K}$-**analytic curve** we understand a subset of $\mathbb{K}^m$ which is locally diffeomorphic to a 1-dimensional (possibly singular) $\mathbb{K}$-analytic subvariety of $\mathbb{K}^m$. Germs of $\mathbb{C}$-analytic parameterized curves can be identified with germs of irreducible $\mathbb{C}$-analytic curves.

We now recall basic properties of algebraic restrictions which are useful for a description of this subset ([DJZ2]).

First we can reduce the dimension of the manifold we consider due to the following propositions.

If the germ of a set $N \subset \mathbb{R}^m$ is contained in a non-singular submanifold $M \subset \mathbb{R}^m$ then the classification of algebraic restrictions to N of p-forms on $\mathbb{R}^m$ reduces to the classification of algebraic restrictions to N of p-forms on M. At first note that the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ can be identified:

Proposition 2.9. *Let N be the germ at 0 of a subset of $\mathbb{R}^m$ contained in a non-singular submanifold $M \subset \mathbb{R}^m$ and let ω_1, ω_2 be p-forms on $\mathbb{R}^m$. Then $[\omega_1]_N = [\omega_2]_N$ if and only if $\left[\omega_1|_{TM}\right]_N = \left[\omega_2|_{TM}\right]_N$.*

The following, less obvious statement, means that the *orbits* of the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ also can be identified.

Proposition 2.10. *Let N_1, N_2 be germs of subsets of $\mathbb{R}^m$ contained in equal-dimensional non-singular submanifolds M_1, M_2 respectively. Let ω_1, ω_2 be two germs of p-forms. The algebraic restrictions $[\omega_1]_{N_1}$ and $[\omega_2]_{N_2}$ are diffeomorphic if and only if the algebraic restrictions $\left[\omega_1|_{TM_1}\right]_{N_1}$ and $\left[\omega_2|_{TM_2}\right]_{N_2}$ are diffeomorphic.*

To calculate the space of algebraic restrictions of 2-forms we will use the following obvious properties.

Proposition 2.11. *If $\omega \in \mathcal{A}_0^k(N, \mathbb{R}^{2n})$ then $d\omega \in \mathcal{A}_0^{k+1}(N, \mathbb{R}^{2n})$ and $\omega \wedge \alpha \in \mathcal{A}_0^{k+p}(N, \mathbb{R}^{2n})$ for any p-form α on $\mathbb{R}^{2n}$.*

The next step of our calculation is the description of the subspace of algebraic restriction of closed 2-forms. The following proposition is very useful for this step.

Proposition 2.12. *Let $a_1, \ldots, a_k$ be a basis of the space of algebraic restrictions of 2-forms to N satisfying the following conditions*

1. $da_1 = \cdots = da_j = 0$,

2. the algebraic restrictions $da_{j+1}, \ldots, da_k$ *are linearly independent.*

Then $a_1, \ldots, a_j$ *is a basis of the space of algebraic restriction of closed* 2*-forms to* N*.*

Then we need to determine which algebraic restrictions of closed 2-forms are realizable by symplectic forms. This is possible due to the following fact.

Proposition 2.13. *Let* $N \subset \mathbb{R}^{2n}$*. Let* r *be the minimal dimension of non-singular submanifolds of* $\mathbb{R}^{2n}$ *containing* N*. Let* M *be one of such* r*-dimensional submanifolds. The algebraic restriction* $[\theta]_N$ *of the germ of closed* 2*-form* θ *is realizable by the germ of a symplectic form on* $\mathbb{R}^{2n}$ *if and only if* $rank(\theta|_{T_0 M}) \geq 2r - 2n$*.*

Let us fix the following notations:

• $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$: the vector space consisting of algebraic restrictions of germs of all 2-forms on $\mathbb{R}^{2n}$ to the germ of a subset $N \subset \mathbb{R}^{2n}$;

• $\left[Z^2(\mathbb{R}^{2n})\right]_N$: the subspace of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ consisting of algebraic restrictions of germs of all closed 2-forms on $\mathbb{R}^{2n}$ to N;

• $[\text{ Symp } (\mathbb{R}^{2n})]_N$: the open set in $\left[Z^2(\mathbb{R}^{2n})\right]_N$ consisting of algebraic restrictions of germs of all symplectic 2-forms on $\mathbb{R}^{2n}$ to N.

3 Discrete symplectic invariants.

We can use some discrete symplectic invariants to characterize symplectic singularity classes. The first one is a symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let N be a germ of a subset of $(\mathbb{R}^{2n}, \omega)$.

Definition 3.1. *The* **symplectic multiplicity** $\mu_{sympl}(N)$ *of* N *is the codimension of a symplectic orbit of* N *in an orbit of* N *with respect to the action of the group of local diffeomorphisms.*

The second one is the index of isotropy [DJZ2].

Definition 3.2. *The* **index of isotropy** $\iota(N)$ *of* N *is the maximal order of vanishing of the 2-forms* $\omega|_{TM}$ *over all smooth submanifolds* M *containing* N.

They can be described in terms of algebraic restrictions.

Proposition 3.3 ([DJZ2]). *The symplectic multiplicity of the germ of a quasi-homogeneous subset* N *in a symplectic space is equal to the codimension of the orbit of the algebraic restriction* $[\omega]_N$ *with respect to the group of local diffeomorphisms preserving* N *in the space of algebraic restrictions of closed 2-forms to* N.

Proposition 3.4 ([DJZ2]). *The index of isotropy of the germ of a quasi-homogeneous subset* N *in a symplectic space* $(\mathbb{R}^{2n}, \omega)$ *is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction* $[\omega]_N$.

3.1 Lagrangian tangency order

There is one more discrete symplectic invariant introduced in [D] following ideas from [A1] which is defined specifically for a parameterized curve. This is the maximal tangency order of a curve $f : \mathbb{R} \to M$ to a smooth Lagrangian submanifold. If $H_1 = ... = H_n = 0$ define a smooth submanifold L in the symplectic space then the tangency order of a curve $f : \mathbb{R} \to M$ to L is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \cdots, H_n \circ f$. We denote the tangency order of f to L by $t(f, L)$.

Definition 3.5. *The* **Lagrangian tangency order** $Lt(f)$ ***of a curve*** f *is the maximum of* $t(f, L)$ *over all smooth Lagrangian submanifolds* L *of the symplectic space.*

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of algebraic restrictions.

Proposition 3.6 ([D]). *Let* f *be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at* 0. *Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve* f *is the maximum of the order of vanishing on* f *over all 1-forms* α *such that* $[\omega]_f = [d\alpha]_f$

We can generalize this invariant for curves which may be parameterized analytically. Lagrangian tangency order is the same for every 'good' analytic parametrization of a curve [W]. Considering only such parameterizations we can choose one and calculate the invariant for it. It is easy to show that this invariant doesn't depend on chosen parametrization.

Proposition 3.7. *Let $f : \mathbb{R} \to M$ and $g : \mathbb{R} \to M$ be good analytic parametrizations of the same curve. Then $Lt(f) = Lt(g)$.*

Proof. There exists a diffeomorphism $\theta : \mathbb{R} \to \mathbb{R}$ such that $g(s) = f(\theta(s))$ and $\frac{d\theta}{ds}|_0 \neq 0$. Let $H_1 = \ldots = H_n = 0$ define a smooth submanifold L in the symplectic space. If $\frac{d^l(H_i \circ f)}{dt^l}|_0 = 0$ for $l = 1, ..., k$ then

$$\frac{d^{k+1}(H_i \circ g)}{ds^{k+1}}|_0 = \frac{d^{k+1}(H_i \circ f \circ \theta)}{ds^{k+1}}|_0 = \frac{d^{k+1}(H_i \circ f)}{dt^{k+1}}|_0 \cdot (\frac{d\theta}{ds})^{k+1}|_0$$

so the orders of vanishing at 0 of functions $H_i \circ f$ and $H_i \circ g$ are equal and hence $t(f, L) = t(g, L)$ what implies that $Lt(f) = Lt(g)$. □

We can generalize Lagrangian tangency order for sets containing parametric curves. Let N be a subset of a symplectic space $(\mathbb{R}^{2n}, \omega)$.

Definition 3.8. *The* **tangency order of the germ of a subset** N **to the germ of a submanifold** L *$t[N, L]$ is equal to the minimum of $t(f, L)$ over all parameterized curve-germs f such that $Imf \subseteq N$.*

Definition 3.9. *The* **Lagrangian tangency order of** N *$Lt(N)$ is equal to the maximum of $t[N, L]$ over all smooth Lagrangian submanifold-germs L of the symplectic space.*

In this paper we consider N which are singular analytic curves. They may be identified with a multi-germ of parametric curves. We define invariants which are special cases of the above definition.
Consider a multi-germ $(f_i)_{i\in\{1,\cdots,r\}}$ of analytically parameterized curves f_i. For any smooth submanifold L in the symplectic space we have r-tuples $(t(f_1, L), \cdots, t(f_r, L))$.

Definition 3.10. *For any $I \subseteq \{1, \cdots, r\}$ we define **the tangency order of the multi-germ** $(f_i)_{i\in I}$ to L:*

$$t[(f_i)_{i\in\ I}, L] = \min_{i\in\ I} t(f_i, L).$$

Definition 3.11. *The* **Lagrangian tangency order** $Lt((f_i)_{i\in\ I})$ ***of a multi-germ*** $(f_i)_{i\in I}$ *is the maximum of* $t[(f_i)_{i\in\ I}, L]$ *over all smooth Lagrangian submanifolds* L *of the symplectic space.*

For multi-germs we can also define relative invariants according to selected branches or collections of branches.

Definition 3.12. *Let* $S \subseteq I \subseteq \{1, \cdots, r\}$. *For* $i \in S$ *let us fix numbers* $t_i \leq Lt(f_i)$. *The* **relative Lagrangian tangency order** $Lt[(f_i)_{i\in I} : (S, (t_i)_{i\in S})]$ *of a multi-germ* $(f_i)_{i\in I}$ *related to* S *and* $(t_i)_{i\in S}$ *is the maximum of* $t[(f_i)_{i\in I\setminus S}, L]$ *over all smooth Lagrangian submanifolds* L *of the symplectic space for which* $t(f_i, L) = t_i$, *if such submanifolds exist, or* $-\infty$ *if there are no such submanifolds.*

We can also define special relative invariants according to selected branches of multi-germ.

Definition 3.13. *For fixed* $j \in I$ *the* ***Lagrangian tangency order related to*** f_j *of a multi-germ* $(f_i)_{i\in I}$ *denoted by* $Lt[(f_i)_{i\in I} : f_j]$ *is the maximum of* $t[(f_i)_{i\in I\setminus\{j\}}, L]$ *over all smooth Lagrangian submanifolds* L *of the symplectic space for which* $t(f_j, L) = Lt(f_j)$,

These invariants have geometric interpretations. If $Lt(f_i) = \infty$ then a branch f_i is included in a smooth Lagrangian submanifold. If $Lt((f_i)_{i\in\ I}) = \infty$ then exists a Lagrangian submanifold containing all curves f_i for $i \in I$.

We may use these invariants for distinguishing symplectic singularities.

4 Symplectic $A - D - E$ classification by Lagrangian tangency orders

A complete symplectic classification of classical $A - D - E$ singularities of planar curves was obtained using a method of algebraic restriction in [DJZ2].

Let $N = \{H(x_1, x_2) = x_{\geq 3} = 0\}$ where $H(x_1, x_2)$ is a function representing one of the classical singularities A_k, D_k, E_6, E_7, E_8, see Table 1. Classification of these singularities is equivalent to classification of algebraic restrictions of the space $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ with respect to the group of symmetries of the curve $\{H = 0\} \subset \mathbb{R}^2$. This classification involves functions and families of functions given in the second column of Table 1.

Let us transfer the normal forms $\mathcal{F}_i = [F_i dx_1 \wedge dx_2]_{\{H=0\}}$ to symplectic normal forms. Fix any symplectic form, for example,

$$\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

If $n \geq 2$ then the algebraic restriction $[F_i(x_1, x_2)dx_1 \wedge dx_2]_N$ can be realized by the symplectic form $\omega_i = F_i dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$ which can be brought to ω_0 by the change of coordinates

$$x_1 = p_1,\ x_2 = p_2,\ x_3 = q_1 - \textstyle\int_0^{p_2} F_i(p_1, t)dt,\ x_4 = q_2,$$
$$x_5 = p_3,\ x_6 = q_3, \ldots,\ x_{2n-1} = p_n,\ x_{2n} = q_n.$$

The given change of coordinates brings N to the form

$$N^i = \left\{ H(p_1, p_2) = q_1 - \int_0^{p_2} F_i(p_1, t)dt = q_{\geq 2} = p_{\geq 3} = 0 \right\} \subset \left(\mathbb{R}^{2n}, \omega_0\right). \quad (1)$$

The complete symplectic classification of the A_k, D_k, E_6, E_7, E_8 singularities is given by the following theorem.

Theorem 4.1 ([DJZ2]). *Fix a function $H = H(x_1, x_2)$ in Table 1. Any curve in the symplectic space $(\mathbb{R}^{2n}, \omega_0)$, $n \geq 2$, which is diffeomorphic to the curve $N : H(x_1, x_2) = x_{\geq 3} = 0$ is symplectically equivalent to one and only one of the normal forms N^i, $i = 0, ..., \mu$, given by (1), where F_i are the functions in Table 1 and μ is the multiplicity of H. The parameters b, b_1, b_2 are symplectic moduli. The codimension of the symplectic singularity class defined by the normal form N^i in the class of all curves diffeomorphic to N is equal to i.*

4.1 Distinguishing normal forms by Lagrangian tangency invariants

A curve N may be described as a parameterized curve or as a union of parameterized components C_i preserved by local diffeomorphisms in symplectic space $(\mathbb{R}^{2n}, \omega_0)$, $n \geq 2$. Lagrangian tangency orders $Lt(N)$ and $Lt(C_i)$ are preserved by local symplectomorphisms. For calculating Lagrange tangency orders we give their parametrization in the coordinate system

$H(x_1,x_2)$	$F_i(x_1,x_2),\ i=0,1,\ldots,\mu$
$A_k: x_1^{k+1}-x_2^2$ $k\geq 1$	$F_0=1,\ \ F_i=x_1^i,\ i=1,\ldots,k-1$ $F_k=0$
$D_k: x_1^2x_2- x_2^{k-1}$ $k\geq 4$	$F_0=1,\ \ F_i=bx_1+x_2^i,\ \ i=1,\ldots,k-4$ $F_{k-3}=(\pm 1)^k x_1+bx_2^{k-3}$, $F_{k-2}=x_2^{k-3},\ F_{k-1}=x_2^{k-2},\ F_k=0$
$E_6: x_1^3-x_2^4$	$F_0=1,\ F_1=\pm x_2+bx_1,\ F_2=x_1+bx_2^2$, $F_3=x_2^2+bx_1x_2,\ F_4=\pm x_1x_2$, $F_5=x_1x_2^2,\ F_6=0$
$E_7: x_1^3-x_1x_2^3$	$F_0=1,\ F_1=x_2+bx_1,\ F_2=\pm x_1+bx_2^2$, $F_3=x_2^2+bx_1x_2,\ F_4=\pm x_1x_2+bx_2^3$, $F_5=x_2^3,\ F_6=x_2^4,\ F_7=0$
$E_8: x_1^3-x_2^5$	$F_0=\pm 1,\ \ F_1=x_2+bx_1,\ \ F_2=x_1+b_1x_2^2+b_2x_2^3$ $F_3=\pm x_2^2+bx_1x_2,\ \ F_4=\pm x_1x_2+bx_2^3$, $F_5=x_2^3+bx_1x_2^2,\ F_6=x_1x_2^2,\ F_7=\pm x_1x_2^3,\ F_8=0$

Table 1: Classification of the algebraic restrictions to A_k, D_k, E_6, E_7, E_8.

$(p_1,q_1,p_2,q_2,\cdots,p_n,q_n)$. Singularity description and comparison of symplectic invariants (Lagrangian tangency orders, the index of isotropy - ind, the symplectic multiplicity - $\mu^{\rm symp}$) is contained in Tables 2 - 6. As we see in Tables 2 - 6, the index of isotropy and the symplectic multiplicity distinguishes all normal forms except for the following two couples: (α) E_6^3 and E_6^4; (β) E_8^5 and E_8^6. Using new invariants - Lagrangian tangency orders we can distinguish them completely.

Normal form	$f(t)$	$Lt(N)$	ind	$\mu^{\rm symp}$
A_k^i, $0\leq i\leq k-1$	$t^{(k+1+2i)\lambda_k}$	$(k+1+2i)\lambda_k$	i	i
A_k^k	0	∞	∞	k

Table 2: Symplectic invariants of A_k singularity. If k is even then $\lambda_k=1$ and N may be described as a parameterized singular curve $C:(t^2,f(t),t^{k+1},0,\cdots,0)$. If k is odd then $\lambda_k=\frac{1}{2}$ and N is a pair of two smooth parameterized branches: $B_\pm:(t,\pm f(t),\pm t^{\frac{k+1}{2}},0,\cdots,0)$. By $Lt(N)$ we denote $Lt(C)$ or $Lt(B_+,B_-)$.

Normal form	$f(t)$	$Lt(N)$	$Lt(C_2)$	ind	$\mu^{\rm symp}$
D_k^0	$t^{2\lambda_k}$	$2\lambda_k$	$(k-2)\lambda_k$	0	0
D_k^1	$bt^{k\lambda_k} + \frac{1}{2}t^{4\lambda_k}$	$k\lambda_k$	$k\lambda_k$	1	2
$D_k^{1<i<k-3}$	$\frac{1}{i+1}t^{2(i+1)\lambda_k} + +bt^{k\lambda_k}$, $b\neq 0$	$k\lambda_k$	$(k-2+2i)\lambda_k$	1	$i+1$
	$\frac{1}{i+1}t^{2(i+1)\lambda_k}$	$(k-2+2i)\lambda_k$	$(k-2+2i)\lambda_k$	i	$i+1$
D_k^{k-3}	$(\pm 1)^k t^{k\lambda_k} + +\frac{b}{k-2}t^{2(k-2)}\lambda_k$	$k\lambda_k$	∞	1	$k-2$
D_k^{k-2}	$\frac{1}{k-2}t^{2(k-2)\lambda_k}$	$(3k-8)\lambda_k$	∞	$k-3$	$k-2$
D_k^{k-1}	$\frac{1}{k-1}t^{2(k-1)\lambda_k}$	$(3k-6)\lambda_k$	∞	$k-2$	$k-1$
D_k^k	0	∞	∞	∞	k

Table 3: Symplectic invariants of D_k singularity. The curve N consists of 2 invariant components: C_1 - smooth and C_2 - singular. The branch C_1 has a form $(t, 0, 0, 0, \cdots, 0)$. If k is odd then C_2 has a form $(t^{k-2}, f(t), t^2, 0, \cdots, 0)$ and $\lambda_k = 1$. If k is even then C_2 consists of two branches: $B_\pm : (\pm t^{(k-2)/2}, f(t), t, 0, \cdots, 0)$ and $\lambda_k = \frac{1}{2}$.

Normal form	$f(t)$	$Lt(N)$	ind.	$\mu^{\rm symp}$
E_6^0	t^3	4	0	0
E_6^1	$\pm\frac{1}{2}t^6 + bt^7$	7	1	2
E_6^2	$t^7 + \frac{b}{3}t^9$	8	1	3
E_6^3	$\frac{1}{3}t^9 + \frac{b}{2}t^{10}$	10	2	4
E_6^4	$\pm\frac{1}{2}t^{10}$	11	2	4
E_6^5	$\frac{1}{3}t^{13}$	14	3	5
E_6^6	0	∞	∞	6

Table 4: Symplectic invariants of E_6 singularity. The curve N has a parametrization $(t^4, f(t), t^3, 0, \cdots, 0)$.

5 Symplectic S_5-singularities

Denote by (S_5) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$S_5 = \{x \in R^{2n \geq 4} : \ x_1^2 - x_2^2 - x_3^2 = x_2 x_3 = x_{\geq 4} = 0.\} \tag{2}$$

This is the classical 1-dimensional isolated complete intersection singularity S_5 ([G], [AVG]). A complete classification of symplectic singularities

Normal form	$f_1(t)$	$f_2(t)$	$Lt(N)$	$Lt(C_2)$	ind.	μ^{symp}
E_7^0	t	t^2	3	3	0	0
E_7^1	$\frac{1}{2}t^2$	$\frac{1}{2}t^4 + bt^5$	5	5	1	2
E_7^2	$\frac{b}{3}t^3$	$\pm t^5 + \frac{b}{3}t^6$	6	∞	1	3
E_7^3	$\frac{1}{3}t^3$	$\frac{1}{3}t^6 + \frac{b}{2}t^7$	7	∞	2	4
E_7^4	$\frac{b}{4}t^4$	$\pm\frac{1}{2}t^7 + \frac{b}{4}t^8$	8	∞	2	5
E_7^5	$\frac{1}{4}t^4$	$\frac{1}{4}t^8$	9	∞	3	5
E_7^6	$\frac{1}{5}t^5$	$\frac{1}{5}t^{10}$	11	∞	4	6
E_7^7	0	0	∞	∞	∞	7

Table 5: Symplectic invariants of E_7 singularity. The curve N consists of two components: the smooth branch - C_1 and the singular branch - C_2. They have the parametrization: $C_1 : (0, f_1(t), t, 0, \cdots, 0)$ and $C_2 : (t^3, f_2(t), t^2, 0, \cdots, 0)$.

in (S_5) was obtained in [DJZ2]. In the section 5.1 we quote these results. In section 5.2 we use alternative geometric conditions to describe symplectic classes and to distinguish them . In section 5.3 we use Lagrangian tangency orders to confirm this classification.

5.1 Algebraic restrictions and their classification

The following description of the space $[Z^2(\mathbb{R}^{2n})]_{S_5}$ was obtained in [DJZ2].

Proposition 5.1. *The space $[Z^2(\mathbb{R}^{2n})]_{S^5}$ has dimension 5. It is spanned by the algebraic restrictions to S_5 of the 2-forms*

$\theta_1 = dx_1 \wedge dx_2,\ \theta_2 = dx_2 \wedge dx_3,\ \theta_3 = dx_3 \wedge dx_1,\ \theta_4 = x_2 dx_1 \wedge dx_2,$

$\theta_5 = x_3 dx_1 \wedge dx_2 - x_1 dx_2 \wedge dx_3.$

consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_5\theta_5]_{S_5}$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$.

The main results were described in the following theorem.

Theorem 5.2.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{S_5}$ can be brought by a symmetry of S_5 to one of the normal forms $[S_5]^i$ given in the second column of Table 7;

(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{S_5}$ of the singularity class corresponding to the normal form $[S_5]^i$ is equal to i;

Normal form	$f(t)$	$Lt(N)$	ind.	μ^{symp}
E_8^0	$\pm t^3$	5	0	0
E_8^1	$\frac{1}{2}t^6 + bt^8$	8	1	2
E_8^2	$t^8 + \frac{b_1}{3}t^9 + \frac{b_2}{4}t^{12}$	10	1	4
E_8^3	$\pm\frac{1}{3}t^9 + \frac{b}{2}t^{11}$	11	2	4
E_8^4	$\pm\frac{1}{2}t^{11} + \frac{b}{4}t^{12}$	13	2	5
E_8^5	$\frac{1}{4}t^{12} + \frac{b}{3}t^{14}$	14	3	6
E_8^6	$\frac{1}{3}t^{14}$	16	3	6
E_8^7	$\pm\frac{1}{4}t^{17}$	19	4	7
E_8^8	0	∞	∞	8

Table 6: Symplectic invariants of E_8 singularity. The curve N has a parametrization $(t^5, f(t), t^3, 0, \cdots, 0)$.

(iii) The singularity classes corresponding to the normal forms are disjoint;
(iv) The parameters c, c_1, c_2 of the normal forms $[S_5]^0, [S_5]^2, [S_5]^3$ are moduli.

Symplectic class	Normal forms for algebraic restrictions	cod	μ^{sym}	ind
$(S_5)^0$ $2n \geq 4$	$[S_5]^0 : [\theta_2 + c_1\theta_1 + c_2\theta_3]_{S_5}$ $(c_1, c_2) \neq (0,0)$	0	2	0
$(S_5)^2$ $2n \geq 4$	$[S_5]^2 : [\theta_2 + c\theta_4]_{S_5}$	2	3	0
$(S_5)^3$ $2n \geq 6$	$[S_5]^3 : [\theta_4 + c\theta_5]_{S_5}$	3	4	1
$(S_5)^5$ $2n \geq 6$	$[S_5]^5 : [0]_{S_5}$	5	5	∞

Table 7: Classification of symplectic S_5 singularities. *cod* – codimension of the classes; μ^{sym}– symplectic multiplicity; *ind* – the index of isotropy.

In the first column of Table 7 by $(S_5)^i$ we denote a subclass of (S_5) consisting of $N \in (S_5)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[S_5]^i$. The classes $(S_5)^i$ are symplectic singularity classes, i.e. they are closed with respect to the action of the group of symplectomorphisms. The class (S_5) is the disjoint union of the classes $(S_5)^0, (S_5)^2, (S_5)^3, (S_5)^5$. The classes $(S_5)^0$ and $(S_5)^2$ are non-empty for any dimension $2n \geq 4$ of the symplectic space; the classes $(S_5)^3$ and $(S_5)^5$ are empty if $n = 2$ and not empty if $n \geq 3$.

Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega), n \geq 3$ (resp. $n = 2$) which is diffeomorphic to S_5 is symplectically equivalent to one and only one of the normal forms $S_5^i, i = 0, 2, 3, 5$ (resp. $i = 0, 2$). The parameters of the normal forms are moduli. If ω is expressed in Darboux coordinates, $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ then one may obtain the following normal forms:
S_5^0: $p_1^2 - p_2^2 - q_2^2 = 0,\ p_2q_2 = 0, q_1 = c_1p_2 + c_2q_2, p_{\geq 3} = q_{\geq 3} = 0, (c_1, c_2) \neq (0,0)$;
S_5^2: $p_1^2 - p_2^2 - q_2^2 = 0,\ p_2q_2 = 0,\ q_1 = cp_2^2,\ p_{\geq 3} = q_{\geq 3} = 0$;
S_5^3: $p_1^2 - p_2^2 - p_3^2 = 0,\ p_2p_3 = 0,\ q_1 = p_2^2/2,\ q_2 = cp_1p_3,\ q_{\geq 3} = p_{\geq 4} = 0$;
S_5^5: $p_1^2 - p_2^2 - p_3^2 = 0,\ p_2p_3 = 0,\ q_{\geq 1} = p_{\geq 4} = 0$.

5.2 Canonical definition of the classes $(S_5)^i$

The classes $(S_5)^i$ were distinguished geometrically (in [DJZ2]), without using any local coordinate system. In this section we propose another geometric description of these singularities which distinguish more cases.

Let $N \in (S_5)$. Then N is the union of 4 non-singular 1-dimensional submanifolds (strata). Denote by $\ell_1, ..., \ell_4$ the tangent lines at 0 to the strata. These lines span a 3-space $W = W(N)$. Equivalently W is the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. The classes $(S_5)^i$ can be distinguished in terms of the restriction $\omega|_W$, where ω is the symplectic form and vectors v_i tangent to branches B_i. For $N = S_5$ =(2) it is easy to calculate

$$\ell_{1,2} = \text{ span } (\partial/\partial x_1 \pm \partial/\partial x_2),\ \ell_{3,4} = \text{ span } (\partial/\partial x_1 \pm \partial/\partial x_3). \qquad (3)$$

Theorem 5.3. *A stratified submanifold $N \in (S_5)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(S_5)^i$ if and only if the couple (N, ω) satisfies the condition in the last column of Table 8, the row of $(S_5)^i$.*

Remark 5.4. *For any $i \neq j$ the set $B_i \cup B_j$ is A_1 singularity. The condition $\omega|_{\ell_i + \ell_j} = 0$ implies that ω has zero algebraic restriction to $B_i \cup B_j$ (see Table 2). Since any triple of branches is a regular union of 3 one-dimensional submanifold then the condition $\omega|_{\ell_i + \ell_j} = 0\ \ \forall i, j \in \{1, 2, 3, 4\}$ implies that any triple of branches B_i, B_j, B_k is contained in a smooth Lagrangian submanifold (see Section 7.2, Table 8 in [DJZ2]).*

Class	Normal form	cod	Geometric conditions
$(S_5)^0$	$[S_5]^0_0 : [\theta_2 + c_1\theta_1 + c_2\theta_3]_{S_5}$ $c_1 \cdot c_2 \neq 0$, $(c_1 \pm c_2)^2 \neq 1$	0	$\omega\|_{\ell_i+\ell_j} \neq 0 \ \forall i \neq j \in \{1,2,3,4\}$
	$[S_5]^0_1 : [\theta_2 + c_1\theta_1]_{S_5}$ $\|c_1\| \neq 1$	1	$\omega\|_{\ell_i+\ell_j} = 0$ for exactly one pair of branches B_i, B_j (this pair is contained in a Lagrangian submanifold)
	$[S_5]^0_2 : [\theta_1 + \theta_2]_{S_5}$	2	$\omega\|_{\ell_i+\ell_j} = 0$ for exactly three pairs of branches B_i, B_j (these pairs are contained in Lagrangian submanifolds)
$(S_5)^2$	$[S_5]^2 : [\theta_2 + c\theta_4]_{S_5}$	2	$\omega\|_{\ell_i+\ell_j} = 0$ for exactly two pairs of branches B_i, B_j (these pairs are contained in Lagrangian submanifolds)
$(S_5)^3$	$[S_5]^3 : [\theta_4 + c\theta_5]_{S_5}$	3	$\omega\|_{\ell_i+\ell_j} = 0 \ \forall i,j \in \{1,2,3,4\}$, all triples of branches are contained in Lagrangian submanifolds
$(S_5)^5$	$[S_5]^5 : [0]_{S_5}$	5	N is contained in a Lagrangian submanifold

Table 8: Geometric interpretation of singularity classes of S_5; W - the tangent space to a non-singular 3-dimensional manifold containing $N \in (S_5)$; ℓ_i - a line tangent to the stratum B_i.

5.3 Distinguishing symplectic classes of S_5 by Lagrangian tangency orders

Lagrangian tangency orders will be used to confirm a more detailed classification of (S_5). A curve $N \in (S_5)$ consists of 4 non-singular 1-dimensional submanifolds (strata) which may be described as parametrical curves B_1, B_2, B_3, B_4. Their parametrization is given in the second column of Table 9. To distinguish the classes of this singularity completely we need following Lagrangian tangency orders:

$$Lt(N) = Lt(B_1, B_2, B_3, B_4) = \\ = \max_L(\min\{t(B_1,L), t(B_2,L), t(B_3,L), t(B_4,L)\});$$

$$Lt(N_{\{i,j,k\}}) = Lt(B_i, B_j, B_k) = \max_L(\min\{t(B_i,L), t(B_j,L), t(B_k,L)\});$$

$$Lt(N_{\{i,j\}}) = Lt(B_i, B_j) = \max_L(\min\{t(B_i,L), t(B_j,L)\}),$$

where L is a smooth Lagrangian submanifold of a symplectic space.

All branches B_i are smooth so $Lt(B_i) = \infty$ for any $i \in \{1,2,3,4\}$ and these numbers are not useful in the classification. We use Lagrangian tangency orders for pairs and triples of branches. Comparing respective numbers we obtain more detailed classification of symplectic singularities of S_5. The obtained subclasses have a natural geometric interpretation (compare Table 8).

Theorem 5.5. *A stratified submanifold $N \in (S_5)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 9. The parameters c, c_1, c_2 are moduli. The Lagrangian tangency orders of the set are characterized in the fourth column of Table 9.*

Class	Parametrization of branches	Conditions for subclasses	Lagrangian tangency orders
$(S_5)^0_0$ $2n \geq 4$	$(t, \mp c_2 t, 0, \pm t, 0, \cdots)$ $(t, \pm c_1 t, \pm t, 0, \cdots)$	$c_1 \cdot c_2 \neq 0$, $(c_1 \pm c_2)^2 \neq 1$	$Lt(N) = 1$, $Lt(N_{\{i,j\}}) = 1$ for all pairs of branches
$(S_5)^0_1$ $2n \geq 4$	$(t, 0, 0, \pm t, 0, \cdots)$ $(t, \pm c_1 t, \pm t, 0, \cdots)$	$\|c_1\| \neq 1$	$Lt(N) = 1$, $Lt(N_{\{i,j\}}) = \infty$ for exactly one pair of branches
$(S_5)^0_2$ $2n \geq 4$	$(t, 0, 0, \pm t, 0, \cdots)$ $(t, \pm t, \pm t, 0, \cdots)$		$Lt(N) = 1$, $Lt(N_{\{i,j\}}) = \infty$ for exactly three pairs of branches
$(S_5)^2$ $2n \geq 4$	$(t, 0, 0, \pm t, 0, \cdots)$ $(t, \frac{c}{2}t^2, \pm t, 0, 0, \cdots)$		$Lt(N) = 1$, $Lt(N_{\{i,j\}}) = \infty$ for exactly two pairs of branches
$(S_5)^3$ $2n \geq 6$	$(t, 0, 0, \mp ct^2, \pm t, 0, \cdots)$ $(t, \frac{1}{2}t^2, \pm t, 0, 0, 0, \cdots)$		$Lt(N) = 2$, $Lt(N_{\{i,j,k\}}) = \infty$ for all triples of branches
$(S_5)^5$ $2n \geq 6$	$(t, 0, 0, 0, \pm t, 0, \cdots)$ $(t, 0, \pm t, 0, 0, 0, \cdots)$		$Lt(N) = \infty$

Table 9: Lagrangian tangency orders for symplectic classes of S_5 singularity.

6 Symplectic T_7-singularities

Denote by (T_7) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$T_7 = \{x \in \mathbb{R}^{2n \geq 4} : x_1^2 + x_2^3 + x_3^3 = x_2 x_3 = x_{\geq 4} = 0\}. \quad (4)$$

This is the classical 1-dimensional isolated complete intersection singularity T_7 ([G], [AVG]). Let $N \in (T_7)$. Then N is quasi-homogeneous with weights $w(x_1) = 3,\ w(x_2) = w(x_3) = 2$.

We use the method of algebraic restrictions to obtain a complete classification of symplectic singularities of (T_7) presented in the following theorem.

Theorem 6.1. *Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dp_i \wedge dq_i)$, where $n \geq 3$ (resp. $n = 2$) which is diffeomorphic to T_7 is symplectically equivalent to one and only one of the normal forms $T_7^i, i = 0, 1, \cdots, 7$ (resp. $i = 0, 1, 2, 4$). The parameters c, c_1, c_2 of the normal forms are moduli.*

T_7^0: $p_1^2 + p_2^3 + q_2^3 = 0,\ p_2q_2 = 0,\ q_1 = c_1q_2 + c_2p_2,\ p_{\geq 3} = q_{\geq 3} = 0,\ c_1 \cdot c_2 \neq 0$;

T_7^1: $p_1^2 + p_2^3 + q_1^3 = 0,\ p_2q_1 = 0,\ q_2 = c_1q_1 - c_2p_1p_2,\ p_{\geq 3} = q_{\geq 3} = 0$;

T_7^2: $p_1^2 + p_2^3 + q_2^3 = 0, p_2q_2 = 0, q_1 = \frac{c_1}{2}q_2^2 + \frac{c_2}{2}p_2^2, p_{\geq 3} = q_{\geq 3} = 0, (c_1, c_2) \neq (0, 0)$;

T_7^4: $p_1^2 + p_2^3 + q_2^3 = 0,\ \ p_2q_2 = 0,\ \ q_1 = \frac{c}{3}q_2^3,\ \ p_{\geq 3} = q_{\geq 3} = 0$;

T_7^3: $p_1^2 + p_2^3 + p_3^3 = 0,\ p_2p_3 = 0,\ q_1 = \frac{c_1}{2}p_2^2 + \frac{1}{2}p_3^2,$
$q_2 = -c_2p_1p_3,\ p_{\geq 4} = q_{\geq 3} = 0$;

T_7^5: $p_1^2 + p_2^3 + p_3^3 = 0,\ p_2p_3 = 0,\ q_1 = \frac{c}{3}p_3^3,\ q_2 = -p_1p_3,\ p_{\geq 4} = q_{\geq 3} = 0$;

T_7^6: $p_1^2 + p_2^3 + p_3^3 = 0,\ p_2p_3 = 0,\ q_1 = \frac{1}{3}p_3^3,\ p_{\geq 4} = q_{\geq 2} = 0$;

T_7^7: $p_1^2 + p_2^3 + p_3^3 = 0,\ p_2p_3 = 0,\ q_{\geq 1} = p_{\geq 4} = 0$.

In section 6.1 we calculate the set $[\mathrm{Symp}(\mathbb{R}^{2n})]_{T_7}$ and classify it by the action of diffeomorphisms preserving T_7. This allows us to decompose (T_7) onto symplectic singularity classes. In section 6.2 we transfer the normal forms of algebraic restrictions to symplectic normal forms to obtain the proof of Theorem 6.1. In section 6.3 we use Lagrangian tangency orders to distinguish more symplectic singularity classes. In section 6.4 we propose a geometric description of these singularities which confirms this more detailed classification. Some of the proofs are presented in section 6.5.

6.1 Algebraic restrictions and their classification

One has the following relations for (T_7)-singularities

$$[d(x_2x_3)]_{T_7} = [x_2dx_3 + x_3dx_2]_{T_7} = 0 \tag{5}$$

$$[d(x_1^2 + x_2^3 + x_3^3)]_{T_7} = [2x_1dx_1 + 3x_2^2dx_2 + 3x_3^2dx_3]_{T_7} = 0 \tag{6}$$

	relations	proof
1.	$[x_2 dx_2 \wedge dx_3]_N = 0$	$(5) \wedge dx_2$
2.	$[x_3 dx_2 \wedge dx_3]_N = 0$	$(5) \wedge dx_3$
3.	$[x_3 dx_1 \wedge dx_2]_N = [x_2 dx_3 \wedge dx_1]_N$	$(5) \wedge dx_1$
4.	$[x_1 dx_1 \wedge dx_2]_N = 0$	$(6) \wedge dx_2$ and row 2.
5.	$[x_1 dx_1 \wedge dx_3]_N = 0$	$(6) \wedge dx_3$ and row 1.
6.	$[x_2^2 dx_1 \wedge dx_2]_N = [x_3^2 dx_3 \wedge dx_1]_N$	$(6) \wedge dx_1$
7.	$[x_1^2 dx_2 \wedge dx_3]_N = 0$	rows 1. and 2. and $[x_1^2]_N = [-x_2^3 - x_3^3]_N$
8.	$[x_3^2 dx_1 \wedge dx_2]_N = 0$	$(5) \wedge x_3 dx_1$ and $[x_2 x_3]_N = 0$

Table 10: Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = T_7$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 10.

Table 10 and Proposition 2.11 easily imply the following proposition:

Proposition 6.2. $[\Lambda^2(\mathbb{R}^{2n})]_{T_7}$ *is a* 8*-dimensional vector space spanned by the algebraic restrictions to* T_7 *of the* 2*-forms*

$\theta_1 = dx_2 \wedge dx_3,\ \ \theta_2 = dx_1 \wedge dx_3,\ \ \theta_3 = dx_1 \wedge dx_2,\ \theta_4 = x_3 dx_1 \wedge dx_3,$

$\theta_5 = x_2 dx_1 \wedge dx_2,\ \sigma_1 = x_3 dx_1 \wedge dx_2,\ \sigma_2 = x_1 dx_2 \wedge dx_3,\ \theta_7 = x_3^2 dx_1 \wedge dx_3.$

Proposition 6.2 and results of section 2 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{T_7}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{T_7}$.

Theorem 6.3. $[Z^2(\mathbb{R}^{2n})]_{T_7}$ *is a* 7*-dimensional vector space spanned by the algebraic restrictions to* T_7 *of the quasi-homogeneous* 2*-forms* θ_i

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6 = \sigma_1 - \sigma_2,\ \theta_7.$$

If $n \geq 3$ *then* $[\mathrm{Symp}(\mathbb{R}^{2n})]_{T_7} = [Z^2(\mathbb{R}^{2n})]_{T_7}$*. The manifold* $[\mathrm{Symp}(\mathbb{R}^4)]_{T_7}$ *is an open part of the* 7*-space* $[Z^2(\mathbb{R}^4)]_{T_7}$ *consisting of algebraic restrictions of the form* $[c_1\theta_1 + \cdots + c_7\theta_7]_{T_7}$ *such that* $(c_1, c_2, c_3) \neq (0, 0, 0)$*.*

Theorem 6.4.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ *can be brought by a symmetry of* T_7 *to one of the normal forms* $[T_7]^i$ *given in the second column of Table 11;*

(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ *of the singularity class corresponding to the normal form* $[T_7]^i$ *is equal to* i*;*

(iii) The singularity classes corresponding to the normal forms are disjoint;

(iv) The parameters c, c_1, c_2 *of the normal forms* $[T_7]^i$ *are moduli.*

Symplectic class	Normal forms for algebraic restrictions	cod	$\mu^{\rm sym}$	ind
$(T_7)^0$ $(2n \geq 4)$	$[T_7]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{T_7}$, $c_1 \cdot c_2 \neq 0$	0	2	0
$(T_7)^1$ $(2n \geq 4)$	$[T_7]^1 : [c_1\theta_1 + \theta_2 + c_2\theta_5]_{T_7}$	1	3	0
$(T_7)^2$ $(2n \geq 4)$	$[T_7]^2 : [\theta_1 + c_1\theta_4 + c_2\theta_5]_{T_7}$, $(c_1, c_2) \neq (0,0)$	2	4	0
$(T_7)^3$ $(2n \geq 6)$	$[T_7]^3 : [\theta_4 + c_1\theta_5 + c_2\theta_6]_{T_7}$	3	5	1
$(T_7)^4$ $(2n \geq 4)$	$[T_7]^4 : [\theta_1 + c\theta_7]_{T_7}$	4	5	0
$(T_7)^5$ $(2n \geq 6)$	$[T_7]^5 : [\theta_6 + c\theta_7]_{T_7}$	5	6	1
$(T_7)^6$ $(2n \geq 6)$	$[T_7]^6 : [\theta_7]_{T_7}$	6	6	2
$(T_7)^7$ $(2n \geq 6)$	$[T_7]^7 : [0]_{T_7}$	7	7	∞

Table 11: Classification of symplectic T_7 singularities. *cod* – codimension of the classes; μ^{sym}– symplectic multiplicity; *ind* – the index of isotropy.

The proof of Theorem 6.4 is presented in section 6.5. In the first column of Table 11 by $(T_7)^i$ we denote a subclass of (T_7) consisting of $N \in (T_7)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[T_7]^i$. Theorem 2.5, Theorem 6.4 and Proposition 6.3 imply the following statement.

Proposition 6.5. *The classes* $(T_7)^i$ *are symplectic singularity classes, i.e. they are closed with respect to the action of the group of symplectomorphisms. The class* (T_7) *is the disjoint union of the classes* $(T_7)^i, i \in \{0, 1, \cdots, 7\}$*. The* *classes* $(T_7)^0, (T_7)^1, (T_7)^2, (T_7)^4$ *are non-empty for any dimension* $2n \geq 4$ *of the symplectic space; the classes* $(T_7)^3, (T_7)^5, (T_7)^6, (T_7)^7$ *are empty if* $n = 2$ *and not empty if* $n \geq 3$.

The following theorem explains why the given stratification of (T_7) is natural.

Theorem 6.6. *Fix* $i \in \{0, 1, \cdots, 7\}$. *All stratified submanifolds* $N \in (T_7)^i$ *have the same (a) symplectic multiplicity and (b) index of isotropy given in Table 11.*

Proof. Part (a) follows from Theorems 3.3 and 6.4 and the fact that the codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ of the orbit of an algebraic restriction $a \in [T_7]^i$ is equal to the sum of the number of moduli in the normal form $[T_7]^i$ and the codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ of the class of algebraic restrictions defined by this normal form.

Part (b) follows from Theorem 2.6 and Propositions 3.4 and 2.7. $\square$

6.2 Symplectic normal forms. Proof of Theorem 6.1

Let us transfer the normal forms $[T_7]^i$ to symplectic normal forms using Theorem 2.12, i.e. realizing the algorithm in section 2. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[T_7]^i$ of algebraic restrictions. We can fix, for example

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}, \quad c_1 \cdot c_2 \neq 0;$

$\omega^1 = c_1\theta_1 + \theta_2 + c_2\theta_5 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^2 = \theta_1 + c_1\theta_4 + c_2\theta_5 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}, \ (c_1, c_2) \neq (0, 0);$

$\omega^3 = \theta_4 + c_1\theta_5 + c_2\theta_6 + \sum_{i=1}^{3} dx_1 \wedge dx_{i+3} + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^4 = \theta_1 + c\theta_7 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^5 = \theta_6 + c\theta_7 + \sum_{i=1}^{3} dx_1 \wedge dx_{i+3} + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^6 = \theta_7 + \sum_{i=1}^{3} dx_1 \wedge dx_{i+3} + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^7 = \sum_{i=1}^{3} dx_1 \wedge dx_{i+3} + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}.$

Let $\omega = \sum_{i=1}^{m} dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0, 1, \cdots, 7$ (resp. for $i = 0, 1, 2, 4$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω: $(\Phi^i)^*\omega^i = \omega$. Consider the families $T_7^i = (\Phi^i)^{-1}(T_7)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega)$

which is diffeomorphic to T_7 is symplectically equivalent to one and only one of the normal forms $T_7^i, i = 0, 1, \cdots, 7$ (resp. $i = 0, 1, 2, 4$) presented in Theorem 6.1. By Theorem 6.4we obtain that parameters c, c_1, c_2 of the normal forms are moduli.

6.3 Distinguishing symplectic classes of T_7 by Lagrangian tangency orders

Lagrangian tangency orders will be used to obtain a more detailed classification of (T_7). A curve $N \in (T_7)$ may be described as a union of two parametrical branches B_1 and B_2. Their parameterization is given in the second column of Table 12. To distinguish the classes of this singularity completely we need following three invariants:

$Lt(N) = Lt(B_1, B_2) = \max_L(\min\{t(B_1, L), t(B_2, L)\})$,

$L_n = \max\{Lt(B_1), Lt(B_2)\} = \max\{\max_L t(B_1, L), \max_L t(B_2, L)\}$,

$L_f = \min\{Lt(B_1), Lt(B_2)\} = \min\{\max_L t(B_1, L), \max_L t(B_2, L)\}$,

where L is a smooth Lagrangian submanifold of the symplectic space.

Branches B_1 and B_2 are diffeomorphic and are not preserved by all symmetries of T_7 so neither $Lt(B_1)$ nor $Lt(B_2)$ can be used as invariants. The new invariants are defined instead: L_n describing the Lagrangian tangency order of the *nearest* branch and L_f representing the Lagrangian tangency order of the *farthest* branch. Considering the triples $(Lt(N), L_n, L_f)$ we obtain more detailed classification of symplectic singularities of T_7 than the classification given in Table 11. Some subclasses appear (see Table 12) having a natural geometric interpretation (Tables 13 and 14).

Theorem 6.7. *A stratified submanifold $N \in (T_7)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 12. The parameters c, c_1, c_2 are moduli. The Lagrangian tangency orders of the curve are presented in the fifth, the sixth and the seventh column of Table 12 and the codimension of the classes is given in the fourth column.*

Remark 6.8. *The numbers L_n and L_f can be easily calculated by using Proposition 3.6 to branches B_1 and B_2 or by direct applying the definition of*

Class	Parametrization of branches	Conditions for subclasses	cod	$Lt(N)$	L_n	L_f
$(T_7)^0$ $2n \geq 4$	$(t^3, -c_1t^2, 0, -t^2, 0, \cdots)$ $(t^3, -c_2t^2, -t^2, 0, 0, \cdots)$	$c_1 \cdot c_2 \neq 0$	0	2	3	3
$(T_7)^1$ $2n \geq 4$	$(t^3, -t^2, 0, -c_1t^2, 0, \cdots)$ $(t^3, 0, -t^2, c_2t^5, 0, \cdots)$	$c_1 \cdot c_2 \neq 0$	1	2	5	3
		$c_1 = 0, c_2 \neq 0$	2	3	5	3
		$c_1 \neq 0, c_2 = 0$	2	2	∞	3
		$c_1 = 0, c_2 = 0$	3	3	∞	3
$(T_7)^2$ $2n \geq 4$	$(t^3, \frac{c_1^2}{2}t^4, 0, -t^2, 0, \cdots)$ $(t^3, \frac{c_2^2}{2}t^4, -t^2, 0, 0, \cdots)$	$c_1 \cdot c_2 \neq 0$	2	2	5	5
		$c_1 \cdot c_2 = 0$, $c_1 + c_2 \neq 0$	3	2	∞	5
$(T_7)^3$ $2n \geq 6$	$(t^3, \frac{1}{2}t^4, 0, c_2t^5, -t^2, 0, \cdots)$ $(t^3, \frac{c_1}{2}t^4, -t^2, 0, 0, 0, \cdots)$	$c_1 \neq 0$	3	5	5	5
		$c_1 = 0$	4	5	∞	5
$(T_7)^4$ $2n \geq 4$	$(t^3, \frac{c}{3}t^6, 0, -t^2, 0, \cdots)$ $(t^3, 0, -t^2, 0, 0, \cdots)$		4	2	∞	∞
$(T_7)^5$ $2n \geq 6$	$(t^3, -\frac{c}{3}t^6, 0, t^5, -t^2, 0, \cdots)$ $(t^3, 0, -t^2, 0, 0, 0, \cdots)$		5	5	∞	∞
$(T_7)^6$ $2n \geq 6$	$(t^3, -\frac{1}{3}t^6, 0, 0, -t^2, 0, \cdots)$ $(t^3, 0, -t^2, 0, 0, 0, \cdots)$		6	7	∞	∞
$(T_7)^7$ $2n \geq 6$	$(t^3, 0, 0, 0, -t^2, 0, \cdots)$ $(t^3, 0, -t^2, 0, 0, 0, \cdots)$		7	∞	∞	∞

Table 12: Lagrangian tangency orders for symplectic classes of T_7 singularity.

the Lagrangian tangency order and finding the nearest Lagrangian submanifold to these branches. Next we calculate $Lt(N)$ by definition knowing that it can not be greater than L_f.

We can compute $Lt(B_1)$ using the algebraic restrictions $[\omega^i]_{B_1}$ where the space $[Z^2(\mathbb{R}^{2n})]_{B_1}$ is spanned only by the algebraic restrictions to B_1 of the 2-forms θ_2, θ_4. For example for the class $(T_7)^1$ we have $[c_1\theta_1 + \theta_2 + c_2\theta_5]_{B_1} = [\theta_2]_{B_1}$ and thus $Lt(B_1) \leq 3$. Applying the definition of $Lt(B_1)$ we find the smooth Lagrangian submanifolds L described by the conditions: $p_i = 0$, $i \in \{1, \ldots, n\}$ and we get $Lt(B_1) = t(B_1, L) = 3$.

We can compute $LT(B_2)$ using the algebraic restrictions $[\omega^i]_{B_2}$ where the space $[Z^2(\mathbb{R}^{2n})]_{B_2}$ is spanned only by the algebraic restrictions to B_2 of the 2-forms θ_3, θ_5. For example for the class $(T_7)^1$ we have $[c_1\theta_1 + \theta_2 + c_2\theta_5]_{B_2} = [c_2\theta_5]_{B_2}$ and thus $Lt(B_2) = 5$ if $c_2 \neq 0$ and $Lt(B_2) = \infty$ if $c_2 = 0$.

Finally for the class $(T_7)^1$ we have $L_n = 5$ if $c_2 \neq 0$ and $L_n = \infty$ if $c_2 = 0$ and $L_f = 3$ so $Lt(N) \leq 3$.

For the smooth Lagrangian submanifolds L described by the conditions:

$p_1 = 0,\ q_2 = 0$ and $p_i = 0$ for $i > 2$ we get $t[N, L] = 3$ if $c_1 = 0$ thus $Lt(N) = 3$ in this case. But if $c_1 \neq 0$ then $t[N, L] = 2$ and it can not be greater for any other smooth Lagrangian submanifold so $Lt(N) = 2$ in this case.

6.4 Geometric conditions for the classes $(T_7)^i$

The classes $(T_7)^i$ can be distinguished geometrically, without using any local coordinate system.

Let $N \in (T_7)$. Then N is the union of two branches – singular 1-dimensional irreducible components diffeomorphic to the cusp singularities. In local coordinates they have the form

$$\mathcal{B}_1 = \{x_1^2 + x_3^3 = 0,\ x_2 = x_{\geq 4} = 0\},$$

$$\mathcal{B}_2 = \{x_1^2 + x_2^3 = 0,\ x_{\geq 3} = 0\}.$$

Denote by ℓ_1, ℓ_2 the tangent lines at 0 to the branches $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. These lines span a 2-space P_1. Let P_2 be 2-space tangent at 0 to the branch $\mathcal{B}_1$ and P_3 be 2-space tangent at 0 to the branch $\mathcal{B}_2$. Define the line $\ell_3 = P_2 \cap P_3$. The lines ℓ_1, ℓ_2, ℓ_3 span a 3-space $W = W(N)$. Equivalently W is the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. The classes $(T_7)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form. For $N = T_7 =$(4) it is easy to calculate

$$\ell_1 = \text{ span } (\partial/\partial x_3),\ \ell_2 = \text{ span } (\partial/\partial x_2),\ \ell_3 = \text{ span } (\partial/\partial x_1). \qquad (7)$$

6.4.1 Geometric conditions for the class $[0]_{T_7}$

The geometric distinguishing of the class $(T_7)^7$ follows from Theorem 2.6: $N \in (T_7)^7$ if and only if N it is contained in a non-singular Lagrangian submanifold. The following theorem gives a simple way to check the latter condition without using algebraic restrictions. Given a 2-form σ on a non-singular submanifold M of $\mathbb{R}^{2n}$ such that $\sigma(0) = 0$ and a vector $v \in T_0M$

we denote by $\mathcal{L}_v\sigma$ the value at 0 of the Lie derivative of σ along a vector field V on M such that $v = V(0)$. The assumption $\sigma(0) = 0$ implies that the choice of V is irrelevant.

Proposition 6.9. *Let $N \in (T_7)$ be a stratified submanifold of a symplectic space $(\mathbb{R}^{2n}, \omega)$. Let M^3 be any non-singular submanifold containing N and let σ be the restriction of ω to TM^3. Let $v_i \in \ell_i$ be non-zero vectors. If the symplectic form ω has zero algebraic restriction to N then the following conditions are satisfied:*
I. $\sigma(0) = 0$,
II. $\mathcal{L}_{v_3}\sigma(v_i, v_j) = 0$ *for* $i, j \in \{1, 2\}$,
III. $\mathcal{L}_{v_i}\sigma(v_3, v_i) = 0$ *for* $i \in \{1, 2\}$,
IV. $\mathcal{L}_{v_i}\sigma(v_3, v_j) = \mathcal{L}_{v_j}\sigma(v_3, v_i)$ *for* $i \neq j \in \{1, 2\}$,

Proof. Any 2-form σ which has zero algebraic restriction to T_7 can be expressed in the following form $\sigma = H_1\alpha + H_2\beta + dH_1 \wedge \gamma + dH_2 \wedge \delta$, where $H_1 = x_1^2 + x_2^3 + x_3^3$, $H_2 = x_2x_3$ and α, β are 2-forms on TM^3 and $\gamma = \gamma_1 dx_1 + \gamma_2 dx_2 + \gamma_3 dx_3$ and $\delta = \delta_1 dx_1 + \delta_2 dx_2 + \delta_3 dx_3$ are 1-forms on TM^3. Since

$$H_1(0) = H_2(0) = 0, \quad dH_1|_0 = dH_2|_0 = 0 \tag{8}$$

we obtain the following equality

$$\mathcal{L}_v\sigma = d(V \rfloor \sigma)|_0 + (V \rfloor d\sigma)|_0 = d(V \rfloor \sigma)|_0.$$

(8) also implies that

$$d(V \rfloor \sigma)|_0 = d(V \rfloor dH_1)|_0 \wedge \gamma|_0 + d(V \rfloor dH_2)|_0 \wedge \delta|_0.$$

By simply calculation we get

$$\mathcal{L}_{v_1}\sigma = dx_2 \wedge \delta|_0 = \delta_3|_0\, dx_2 \wedge dx_3 - \delta_1|_0\, dx_1 \wedge dx_2,$$

$$\mathcal{L}_{v_2}\sigma = dx_3 \wedge \delta|_0 = \delta_1|_0\, dx_3 \wedge dx_1 - \delta_2|_0\, dx_2 \wedge dx_3,$$

$$\mathcal{L}_{v_3}\sigma = 2dx_1 \wedge \gamma|_0 = 2\gamma_2|_0\, dx_1 \wedge dx_2 - 2\gamma_3|_0\, dx_3 \wedge dx_1.$$

Finally we obtain

$$\mathcal{L}_{v_1}\sigma(v_3, v_1) = 0, \quad \mathcal{L}_{v_2}\sigma(v_3, v_2) = 0, \quad \mathcal{L}_{v_3}\sigma(v_1, v_2) = 0,$$

$$\mathcal{L}_{v_1}\sigma(v_3, v_2) = -\delta_1|_0 = \mathcal{L}_{v_2}\sigma(v_3, v_1).$$

□

Theorem 6.10. *A stratified submanifold $N \in (T_7)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(T_7)^i$ if and only if the couple (N, ω) satisfies corresponding conditions in the last column of Table 13 or 14.*

Class	Normal form	Geometric conditions
$(T_7)^0$	$[T_7]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{T_7}$ $c_1 \cdot c_2 \neq 0$	$\omega\|_{\ell_i+\ell_j} \neq 0 \;\; \forall i,j \in \{1,2,3\}$, so 2-spaces tangent to branches are not isotropic
$(T_7)^1$		$\exists i \neq j \in \{1,2\} \;\; \omega\|_{\ell_i+\ell_3} = 0$ and $\omega\|_{\ell_j+\ell_3} \neq 0$ (exactly one branch has tangent 2-space isotropic)
	$[T_7]^1_a : [c_1\theta_1 + \theta_2 + c_2\theta_5]_{T_7}$ $c_1 \cdot c_2 \neq 0$	$\omega\|_{\ell_1+\ell_2} \neq 0$ and no branch is contained in a Lagrangian submanifold
	$[T_7]^1_b : [\theta_2 + c_2\theta_5]_{T_7}$, $c_2 \neq 0$	$\omega\|_{\ell_1+\ell_2} = 0$ and no branch is contained in a Lagrangian submanifold
	$[T_7]^1_c : [c_1\theta_1 + \theta_2]_{T_7}$, $c_1 \neq 0$	$\omega\|_{\ell_1+\ell_2} \neq 0$ exactly one branch is contained in a Lagrangian submanifold
	$[T_7]^1_d : [\theta_2]_{T_7}$	$\omega\|_{\ell_1+\ell_2} = 0$ and exactly one branch is contained in a Lagrangian submanifold
$(T_7)^2$		$\omega\|_{\ell_1+\ell_2} \neq 0, \omega\|_{\ell_i+\ell_3} = 0 \; \forall i \in \{1,2\}$
	$[T_7]^2_a : [\theta_1 + c_1\theta_4 + c_2\theta_5]_{T_7}$ $c_1 \cdot c_2 \neq 0$	no branch is contained in a Lagrangian submanifold
	$[T_7]^2_b : [\theta_1 + c_1\theta_4 + c_2\theta_5]_{T_7}$ $c_1 \cdot c_2 = 0, \; c_1 + c_2 \neq 0$	exactly one branch is contained in a Lagrangian submanifold
$(T_7)^4$	$[T_7]^4 : [\theta_1 + c\theta_7]_{T_7}$	$\omega\|_{\ell_1+\ell_2} \neq 0, \omega\|_{\ell_i+\ell_3} = 0 \; \forall i \in \{1,2\}$, and branches are contained in different Lagrangian submanifolds

Table 13: Geometric interpretation of singularity classes of T_7 when $\omega|_W \neq 0$; W - the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n \geq 4}, \omega)$ containing $N \in (T_7)$.

Proof of Theorem 6.10. The conditions on the pair (ω, N) in the last column of Table 13 and Table 14 are disjoint. It suffices to prove that these conditions the row of $(T_7)^i$, are satisfied for any $N \in (T_7)^i$. This is a corollary of the following claims:

1. Each of the conditions in the last column of Tables 13, 14 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs (ω, N);

Class	Normal form	Geometric conditions
$(T_7)^3$	$[T_7]^3_a : [\theta_4 + c_1\theta_5 + c_2\theta_6]_{T_7}$ $c_1 \neq 0$	III is not satisfied and no branch is contained in a Lagrangian submanifold
	$[T_7]^3_b : [\theta_4 + c_2\theta_6]_{T_7}$	III is not satisfied and exactly one branch is contained in a Lagrangian submanifold
$(T_7)^5$	$[T_7]^5 : [\theta_6 + c\theta_7]_{T_7}$	III is satisfied but II is not and branches are contained in different Lagrangian submanifolds.
$(T_7)^6$	$[T_7]^6 : [\theta_7]_{T_7}$	I - IV are satisfied and branches are contained in different Lagrangian submanifolds.
$(T_7)^7$	$[T_7]^7 : [0]_{T_7}$	I - IV are satisfied and N is contained in a Lagrangian submanifold

Table 14: Geometric interpretation of singularity classes of T_7 when $\omega|_W = 0$; W - the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n\geq 6}, \omega)$ containing $N \in (T_7)$; $I - IV$ – conditions of Proposition 6.9.

2. Each of these conditions depends only on the algebraic restriction $[\omega]_N$;

3. Take the simplest 2-forms ω^i representing the normal forms $[T_7]^i$ for algebraic restrictions: ω^0, ω^1, ω^2, ω^3, ω^4, ω^5, ω^6, ω^7. The pair $(\omega = \omega^i, T_7)$ satisfies the condition in the last column of Table 13 or Table 14, the row of $(T_7)^i$.

The first statement is obvious, the second one follows from Lemma 2.7. To prove the third statement we note that in the case $N = T_7 = (4)$ one has $W =$ span $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and $v_1 \in \ell_1 =$ span $(\partial/\partial x_3)$, $v_2 \in \ell_2 =$ span $(\partial/\partial x_2)$, $v_3 \in \ell_3 =$ span $(\partial/\partial x_1)$. By simply calculation and observation of Lagrangian tangency orders we obtain that following statements are true:

(T^0) $\omega^0|_{\ell_1+\ell_2} \neq 0$ and $\omega^0|_{\ell_1+\ell_3} \neq 0$ and also $\omega^0|_{\ell_2+\ell_3} \neq 0$, and $L_n < \infty$ and $L_f < \infty$ hence no branch is contained in a smooth Lagrangian submanifold.

(T^1) For any c_1, c_2 $\omega^1|_{\ell_1+\ell_3} = 0$ and $\omega^1|_{\ell_2+\ell_3} \neq 0$ or $\omega^1|_{\ell_1+\ell_3} \neq 0$ and $\omega^1|_{\ell_2+\ell_3} = 0$. If $c_2 = 0$ then and $L_n = \infty$ and $L_f < \infty$ hence exactly one branch is contained in some smooth Lagrangian submanifold. For $c_2 \neq 0$ $L_n < \infty$ and $L_f < \infty$ so no branch is contained in a smooth Lagrangian submanifold. $\omega^1|_{\ell_1+\ell_2} = 0$ if and only if $c_1 = 0$.

(T^2) For any c_1, c_2 $\omega^2|_{\ell_1+\ell_2} \neq 0$ and $\omega^2|_{\ell_1+\ell_3} = 0$ and also $\omega^2|_{\ell_2+\ell_3} = 0$. If $c_1 \cdot c_2 \neq 0$ then $L_n < \infty$ and $L_f < \infty$ so no branch is contained in a Lagrangian submanifold. If $c_1 = 0$ and $c_2 \neq 0$ or $c_1 \neq 0$ and $c_2 = 0$ then and $L_n = \infty$ and $L_f < \infty$ hence exactly one branch is contained in some smooth Lagrangian submanifold.

(T^3) The Lie derivative of $\omega^3 = \theta_4 + c_1\theta_5 + c_2\theta_6$ along a vector field $V = \partial/\partial x_3$ is not equal to 0, so condition III of Proposition 6.9 is not satisfied. If $c_1 \neq 0$ then $L_n < \infty$ and $L_f < \infty$ hence no branch is contained in a Lagrangian submanifold. If $c_1 = 0$ then $L_n = \infty$ and $L_f < \infty$ hence only one branch is contained in some Lagrangian submanifold.

(T^4) For any c $\omega^4|_{\ell_1+\ell_2} \neq 0$ and $\omega^4|_{\ell_1+\ell_3} = 0$ and also $\omega^4|_{\ell_2+\ell_3} = 0$. Both branches are contained in different Lagrangian submanifolds since $L_n = L_f = \infty$ and $Lt(N) < \infty$.

(T^5) We can calculate the Lie derivatives of $\omega^5 = \theta_6 + c\theta_7$ along a vector fields $V_1 = \partial/\partial x_3$ and $V_2 = \partial/\partial x_2$ and $V_3 = \partial/\partial x_3$: $\mathcal{L}_{V_1}\omega^5(V_3, V_1) = 0$ and $\mathcal{L}_{V_2}\omega^5(V_3, V_2) = 0$, so condition III of Proposition 6.9 is satisfied, but the Lie derivative $\mathcal{L}_{V_3}\omega^5(V_1, V_2)$ is not equal to 0, so condition II of Proposition 6.9 is not satisfied. We have $Lt(N) < \infty$ and $L_n = L_f = \infty$ hence branches are contained in different Lagrangian submanifolds.

(T^6) The Lie derivatives of $\omega^6 = \theta_7$, $\mathcal{L}_{V_i}\omega^6(V_j, V_k) = 0$ for $i, j, k \in \{1, 2, 3\}$, so conditions II, III and IV of Proposition 6.9 are satisfied. We have $Lt(N) < \infty$ and $L_n = L_f = \infty$ hence branches are contained in different Lagrangian submanifolds.

(T^7) For $\omega^7 = 0$ we have $\mathcal{L}_{V_i}\omega^7(V_j, V_k) = 0$ for $i, j, k \in \{1, 2, 3\}$, so conditions II, III and IV of Proposition 6.9 are satisfied. The condition $Lt(N) = \infty$ implies the curve N is contained in a smooth Lagrangian submanifold. □

6.5 Proof of Theorem 6.4

In our proof we use vector fields tangent to $N \in (T_7)$. A Hamiltonian vector field is an example of such a vector field. We recall by [AGLV] a suitable definition and facts.

Definition 6.11. *Let $H = \{H_1 = \cdots = H_p = 0\} \subset \mathbb{R}^n$ be a complete intersection. Consider a set of $p+1$ integers $1 \leq i_1 < \cdots < i_{p+1} \leq n$. A Hamiltonian vector field $X_H(i_1, \ldots, i_{p+1})$ on a complete intersection H is the determinant obtained by expansion with respect to the first row of the symbolic $(p+1) \times (p+1)$ matrix*

$$X_H(i_1,\ldots,i_{p+1}) = det \begin{bmatrix} \partial/\partial x_{i_1} & \cdots & \partial/\partial x_{i_{p+1}} \\ \partial H_1/\partial x_{i_1} & \cdots & \partial H_1/\partial x_{i_{p+1}} \\ \vdots & \cdots & \vdots \\ \partial H_p/\partial x_{i_1} & \cdots & \partial H_p/\partial x_{i_{p+1}} \end{bmatrix} \quad (9)$$

Theorem 6.12 ([Wa]). *Let $H = \{H_1 = \cdots = H_p = 0\} \subset \mathbb{R}^n$ be a positive dimensional complete intersection with an isolated singularity. If $H_1,\ldots,H_p$ are quasi-homogeneous with positive weights $\lambda_1,\ldots,\lambda_n$ than the module of vector fields tangent to H is generated by the Euler field $E = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$ and the Hamiltonian vector fields $X_H(i_1,\ldots,i_{p+1})$ where the numbers i_1 , ... , i_{p+1} run through all possible sets $1 \le i_1 < \cdots < i_{p+1} \le n$.*

Proposition 6.13. *Let $H = \{H_1 = \cdots = H_{n-1} = 0\} \subset \mathbb{R}^n$ be a 1-dimensional complete intersection. If X_H is the Hamiltonian vector field on H then $[\mathcal{L}_{X_H}(\alpha)]_H = [0]_H$ for any closed 2-form α.*

Proof. Note that $X_H \rfloor dx_1 \wedge \ldots \wedge dx_n = dH_1 \wedge \ldots \wedge dH_p$. This implies for $i < j$

$$X_H \rfloor dx_i \wedge dx_j = (-1)^{i+j+1}(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{n-2}}}) \rfloor (dH_1 \wedge \cdots \wedge dH_{n-1}) =$$

$$= \sum_{k=1}^{n-1} (-1)^{k+i+j}(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{n-2}}}) \rfloor (dH_{l_{1,k}} \wedge \cdots \wedge dH_{l_{n-2,k}}) dH_k =$$

$$= \sum_{k=1}^{n-1} f_k dH_k$$

where $(i_1,\cdots,i_{n-2}) = (1,\cdots,i-1,i+1,\cdots,j-1,j+1,\cdots,n)$ and for $k \in \{1,\cdots,n-1\}$ we take a sequence $(l_{1,k},\cdots,l_{n-2,k}) = (1,\cdots,k-1,k+1,\cdots,n-1)$.
Thus $[X_H \rfloor dx_i \wedge dx_j]_H = [\sum_{k=1}^{n-1} f_k dH_k]_H = [0]_H$. If $\alpha = \sum_{i<j} g_{i,j} dx_i \wedge dx_j$ is a closed 2-form then $[\mathcal{L}_{X_H}\alpha]_H = [d(X_H \rfloor \alpha)]_H$. It implies that

$$[\mathcal{L}_{X_H}\alpha]_H = \sum_{i<j} g_{i,j}[d(X_H \rfloor dx_i \wedge dx_j)]_H + [dg_{i,j} \wedge (X_H \rfloor dx_i \wedge dx_j)]_H = [0]_H.$$

□

As a corollary of the above facts we obtain that the germ of a vector field tangent to T_7 of non trivial action on algebraic restriction of closed 2-forms to T_7 may be described as a linear combination germs of vector fields: $X_1 = E$, $X_2 = x_3E$, $X_3 = x_2E$, $X_4 = x_1E$, $X_5 = x_2^2E$, $X_6 = x_3^2E$ where E is the Euler vector field $E = 3x_1\partial/\partial x_1 + 2x_2\partial/\partial x_2 + 2x_3\partial/\partial x_3$.

Proposition 6.14. *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to N on the basis of the vector space of algebraic restrictions of closed 2-forms to N is presented in Table 15.*

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_5]$	$[\theta_6]$	$[\theta_7]$
$X_1 = E$	$4[\theta_1]$	$5[\theta_2]$	$5[\theta_3]$	$7[\theta_4]$	$7[\theta_5]$	$7[\theta_6]$	$9[\theta_7]$
$X_2 = x_3E$	$[0]$	$7[\theta_4]$	$3[\theta_6]$	$9[\theta_7]$	$[0]$	$[0]$	$[0]$
$X_3 = x_2E$	$[0]$	$-3[\theta_6]$	$7[\theta_5]$	$[0]$	$-9[\theta_7]$	$[0]$	$[0]$
$X_4 = x_1E$	$-4[\theta_6]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_5 = x_2^2E$	$[0]$	$[0]$	$-9[\theta_7]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_6 = x_3^2E$	$[0]$	$9[\theta_7]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$

Table 15: Infinitesimal actions on algebraic restrictions of closed 2-forms to T_7. $E = 3x_1\partial/\partial x_1 + 2x_2\partial/\partial x_2 + 2x_3\partial/\partial x_3$

Let $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 6.4 follows from the following lemmas.

Lemma 6.15. *If $c_1 \cdot c_2 \cdot c_3 \neq 0$ then the algebraic restriction $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{T_7}$.*

Proof of Lemma 6.15.

We use the homotopy method to prove that $\mathcal{A}$ is diffeomorphic to $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{T_7}$.

Let $\mathcal{B}_t = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + (1-t)c_4\theta_4 + (1-t)c_5\theta_5 + (1-t)c_6\theta_6 + (1-t)c_7\theta_7]_{T_7}$ for $t \in [0;1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{T_7}$. We prove that there exists a family $\Phi_t \in Symm(T_7),\ t \in [0;1]$ such that

$$\Phi_t^* \mathcal{B}_t = \mathcal{B}_0,\ \Phi_0 = id. \tag{10}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt} = V_t(\Phi_t)$. Then differentiating (10) we obtain

$$\mathcal{L}_{V_t} \mathcal{B}_t = c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7. \tag{11}$$

We are looking for V_t in the form $V_t = (b_1x_2 + b_2x_3 + b_3x_2^2 + b_4x_3^2 + b_5x_1)E$ where $b_1, b_2, b_3, b_4, b_5 \in \mathbb{R}$. Then by Proposition 6.14 equation (11) has a form

$$\begin{bmatrix} 0 & 4c_2 & 0 & 0 & 0 \\ 7c_3 & 0 & 0 & 0 & 0 \\ -3c_2 & 3c_3 & 0 & 0 & -4c_1 \\ -9c_5 & 9c_4 & -9c_3 & 9c_2 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} \tag{12}$$

If $c_1 \cdot c_2 \cdot c_3 \neq 0$ we can solve (12) and Φ_t may be obtained as a flow of vector field V_t. The family Φ_t preserves T_7, because V_t is tangent to T_7 and $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments we have $\mathcal{A}$ diffeomorphic to $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{T_7}$. By the condition $c_1 \neq 0$ we have a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_1|^{-\frac{3}{4}}x_1, |c_1|^{-\frac{1}{2}}x_2, |c_1|^{-\frac{1}{2}}x_3) \tag{13}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [\frac{c_1}{|c_1|}\theta_1 + c_2|c_1|^{-\frac{5}{4}}\theta_2 + c_3|c_1|^{-\frac{5}{4}}\theta_3]_{T_7} = [\pm\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{T_7}.$$

By the following symmetry of T_7: $(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we have that $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{T_7}$ and $[-\theta_1 + \widetilde{c}_3\theta_2 + \widetilde{c}_2\theta_3]_{T_7}$ are diffeomorphic. □

Lemma 6.16. *If* $c_2 \cdot c_3 = 0$ *and* $c_2 + c_3 \neq 0$ *then the algebraic restriction of the form* $[c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ *can be reduced by a symmetry of* T_7 *to an algebraic restriction* $[\widetilde{c}_1\theta_1 + \theta_2 + \widetilde{c}_5\theta_5]_{T_7}$.

Proof of Lemma 6.16. We use similar methods as above to prove that if $c_2 \cdot c_3 = 0$ and $c_2 + c_3 \neq 0$ then $\mathcal{A}$ is diffeomorphic to $[\widetilde{c}_1\theta_1 + \theta_2 + \widetilde{c}_5\theta_5]_{T_7}$. If $c_3 = 0$ then $c_2 \neq 0$ and $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ Let $\mathcal{B}_t = [c_1\theta_1 + c_2\theta_2 + (1-t)c_4\theta_4 + c_5\theta_5 + (1-t)c_6\theta_6 + (1-t)c_7\theta_7]_{T_7}$ for $t \in [0;1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_5\theta_5]_{T_7}$. We prove that there exists a family $\Phi_t \in Symm(T_7)$, $t \in [0;1]$ such that

$$\Phi_t^*\mathcal{B}_t = \mathcal{B}_0, \ \Phi_0 = id. \tag{14}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt} = V_t(\Phi_t)$. Then differentiating (14) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t = c_4\theta_4 + c_6\theta_6 + c_7\theta_7. \tag{15}$$

We are looking for V_t in the form $V_t = (b_1x_2 + b_2x_3 + b_4x_3^2 + b_5x_1)E$ where $b_1, b_2, b_4, b_5 \in \mathbb{R}$. Then by Proposition 6.14 equation (15) has a form

$$\begin{bmatrix} 0 & 4c_2 & 0 & 0 \\ -3c_2 & 0 & 0 & -4c_1 \\ -9c_5 & 9c_4 & 9c_2 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_6 \\ c_7 \end{bmatrix} \tag{16}$$

If $c_2 \neq 0$ we can solve (16) and Φ_t may be obtained as a flow of vector field V_t. The family Φ_t preserves T_7, because V_t is tangent to T_7 and $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_5\theta_5]_{T_7}$. By the condition $c_2 \neq 0$ we have a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : \ (x_1, x_2, x_3) \mapsto (c_2^{-\frac{3}{5}}x_1, c_2^{-\frac{2}{5}}x_2, c_2^{-\frac{2}{5}}x_3) \tag{17}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [c_1c_2^{-\frac{4}{5}}\theta_1 + \theta_2 + c_5c_2^{-\frac{7}{5}}\theta_5]_{T_7} = [\widetilde{c}_1\theta_1 + \theta_2 + \widetilde{c}_5\theta_5]_{T_7}.$$

If $c_2 = 0$ then $c_3 \neq 0$ and by the diffeomorphism $\Theta \in Symm(T_7)$ of the form: $(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we obtain $\Theta^*[c_1\theta_1 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7} = [-c_1\theta_1 + c_3\theta_2 + c_4\theta_5 + c_5\theta_4 - c_6\theta_6 - c_7\theta_7]_{T_7}$ and we may use the homotopy method now. □

Lemma 6.17. *If* $c_1 \neq 0$ *and* $(c_4, c_5) \neq (0,0)$ *then the algebraic restriction of the form* $[c_1\theta_1 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ *can be reduced by a symmetry of* T_7 *to an algebraic restriction* $[\theta_1 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{T_7}$.

Proof of Lemma 6.17.

We prove that if $(c_2, c_3) = (0,0)$, $c_1 \neq 0$ and $(c_4, c_5) \neq (0,0)$ then $\mathcal{A} = [c_1\theta_1 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ is diffeomorphic to $[\theta_1 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{T_7}$. Let $\mathcal{B}_t = [c_1\theta_1 + c_4\theta_4 + c_5\theta_5 + (1-t)c_6\theta_6 + (1-t)c_7\theta_7]_{T_7}$ for $t \in [0;1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_1\theta_1 + c_4\theta_4 + c_5\theta_5]_{T_7}$. We must find a vector field V_t satisfying equation

$$\mathcal{L}_{V_t}\mathcal{B}_t = c_6\theta_6 + c_7\theta_7. \tag{18}$$

This vector field V_t has the form $V_t = (b_1x_2 + b_2x_3 + b_5x_1)E$ where $b_1, b_2, b_5 \in \mathbb{R}$. Then by Proposition 6.14 equation (18) has a form

$$\begin{bmatrix} 0 & 0 & -4c_1 \\ -9c_5 & 9c_4 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_6 \\ c_7 \end{bmatrix} \tag{19}$$

If $c_1 \neq 0$ and c_4 or c_5 is not equal 0 we can solve (19). Then for family Φ_t obtained as a flow of vector field V_t we have $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_5\theta_5]_{T_7}$. By the condition $c_1 \neq 0$ we have a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_1|^{-\frac{3}{4}}x_1, |c_1|^{-\frac{1}{2}}x_2, |c_1|^{-\frac{1}{2}}x_3) \tag{20}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [\frac{c_1}{|c_1|}\theta_1 + c_4|c_1|^{-\frac{7}{4}}\theta_4 + c_5|c_1|^{-\frac{7}{4}}\theta_3]_{T_7} = [\pm\theta_1 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{T_7}.$$

By the following symmetry of T_7: $(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we have that $[\theta_1 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{T_7}$ is diffeomorphic to $[-\theta_1 + \widetilde{c}_4\theta_5 + \widetilde{c}_5\theta_4]_{T_7}$. □

Lemma 6.18. *If* $c_1 \neq 0$ *then the algebraic restriction of the form* $[c_1\theta_1 + c_6\theta_6 + c_7\theta_7]_{T_7}$ *can be reduced by a symmetry of* T_7 *to an algebraic restriction* $[\theta_1 + \widetilde{c}_7\theta_7]_{T_7}$.

Proof of Lemma 6.18.

We prove now that if $(c_2, c_3, c_4, c_5) = (0,0,0,0)$, $c_1 \neq 0$ then $\mathcal{A} = [c_1\theta_1 + c_6\theta_6 + c_7\theta_7]_{T_7}$ is diffeomorphic to $[\theta_1 + \widetilde{c}_7\theta_7]_{T_7}$.
Let $\mathcal{B}_t = [c_1\theta_1 + (1-t)c_6\theta_6 + c_7\theta_7]_{T_7}$ for $t \in [0;1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_1\theta_1 + c_7\theta_7]_{T_7}$. We must find a vector field V_t satisfying equation

$$\mathcal{L}_{V_t}\mathcal{B}_t = c_6\theta_6. \tag{21}$$

By Proposition 6.14 we have $\mathcal{L}_{x_1E}\mathcal{B}_t = -4c_1\theta_6$ so we can use $V_t = \frac{-c_6}{4c_1}x_1E$ and for family Φ_t obtained as a flow of vector field V_t we have $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. So $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_1\theta_1 + c_7\theta_7]_{T_7}$. By the condition $c_1 \neq 0$ we have a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_1|^{-\frac{3}{4}}x_1, |c_1|^{-\frac{1}{2}}x_2, |c_1|^{-\frac{1}{2}}x_3) \tag{22}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [\frac{c_1}{|c_1|}\theta_1 + c_7|c_1|^{-\frac{9}{4}}\theta_7]_{T_7} = [\pm\theta_1 + \widetilde{c}_7\theta_7]_{T_7}.$$

By the following symmetry of T_7: $(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we have that $[\theta_1 + \widetilde{c}_7\theta_7]_{T_7}$ is diffeomorphic to $[-\theta_1 - \widetilde{c}_7\theta_7]_{T_7}$. □

Lemma 6.19. *If* $(c_4, c_5) \neq (0,0)$ *then the algebraic restriction of the form* $[c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ *can be reduced by a symmetry of* T_7 *to an algebraic restriction* $[\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{T_7}$.

Proof of Lemma 6.19.

We prove that if $c_1 = c_2 = c_3 = 0$ and $(c_4, c_5) \neq (0,0)$ then $\mathcal{A} = [c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$ is diffeomorphic to $[\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{T_7}$.

By Proposition 6.14 $\mathcal{L}_{x_3E}[\theta_4] = 9[\theta_7]$. If $c_4 \neq 0$ we may use $V_t = \frac{c_6}{9c_4}x_3E$ and reduce $\mathcal{A}$ to $\mathcal{B}_1 = [c_4\theta_4 + c_5\theta_5 + c_6\theta_6]_{T_7}$. By the condition $c_4 \neq 0$ we have a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_4^{-\frac{3}{7}}x_1, c_4^{-\frac{2}{7}}x_2, c_4^{-\frac{2}{7}}x_3) \tag{23}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [\theta_4 + \frac{c_5}{c_4}\theta_5 + \frac{c_6}{c_4}\theta_6]_{T_7} = [\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{T_7}.$$

If $c_4 = 0$ then $c_5 \neq 0$ and using the diffeomorphism $\Theta \in Symm(T_7)$ of the form: $(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we obtain $\Theta^*[c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7} = [c_5\theta_4 + c_4\theta_5 - c_6\theta_6 - c_7\theta_7]_{T_7}$ and we may use previous method. □

Lemma 6.20. *If* $c_6 \neq 0$ *then the algebraic restriction* $\mathcal{A} = [c_6\theta_6 + c_7\theta_7]_{T_7}$ *can be reduced by a symmetry of* T_7 *to an algebraic restriction* $[\theta_6 + \widetilde{c}_7\theta_7]_{T_7}$.

Proof of Lemma 6.20. Because $c_6 \neq 0$ we may use a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_6^{-\frac{3}{7}} x_1, c_6^{-\frac{2}{7}} x_2, c_6^{-\frac{2}{7}} x_3) \tag{24}$$

and we obtain

$$\Psi^*(\mathcal{A}) = [\theta_6 + c_7 c_6^{-\frac{9}{7}} \theta_7]_{T_7} = [\theta_6 + \widetilde{c}_7\theta_7]_{T_7}.$$

□

Lemma 6.21. *If* $c_7 \neq 0$ *then the algebraic restriction* $[c_7\theta_7]_{T_7}$ *can be reduced by a symmetry of* T_7 *to an algebraic restriction* $[\theta_7]_{T_7}$.

Proof of Lemma 6.21. Because $c_6 \neq 0$ we may use a diffeomorphism $\Psi \in Symm(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_7^{-\frac{3}{9}} x_1, c_7^{-\frac{2}{9}} x_2, c_7^{-\frac{2}{9}} x_3) \tag{25}$$

and we obtain

$$\Psi^*([c_7\theta_7]_{T_7}) = [\theta_7]_{T_7}.$$

□

Statement *(ii)* of Theorem 6.4 follows from conditions in the proof of part (i) and *(iii)* follows from Theorem 6.10 which was proved in section 6.4.

Now we prove that the parameters c, c_1, c_2 are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$. From Table 15 we see that the tangent space to the orbit of $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$ at $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$ is spanned by the linearly independent algebraic restrictions $[4c_1\theta_1 + 5\theta_2 + 5c_2\theta_3]_{T_7}$, $[\theta_4]_{T_7}, [\theta_5]_{T_7}, [\theta_6]_{T_7}, [\theta_7]_{T_7}$. Hence the algebraic restrictions $[\theta_1]_{T_7}$ and $[\theta_3]_{T_7}$ do not belong to it. Therefore the parameters c_1 and c_2 are independent moduli in the normal form $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$. □

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Differential structures on natural bundles connected with a differential space

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Abstract

For a given differential space differential structures on tangent and cotangent space are described and investigated. It is proved that: (i) for any point the tangent space to a differential subspace is closed in the whole tangent space; (ii) any co-vector is a differential of a smooth function.

Key words and phrases: differential space, differential structure.
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1 Introduction

This article is the first of a series of papers concerning integration of differential forms and densities on differential spaces. We describe natural differential structures defined on tangent and cotangent spaces by a given differential structure on the basic space. We also investigate properties of so obtained differential spaces.

Section 2 of the paper contains basic definitions and the description of preliminary facts concerning the theory of differential spaces. In Section 3 we describe the standard differential structure on the space tangent to a given differential space and show new results about topological properties of this structure (Theorems 3.1, 3.2 and 3.3). Section 4 is devoted to the investigation of the space cotangent to a given differential space. We prove that any co-vector is a differential of some smooth function (Proposition 4.4 and Theorem 4.3). We also propose the quite new definition of the

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differential structure on the cotangent space (remarks after Theorem 4.3) and give some basic properties of this differential structure (Proposition 4.5 and 4.6). Without any other explanation we use the following symbols: **N**-the set of natural numbers; **R**-the set of reals.

2 Differential spaces

Let M be a nonempty set and let $\mathcal{C}$ be a family of real valued functions on M. Denote by $\tau_{\mathcal{C}}$ the weakest topology on M with respect to which all functions of $\mathcal{C}$ are continuous.

A basis of the topology $\tau_{\mathcal{C}}$ consists of sets:

$$(\alpha_1, \ldots, \alpha_n)^{-1}(P) = \bigcap_{i=1}^{n} \{m \in M : a_i < \alpha_i(m) < b_i\},$$

where $n \in \mathbf{N}$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}$, $a_i < b_i$, $\alpha_1, \ldots, \alpha_n \in \mathcal{C}$, $P = \{(x_1, \ldots, x_n) \in \mathbf{R}^n; a_i < x_i < b_i, i = 1, \ldots, n\}$.

Definition 2.1 A function $f : M \to \mathbf{R}$ is called *a local $\mathcal{C}$-function on M* if for every $m \in M$ there is a neighbourhood V of m and $\alpha \in \mathcal{C}$ such that $f_{|V} = \alpha_{|V}$. The set of all local $\mathcal{C}$-functions on M is denoted by $\mathcal{C}_M$.

Note that any function $f \in \mathcal{C}_M$ is continuous with respect to the topology $\tau_{\mathcal{C}}$. In fact, if $\{V_i\}_{i \in I}$ is such an open (with respect to $\tau_{\mathcal{C}}$) covering of M that for any $i \in I$ there exists $\alpha_i \in \mathcal{C}$ satisfying $f_{|V_i} = \alpha_{i|V_i}$ and U is an open subset of $\mathbf{R}$ then

$$f^{-1}(U) = \bigcup_{i \in I} (\alpha_{i|V_i})^{-1}(U).$$

Since $(\alpha_{i|V_i})^{-1}(U)$ is open in V_i and $V_i \in \tau_{\mathcal{C}}$ we obtain $(\alpha_{i|V_i})^{-1}(U) \in \tau_{\mathcal{C}}$ for any $i \in I$. Hence $f^{-1}(U) \in \tau_{\mathcal{C}}$. Bearing in mind that U is an arbitrary open set in $\mathbf{R}$ we obtain that f is continuous with respect to $\tau_{\mathcal{C}}$.

We have $\mathcal{C} \subset \mathcal{C}_M$ which implies $\tau_{\mathcal{C}} \subset \tau_{\mathcal{C}_M}$. On the other hand any element of $\mathcal{C}_M$ is a function continuous with respect to $\tau_{\mathcal{C}}$. Then $\tau_{\mathcal{C}_M} \subset \tau_{\mathcal{C}}$ and consequently $\tau_{\mathcal{C}_M} = \tau_{\mathcal{C}}$.

Definition 2.2 A function $f : M \to \mathbf{R}$ is called *a $\mathcal{C}$-smooth function on M* if there exist $n \in \mathbf{N}$, $\omega \in C^\infty(\mathbf{R}^n)$ and $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ such that

$$f = \omega \circ (\alpha_1, \dots, \alpha_n).$$

The set of all $\mathcal{C}$-smooth functions on M is denoted by sc$\mathcal{C}$.

We have $\mathcal{C} \subset sc\mathcal{C}$, which implies $\tau_\mathcal{C} \subset \tau_{sc\mathcal{C}}$. On the other hand any superposition $\omega \circ (\alpha_1, \dots, \alpha_n)$ is continuous with respect to $\tau_\mathcal{C}$, which gives $\tau_{sc\mathcal{C}} \subset \tau_\mathcal{C}$. Consequently $\tau_{sc\mathcal{C}} = \tau_\mathcal{C}$.

Definition 2.3 A set $\mathcal{C}$ of real functions on M is said to be a *(Sikorski's) differential structure* if: (i) $\mathcal{C}$ is *closed with respect to localization* i.e. $\mathcal{C} = \mathcal{C}_M$; (ii) $\mathcal{C}$ is *closed with respect to superposition with smooth functions* i.e. $\mathcal{C} = $ sc$\mathcal{C}$.

In this case the pair $(M, \mathcal{C})$ is said to be a *(Sikorski's) differential space* (see [2]). Any element of $\mathcal{C}$ is called a *smooth function on M* (with respect to $\mathcal{C}$).

Proposition 2.1 *The intersection of differential structures defined on a set $M \neq \emptyset$ is a differential structure on M.*

Proof. Let $\{\mathcal{C}_i\}_{i \in I}$ be a family of differential structures defined on a set M and let $\mathcal{C} := \bigcap_{i \in I} \mathcal{C}_i$. Then $\mathcal{C}$ is a nonempty family of real-valued functions on M (it contains all constant functions). If $n \in \mathbf{N}$, $\omega \in C^\infty(\mathbf{R}^n)$ and $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ then for any $i \in I$ $\alpha_1, \dots, \alpha_n \in \mathcal{C}_i$ and consequently $\omega \circ (\alpha_1, \dots, \alpha_n) \in \mathcal{C}_i$. Hence $\omega \circ (\alpha_1, \dots, \alpha_n) \in \mathcal{C}$, which means that $sc\mathcal{C} = \mathcal{C}$.

Since $\mathcal{C} \subset \mathcal{C}_i$ for any $i \in I$ we have $\tau_\mathcal{C} \subset \tau_{\mathcal{C}_i}$. It means that any subset of M open with respect to $\tau_\mathcal{C}$ is open with respect to $\tau_{\mathcal{C}_i}$ for $i \in I$.

Let $\beta \in \mathcal{C}_M$. Choose for any $m \in M$ a set $U_m \in \tau_\mathcal{C}$ and a function $\alpha_m \in \mathcal{C}$ such that $m \in U_m$ and $\beta_{|U_m} = \alpha_{m|U_m}$. Since $\alpha_m \in \mathcal{C}_i$ and $U_m \in \tau_{\mathcal{C}_i}$ we obtain $\beta \in (\mathcal{C}_i)_M = \mathcal{C}_i$ for any $i \in I$. Then $\beta \in \mathcal{C}$ and consequently $\mathcal{C}_M = \mathcal{C}$.

Equalities $\mathcal{C}_M = \mathcal{C} = sc\mathcal{C}$ mean that $\mathcal{C}$ is a differential structure on M.□

Let $\mathcal{F}$ be a set of real functions on M. Then, by Proposition 2.1, the intersection $\mathcal{C}$ of all differential structures on M containing $\mathcal{F}$ is a differential structure on M. It is the smallest differential structure on M containing $\mathcal{F}$.

One can easily prove that $\mathcal{C} = (sc\mathcal{F})_M$ (see [3]). This structure is called *the differential structure generated by* $\mathcal{F}$. Functions of $\mathcal{F}$ are called *generators* of the differential structure $\mathcal{C}$. We also have $\tau_{(sc\mathcal{F})_M} = \tau_{sc\mathcal{F}} = \tau_{\mathcal{F}}$ (see remarks after Definitions 2.1 and 2.2).

Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be differential spaces. A map $F : M \to N$ is said to be *smooth* if for any $\beta \in \mathcal{D}$ the superposition $\beta \circ F \in \mathcal{C}$. We will denote the fact that $\mathcal{F}$ is smooth writing

$$F : (M, \mathcal{C}) \to (N, \mathcal{D}).$$

If $F : (M, \mathcal{C}) \to (N, \mathcal{D})$ is a bijection and $F^{-1} : (N, \mathcal{D}) \to (M, \mathcal{C})$ then F is called *a diffeomorphism*

If A is a nonempty subset of M and $\mathcal{C}$ is a differential structure on M then $\mathcal{C}_A$ denotes the differential structure on A generated by the family of restrictions $\{\alpha_{|A} : \alpha \in \mathcal{C}\}$. The differential space $(A, \mathcal{C}_A)$ is called *a differential subspace* of $(M, \mathcal{C})$. One can easily prove the following

Proposition 2.2 *Let* $(M, \mathcal{C})$ *and* $(N, \mathcal{D})$ *be differential spaces and let* $F : M \to N$. *Then* $F : (M, \mathcal{C}) \to (N, \mathcal{D})$ *iff* $F : (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$.

If the map $F : (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$ is a diffeomorphism then we say that $F : M \to N$ is *a diffeomorphism onto its range* (in $(N, \mathcal{D})$). In particular the natural embedding $A \ni m \mapsto i(m) := m \in M$ is a diffeomorphism of $(A, \mathcal{C}_A)$ onto its range in $(M, \mathcal{C})$.

If $\{(M_i, \mathcal{C}_i)\}_{i \in I}$ is an arbitrary family of differential spaces then we consider the Cartesian product $\prod_{i \in I} M_i$ as a differential space with the differential structure $\hat{\bigotimes}_{i \in I} \mathcal{C}_i$ generated by the family of functions $\mathcal{F} := \{\alpha_i \circ pr_i : i \in I, \alpha_i \in \mathcal{C}_i\}$, where $\prod_{i \in I} M_i \ni (m_i) \mapsto pr_j((m_i)) =: m_j \in M_j$ for any $j \in I$. The topology $\tau_{\hat{\bigotimes}_{i \in I} \mathcal{C}_i}$ coincides with the standard product topology on $\prod_{i \in I} M_i$. We will denote the differential structure $\hat{\bigotimes}_{i \in I} C^\infty(\mathbf{R})$ on $\mathbf{R}^I$ by $C^\infty(\mathbf{R}^I)$. In the case when I is an n-element finite set the differential structure $C^\infty(\mathbf{R}^I)$ coincides with the ordinary differential structure $C^\infty(\mathbf{R}^n)$ of all real-valued functions on $\mathbf{R}^n$ which possess partial derivatives of any order (see [3]). In any case a function $\alpha : \mathbf{R}^I \to \mathbf{R}$ is an element of $C^\infty(\mathbf{R}^I)$ iff for any $a = (a_i) \in \mathbf{R}^I$ there are $n \in \mathbf{N}$, elements $i_1, i_2, \ldots, i_n \in I$, a set U open in $\mathbf{R}^n$ and a function $\omega \in C^\infty(\mathbf{R}^n)$ such that $a \in U[i_1, i_2, \ldots, i_n] := \{(x_i) \in \mathbf{R}^I : (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in U\}$ and for any $x = (x_i) \in U[i_1, i_2, \ldots, i_n]$ we have

$$\alpha(x) = \omega(x_{i_1}, x_{i_2}, \ldots, x_{i_n}).$$

Let M be a group (a ring, a field, a vector space over the field $\mathbf{K}$). A differential structure $\mathcal{C}$ on M is said to be *a group* (*ring, field, vector space*) *differential structure* if the suitable group (ring, field, vector space) operations are smooth with respect to $\mathcal{C}$, $\mathcal{C}\hat{\otimes}\mathcal{C}$ and $\mathcal{C}_{\mathbf{K}}$, where $\mathcal{C}_{\mathbf{K}}$ is a field differential structure on $\mathbf{K}$. In this case the pair $(M, \mathcal{C})$ is called *a differential group* (*ring field, vector space*). If $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$ we take $\mathcal{C}_{\mathbf{K}} = C^\infty(\mathbf{K})$ as a standard field differential structure (see [1]).

Proposition 2.3 *Let V be a vector space over $\mathbf{R}$ and let $\mathcal{F}$ be a family of constant functions and linear functionals defined on V. Then the differential structure $\mathcal{C}$ generated by $\mathcal{F}$ on V is a vector space differential structure.*

Proof. It is enough to show that for any $\alpha \in \mathcal{F}$ there exist functions $\beta \in \mathcal{F}$, $\gamma \in C^\infty(\mathbf{R})$ and $\omega_1, \omega_2 \in C^\infty(\mathbf{R}^2)$ such that for any $v, w \in V$ and $t \in \mathbf{R}$

$$\alpha(v + w) = \omega_1(\beta(v), \beta(w)), \quad \alpha(tv) = \omega_2(\gamma(t), \beta(v)).$$

If $\alpha = a = const$ then

$$\alpha(v + w) = \alpha(v) = a, \quad \alpha(tv) = \alpha(v) = a$$

and we can take $\beta = \alpha$, $\gamma = 1 = const$,

$$\omega_1(x, y) = x, \quad \omega_2(x, y) = y, \quad (x, y) \in \mathbf{R}^2.$$

If α is a linear functional on V we have

$$\alpha(v + w) = \alpha(v) + \alpha(w), \quad \alpha(tv) = t\alpha(v).$$

Then we put $\beta = \alpha$, $\gamma = id_{\mathbf{R}}$, $\omega_1(x, y) = x + y$, $\omega_2(x, y) = xy$, $(x, y) \in \mathbf{R}^2$.

□

Note that if $\mathcal{C}$ is a vector space differential structure on a vector space V then V endowed with the topology $\tau_{\mathcal{C}}$ is a topological vector space.

Let $\mathcal{F}$ be a family of generators of a differential structure $\mathcal{C}$ on a set M. *The generator embedding* of the differential space $(M, \mathcal{C})$ into the Cartesian space defined by $\mathcal{F}$ is a mapping $\phi_{\mathcal{F}} : (M, \mathcal{C}) \to (\mathbf{R}^{\mathcal{F}}, C^{\infty}(\mathbf{R}^{\mathcal{F}}))$ given by the formula

$$\phi_{\mathcal{F}}(m) = (\alpha(m))_{\alpha \in \mathcal{F}}$$

(for example if $\mathcal{F} = \{\alpha_1, \alpha_2, \alpha_3\}$ then $\phi_{\mathcal{F}}(m) = (\alpha_1(m), \alpha_2(m), \alpha_3(m)) \in \mathbf{R}^3 \cong \mathbf{R}^{\mathcal{F}}$). If $\mathcal{F}$ separates points of M, the generator embedding is a diffeomorphism onto its image. On that image we consider the differential structure of a subspace of $(\mathbf{R}^{\mathcal{F}}, C^{\infty}(\mathbf{R}^{\mathcal{F}}))$.

3 The differential structure on the tangent space

Definition 3.1 By *a tangent vector* to a differential space $(M, \mathcal{C})$ at a point $m \in M$ we mean an $\mathbf{R}$-linear mapping $v : \mathcal{C} \to \mathbf{R}$ satisfying the Leibnitz condition: $v(\alpha \cdot \beta)(m) = \alpha(m)v(\beta) + \beta(m)v(\alpha)$ for any $\alpha, \beta \in \mathcal{C}$. We denote by $T_m M$ the set of all vectors tangent to $(M, \mathcal{C})$ at the point $m \in M$ and call it *the tangent space to* $(M, \mathcal{C})$ *at the point* m. The union $TM := \bigcup_{m \in M} T_m M$ is called *the tangent space to* $(M, \mathcal{C})$.

The set TM can be endowed with a differential structure in the following standard way. We define *the projection* $\pi : TM \to M$ such that for any $m \in M$ and any $v \in T_m M$

$$\pi(v) = m.$$

For any $\alpha \in \mathcal{C}$ we define *the differential* (or *the exterior derivative*) of α as the map $d\alpha : TM \to \mathbf{R}$ given by the following formula

$$d\alpha(v) := v(\alpha), \qquad v \in TM.$$

Then we define $\mathcal{TC}$ as the differential structure on TM generated by the family of functions $\mathcal{TC}_0 := \{\alpha \circ \pi : \alpha \in \mathcal{C}\} \cup \{d\alpha : \alpha \in \mathcal{C}\}$. From now on we will consider TM as a differential space with the differential structure $\mathcal{TC}$.

For any $m \in M$ we will denote by $d\alpha_m$ the restriction $d\alpha_{|T_m M}$. It is clear that $d\alpha_m$ is a linear functional on $T_m M$.

We also have that $\pi : (TM, \mathcal{TC}) \to (M, \mathcal{C})$. Then π is continuous and for any $U \in \tau_{\mathcal{C}}$ the set $TU := \bigcup_{m \in U} T_m M = \pi^{-1}(U)$ is open in TM ($TU \in \tau_{\mathcal{TC}}$). It can be proved that TU is a tangent space to the differential space $(U, \mathcal{C}_U)$.

Theorem 3.1 *If $(M, \mathcal{C})$ is a differential space then for any $m \in M$ the pair $(T_m M, \mathcal{TC}_{T_m M})$ is a differential vector space and $T_m M$ is a Hausdorff space (with respect to the topology induced by $\mathcal{TC}_{T_m M}$).*

Proof. The differential structure $\mathcal{TC}_{T_m M}$ is generated by the family of functions $\mathcal{TC}_{0|T_m M} := \{\beta_{|T_m M} : \beta \in \mathcal{TC}_0\}$. If $\beta = \alpha \circ \pi$, where $\alpha \in \mathcal{C}$, then $\beta_{|T_m M} = \alpha(m) = const.$ In the opposite case $\beta_{|T_m M}(v) = d\alpha_m(v)$ is a linear functional on $T_m M$. Hence by Proposition 2.3 $\mathcal{TC}_{T_m M}$ is a vector space differential structure.

If $v_1, v_2 \in T_m M$ and for any $\alpha \in \mathcal{C}$ equalities $v_1(\alpha) = d\alpha_m(v_1) = d\alpha_m(v_2) = v_2(\alpha)$ hold then $v_1 = v_2$ (v_1 and v_2 are linear functionals on $\mathcal{C}$). It means that the family $\mathcal{TC}_{0|T_m M}$ separates points in $T_m M$. Consequently the topology defined by this family is a Hausdorff topology. □

Let us consider the differential space $(\mathbf{R}^I, C^\infty(\mathbf{R}^I))$. The differential structure $C^\infty(\mathbf{R}^I)$ is generated by the family of projections $\mathcal{F} := \{\pi_i\}_{i \in I}$, where

$$\pi_j((x_i)) := x_j \qquad (x_i) \in \mathbf{R}^I, \;\; j \in I.$$

For any $x = (x_i),\ v = (v_i) \in \mathbf{R}^I$ the functional $\vec{v} : C^\infty(\mathbf{R}^I) \to \mathbf{R}$ given by the formula

$$\vec{v}(\alpha) := \sum_{i \in I} v_i \frac{\partial \alpha}{\partial x_i}(x)$$

is well defined (in some neighbourhood of x the function α depends on a finite number of variables x_i) and is a vector tangent to $\mathbf{R}^I$ at x. On the other hand, if $u \in T_x\mathbf{R}^I$ and for any $i \in I$ we denote $v_i := u(\pi_i)$, then for any $\alpha \in C^\infty(\mathbf{R}^I)$ we have $\vec{v}(\alpha) = u(\alpha)$. Then we identify the set $T_x\mathbf{R}^I$ with $\{x\} \times \mathbf{R}^I$. Consequently we identify the set $T\mathbf{R}^I$ with $\mathbf{R}^I \times \mathbf{R}^I$. In this case the differential structure $\mathcal{T}C^\infty(\mathbf{R}^I)$ is generated by the family of functions $\mathcal{TF} := \{\pi_i \circ \pi\}_{i \in I} \cup \{d\pi_i\}_{i \in I}$, where

$$\pi(x, v) = x, \qquad (x, v) \in \mathbf{R}^I \times \mathbf{R}^I.$$

Hence for any $j \in I$

$$\pi_j \circ \pi((x_i),(v_i)) = x_j \quad \text{and} \quad d\pi_j((x_i),(v_i)) = v_j.$$

It means that $\mathcal{T}C^\infty(\mathbf{R}^I) = C^\infty(\mathbf{R}^I \times \mathbf{R}^I)$ and consequently for any $x \in \mathbf{R}^I$ the differential structure $\mathcal{T}C^\infty(\mathbf{R}^I)_{T_x\mathbf{R}^I}$ is generated by the family of projections $\{\pi'_i : \{x\} \times \mathbf{R}^I \to \mathbf{R}\}_I$, where

$$\pi'_j(x,(v_i)) = v_j.$$

Then we can identify $\mathcal{T}C^\infty(\mathbf{R}^I)_{T_x\mathbf{R}^I}$ with $C^\infty(\mathbf{R}^I)$.

Let $\phi_{\mathcal{F}} : (M,\mathcal{C}) \to (\mathbf{R}^{\mathcal{F}}, C^\infty(\mathbf{R}^{\mathcal{F}}))$ be the generator embedding of the differential Hausdorff space $(M,\mathcal{C})$ defined by some family of generators $\mathcal{F}$. Then we can identify differential spaces $(M,\mathcal{C})$ and $(\phi_{\mathcal{F}}(M), C^\infty(\mathbf{R}^{\mathcal{F}})_{\phi_{\mathcal{F}}(M)})$ ($\phi_{\mathcal{F}}$ is a diffeomorphism). We also identify the tangent spaces T_mM and $T_{\phi_{\mathcal{F}}(m)}\phi_{\mathcal{F}}(M)$ using *the tangent map* $T\phi_{\mathcal{F}}$ (for any $\alpha \in C^\infty(\mathbf{R}^{\mathcal{F}})_{\phi_{\mathcal{F}}(M)}$).

Theorem 3.2 *Let I be an arbitrary nonempty set and let X be a nonempty subset of the Cartesian space $\mathbf{R}^I$. Then for any $x = (x_i) \in X$ the space T_xX tangent to the differential space $(X, C^\infty(\mathbf{R}^I)_X)$ at the point x is a closed subspace of the space $T_x\mathbf{R}^I$ tangent to the differential space $(\mathbf{R}^I, C^\infty(\mathbf{R}^I))$ at x.*

Proof. Let $x = (x_i) \in X$ and let $(v^{(n)}) = ((v_i^{(n)}))$ be a sequence of elements of T_xX convergent in $T_x\mathbf{R}^I$ to a vector $v^{(0)} = (v_i^{(0)})$. Then for any $i \in I$ we have

$$\lim_{n\to\infty} v_i^{(n)} = v_i^{(0)}.$$

Suppose that $\alpha \in C^\infty(\mathbf{R}^I)_X$. Then there exist: a number $n \in \mathbf{N}$, elements $i_1, i_2, \ldots, i_n \in I$, a nonempty set U open in $\mathbf{R}^n$ and a function $\omega \in C^\infty(\mathbf{R}^n)$ such that for any $y \in U[i_1, i_2, \ldots, i_n] \cap X$ we have $\alpha(y) = \omega(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$ (see remarks after Proposition 2.2). Moreover

$$v^{(n)}(\alpha) = \sum_{j=1}^{n} v_{i_j}^{(n)} \frac{\partial \alpha}{\partial x_{i_j}}(x_{i_1}, x_{i_2}, \ldots, x_{i_n}). \tag{1}$$

This implies that

$$\lim_{n\to\infty} v^{(n)}(\alpha) = \sum_{j=1}^{n} v_{i_j}^{(0)} \frac{\partial \alpha}{\partial x_{i_j}}(x_{i_1}, x_{i_2}, \ldots, x_{i_n}).$$

Then the left hand side of this equality does not depend on the choice of ω whereas the right hand side is a functional which can be identified with $v^{(0)}$. $\square$

Remark 3.1 Using similar arguments as in the proof of Theorem 3.2 (formula (1) and the limit in $\mathbf{R}^I \times \mathbf{R}^I$) one can show that if X is a nonempty closed subset of the Cartesian space $\mathbf{R}^I$ then the space TX tangent to the differential space $(X, C^\infty(\mathbf{R}^I)_X)$ is a closed subspace of the space $T\mathbf{R}^I$.

Theorem 3.3 *Let $(M, \mathcal{C})$ be a differential Hausdorff space and let A be a nonempty subset of M. Then for any $m \in A$ the space T_mA tangent to the differential space $(A, \mathcal{C}_A)$ at the point m is a closed subspace of the space T_mM tangent $(M, \mathcal{C})$ at m.*

Proof. Let $\phi_{\mathcal{C}} : (M, \mathcal{C}) \to (\mathbf{R}^{\mathcal{C}}, C^\infty(\mathbf{R}^{\mathcal{C}}))$ be the generator embedding of $(M, \mathcal{C})$ defined by the family of generators $\mathcal{C}$ and let $\phi_{\mathcal{C}_{|A}} : (A, \mathcal{C}_A) \to (\mathbf{R}^{\mathcal{C}}, C^\infty(\mathbf{R}^{\mathcal{C}}))$ be the generator embedding of $(A, \mathcal{C}_A)$ defined by the family of generators $\{\alpha_{|A}\}_{\alpha \in \mathcal{C}}$. Then $\phi_{\mathcal{C}_{|A}} = (\phi_{\mathcal{C}})_{|A}$ and we can identify: (i) $(M, \mathcal{C})$ and $(\phi_{\mathcal{C}}(M), C^\infty(\mathbf{R}^{\mathcal{C}})_{\phi_{\mathcal{C}}(M)})$; (ii) $(A, \mathcal{C}_A)$ and $(\phi_{\mathcal{C}}(A), C^\infty(\mathbf{R}^{\mathcal{C}})_{\phi_{\mathcal{C}}(A)})$. For any $m \in A$ we also identify: (i) tangent spaces T_mM and $T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(M)$; (ii) tangent spaces T_mA and $T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(A)$. Since by Theorem 2.2 $T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(A)$ is a closed subspace of $T_{\phi_{\mathcal{C}}(m)}\mathbf{R}^{\mathcal{C}}$ and $T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(A) \subset T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(M) \subset T_{\phi_{\mathcal{C}}(m)}\mathbf{R}^{\mathcal{C}}$ we obtain that $T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(A)$ is a closed subspace of $T_{\phi_{\mathcal{C}}(m)}\phi_{\mathcal{C}}(M)$. It means that T_mA is a closed subspace of T_mM. $\square$

Remark 3.2 Using Remark 2.1 and similar arguments as in the proof of Theorem 2.3 one can prove that if $(M, \mathcal{C})$ is a differential space and A is a nonempty closed subset of M then the space TA tangent to A (in the sense of differential space $(A, \mathcal{C}_A)$) is a closed subset of the space TM tangent to M.

Definition 3.2 A map $X : M \to TM$ such that for any $m \in M$ the value $X(m) \in T_mM$ is called *a vector field* on M. A vector field X on M is *smooth* if $X : (M, \mathcal{C}) \to (TM, \mathcal{TC})$.

4 The differential structure on the cotangent space

For a map $f : M \to N$ we denote by $f^* : \mathbf{R}^N \to \mathbf{R}^M$ the map given by the formula: $f^*(\beta) = \beta \circ f$.

Theorem 4.1 *If $\{(M_i, \mathcal{C}_i)\}_{i \in I}$ is a family of differential spaces, then for any family of mappings $\mathcal{F} = \{f_i\}_{i \in I}$, where $f_i : M_i \to N$, the pair $(N, \bigcap_{i \in I} (f_i^*)^{-1}(\mathcal{C}_i))$ is a differential space and the set $\bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$ is the greatest of differential structures $\mathcal{D}$ such that for any $i \in I$ the map $f_i : (M_i, \mathcal{C}_i) \to (N, \mathcal{D})$.*

Proof. It is proved in [3] that for any $i \in I$ the family of functions $(f_i^*)^{-1}(\mathcal{C}_i)$ is the greatest differential structure on N for which f_i is a smooth map. Then by Proposition 2.1 the family $\bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$ is a differential structure on N.

Let $\beta \in \bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$. Then for any $i \in I$ we have $\beta \in (f_i^*)^{-1}(\mathcal{C}_i)$, which means that $f_i^*(\beta) = \beta \circ f_i \in \mathcal{C}_i$. Hence $f_i : (M_i, \mathcal{C}_i) \to \bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$.

Let $\mathcal{D}$ be such a differential structure on N that for any $i \in I$ the map $f_i : M_i \to N$ is smooth with respect to $\mathcal{C}_i$ and $\mathcal{D}$. Let $\gamma \in \mathcal{D}$. Then for any $i \in I$ we have $f_i^*(\gamma) = \gamma \circ f_i \in \mathcal{C}_i$ so $\gamma \in (f_i^*)^{-1}(\mathcal{C}_i)$. It means that $\gamma \in \bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$. Since γ is an arbitrary element of $\mathcal{D}$ we obtain that $\mathcal{D} \subset \bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$. □

Definition 4.1 The differential structure $\bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i)$ on the set N described in Theorem 2.2 is said to be *co-induced* by the family $\mathcal{F}$ and the family $\{\mathcal{C}_i\}_{i \in I}$.

Theorem 4.2 *Let $\mathcal{D}$ be a differential structure on a set N co-induced by the family of mappings $\mathcal{F} = \{f_i : M_i \to N\}_{i \in I}$ and the family of differential structures $\{\mathcal{C}_i\}_{i \in I}$. Let $(P, \mathcal{G})$ be a differential space. Then the map $g : N \to P$ is smooth with respect to $\mathcal{D}$ and $\mathcal{G}$ iff for any $i \in I$ the map $g \circ f_i$ is smooth with respect to $\mathcal{C}_i$ and $\mathcal{G}$.*

Proof. ($\Rightarrow$) It follows from the fact that for any $i \in I$ the map f_i is smooth and the superposition of smooth maps is a smooth map.
($\Leftarrow$) It was proved in [3] that if $g \circ f_i$ is a smooth map with respect to $\mathcal{C}_i$ and

$\mathcal{G}$ then f_i is smooth with respect to $(f_i^*)^{-1}(\mathcal{C}_i)$ and $\mathcal{G}$, where $i \in I$. Hence for any $i \in I$ and any $\alpha \in \mathcal{G}$ we have $\alpha \circ g \in (f_i^*)^{-1}(\mathcal{C}_i)$. Consequently $\alpha \circ g \in \bigcap_{i \in I}(f_i^*)^{-1}(\mathcal{C}_i) = \mathcal{D}$, which means that g is smooth. □

Definition 4.2 The *cotangent space* T_m^*M to $(M, \mathcal{C})$ at a point $m \in M$ is the dual space to the tangent space T_mM (it is the space of all continuous linear functionals defined on T_mM). The union $T^*M := \bigcup\limits_{m \in M} T_m^*M$ is called *the cotangent space to* $(M, \mathcal{C})$.

Proposition 4.1 *If α is a smooth function on a differential space $(M, \mathcal{C})$ and $m \in M$ then the differential $d\alpha_m$ is an element of the cotangent space T_m^*M.*

Proof. Since the linear functional $d\alpha_m$ is an element of the set $\mathcal{TC}_{0|T_mM}$ of generators of differential structure $\mathcal{TC}_{T_mM}$ on T_mM we obtain that $d\alpha_m$ is continuous (see Theorem 2.1). □

If I is a nonempty set and $x = (x_i) = (x_i)_{i \in I}$ is an element of $\mathbf{R}^I$ then we identify the tangent space $T_x\mathbf{R}^I$ with $\{x\} \times \mathbf{R}^I \cong \mathbf{R}^I$ endowed with the standard product topology. Then the cotangent space $T_x^*\mathbf{R}^I$ should be identified with the dual space $(\mathbf{R}^I)^*$. For any $j \in I$ we denote by e_j the element $(x, (v_i))$ of $T_x\mathbf{R}^I$ such that $v_i = 0$ for $i \neq j$ and $v_i = 1$ for $i = j$. Any functional $p \in T_x^*\mathbf{R}^I$ defines the element $(p_i) \in \mathbf{R}^I$ by the following formula

$$p_i = p(e_i), \qquad i \in I. \tag{2}$$

Proposition 4.2 *For any $x \in \mathbf{R}^I$ and any $p \in T_x^*\mathbf{R}^I$ there exists $n \in \mathbf{N}$ and elements $i_1, i_2, \ldots, i_n \in I$ such that $p_i = 0$ for $i \in I \setminus \{i_1, i_2, \ldots, i_n\}$, where numbers p_i are given by the formula (2).*

Proof. Suppose that the statement is not true. Then there exists an infinite sequence $(i_1, i_2, \ldots)$ of different elements of I such that for any $n \in \mathbf{N}$ we have $p_{i_n} \neq 0$. Let the element $(x, (v_i)) \in T_x\mathbf{R}^I$ be such that for any $n \in \mathbf{N}$

$$v_{i_n} = \frac{1}{p_{i_n}}$$

and $v_i = 0$ for $i \in I \setminus \{i_n : n \in \mathbf{N}\}$. Let $(x, (v_i^{(n)}))$ be the sequence of elements of $T_x\mathbf{R}^I$ such that $v_{i_k}^{(n)} = v_{i_k}$ for $k \leq n$ and $v_i^{(n)} = 0$ for $i \in I \setminus \{i_1, i_2, \ldots, i_n\}$.

Then the sequence converges to $(x,(v_i))$ in $T_x\mathbf{R}^I$. On the other hand we have

$$(x,(v_i^{(n)})) = \sum_{k=1}^{n} v_{i_k} e_{i_k},$$

which implies that

$$p(x,(v_i^{(n)})) = \sum_{k=1}^{n} v_{i_k} p(e_{i_k}) = \sum_{k=1}^{n} \frac{1}{p_{i_k}} p_{i_k} = n.$$

Hence

$$p((x,(v_i^{(n)}))) = \lim_{n\to\infty} p(x,(v_i^{(n)})) = \infty,$$

which is a contradiction. □

Proposition 4.3 *For any $x \in \mathbf{R}^I$ and any $p \in T_x^*\mathbf{R}^I$ there exists $n \in \mathbf{N}$ and elements $i_1, i_2, \ldots, i_n \in I$ such that for any $v = (x,(v_i)) \in T_x\mathbf{R}^I$*

$$p(v) = \sum_{k=1}^{n} p_{i_k} v_{i_k}, \tag{3}$$

where numbers p_i are given by the formula (2).

Proof. Let $n \in \mathbf{N}$ and $i_1, i_2, \ldots, i_n \in I$ be such as in Proposition 4.2. For any nonempty set $J \subset I$ denote by V_J the vector space consisting of such $v = (x,(v_i)) \in T_x\mathbf{R}^I$ that

$$v_i = 0 \quad \text{for} \quad i \in I \setminus J. \tag{4}$$

If J is finite, say $J = \{j_1, j_2, \ldots, j_m\}$ for some $m \in \mathbf{N}$, and $v = (x,(v_i)) \in V_J$ then $(x,(v_i)) = \sum_{k=1}^{m} v_{j_k} e_{j_k}$. Hence

$$p(x,(v_i)) = \sum_{k=1}^{m} v_{j_k} p(e_{j_k}) = \sum_{k=1}^{m} v_{j_k} p_{j_k} = \sum_{k=1}^{n} p_{i_k} v_{i_k},$$

which means that the equality (3) holds.

Let us consider the family 2^I of all subsets of the set I as a set which is ordered by the ordinary inclusion. If $\mathcal{J}$ is such a linearly ordered subfamily of 2^I (for any $J_1, J_2 \in \mathcal{J}$ we have $J_1 \subset J_2$ or $J_2 \subset J_1$ and consequently

$V_{J_1} \subset V_{J_2}$ or $V_{J_2} \subset V_{J_1}$) that for any $J \in \mathcal{J}$ and any $v = (x, (v_i)) \in V_J$ the equality (3) holds then this equality holds for any $v \in \mathrm{span}(\bigcup_{J \in \mathcal{J}} V_J)$. Since any element u of $V_{\bigcup \mathcal{J}}$ is a limit of some (generalized) sequence of elements of $\mathrm{span}(\bigcup_{J \in \mathcal{J}} V_J)$ and p is a continuous functional we obtain that (3) holds for $v = u$. Hence, by Kuratowski-Zorn lemma, in the family $\mathcal{P}$ of all subsets J of I for which all elements v of V_J fulfil (3) there exists some maximal element J_0.

Suppose that $J_0 \neq I$ and that $i_0 \in I \setminus J_0$. Then $J_1 = J_0 \cup \{i_0\} \neq J_0$ and any element $v \in J_1$ is of the form $v = v_0 + v_{i_0} e_{i_0}$, where $v_0 \in J_0$ and $v_{i_0} \in \mathbf{R}$. Since $p(v) = p(v_0) + p(v_{i_0} e_{i_0}) = p(v_0) + v_{i_0} p_{i_0}$ and both v_0 and $v_{i_0} e_{i_0}$ fulfil (3) then v also fulfils (3). This leads to the contradiction. □

Proposition 4.4 *For any $x \in \mathbf{R}^I$ and any $p \in T^*_x \mathbf{R}^I$ there exists a function $\alpha \in C^\infty(\mathbf{R}^I)$ such that $p = d\alpha_x$.*

Proof. For $x \in \mathbf{R}^I$ and $p \in T^*_x \mathbf{R}^I$ choose $n \in \mathbf{N}$ and $i_1, i_2, \ldots, i_n \in I$ such as in Proposition 4.3. Then it is enough to take

$$\alpha((y_i)_{i \in I}) := \sum_{k=1}^{n} p_{i_k} y_{i_k}, \qquad (y_i)_{i \in I} \in \mathbf{R}^I.$$

□

Theorem 4.3 *Let $(M, \mathcal{C})$ be a differential Hausdorff space. Then for any $m \in M$ and any $p \in T^*_m M$ there exists a function $\omega \in \mathcal{C}$ such that $p = d\omega_m$.*

Proof. Let $\phi_{\mathcal{F}} : (M, \mathcal{C}) \to (\mathbf{R}^{\mathcal{F}}, C^\infty(\mathbf{R}^{\mathcal{F}}))$ be the generator embedding of $(M, \mathcal{C})$ defined by the family of generators $\mathcal{F}$ (we can take $\mathcal{F} = \mathcal{C}$). Then we can identify $(M, \mathcal{C})$ and $(\phi_{\mathcal{F}}(M), C^\infty(\mathbf{R}^{\mathcal{F}})_{\phi_{\mathcal{F}}(M)})$. Hence we assume that $M \subset \mathbf{R}^{\mathcal{F}}$ and for any $m \in M$ the tangent space $T_m M$ is a closed subspace of the topological vector space $T_m \mathbf{R}^{\mathcal{F}} = \{m\} \times \mathbf{R}^{\mathcal{F}} \cong \mathbf{R}^{\mathcal{F}}$ (see Theorem 3.2). The topology of $\alpha \in \mathcal{F}$ is defined by a family $\{\rho_\alpha\}_{\alpha \in \mathcal{F}}$ of semi-norms such that

$$\rho_\alpha((x_\beta)_{\beta \in \mathcal{F}}) := |x_\alpha|, \qquad \alpha \in \mathcal{F}.$$

Hence the topology of $T_m M$ is defined by restriction of semi-norms ρ_α to $T_m M$.

Let $p \in T^*_m M$. Then there exists $\beta \in \mathcal{F}$ and $C > 0$ such that for any $v = (m, (v_\alpha)) \in T_m M$

$$|p(v)| \leq C \rho_\beta(v) = |v_\beta|.$$

(see [4], I.6, Theorem 1). By the famous Hahn-Banach extension theorem the functional p can be extended to such a continuous linear functional p_0 on $T_m\mathbf{R}^{\mathcal{F}}$ that

$$|p_0(v)| \leq C\rho_\beta(v), \qquad v \in T_m\mathbf{R}^{\mathcal{F}}.$$

(see [4], IV.5, Theorem 1). Using now Proposition 3.4 we obtain that $p_0 = d\gamma$, where $\gamma \in C^\infty(\mathbf{R}^{\mathcal{F}})$. Then $p = p_{0|T_mM} = d\gamma_{|T_mM} = d\omega$, where $\omega := \gamma_{|M}$. □

We endow the cotangent space T^*M with the differential structure $\mathcal{T}^*\mathcal{C}$ co-induced by the family of maps $\{f_\alpha : \mathbf{R} \times M \to T^*M\}_{\alpha\in\mathcal{C}}$, where

$$f_\alpha(t, m) := td\alpha_m, \qquad (t, m) \in \mathbf{R} \times M$$

and $\mathbf{R} \times M$ is considered as a differential space with the differential structure $C^\infty(\mathbf{R})\hat{\otimes}\mathcal{C}$.

Let $\tilde{\pi} : T^*M \to M$ be a map such that for any $m \in M$ and any $p \in T^*_mM$

$$\tilde{\pi}(p) := m.$$

We call $\tilde{\pi}$ *the natural projection* of the cotangent space T^*M onto its *base* M.

Proposition 4.5 *The natural projection $\tilde{\pi} : T^*M \to M$ is a smooth map.*

Proof. For any $\alpha \in \mathcal{C}$ we have

$$\tilde{\pi} \circ f_\alpha(t, m) = \tilde{\pi}(td\alpha_m) = m, \qquad (t, m) \in \mathbf{R} \times M.$$

Hence $\tilde{\pi} \circ f_\alpha$ is a natural projection of $\mathbf{R} \times M$ onto M which is a smooth map. It now follows from Theorem 4.2 that $\tilde{\pi}$ is a smooth map. □

Proposition 4.6 *For any smooth vector field X on M the function $T^*M \ni p \mapsto \beta_X(p) := p(X(p)) \in \mathbf{R}$ is smooth on T^*M.*

Proof. It is enough to show that for any $\alpha \in \mathcal{C}$ the superposition $\beta_X \circ f_\alpha \in C^\infty(\mathbf{R})\hat{\otimes}\mathcal{C}$ (see Theorem 4.2). We have

$$\beta_X \circ f_\alpha(t, x) = \beta_X(td\alpha_x) = td\alpha(X(x)) = tX(x)\alpha, \quad (t, x) \in \mathbf{R} \times M.$$

Since X is smooth we obtain that the function $M \ni x \mapsto X(x)\alpha \in \mathbf{R}$ is smooth on M. Then

$$\beta_X \circ f_\alpha(t, x) = \omega(t, X(x)\alpha), \qquad (t, x) \in \mathbf{R} \times M$$

for

$$\omega(t, s) = ts, \qquad (t, s) \in \mathbf{R}^2,$$

which means that $\beta_X \circ f_\alpha$ is smooth on $\mathbf{R} \times M$. $\square$

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Integrability of Hamiltonian systems on varieties

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1 Introduction

Let $(\mathbf{R}^{2n}, \omega)$ be a symplectic manifold. Then the tangent bundle $T\mathbf{R}^{2n}$ is isomorphic to the cotangent bundle $T^*\mathbf{R}^{2n}$. The isomorphism is established by vector bundle morphism $\beta : T\mathbf{R}^{2n} \ni u \mapsto \omega(u, \cdot) \in T^*\mathbf{R}^{2n}$. Thus the tangent bundle $T\mathbf{R}^{2n}$ is endowed with the canonical symplectic structure $\dot{\omega} = \beta^* d\theta$ where θ is a Liouville form on $T^*\mathbf{R}^{2n}$. Let C be a submanifold of $\mathbf{R}^{2n}$ and $H : C \to \mathbf{R}$ a smooth function on C. The usual notion of Hamiltonian system (generalized after P.A.M. Dirac [1]) is defined as a subbundle of $T\mathbf{R}^{2n}$ over C, being a Lagrangian submanifold of $(T\mathbf{R}^{2n}, \dot{\omega})$,(cf. [7])

$$L_H = \{v \in T\mathbf{R}^{2n} : \omega(v, u) = -dH(u) \quad \forall_{u \in TC}\}. \tag{1}$$

If C is an open domain of $\mathbf{R}^{2n}$ then L_H is a smooth section of $\pi : T\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ and its local integrability is a characteristic property, i.e. at each point $v \in L_H$ there is a smooth curve $\alpha : (-\epsilon, \epsilon) \to \mathbf{R}^{2n}$ such that $(\alpha(0), \dot{\alpha}(0)) = v$ and $(\alpha(t), \dot{\alpha}(t)) \in L_H$ for every $t \in (-\epsilon, \epsilon)$. The curve α is called an integral curve of L_H with initial value v and v is called an integrable point of L_H. Since L_H is introduced to describe dynamics (of mechanical, biological, etc. systems) the existence of such α for L_H should not be an exceptional property and that for each $v \in L_H$ there should exist a neighborhood U of v in L_H and $\epsilon > 0$ such that the mapping $U \times (-\epsilon, \epsilon) \ni (\bar{v}, t) \mapsto \alpha_{\bar{v}}(t)$, $\alpha_{\bar{v}}(o) = v$ is defined and at least continuous. The general Hamiltonian system (1) is called integrable if it consists only of integrable points (cf. [1, 3, 4, 6, 9]). It is called smoothly integrable if moreover it consists of smoothly integrable

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points, i.e. around each $v \in L_H$ there exists a smooth family $\alpha : U\times(-\epsilon,\epsilon) \ni (\bar{v},t) \mapsto \mathbf{R}^{2n}$ of solutions of L_H such that $(\alpha_{\bar{v}}(0),\dot{\alpha}_{\bar{v}}(0)) = v$.

In local Darboux coordinates $\omega = \sum_{i=1}^n dy_i \wedge dx_i$ and

$$\dot{\omega} = \sum_{i=1}^{n}(d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i).$$

The generalized Hamiltonian system (1) can be written by a generalized Hamiltonian function $F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}$,

$$\dot{x}_i = \frac{\partial F}{\partial y_i}(x,y,\lambda), i = 1,\dots,n \tag{2}$$

$$\dot{y}_i = -\frac{\partial F}{\partial x_i}(x,y,\lambda), i = 1,\dots,n \tag{3}$$

$$0 = a_\ell(x,y), \ell = 1,\dots,k, \quad \lambda \in \mathbf{R}^k, \tag{4}$$

where $F(x,y,\lambda) = b(x,y) + \sum_{l=1}^k \lambda_\ell a_\ell(x,y)$, C is defined as a zero-level set of the mapping $(x,y) \to (a_1(x,y),\dots,a_k(x,y))$ and $b(x,y)$ is an arbitrary smooth extension of the function $H : C \to \mathbf{R}$.

The aim of this paper is to investigate integrability of Hamiltonian systems on varieties. We find conditions that L_F is smoothly integrable for various properties of C and a general function on C.

2 Formulation of results

Throughout this paper, unless otherwise stated, we consider only implicit Hamiltonian systems $L_F \subset T\mathbf{R}^{2n}$ generated by Morse families $F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}$ of the form

$$F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + b(x,y),$$

where F satisfies the rank condition:

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial\lambda\partial x}(x,y,\lambda), \frac{\partial^2 F}{\partial\lambda\partial y}(x,y,\lambda)\right) = k$$

at every point (x, y, λ) of the critical manifold

$$C_F = \{(x, y, \lambda) \in \mathbf{R}^{2n} \times \mathbf{R}^k \mid \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0\}.$$

of F.

2.1 The problem

Concerning the integrability of the implicit Hamiltonian system L_F, we already have the following result proved in [3].

Theorem 1. *([3]) An implicit Hamiltonian system $L_F \subset T\mathbf{R}^{2n}$ generated by a Morse family*

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x, y, \lambda) = \sum_{\ell=1}^{k} a_\ell(x, y)\lambda_\ell + b(x, y)$$

is smoothly integrable if and only if

$$\{a_i, a_\ell\} = 0 \quad \text{and} \quad \{b, a_\ell\} = 0, \quad 1 \leq i, \ell \leq k,$$

$$\text{on} \quad C = \{(x, y) \in \mathbf{R}^{2n} \mid a_i(x, y) = 0, \quad 1 \leq i \leq k\}.$$

where $\{f, g\}$ denotes the Poisson bracket of f and g.

In what follows we investigate the following problem:

Problem 1. *In the case if L_F is not smoothly integrable, which part of L_F is integrable?*

2.2 Results

Let $\pi : T\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ and $\widetilde{\pi} : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}^{2n}$ denote the canonical projections respectively,

$$\pi(x, y, \dot{x}, \dot{y}) = (x, y), \qquad \widetilde{\pi}(x, y, \lambda) = (x, y).$$

Let $\phi : C_F \to L_F$ denote the map defined by

$$\phi(x, y, \lambda) = (x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)), \quad (x, y, \lambda) \in C_F.$$

Since

$$\frac{\partial F}{\partial \lambda_\ell}(x, y, \lambda) = a_\ell(x, y),$$

setting

$$C = \{(x, y) \in \mathbf{R}^{2n} \mid a_1(x, y) = \cdots = a_k(x, y) = 0\},$$

we have

$$C_F = C \times \mathbf{R}^k.$$

Then the implicit Hamiltonian system $L_F \subset T\mathbf{R}^{2n}$ generated by F is given by

$$\begin{aligned} L_F &= \phi(C_F) \\ &= \{(x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda) \in T\mathbf{R}^{2n} \mid (x, y, \lambda) \in C_F = C \times \mathbf{R}^k\} \\ &= \{(x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda) \in T\mathbf{R}^{2n} \mid \\ &\qquad a_1(x, y) = \cdots = a_k(x, y) = 0, \lambda \in \mathbf{R}^k\}. \end{aligned}$$

In this paper we find conditions for a submanifold of L_F to be smoothly integrable in the case where the Morse family does not satisfy the condition in Theorem 1, i.e. $\{a_i, a_\ell\} = 0$ and $\{b, a_\ell\} = 0$ on C, $1 \leq i, \ell \leq k$.

Consider the $k \times k$ skew-symmetric matrix $(\{a_\ell, a_m\}(x, y))$ and the linear equation

$$A(x, y) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = (\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}.$$

Set

$$\widetilde{S}_F = \left\{ (x, y, \lambda) \in C_F \mid (\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix} \right\},$$

$$S_F = \phi(\widetilde{S}_F) \subset L_F.$$

First we have the following basic result.

Theorem 2. *1) If a submanifold M of L_F is an integrable submanifold of the implicit Hamiltonian system L_F, then it is an integrable submanifold of the tangent bundle TC of C.*

2) If the linear equation

$$(\{a_\ell, a_m\}(x,y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix}$$

has a smooth solution $(\lambda_1(x,y), \cdots, \lambda_k(x,y))$ defined on C, then the image

$$G_\lambda = \phi(\widetilde{G}_\lambda)$$

by ϕ of the graph of the solution

$$\widetilde{G}_\lambda = \{(x, y, \lambda_1(x,y), \ldots, \lambda_k(x,y)) \mid (x,y) \in C\}$$

is a smoothly integrable submanifold of L_F.

Remark 1. *From Theorem 2. 1), in order to check that M is smoothly integrable, it is enough to check that*

1) M is a submanifold of TC

and that

2) M is smoothly integrable as an implicit differential system,

to which we can apply the results in [3].

Theorem 2. 1) is a direct consequence of Lemmas 2 and 3 given in the next section.

A situation diametrically opposite to the Theorem 1 is in the case if

$$\det(\{a_\ell, a_m\}(x,y)) \neq 0.$$

Under this condition we have

Theorem 3. *Let $L_F \subset T\mathbf{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family*

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + b(x,y)$$

Suppose that

$$k \quad \textit{is even and} \quad \det\left(\{a_\ell, a_m\}(x,y)\right) \neq 0.$$

Then S_F is a smoothly integrable submanifold of L_F and it is the maximal integrable submanifold of L_F in the sense that any other smoothly integrable submanifold of L_F is a submanifold of S_F. Moreover, the projection $\pi_{|S_F} : S_F \to C$ is a diffeomorphism and has no singular points. Consequently, S_F is a unique smoothly integrable submanifold of L_F such that $\pi(S_F) = C$.

When k is odd we have $\det A(x,y) = 0$ everywhere. As a result corresponding to Theorem 3, we have

Theorem 4. *Let $L_F \subset T\mathbf{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family*

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + b(x,y).$$

Suppose that k is odd and the rank of $(\{a_\ell, a_m\}(x,y))$ is constant and equal to $k-1$.

Suppose also that the linear equation

$$A(x,y)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_k\end{pmatrix} = (\{a_\ell, a_m\}(x,y))\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_k\end{pmatrix} = \begin{pmatrix}\{b, a_1\}(x,y)\\ \vdots \\ \{b, a_k\}(x,y)\end{pmatrix}$$

has a smooth solution $\lambda(x,y) = (\lambda_1(x,y), \ldots, \lambda_1(x,y))$ on C. Then

1) S_F is a smoothly integrable submanifold of L_F and it is the maximal integrable submanifold in the sense that any other smoothly integrable submanifold of L_F is a submanifold of S_F.

2) Moreover, S_F is a line bundle over C with the projection map $\pi_{|S_F} : S_F \to C$ and the projection map $\pi_{|S_F} : S_F \to C$ has no singular points.

The maximality of S_F, both in Theorems 3 and 4, follows from Lemma 3 given in the next section.

Theorem 4. 1) is a direct consequence of Theorem 4. 2), Lemma 3 and the following more general theorem.

Theorem 5. *Let $L_F \subset T\mathbf{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family*

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y)\lambda_i + b(x, y).$$

Let M be a submanifold of L_F such that the projection $\pi_{|M} : M \to C$ is a submersion.

Then M is smoothly integrable if and only if $M \subset S_F$.

As a direct corollary of Theorem 5, we have the following theorem which is a generalization of Theorem 4.

Theorem 6. *Let $L_F \subset T\mathbf{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family*

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y)\lambda_i + b(x, y)$$

Suppose that the linear equation

$$(\{a_i, a_j\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}$$

has a smooth solution $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ on C. Suppose also that the kernel set

$$\widetilde{K}_F = \ker(\{a_i, a_j\}) = \{(x, y, \lambda) \in C \times \mathbf{R}^k \mid (\{a_i, a_j\}(x, y))\, \lambda = 0\}$$

contains an m dimensional smooth vector subbundle $\widetilde{K}$ of the vector bundle $C \times \mathbf{R}^k$ over C. Then

$$S = \{(x, y, \frac{\partial F}{\partial y}(x, y, \lambda(x, y) + \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda(x, y) + \lambda)) \mid (x, y, \lambda) \in \widetilde{K}\}$$

is a $(2n - k + m)$ dimensional smoothly integrable submanifold of L_F.

The condition in Theorem 6 that the kernel set $\widetilde{K}_F$ contains an m dimensional smooth vector subbundle is not a generic condition if $m > 0$ for k even, and if $m > 1$ for k odd. Because in general if k is even, $\det(\{a_\ell, a_m\}(x,y)) \neq 0$ almost everywhere, and if k is odd, $\operatorname{rank}(\{a_\ell, a_m\}(x,y)) = k-1$ almost everywhere. For k even we define

$$C_{reg} = \{(x,y) \in \mathbf{R}^{2n} \mid \det(\{a_\ell, a_m\}(x,y)) \neq 0\},$$

for k odd we have

$$C_{k-1} = \{(x,y) \in \mathbf{R}^{2n} \mid \operatorname{rank}(\{a_\ell, a_m\}(x,y)) = k-1\}.$$

In the generic situation, we have

Theorem 7. *Suppose that k is even. Suppose also that*

$$\det(\{a_\ell, a_m\}(x,y)) \neq 0$$

almost everywhere but

$$\det(\{a_\ell, a_m\}(0,0)) = 0.$$

Then $L_F \cap \pi^{-1}(C_{reg})$ is smoothly integrable implicit differential system of TC_{reg}. Moreover there exists a smoothly integrable differential system M such that $\pi(M) = C$ if and only if the linear equation

$$(\{a_\ell, a_m\}(x,y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix}$$

has a smooth solution. Such a smoothly integrable differential system M is unique and it has the properties that $M \cap \pi^{-1}(C_{reg}) = L_F \cap \pi^{-1}(C_{reg})$ and that $\pi_M : M \to C$ is a diffeomorphism.

Remark 2. *A necessary and sufficient condition for the linear equation*

$$(\{a_\ell, a_m\}(x,y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix}$$

to have a smooth solution is already investigated in [3]. For k even we can apply this condition to the linear equation and we can have a corollary of Theorem 7 translating the condition in terms of $a_i(x,y)$'s and $b(x,y)$.

Remark 3. *In the case where k is odd we can have a similar result. However, when k is odd the rank of the matrix $(\{a_i, a_j\}(0,0))$ is less than $k-1$,*

1) There is a question, in a generic situation, whether the kernel set

$$\widetilde{K}_F = \ker(\{a_i, a_j\}) = \{(x,y,\lambda) \in C \times \mathbf{R}^k \mid (\{a_i, a_j\}(x,y))\,\lambda = 0\}$$

contains or not a smooth line bundle over C appeared in Theorem 6.

2) Moreover when k is odd, we can not apply our condition for the linear equation to have a smooth solution. Since $\det(\{a_i, a_j\}(x,y)) = 0$, the product of the matrix $(\{a_i, a_j\}(x,y))$ and its cofactor matrix is always the zero matrix. Thus we can not apply our method.

Theorems 3, 4, 5 and 6 are obtained by reducing the fibers of the bundle $\pi : L_F \to C$. Reducing the base space C, we obtain

Theorem 8. *Suppose that L_F is not smoothly integrable. Let*

$$g_1, \ldots, g_s : \mathbf{R}^{2n} \to \mathbf{R}$$

be smooth functions such that the Jacobian matrix of the map $(a, g) = (a_1, \ldots, a_k, g_1, \ldots, g_s) : \mathbf{R}^{2n} \to \mathbf{R}^{k+s}$ has the maximal rank $k+s$. Let $C_g \subset C$ be a submanifold defined by

$$C_g = \{(x,y) \in C \mid g_1(x,y) = \cdots = g_s(x,y) = 0\}.$$

Then $\phi(C_g \times \mathbf{R}^k)(\subset L_F)$ is smoothly integrable if and only if

$$\{a_\ell, a_m\} = \{b, a_m\} = 0, \{a_\ell, g_t\} = \{b, g_t\} = 0 \qquad \text{on} \quad C_g,$$

$$1 \leq \ell, m \leq k, \quad 1 \leq t \leq s.$$

3 Basic lemmas

The implicit Hamiltonian system L_F we consider in this paper, generated by a Morse family of the form

$$F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + b(x,y),$$

is of a special kind in the sense that the projection map $\pi \mid L_F : L_F \to \mathbf{R}^{2n}$ has no regular points, while the regular points are dense in generic implicit Hamiltonian system.

We can easu to see that the following three properties still hold in the present irregular case.

Lemma 1. *1) L_F is a Lagrangian submanifold of $T\mathbf{R}^{2n}$.*

2) $\phi : C_F \to L_F$ is a diffeomorphism.

3) A submanifold M of L_F is integrable if and only if there exists a smooth vector field ξ tangent to M such that

$$d\pi(\xi(x,y,\dot{x},\dot{y})) = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y},$$

equivalently if and only there exists a smooth vector field $\widetilde{\xi}$ tangent to $\widetilde{M} = \phi^{-1}(M)$ such that

$$d\widetilde{\pi}(\widetilde{\xi}(x,y,\lambda)) = \frac{\partial F}{\partial y}(x,y,\lambda)\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x,y,\lambda)\frac{\partial}{\partial y}.$$

Consider the $k \times k$ skew-symmetric matrix $(\{a_\ell, a_m\}(x,y))$ and the linear equation

$$A(x,y)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_k\end{pmatrix} = (\{a_\ell, a_m\}(x,y))\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_k\end{pmatrix} = \begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots \\ \{b,a_k\}(x,y)\end{pmatrix}.$$

Set

$$\widetilde{S}_F = \left\{ (x,y,\lambda) \in C_F \mid (\{a_\ell, a_m\}(x,y))\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_k\end{pmatrix} = \begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots \\ \{b,a_k\}(x,y)\end{pmatrix} \right\},$$

$$S_F = \phi(\widetilde{S}_F) \subset L_F.$$

Lemma 2. *1) For a point $(x,y,\lambda) \in C_F$, the vector*

$$d\widetilde{\pi}(\frac{\partial F}{\partial y}(x,y,\lambda)\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x,y,\lambda)\frac{\partial}{\partial y})$$

is tangent to C if and only if

$$(\{a_\ell, a_m\}(x,y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix}.$$

2) Equivalently, for a point $(x, y, \dot{x}, \dot{y}) \in L_F$, the vector $\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}$ is tangent to C at (x, y) if and only if $(x, y, \dot{x}, \dot{y}) \in S_F$.

3) Consequently S_F is contained in TC: $\quad S_F = TC \cap L_F$.

Lemma 3. *Let $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in L_F$ and let*

$$(x_0, y_0, \lambda_0) = \phi^{-1}(x_0, y_0, \dot{x}_0, \dot{y}_0) \in C_F.$$

If $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ is an integrable point of L_F, then $\lambda_0 = (\lambda_{01}, \ldots, \lambda_{0k})$ is a solution of the linear equation

$$(\{a_i, a_j\}(x_0, y_0)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x_0, y_0) \\ \vdots \\ \{b, a_k\}(x_0, y_0) \end{pmatrix},$$

which means that

$$(x_0, y_0, \lambda_0) \in \widetilde{S}_F \quad \text{and} \quad (x_0, y_0, \dot{x}_0, \dot{y}_0) \in S_F.$$

Consequently any integrable submanifold of L_F is a subset of $S_F = TC \cap L_F$.

3.1 Proof of Lemma 2

Since C is defined by the equations $a_1(x, y) = a_2(x, y) = \cdots = a_k(x, y) = 0$,

$$d\widetilde{\pi}(\frac{\partial F}{\partial y}(x, y, \lambda)\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda)\frac{\partial}{\partial y})$$

is tangent to C if and only if

$$d\widetilde{\pi}(\frac{\partial F}{\partial y}(x, y, \lambda)\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda)\frac{\partial}{\partial y})(a_i(x, y)) = 0, \quad i = 1, \ldots, k,$$

which holds if and only if

$$(\frac{\partial F}{\partial y}(x,y,\lambda)\frac{\partial}{\partial x}-\frac{\partial F}{\partial x}(x,y,\lambda)\frac{\partial}{\partial y})(a_j(x,y))=0,\quad j=1,\ldots,k,$$

which holds if and only if

$$\{F,a_j\}(x,y,\lambda)=0\quad j=1,\ldots,k,.$$

Since

$$F(x,y,\lambda)=\sum_{i=1}^{k}a_i(x,y)\lambda_i+b(x,y),$$

the last equality holds if and only if

$$\sum_{i=1}^{k}\{a_i,a_j\}(x,y)\lambda_i+\{b,a_j\}(x,y)=0,\quad j=1,\ldots,k.$$

which holds if and only if

$${}^t\left(\{a_i,a_j\}(x,y)\right)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}+\begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots\\ \{b,a_k\}(x,y)\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix},$$

which holds if and only if

$$(\{a_i,a_j\}(x,y))\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}=\begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots\\ \{b,a_k\}(x,y)\end{pmatrix}.$$

Here recall that the matrix $(\{a_i,a_j\}(x,y))$ is skewsymmetric. This completes the proof of Lemma 2. □

3.2 Proof of Lemma 3

Since $(x_0,y_0,\dot{x}_0,\dot{y}_0)\in L_F$ is an integrable point of L_F, there exists a smooth curve

$$\gamma(t)=(x(t),y(t))\in\mathbf{R}^{2n},\quad -\epsilon<t<\epsilon$$

such that

$$(x(t), y(t), \frac{dx}{dt}(t), \frac{dy}{dt}(t)) \in L_F, \quad -\epsilon < t < \epsilon$$

and

$$(x(0), y(0), \frac{dx}{dt}(0), \frac{dy}{dt}(0)) = (x_0, y_0, \dot{x}_0, \dot{y}_0).$$

Let $\widetilde{\gamma} : (\epsilon, \epsilon) \to C_F$ be the curve defined by

$$\widetilde{\gamma}(t) = \phi(x(t), y(t), \frac{dx}{dt}(t), \frac{dy}{dt}(t)).$$

Denote $\widetilde{\gamma}(t) = (x(t), y(t), \lambda(t))$. Since $(x_0, y_0, \lambda_0) = \phi^{-1}(x_0, y_0, \dot{x}_0, \dot{y}_0)$, we see that $\lambda(0) = \lambda_0$.

Since $\widetilde{\gamma}(t) \in C_F, \quad -\epsilon < t < \epsilon$, we see that

$$\frac{d\widetilde{\gamma}}{dt}(0) = \dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y} + \frac{d\lambda}{dt}(0) \frac{\partial}{\partial \lambda}$$

is tangent to L_F. Since L_F is defined by $a_1(x, y) = 0, \ldots, a_k(x, y) = 0$, we have

$$(\dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y} + \frac{d\lambda}{dt}(0) \frac{\partial}{\partial \lambda})(a_j) = 0, \quad j = 1, \ldots, k.$$

Thus

$$0 = \dot{x}_0 \frac{\partial a_j}{\partial x}(0) + \dot{y}_0 \frac{\partial a_j}{\partial y}(0) =$$

$$= \frac{\partial F}{\partial y}(x_0, y_0, \lambda_0) \frac{\partial a_j}{\partial x}(0) - \frac{\partial F}{\partial x}(x_0, y_0, \lambda_0) \frac{\partial a_j}{\partial y}(0) = \{F, a_j\}(x_0, y_0, \lambda_0).$$

Since

$$F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y) \lambda_i + b(x, y),$$

we have

$$\sum_{i=1}^{k} \{a_i, a_j\}(x_0, y_0) \lambda_{0i} + \{b, a_j\}(x_0, y_0) = 0, \quad j = 1, \ldots, k.$$

Hence

$$^t\left(\{a_i, a_j\}(x_0, y_0)\right) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} + \begin{pmatrix} \{b, a_1\}(x_0, y_0) \\ \vdots \\ \{b, a_k\}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus $\lambda_0 = (\lambda_{01}, \ldots, \lambda_{0k})$ is a solution of the linear equation

$$(\{a_i, a_j\}(x_0, y_0)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x_0, y_0) \\ \vdots \\ \{b, a_k\}(x_0, y_0) \end{pmatrix}.$$

Here recall that the matrix $(\{a_i, a_j\}(x_0, y_0))$ is skewsymmetric. This completes the proof of Lemma 3. □

4 Proofs of Theorems

4.1 Proof of Theorem 2

Theorem 2. 1) is immediate from Lemma 3.
Proof of Theorem 2. 2) Suppose that the linear equation

$$(\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}$$

has a smooth solution $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ defined on C. Consider the image

$$G_\lambda = \phi(\widetilde{G}_\lambda)$$

by ϕ of the graph

$$\widetilde{G}_\lambda = \{(x, y, \lambda_1(x, y), \ldots, \lambda_k(x, y)) \mid (x, y) \in C\}$$

of the solution $(\lambda_1(x, y), \ldots, \lambda_k(x, y))$.

Since $\lambda(x, y)$ is a solution of the linear equation, from Lemma 2, we see that the vector

$$d\widetilde{\pi}(\frac{\partial F}{\partial y}(x, y, \lambda(x, y))\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda(x, y))\frac{\partial}{\partial y})$$

is tangent to C. Since $\lambda(x, y)$ is smooth, the vector

$$d\widetilde{\pi}(\frac{\partial F}{\partial y}(x, y, \lambda(x, y))\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda(x, y))\frac{\partial}{\partial y})$$

depends smoothly on (x, y). Since $\widetilde{\pi}\mid_{\widetilde{G}_\lambda}: \widetilde{G}_\lambda \to C$ is a diffeomorphism then there exists a smooth vector field $\widetilde{\xi}$ tangent to $\widetilde{G}_\lambda$ such that

$$d\widetilde{\pi}(\widetilde{\xi}(x, y, \lambda(x, y))) = \frac{\partial F}{\partial y}(x, y, \lambda(x, y))\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda(x, y))\frac{\partial}{\partial y}.$$

Then, from Lemma 1. 3), the image $G_\lambda = \phi(\widetilde{G}_\lambda)$ is a smoothly integrable submanifold of L_F. This completes the proof of Theorem 2. □

4.2 Proof of Theorem 3

Consider the $k \times k$ matrix $(\{a_\ell, a_m\}(x, y))$ and the linear equation

$$(\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}$$

and set

$$\widetilde{S}_F =$$

$$\{(x, y, \lambda) \in \mathbf{R}^{2n} \times \mathbf{R}^k \mid (\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}\}.$$

Since $\det(\{a_\ell, a_m\}(x, y)) \neq 0$ on C, the linear equation

$$(\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}$$

has a unique smooth solution $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ on C. Then we have

$$\widetilde{S}_F = \{(x, y, \lambda) \in \mathbf{R}^{2n} \times \mathbf{R}^k \mid \lambda = \lambda(x, y), \quad (x, y) \in C\}.$$

Thus $\widetilde{S}_F$ is the graph of the map $\lambda : C \to \mathbf{R}^k$. Therefore the projection map $\widetilde{\pi}\mid_{\widetilde{S}_F}: \widetilde{S}_F \to C$ is a submersion and so is $\pi_{|S_F} : S_F \to C$. Moreover, from Lemma 2, S_F is an implicit differential system as a submanifold of TC. Thus S_F is an smoothly integrable implicit differential system and it is a smoothly i integrable submanifold of L_F.

Now the maximality of S follows from Lemma 4. This completes the proof of Theorem 3. □

4.3 Proof of Theorem 4 by using Theorem 5

Let $L_F \subset T\mathbf{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x, y, \lambda) = \sum_{\ell=1}^{k} a_\ell(x, y)\lambda_\ell + b(x, y)$$

Suppose that

$$k \quad \text{is odd and the rank of} \quad (\{a_i, a_j\}(x, y)) \quad \text{is constantly} \quad k - 1.$$

Suppose also that the linear equation

$$A(x,y)\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = (\{a_i, a_j\}(x, y))\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}.$$

has a smooth solution $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_1(x, y))$ on C.

Since the matrix $(\{a_i, a_j\}(x, y))$ depends smoothly on $(x, y) \in C$ and has a constant rank $k - 1$, the kernel set

$$\widetilde{K}_F = \{(x, y, \lambda) \in C_F \mid (\{a_i, a_j\}(x, y))\, \lambda = 0\}$$

is a smooth line bundle over C and we see that

$$\widetilde{S}_F = \{(x, y, \lambda(x, y) + \lambda) \mid (x, y) \in C, (x, y, \lambda) \in \widetilde{K}_F\}.$$

Therefore $\widetilde{S}_F$ is also a line bundle over C and so is $S_F = \phi(\widetilde{S}_F)$. Thus, S_F is a smooth manifold and the projection $\pi : S_F \to C$ is a submersion. From Theorem 5, $S_F = \phi(\widetilde{S}_F)$ is a smoothly integrable submanifold of L_F. The maximality of S_F follows from Lemma 3. This completes the proof of Theorem 3. □

4.4 Proof of Theorem 5 and Theorem 6

Let $L_F \subset T\mathbf{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family

$$F : \mathbf{R}^{2n} \times \mathbf{R}^k \to \mathbf{R}, \qquad F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y)\lambda_i + b(x, y).$$

Suppose that M is a submanifold of L_F such that the projection $\pi\mid_M: M \to C$ is a submersion.

If M is smoothly integrable, then, from Lemma 3, we have $M \subset S_F$.

Conversely, suppose that $M \subset S_F$. Let

$$(x_0, y_0, \dot{x}_0, \dot{y}_0) \in M \quad \text{and} \quad (x_0, y_0, \lambda_0) = \phi^{-1}(x_0, y_0, \dot{x}_0, \dot{y}_0).$$

Since

$$(x_0, y_0, \dot{x}_0, \dot{y}_0) \in S_F \quad \text{and} \quad (x_0, y_0, \lambda_0) \in \widetilde{S}_F,$$

from the definition of S_F and from Lemma 2, the vector

$$\dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y} = \frac{\partial F}{\partial y}(x_0, y_0, \lambda_0) \frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x_0, y_0, \lambda_0) \frac{\partial}{\partial y}$$

is tangent to C at (x_0, y_0) and smoothly depends on $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in M$. Since $\pi\mid_M: M \to C$ is a submersion, there exists a smooth vector field ξ tangent to M such that

$$d\pi(\xi(x_0, y_0, \dot{x}_0, \dot{y}_0)) = \dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y}, \quad \forall (x_0, y_0, \dot{x}_0, \dot{y}_0) \in M.$$

Thus, from Lemma 1, M is smoothly integrable. This completes the proof of Theorem 5. □

Now Theorem 6 is a direct corollary of Theorem 5.

4.5 Proof of Theorem 7

The fact that $L_F \cap \pi^{-1}(C_{reg})$ is a smoothly integrable implicit differential system of TC_{reg} is a direct corollary of Theorem 3.

Now suppose that the linear equation

$$(\{a_\ell, a_m\}(x, y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}$$

has a smooth solution $(\lambda_1(x, y), \ldots, \lambda_k(x, y))$. Then, by Theorem 2. 2), the image $G_\lambda = \phi(\widetilde{G}_\lambda)$ of the graph $\widetilde{G}_\lambda$ of the solution

$$(\lambda_1(x, y), \ldots, \lambda_k(x, y))$$

is a smoothly integrable submanifold of L_F. Take G_λ as M we seek. Then, by Theorem 3, $M \cap TC_{reg} = G_\lambda \cap TC_{reg}$ and $S_F \cap TC_{reg}$ must coincide. Since C_{reg} is dense in C, the uniqueness of such M follows.

Conversely suppose that there exists a smoothly integrable differentiable system M such that $\pi(M) = C$. Then, again by Theorem 3, $M \cap TC_{reg}$ must coincide with $S_F \cap TC_{reg}$. Consider the inverse image $\widetilde{M} = \phi^{-1}(M) \subset C_F \subset C \times \mathbf{R}^k$. Since, by Theorem 3, $\widetilde{S}_F \cap (C_{reg} \times \mathbf{R}^k)$ is the graph of a smooth solution $\lambda : C_{reg} \to \mathbf{R}^k$ of the linear equation

$$(\{a_\ell, a_m\}(x,y)) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix}, \quad (x,y) \in C_{reg},$$

$\widetilde{M} \cap (C_{reg} \times \mathbf{R}^k)$ must coincide with the graph of this smooth solution $\lambda(x,y)$, $(x,y) \in C_{reg}$. Since C_{reg} is dense in C and $\widetilde{M}$ is a smooth submanifold such that $\widetilde{\pi}(\widetilde{M}) = C$, $\lambda(x,y)$ can be extended to a smooth solution of the linear equation. Thus the linear equation has a smooth solution. This completes the proof of Theorem 7. $\square$

4.6 Proof of Theorem 8

Theorem 8 can be proved in the same way as Theorem 1. We repeat it below.

Let

$$F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + b(x,y)$$

be a Morse family. Then we have

$$\frac{\partial F}{\partial \lambda_\ell}(x,y,\lambda) = a_\ell(x,y).$$

Set

$$C = \{(x,y) \in \mathbf{R}^{2n} \mid a_1(x,y) = \ldots = a_k(x,y) = 0\},$$

$$C_g = \{(x,y) \in C \mid g_1(x,y) = \cdots = g_s(x,y) = 0\},$$

$$C_F = \{(x,y,\lambda) \in \mathbf{R}^{2n} \times \mathbf{R}^k \mid \frac{\partial F}{\partial \lambda_1}(x,y,\lambda) = \cdots = \frac{\partial F}{\partial \lambda_k}(x,y,\lambda) = 0\} =$$

$$= \{(x, y, \lambda) \in \mathbf{R}^{2n} \times \mathbf{R}^k \mid a_1(x, y) = \cdots = a_k(x, y) = 0\} = C \times \mathbf{R}^k,$$

$$C_{F,g} = \{(x, y, \lambda) \in C_F \mid g_1(x, y) = \cdots = g_s(x, y) = 0\} = C_g \times \mathbf{R}^k,$$

$$L_{F,g} = \phi(C_{F,g}).$$

Now $L_{F,g} = \phi(C_{F,g})$ is smoothly integrable if and only if there exists a smooth tangent vector filed ξ on $L_{F,g} = \phi(C_{F,g})$ such that

$$d\pi(\xi(x, y, \dot{x}, \dot{y})) = \sum_{i=1}^{n} \dot{x}_i \frac{\partial}{\partial x_i} + \dot{y}_i \frac{\partial}{\partial y_i}$$

where $\pi : T\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is the projection of the tangent bundle
$\Longleftrightarrow$ there exist smooth functions $\mu_\ell(x, y, \lambda), \ell = 1, \ldots, k$, such that

$$\text{the vector field} \quad \widetilde{\xi}(x, y, \lambda) = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda)\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}(x, y, \lambda)\frac{\partial}{\partial y_i} +$$

$$+ \sum_{\ell=1}^{k} \mu_\ell(x, y, \lambda)\frac{\partial}{\partial \lambda_\ell} \text{ is tangent to } C_{F,g} = C_g \times \mathbf{R}^k$$

$\Longleftrightarrow$

$$\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda)\frac{\partial}{\partial x} - \frac{\partial F}{\partial x_i}(x, y, \lambda)\frac{\partial}{\partial y} \quad \text{is tangent to} \quad C_{F,g}$$

$\Longleftrightarrow$

$$(\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial y_i})a_\ell = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le \ell \le k,$$

$$(\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial y_i})g_t = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le t \le s$$

$\Longleftrightarrow$

$$\{F, a_\ell\} = \sum_{i=1}^{k} \{a_i, a_\ell\}\lambda_i + \{b, a_\ell\} = 0 \quad \text{on} \quad C_F, \quad 1 \le \ell \le k.$$

$$\{F, g_t\} = \sum_{i=1}^{k} \{a_i, g_t\}\lambda_i + \{b, g_t\} = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le t \le s.$$

Differentiating the equalities with respect to λ_i, we have

$$\{a_i, a_\ell\} = \{a_i, g_t\} = 0, \quad \text{and then} \quad \{b, a_\ell\} = \{b, g_t\} = 0,$$

$$\text{on} \quad C_{F,g}, \quad 1 \le \ell \le k, 1 \le t \le s.$$

Conversely, if

$$\{a_i, a_\ell\} = \{a_i, g_t\} = 0, \quad \text{and then} \quad \{b, a_\ell\} = \{b, g_t\} = 0,$$

$$\text{on} \quad C_{F,g}, \quad 1 \le \ell \le k, 1 \le t \le s,$$

ten trivially we have

$$\{F, a_\ell\} = \sum_{i=1}^{k} \{a_i, a_\ell\}\lambda_i + \{b, a_\ell\} = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le \ell \le k.$$

$$\{F, g_t\} = \sum_{i=1}^{k} \{a_i, g_t\}\lambda_i + \{b, g_t\} = 0 \quad \text{on} \quad C_{F,g} \quad 1 \le t \le s,$$

and $L_{F,g} = \phi(C_{F,g})$ is smoothly integrable. This completes the proof of Theorem 8. □

4.7 Example for Theorem 8

Example 1. *Consider the following function.*

$$F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y)\lambda_i + b(x, y)$$

$$= \sum_{i=1}^{k} x_i\lambda_i + b_1(y_1, \ldots, y_k)b_2(x_{k+1}, \ldots, x_m),$$

$$k + 1 \le m \le n, \quad b_1(0) = b_2(0) = 0, \quad b_1, b_2 \quad \textit{are not constantly} \quad 0.$$

Then

$$\{a_\ell, a_m\} = \{x_\ell, x_m\} = 0, \quad 1 \le \ell, m \le k.$$

However

$$\{a_\ell, b\} = \{x_\ell, b\} = -\frac{\partial b_1}{\partial y_\ell} \cdot b_2 \ne 0$$

$$on \quad C = \{a_1 = \cdots = a_k = 0\} = \{x_1 = \cdots = x_k = 0\}.$$

Thus L_F itself is not smoothly integrable.

Now consider the functions

$$g_1(x,y) = x_{k+1}, \ldots, g_s(x,y) = x_{k+s} = x_m, \quad where \quad s = m-k,$$

and set

$$S = \big\{(x,y) \in \mathbf{R}^{2n} \mid$$
$$a_1(x,y) = \cdots = a_k(x,y) = g_1(x,y) = \cdots = g_s(x,y) = 0\big\}.$$

Then

$$\{a_\ell, b\} = -\frac{\partial b_1}{\partial y_\ell} b_2(x_{x+1}, \ldots, x_m) = 0,$$
$$\{a_\ell, g_t\} = \{x_\ell, x_{k+t}\} = 0, \quad \{b, g_t\} = \{b, x_{k+t}\} = 0,$$
$$1 \le \ell \le k, \quad 1 \le t \le s = m-k,$$
$$on \quad S = \{a_1 = \cdots = a_k = g_1 = \cdots = g_s = 0\}.$$

Then, by Theorem 8, $L_F \cap (S \times \mathbf{R}^{2n})$ is smoothly integrable.

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Properties of reachable sets in sub-Lorentzian geometry

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Abstract

The aim of this paper is to present basic facts concerning future timelike, nonspacelike and null reachable sets from a given point q_0 in the sub-Lorentzian geometry. In particular we prove that the three sets have identical interiors and boundaries. Further, among other things, we show that for Lorentzian metrics on contact distributions on $\mathbf{R}^{2n+1}$, $n \geq 1$, the boundary of reachable sets from q_0 is made up of null future directed curves starting from q_0. Every such curve has only a finite number of non-smooth points; smooth pieces of every such curve are Hamiltonian geodesics. For general sub-Lorentzian structures, contrary to the Lorentzian case, timelike curves may appear on the boundary. It turns out that such curves are always Goh curves. We also generalize the classical result on null geodesics: every null future directed Hamiltonian geodesic initiating at q_0 is contained in the boundary of the reachable set from q_0. At the end, in the appendix, reachable sets for the sub-Lorentzian Martinet flat structure are computed.

Keywords: sub-Lorentzian manifolds, geodesics, reachable sets, geometric optimality

1 Introduction

1.1 Motivation

Suppose that (M, g) is a time-oriented Lorentzian manifold (all definitions may be found in Section 2). Take a point $q_0 \in M$ and fix its neighbourhood

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U. Denote by $I^+(q_0, U)$ (resp. $J^+(q_0, U)$) the chronological (resp. causal) future of a point q_0. In the sequel $I^+(q_0, U)$ (resp. $J^+(q_0, U)$) will be called the future timelike (resp. nonspacelike) reachable set from q_0. It can be proved (see [11], [2]) that if U is a normal neighbourhood of q_0, then

$$I^+(q_0, U) = \exp_{q_0}(\{v \in T_{q_0}M : \quad g(v,v) < 0,\, g(v, X(q_0)) < 0\}) \cap U, \quad (1)$$

$$J^+(q_0, U) = \exp_{q_0}(\{v \in T_{q_0}M : \quad g(v,v) \le 0,\, g(v, X(q_0)) < 0\}) \cap U, \quad (2)$$

where $\exp_{q_0}$ is the (Lorentzian) exponential mapping with the pole at q_0, and X is a time orientation of (M, g) defined on U. In particular, $I^+(q_0, U)$ is open, $J^+(q_0, U)$ is closed relative to U, and both sets have identical interiors and boundaries. Moreover,

$$\partial J^+(q_0, U) \backslash \partial U =$$

$$= \exp_{q_0}(\{v \in T_{q_0}M : \quad g(v,v) = 0,\, g(v, X(q_0)) < 0\}) \cap U, \qquad (3)$$

from which it is seen that the boundary $\partial J^+(q_0, U) \backslash \partial U$ is formed by maximizing null future directed geodesics starting from q_0. More precisely, if $x^0, x^1, ..., x^n$ are exponential coordinates on U centered at q_0 with a time orientation $\frac{\partial}{\partial x^0}$, then

$$I^+(q_0, U) = \left\{-(x^0)^2 + (x^1)^2 + ... + (x^n)^2 < 0,\, x^0 > 0\right\},$$

$$J^+(q_0, U) = \left\{-(x^0)^2 + (x^1)^2 + ... + (x^n)^2 \le 0,\, x^0 \ge 0\right\},$$

and

$$\partial J^+(q_0, U) \backslash \partial U = \left\{-(x^0)^2 + (x^1)^2 + ... + (x^n)^2 = 0,\, x^0 \ge 0\right\}.$$

Let $\Phi(x^0, ..., x^n) = -(x^0)^2 + (x^1)^2 + ... + (x^n)^2$; the gradient $\nabla\Phi$, computed with respect to g, is a null vector field when restricted to $\partial J^+(q_0, U) \backslash (\partial U \cup \{q_0\})$. It follows that the latter set is smooth and each tangent space to it contains a single nonspacelike direction, namely the one of $\nabla\Phi$ - cf. Lemma 5.1.

The aim of this paper is to establish some partial results of above-mentioned type for reachable sets in the sub-Lorentzian geometry.

1.2 Organization of the paper

Section 2 contains a review of basic notions and facts on the sub-Lorentzian geometry. The reader familiar with these notions can omit this section. Proposition 2.1 is new; it gives necessary and sufficient conditions for existence of Lorentzian metrics on distributions.

In Section 3 we summarize all what we know about reachable sets from a point for general sub-Lorentzian structures. In particular we prove that null, timelike and nonspacelike reachable sets have identical interiors and boundaries - Theorems 3.1, 3.2. At the end of Section 3 some examples of reachable sets are given.

In Section 4 we present a notion of geometric optimality and recall the Pontryagin maximum principle in the geometric version.

Section 5 presents a generalization of a classical result concerning local optimality of null Lorentzian geodesics. Namely we prove that sub-Lorentzian null future directed Hamiltonian geodesics are geometrically optimal. Moreover, they are also locally optimal with respect to a given sub-Lorentzian metric - Theorem 5.1.

In Section 6 we study the boundary of reachable sets. Among other things we prove that timelike curves contained in $\partial J^+(q_0, U)\backslash\partial U$, q_0 being a point and U its normal neighbourhood, are so-called Goh curves (Lemma 6.1) so, for instance, they do not exist for sub-Lorentzian metrics (H, g), where rank $H \geq 3$ and H is generic. In such cases timelike reachable sets $I^+(q_0, U)$ are open. Moreover, if we strengthen assumptions imposed on H, we can ensure that the boundary $\partial J^+(q_0, U)\backslash\partial U$ is made up of null future directed curves, and that the sub-Lorentzian distance, $f[U]$, from q_0 is continuous at every point $q \in \partial J^+(q_0, U)\backslash\partial U$. Further, we also prove that if (H, g) is a sub-Lorentzian structure on $\mathbf{R}^{2n+1}$ such that H is contact, then $\partial J^+(q_0, U)\backslash\partial U$ consists of piecewise smooth null future directed curves starting from q_0; smooth pieces of each such curve are Hamiltonian geodesics - Theorem 6.2. We also give some partial results in rank-two case: Propositions 6.2, 6.3, 6.4.

Finally, in Section 7 we compute reachable sets in the Martinet flat case.

Note that, as is explained in Section 6, all above results concerning reachable sets can be applied to control affine systems with controls taking values in the unit closed ball centered at zero.

2 Review of Basic Notions in the sub-Lorentzian Geometry

All proofs of the results presented in this section may be found in [6], [9].

2.1 Horizontal curves

Let M be a smooth (i.e. of class C^∞) connected $(n+1)$-dimensional manifold. Let H be a smooth distribution on M of constant rank $k+1$. For a point $q \in M$ and a positive integer i let us define H_q^i to be the vector space generated by all vectors of the form

$$[X_1, [X_2, \ldots [X_{k-1}, X_k] \ldots]]\,(q), \tag{4}$$

where $X_1, \ldots, X_k$ are local sections of H defined near q and $1 \leq k \leq i$. H is said to be *bracket generating*, if for every $q \in M$ there is an $i = i(q) \in \mathbf{N}$ such that $H_q^{i(q)} = T_qM$. H is said to be *2-generating* if $H_q^2 = T_qM$ for each q in M. In the sequel we suppose H to be bracket generating.

The geometry of the couple (M, H) is determined by *horizontal* or *admissible* curves, that is such curves $\gamma : [a, b] \longrightarrow M$ that (i) γ is absolutely continuous, (ii) $\dot{\gamma}(t) \in H_{\gamma(t)}$ a.e. on $[a, b]$, (iii) the derivative $\dot{\gamma}$ is square integrable relative to some Riemannian metric on M. Denote by Ω_q^T the set of all horizontal curves $\gamma : [0, T] \longrightarrow M$ starting from $\gamma(0) = q$ and consider the *endpoint mapping*

$$end_q^T : \Omega_q^T \longrightarrow M, \qquad \gamma \longrightarrow \gamma(T).$$

It turns out that Ω_q^T is a Hilbert manifold and end_q^T is smooth (see for instance [3]). Notice that, since H is bracket generating, $end_q^T(\Omega_q^T) = M$ for every $q \in M$ - it is the classical Chow-Rashevski theorem.

A curve $\gamma \in \Omega_q^T$ is said to be *abnormal* (or *singular*) if the differential $d_\gamma end_q^T : T_\gamma \Omega_q^T \longrightarrow T_{\gamma(T)}M$ is not surjective. In case $H = TM$ the differential $d_\gamma end_q^T$ is surjective for every $\gamma \in \Omega_q^T$, so abnormal curves do not exist. In case of a contact distribution $d_\gamma end_q^T$ degenerates only for constant curves $\gamma(t) \equiv q$, so in this case non-trivial abnormal curves do not exist. Let us note here that the formula for the differential of end_q^T yields: $\gamma \in \Omega_q^T$ is abnormal if and only if there exists an absolutely continuous curve $\lambda : [0, T] \longrightarrow T^*M$

such that $\lambda(t) \in T^*_{\gamma(t)}M\backslash\{0\}$ annihilates $H_{\gamma(t)}$ for every t. It follows that any sub-arc of an abnormal curve is abnormal.

At the end let us state one more definition: a horizontal curve $\gamma : [0,T] \longrightarrow M$ will be called *a Goh curve* if it is abnormal and has an absolutely continuous lift $\lambda : [0,T] \longrightarrow T^*M\backslash\{0\}$ such that for every $t \in [0,T]$

$$\langle \lambda(t), [X,Y](\gamma(t)) \rangle = 0,$$

where X, Y are arbitrary horizontal vector fields defined around $\gamma([0,T])$.

For simplicity we adopt the following convention:

all curves, vectors and vector fields are supposed to be horizontal.

2.2 Sub-Lorentzian metrics

Let g be a Lorentzian metric on H, i.e. g is a global section of the vector bundle $H^* \otimes H^* \longrightarrow M$ such that $g_q : H_q \times H_q \longrightarrow \mathbf{R}$ is a nondegenerate symmetric bilinear form of index one for every $q \in M$. For $v, w \in H_q$ we shall write $g(v,w)$ instead of $g_q(v,w)$. The couple (H,g) is called a *sub-Lorentzian metric* on M, and the triple (M,H,g) - a *sub-Lorentzian manifold.*

As in the Lorentzian geometry we say that a vector $v \in H_q$ is *timelike*, if $g(v,v) < 0$, is *nonspacelike*, if $g(v,v) \leq 0$ and $v \neq 0$, is *null* if $g(v,v) = 0$ and $v \neq 0$, and finally is *spacelike* if $g(v,v) > 0$ or $v = 0$.

Define a *time orientation* of (M,H,g) to be a continuous timelike vector field X on M. We say that a sub-Lorentzian metric (H,g) is *time-orientable* if (M,H,g) admits a time orientation. Using similar arguments as in [19] and [18] one can prove the following

Proposition 2.1. *Let M be a smooth manifold and H a smooth distribution on M of constant rank. Then H admits a metric of signature l, if and only if H possesses an l-dimensional subdistribution. In particular, for $l = 1$, the following conditions are equivalent:*
(i) H admits a Lorentzian metric;
(ii) H admits a Lorentzian metric which is time-oriented;
(iii) H possesses a 1-dimensional subdistribution.

As an example consider S^5, a 5-dimensional sphere. Let X be a non-vanishing vector field on S^5, and take ω to be a 1-form satisfying $\langle \omega, X \rangle = 1$

everywhere on S^5. Now if we define $H = \ker \omega$, then H is a distribution of rank 4 on S^5. We will show that H does not admit Lorentzian metrics. Indeed, suppose the converse. Then by Proposition 2.1 there exists a nonvanishing vector field Y with $\langle \omega, Y \rangle = 0$ everywhere. Thus $Span\{X, Y\}$ is a distribution of rank 2 on S^5 which is impossible (cf. [19]).

From now on we suppose our (M, H, g) to be time-oriented by a vector field X. A nonspacelike $v \in H_q$ is said to be *future-directed* (resp. *past-directed*) if $g(v, X(q)) < 0$ (resp. $g(v, X(q)) > 0$). Now a curve $\gamma : [a, b] \longrightarrow M$ is timelike (resp. timelike future directed, nonspacelike, nonspacelike future directed, null, null future directed) if so is $\dot{\gamma}(t)$ a.e. on $[a, b]$.

We will use the following abbreviations: "t." for "timelike", "nspc." for "nonspacelike", and "f.d." for "future directed". So for instance a t.f.d. curve is a (horizontal) curve which is timelike future directed.

2.3 Normal neighbourhoods. Convergence of sequences of curves

Up to the end of this section (M, H, g) is a fixed sub-Lorentzian time-oriented manifold.

We will introduce a concept of so-called normal neighbourhoods. Take a point $q_0 \in M$ and let U be its arbitrary neighbourhood. Replacing U with possibly smaller open set containing q_0 we can assume that the closure $\overline{U}$ is compact and that there exists an orthonormal frame $X_0, ..., X_k$ of H defined on $\overline{U}$; here X_0 is a time orientation. Extend this frame to the basis $X_0, ..., X_k, ..., X_n$ of TM again defined on $\overline{U}$. Now we can define a time-oriented Lorentzian metric h on U by assuming the basis $X_0, ..., X_n$ to be orthonormal relative to h with a time orientation X_0. Next, possibly shrinking U again, we suppose that U is a normal convex neighbourhood relative to h and its closure $\overline{U}$ is contained in some bigger normal convex (relative to h) set. Such a U just obtained is called a *normal neighbourhood* of q_0. Obviously, each point of M possesses arbitrarily small normal neighbourhoods.

Normal neighbourhoods are very useful, particularly because they have good properties according to convergence of sequences of nspc. curves. To be more precise, let $\gamma, \gamma_\nu : [a, b] \longrightarrow M$, $\nu = 1, 2, ...$, be curves in M. We say that $\{\gamma_\nu\}$ *is convergent to* γ *in the* C^0 *topology on curves*, if $\gamma_\nu(a) \longrightarrow \gamma(a)$,

$\gamma_\nu(b) \longrightarrow \gamma(b)$ as $\nu \longrightarrow \infty$, and for every open set V containing $\gamma([a,b])$ there is an integer Λ such that $\gamma_\nu([a,b]) \subset V$ for all $\nu > \Lambda$. Suppose now that U is a normal neighbourhood of a point q_0; let $\gamma_\nu : [0,T] \longrightarrow U$ be a nspc.f.d. curve starting from $\gamma_\nu(0) = q_0$, $\nu = 1, 2, ...$ If $\gamma_\nu(T) \longrightarrow q$ for a $q \in U$, then one can prove that, after passing to a subsequence, $\{\gamma_\nu\}$ converges in the C^0 topology on curves to a nspc.f.d. $\gamma : [0,T] \longrightarrow U$; of course $\gamma(0) = q_0$, $\gamma(T) = q$.

2.4 Sub-Lorentzian geodesics, reachable sets and local distance functions

Let $\gamma : [a,b] \longrightarrow M$ be a nspc. curve; we define its *length* in the usual manner to be

$$L(\gamma) = \int_a^b |g(\dot{\gamma}(t), \dot{\gamma}(t))|^{1/2} dt.$$

The operation L is upper semicontinuous in the following sense: if $\{\gamma_\nu\}$ is a sequence of nspc.f.d. curves which converges in the C^0 topology on curves to a (nspc.f.d.) curve γ then $\limsup_{\nu \longrightarrow \infty} L(\gamma_\nu) \leq L(\gamma)$.

If U is an open subset of M and $\gamma : [a,b] \longrightarrow M$ is a nspc.f.d. curve contained in U, then γ is called a *U-maximizer* if it is longest curve among all nspc.f.d. curves contained in U and joining $\gamma(a)$ to $\gamma(b)$. Curves in U which are locally U-maximizers are called *U-geodesics.*

For a given point q_0 and its neighbourhood U we defined the *future timelike reachable set from* q_0 to be the set $I^+(q_0, U)$ of all points in U that can be reached from q_0 by a t.f.d. curve contained in U. Analogously we define the *future nonspacelike reachable set from* q_0 to be the set $J^+(q_0, U)$ of all points in U that can be reached from q_0 by a nspc.f.d. curve contained in U.

For $q_0, q \in U$ let $\Omega^{nspc}_{q_0,q}(U)$ be the set of all nspc.f.d. curves in U joining q_0 to q. We define

$$f[U] : U \longrightarrow \mathbf{R},$$

the *(local) sub-Lorentzian distance from* q_0 *relative to the set* U, by formula

$$f[U](q) = \begin{cases} \sup\left\{L(\gamma) : \gamma \in \Omega^{nspc}_{q_0,q}(U)\right\} : & q \in J^+(q_0, U) \\ 0 : & q \notin J^+(q_0, U) \end{cases}.$$

Now suppose that U is a normal neighbourhood of q_0. It turns out that if $q \in J^+(q_0, U)$, then U-maximizers connecting q_0 to q exist. As a corollary one can prove that $f[U]$ is upper semicontinuous, and that it is continuous along smooth timelike U-maximizers contained in *int* $I^+(q_0, U)$.

2.5 Horizontal gradient

Let $U \subset M$ be an open subset and let $\varphi : U \longrightarrow \mathbf{R}$ be a smooth function. By the *horizontal gradient of* φ we mean the vector field $\nabla_H \varphi$ which is defined by condition $(\partial_v \varphi)(q) = g(\nabla_H \varphi(q), v)$ for every $q \in U$ and $v \in H_q$. It can be proved that if $\nabla_H \varphi$ is a timelike past directed vector field on U such that $g(\nabla_H \varphi, \nabla_H \varphi) \equiv \mathit{const}$ on U, then trajectories of $-\nabla_H \varphi$ are unique U-maximizers.

2.6 Hamiltonian geodesics and the exponential mapping

To every sub-Lorentzian metric (H, g) on M we can canonically associate the vector bundle morphism $G : T^*M \longrightarrow H$ covering identity, such that $\mathrm{Im}\, G = H$ and $g(v, w) = \langle \xi, G\eta \rangle = \langle \eta, G\xi \rangle$ for every $\xi \in G^{-1}(v)$ and $\eta \in G^{-1}(w)$. This permits us to define the so-called *geodesic Hamiltonian* $\mathcal{H} : T^*M \longrightarrow \mathbf{R}$,

$$\mathcal{H}(\lambda) = \frac{1}{2} \langle \lambda, G\lambda \rangle .$$

If $X_0, X_1, ..., X_k$ is an orthonormal basis of H defined on an open set U with X_0 timelike, then on $T^*M_{|U}$ we have

$$\mathcal{H}(q, p) = -\frac{1}{2} \langle p, X_0(q) \rangle^2 + \frac{1}{2} \sum_{j=1}^{k} \langle p, X_j(q) \rangle^2 .$$

By $\overrightarrow{\mathcal{H}}$ we denote the Hamiltonian vector field corresponding to it and by Φ_t its (local) flow on T^*M. Now a curve $\gamma : [a, b] \longrightarrow M$ is called a *Hamiltonian geodesic* if there is a $\Gamma : [a, b] \longrightarrow T^*M$ such that $\dot{\Gamma}(t) = \overrightarrow{\mathcal{H}}(\Gamma(t))$ and $\gamma(t) = \pi \circ \gamma(t)$, on $[a, b]$, $\pi : T^*M \longrightarrow M$ being the canonical projection. Note that Hamiltonian geodesics preserve their causal character.

To state our last definition, for a $q \in M$, denote by D_q the set of all $\lambda \in T_q^*M$ such that the curve $t \longrightarrow \Phi_t(\lambda)$ is defined on $[0, 1]$. The mapping

$$\exp_q : D_q \longrightarrow M, \quad \exp_q(\lambda) = \pi \circ \Phi_1(\lambda)$$

is called *exponential mapping (with the pole at q).* Of course D_q is open and $\exp_q$ is smooth. Contrary to the Lorentzian geometry $\exp_q$ is not a diffeomorphism at 0; moreover, at least for rank 2 distributions, it is not 'onto' a neighbourhood of q.

3 Reachable Sets in the sub-Lorentzian Geometry

3.1 Basic properties

This section is devoted to the study of reachable sets for general sub-Lorentzian structures. Lemma 3.1 was already obtained in [9]. However, for completeness of the exposition, we recall all the proofs.

Let (M, H, g) be a fixed sub-Lorentzian time-oriented manifold. We start with the remark concerning smooth t.f.d. approximations to nspc.f.d. curves (the existence of such approximations to t.f.d. curves is clear). So let $\gamma : [a, b] \longrightarrow M$ be a nspc.f.d. curve, i.e. $\dot\gamma(t) = Z(t, \gamma(t))$, where

$$Z(t, q) = \sum_{\alpha=0}^{k} u_\alpha(t) X_\alpha(q), \quad -u_0(t)^2 + \sum_{i=1}^{k} u_i(t)^2 \leq 0, \quad u_0(t) > 0,$$

a.e. on $[a, b]$, $(u_0, ..., u_k) \in L^2([a, b], \mathbf{R}^{k+1})$, and $X_0, ..., X_k$ is a smooth orthonormal basis of H defined in a neighbourhood of γ with a time orientation X_0 (if such a basis do not exist, we divide γ into a finite number of smaller pieces). Now take a sequence a_ν such that $0 < a_\nu \nearrow 1$ and write

$$Z_\nu(t, q) = u_0(t) X_0(q) + a_\nu \sum_{i=1}^{k} u_i(t) X_i(\gamma(t)).$$

Let γ_ν be a solution to the following Cauchy problem

$$\dot\gamma_\nu(t) = Z_\nu(t, \gamma_\nu(t)), \quad \gamma_\nu(a) = \gamma(a),$$

which is defined on the whole $[a,b]$, provided ν is sufficiently large. Of course each γ_ν is timelike and, since

$$(u_0, a_\nu u_1, ..., a_\nu u_k) \longrightarrow (u_0, u_1, ..., u_k)$$

in $L^2([a,b], \mathbf{R}^{k+1})$ as $\nu \longrightarrow \infty$, $\gamma_\nu \longrightarrow \gamma$ uniformly on $[a,b]$ (cf. [6]). At the end it suffices to notice that each γ_ν can be approximated by a smooth t.f.d. curve.

Now let us fix a point $q_0 \in M$ and its normal neighbourhood U. Recall that in Section 2.4 we defined two sets $I^+(q_0, U)$ and $J^+(q_0, U)$. Introduce two other sets, namely let $I_0^+(q_0, U)$ (resp. $J_0^+(q_0, U)$) be the reachable set from q_0 for a family of all smooth t.f.d. (resp. nspc.f.d.) vector fields on U. Properties of reachable sets of this type are summarized for instance in [13].

Let $X_0, X_1, ..., X_k$ be an orthonormal frame for H defined on $\overline{U}$. Consider the control system

$$\dot{q}(t) = \sum_{\alpha=0}^{k} u_\alpha(t) X_\alpha(q(t)), \;\; t \in [0,T]. \tag{5}$$

Moreover let us define two sets:

$$C_0 = \left\{ (u_0, ..., u_k) \in \mathbf{R}^{k+1} : -u_0^2 + \sum_{i=1}^{k} u_i^2 < 0, \; u_0 > 0 \right\}$$

and

$$C = \left\{ (u_0, ..., u_k) \in \mathbf{R}^{k+1} : -u_0^2 + \sum_{i=1}^{k} u_i^2 \leq 0, \; u_0 > 0 \right\}. \tag{6}$$

Then the set $I^+(q_0, U)$ (resp. $J^+(q_0, U)$) corresponds to enpoints of trajectories of (5) starting from q_0, where the set of admissible controls is the set of square integrable mappings $u : [0, T(u)] \longrightarrow C_0$ (resp. $u : [0, T(u)] \longrightarrow C$), where final time $T(u) > 0$ depends on a control, while $I_0^+(q_0, U)$ (resp. $J_0^+(q_0, U)$) is generated by piecewise smooth controls $u : [0, T(u)] \longrightarrow C_0$ (resp. $u : [0, T(u)] \longrightarrow C$); here $T(u) > 0$ again depends on a control.

First of all let us note the following

Lemma 3.1. *$J^+(q_0, U)$ is closed with respect to U.*

Proof. Let $q_\nu \in J^+(q_0, U)$ be such that $q_\nu \longrightarrow q$ with $q \in U$. Let γ_ν be a nspc.f.d. curve connecting q_0 to q_ν. From Section 2.3 we know that, after passing to a subsequence, $\gamma_\nu \longrightarrow \gamma$ in the C^0 topology on curves, where γ is nspc.f.d. and joins q_0 with q. Thus $q \in J^+(q_0, U)$. □

As a corollary

$$cl_U(I_0^+(q_0, U)) \subset J^+(q_0, U),$$

where cl_U stands for the closure with respect to U. Moreover from the remark at the beginning of this section,

$$J^+(q_0, U) \subset cl_U(I_0^+(q_0, U)),$$

from which it follows that

$$J^+(q_0, U) = cl_U(I_0^+(q_0, U)) = cl_U(I^+(q_0, U)).$$

Next, Krener's theorem [14] yields

$$I_0^+(q_0, U) \subset cl_U(int I_0^+(q_0, U)),$$

which in turn gives

$$J^+(q_0, U) = cl_U(int\ I_0^+(q_0, U)) = cl_U(int\ I^+(q_0, U)). \tag{7}$$

As the next step we will prove

Lemma 3.2. *int* $J^+(q_0, U) =$ *int* $I^+(q_0, U)$.

Proof. Obviously *int* $I^+(q_0, U) \subset$ *int* $J^+(q_0, U)$. Take a point $q \in intJ^+(q_0, U)$ and fix an open V such that $q \in V \subset int\ J^+(q_0, U)$. Consider the family $\mathcal{F}$ of all smooth timelike past directed vector fields on V. Clearly $\mathcal{F}$ is bracket generating, so its reachable set $\mathcal{A}_\mathcal{F}(q)$ from q has a non-empty interior. Now, because of (7), there is a point $q_1 \in int\ \mathcal{A}_\mathcal{F}(q) \cap int\ I^+(q_0, U) \cap V$. In this way we have established the existence of t.f.d. curves σ_1, σ_2 in U, such that σ_1 joins q_0 to q_1, and σ_2 joins q_1 to q. The curve $\sigma_1 \cup \sigma_2$ joins q_0 to q and is contained in *int* $I^+(q_0, U)$. This last statement follows from a standard fact from control theory: any curve starting from q_0 which enters the interior of the reachable set from q_0 cannot leave this interior. □

We sum up our considerations as follows.

Theorem 3.1. *For every q_0 and every normal neighbourhood U of q_0*
(a) $cl_U(intI^+(q_0,U)) = J^+(q_0,U)$;
(b) $int\ I^+(q_0,U) = int\ J^+(q_0,U)$;
(c) $\tilde{\partial} I^+(q_0,U) = \tilde{\partial} J^+(q_0,U)$,
where $\tilde{\partial} A$ is a boundary of a set A relative to U.

Next we will investigate some properties of the set $N^+(q_0,U)$ which is defined to be the set of all points that can be reached from q_0 by a null f.d. curve contained in U. $N^+(q_0,U)$ is called a *(future) null reachable set from* q_0. Our aim is to prove

Theorem 3.2. *For every q_0 and every normal neighbourhood U of q_0*
(a) $cl_U(intN^+(q_0,U)) = J^+(q_0,U)$;
(b) $intN^+(q_0,U) = intJ^+(q_0,U)$;
(c) $\tilde{\partial} N^+(q_0,U) = \tilde{\partial} J^+(q_0,U)$.

Proof. Let $\gamma : [0,T] \longrightarrow U$ be a smooth t.f.d. curve, $\gamma(0) = q_0$. Assuming that γ is parameterized by arc length we can find an orthonormal frame $Z_0, ..., Z_k$ for H defined on U such that $\dot{\gamma} = Z_0$. Using for instance [1] or [12] we know that γ can be approximated by a sequence of null curves $\gamma_\nu : [0,T] \longrightarrow U$ such that $\gamma_\nu(0) = q_0$ and $\dot{\gamma}_\nu(t) = Y_{t,\nu}(\gamma_\nu(t))$, where $Y_{t,\nu}$ is a non-autonomous vector field satisfying $Y_{t,\nu} \in \{Z_0 + Z_1, Z_0 - Z_1\}$ for every t and ν. One can assume that every γ_ν is piecewise smooth. If we denote by $N_0^+(q_0,U)$ the reachable set from q_0 for the family of all smooth null f.d. vector fields on U, then in view of the presented argument and the remark from the begining of this section

$$cl_U(N_0^+(q_0,U)) = J^+(q_0,U).$$

Next let us notice that if $X_0, ..., X_k$ is any orthonormal basis for $H_{|U}$ which is bracket generating, then so is the family

$$\{X_0 \pm X_i : \ i = 1, ..., k\}$$

(resp. $\{-X_0 \pm X_i : \ i = 1, ..., k\}$) of smooth null future (resp. past) directed vector fields on U. Thus *int* $N^+(q_0,U) \neq \emptyset$ by Krener's theorem, and the rest of the proof is similar to the proof of Theorem 3.1. □

At the end let us notice that, unlike the classical Lorentzian geometry, in general $N^+(q_0,U) \neq J^+(q_0,U)$ - see 3.2.3.

3.2 Examples of reachable sets

3.2.1 The Heisenberg case

Suppose that $M = \mathbf{R}^3$ and let

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial z}.$$

We define a rank-two distribution $H = Span\{X, Y\}$ and a Lorentzian metric g on it by declaring the basis X, Y to be orthonormal with a time orientation X. It can be computed (see [9]) that

$$I^+(0, \mathbf{R}^3) = \{(x, y, z) : \ -x^2 + y^2 + 4|z| < 0, \ x > 0\}$$

and $J^+(0, \mathbf{R}^3)$ is the closure of $I^+(0, \mathbf{R}^3)$. Next, if U is any normal neighbourhood of 0 then

$$I^+(0, U) = I^+(0, \mathbf{R}^3) \cap U, \quad J^+(0, U) = J^+(0, \mathbf{R}^3) \cap U.$$

Note that the set $\tilde{\partial} J^+(0, U) \cap \{z \neq 0\}$ is smooth, and if q is its arbitrary point, then

$$T_q(\tilde{\partial} J^+(0, U) \cap \{z \neq 0\}) \cap H_q \tag{8}$$

is a 1-dimensional subspace generated by a null direction.

3.2.2 Generalization

The above example can be generalized as follows. Let $\varphi = \varphi(x, y, z)$, $\psi = \psi(z)$ be smooth functions defined near 0 in $\mathbf{R}^3$, $\psi(0) = 0$. Let us define

$$\begin{aligned} X &= \tfrac{\partial}{\partial x} + y\varphi(x, y, z)(y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}) + \tfrac{1}{2}y(1 + \psi(z))\tfrac{\partial}{\partial z} \\ Y &= \tfrac{\partial}{\partial y} - x\varphi(x, y, z)(y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}) - \tfrac{1}{2}x(1 + \psi(z))\tfrac{\partial}{\partial z} \end{aligned}$$

(cf. [7]). Suppose that $H = Span\{X, Y\}$ and that g is the Lorentzian metric on H determined by the condition that X, Y is an orthonormal frame and X is a time orientation. In [10] reachable sets for such a structure (H, g) were computed:

$$I^+(0, U) = \left\{ -x^2 + y^2 + 4\left| \int_0^z \frac{d\zeta}{1 + \psi(\zeta)} \right| < 0, \ x > 0 \right\} \cap U \tag{9}$$

and $J^+(0,U) = cl_U I^+(0,U)$, where U is a sufficiently small normal neighbourhood of 0. Again all spaces of the form (8) are 1-dimensional and are generated by a null direction. Moreover $N^+(0,U) = J^+(0,U)$ as it can be seen from [10].

3.2.3 The Martinet case

Again $M = \mathbf{R}^3$; let us set

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{1}{2}xy\frac{\partial}{\partial z}.$$

We define $H = Span\{X, Y\}$. Obviously H is not contact. As above we define a Lorentzian metric g on H by supposing the family X, Y to be orthonormal with respect to g with a time orientation X. There occurs a new phenomenon here, as compared to the previous cases, namely there is a timelike curve on the boundary $\tilde{\partial} J^+(q,U)$ for certain q's. To see this let $\gamma(t) = (t,0,0)$, $t_1 \leq t \leq t_2$. Suppose that $\eta : [\alpha, \beta] \longrightarrow \mathbf{R}^3$ is a nspc.f.d. curve such that $\eta(\alpha) = \gamma(t_1)$, $\eta(\beta) = \gamma(t_2)$. Notice that $H = \ker \omega$ with

$$\omega = dz - \frac{1}{2}y(ydx - xdy). \tag{10}$$

Now, if $\eta = (\eta_1, \eta_2, \eta_3)$, then

$$0 = \eta_3(\beta) = \eta_3(\alpha) + \frac{1}{2}\int_\eta y(ydx - xdy).$$

Using $\eta_3(\alpha) = 0$ and $d(y^2dx - xydy) = -\frac{3}{2}ydx \wedge dy$ one can see that $\eta_3(\beta)$ is strictly positive unless $\eta_2(t) \equiv 0$. But then (10) implies that $\eta_3(t) \equiv 0$ so η is a reparameterization of γ. It means that the set of all nspc.f.d. curves joining $(t_1,0,0)$ to $(t_2,0,0)$ is made up of a single, up to a change of parameter, curve γ. In view of Theorem 3.2, $\gamma([t_1,t_2]) \subset \tilde{\partial} J^+(\gamma(t_1),U)$ for every $t_1 < t_2$ and every normal neighbourhood U of $\gamma(t_1)$. Remark moreover that γ is a Goh curve - cf. Lemma 6.1 below.

The explicit formulas for reachable sets in the Martinet case are computed in the appendix, at the end of this paper.

Let us notice that in all above examples, unlike classical Lorentzian geometry, $J^+(0,U)$ is not the image under exponential mapping $\exp_0$.

4 Geometric optimality

Notions and facts presented in this section may be found for instance in [1].

Let M be a smooth manifold and let $X(\cdot, u)$ be a family of vector fields $X(\cdot, u) : M \longrightarrow TM$ on M, where $u \in B$, B being an arbitrary subset of $\mathbf{R}^m$. We assume X to be smooth with respect to (q, u) on $M \times \overline{B}$. Consider a control system

$$\dot{q}(t) = X(q(t), u(t)), \tag{11}$$

where controls are supposed to be measurable and bounded with values in B. The set of all such controls is denoted by $\mathcal{B}$.

Fix a point q_0. If $u(\cdot) \in \mathcal{B}$, $u : [0, t_1] \longrightarrow B$ (final time is not fixed), then by $q_u : [0, t_1] \longrightarrow B$ we denote the trajectory of the system (11) corresponding to the control $u(\cdot)$ and starting from q_0. The set of endpoints of all trajectories of the system (11) corresponding to controls from $\mathcal{B}$ and starting from q_0 will be denoted by $\mathcal{A}(q_0)$. $\mathcal{A}(q_0)$ is called *reachable* (or *accessible*) *set from* q_0.

A control $u : [0, t_1] \longrightarrow B$ from $\mathcal{B}$ (resp. the trajectory $q_u : [0, t_1] \longrightarrow M$ corresponding to it) is called *geometrically optimal* if $q_u(t_1) \in \partial\mathcal{A}(q_0)$. Clearly, if $q_u(t_1) \in \partial\mathcal{A}(q_0)$ then $q_u(t) \in \partial\mathcal{A}(q_0)$ for any $t \in [0, t_1]$ (this last remark follows from the known fact saying that if $q_u(t_0) \in int\mathcal{A}(q_0)$ for a certain t_0 then $q_u(t) \in int\mathcal{A}(q_0)$ for any $t > t_0$ belonging to the domain of q_u).

Necessary conditions for a control to be geometrically optimal are given by the well-known Pontryagin maximum principle (PMP for short) which we are going to formulate now. To this end we need to introduce a parameter-dependent Hamiltonian

$$h_u : T^*M \longrightarrow M, \quad h_u(q, p) = \langle p, X(q, u) \rangle\,, \quad q \in M, p \in T_q^*M.$$

As usual $\overrightarrow{h_u}$ stands for the Hamiltonian vector field on T^*M determined by h_u.

Theorem 4.1 (PMP)**.** *Consider the control system (11) and let* $u : [0, t_1] \longrightarrow B$ *be a control. The necessary condition for* $u(\cdot)$ *to be geometrically optimal (i.e. a necessary condition for* $q_u(t_1) \in \partial\mathcal{A}(q_0)$*) is the existence of an absolutely continuous curve* $\lambda : [0, t_1] \longrightarrow T^*M$ *such that the following conditions are satisfied:*

(i) $\lambda(t) \in T^*_{q_u(t)}M$ *and* $\lambda(t) \neq 0$ *on* $[0, t_1]$*;*
(ii) $\dot{\lambda}(t) = \overrightarrow{h_{u(t)}}(\lambda(t))$ *a.e. on* $[0, t_1]$*;*
(iii) $h_{u(t)}(\lambda(t)) = \max_{v \in B} h_v(\lambda(t))$ *a.e. on* $[0, t_1]$*;*
(iv) $\max_{v \in B} h_v(\lambda(t)) = 0$ *everywhere on* $[0, t_1]$*.*

A curve $\lambda : [0, t_1] \longrightarrow T^*M$ described by PMP will be called a *biextremal* covering q_u. Remark at the and of this section that geometric optimality of a curve is invariant under changes of parameterization. The analogous statement for optimal problems with costs is in general not true.

5 Geometrical optimality of null Hamiltonian geodesics

In this section we mearly prove that null f.d. Hamiltonian geodesics starting from a point q_0 are geometrically optimal, and are unique U-maximizers, provided U is a sufficiently small neighbourhood of q_0.

If V is a vector space with a scalar product α, then for a subspace W by W^α we denote the orthogonal complement of W with respect to α:

$$W^\alpha = \{v \in V : \ \alpha(v, w) = 0 \, \textit{for every} \, w \in W\}.$$

Lemma 5.1. *Let* φ *be a smooth function defined on a sub-Lorentzian manifold* (M, H, g)*. Let* $N = \{\varphi = 0\}$ *be nonempty and suppose that* $\nabla_H \varphi \neq 0$ *on* N*. Then the following conditions are equivalent:*
(a) $\nabla_H \varphi$ *is a null field on* N*;*
(b) $T_qN \cap (H_q \cap T_qN)^g \neq \{0\}$ *for every* $q \in N$ *(i.e.* g *is degenerate on* $T_qN \cap (H_q \cap T_qN)^g$*, and hence the dimension of the latter space is* 1*).*

Proof. (a)$\Longrightarrow$(b) Clearly $\nabla_H\varphi(q) \in (H_q \cap T_qN)^g$ for any $q \in N$, and since $\nabla_H\varphi(q)$ is null, we also have $\nabla_H\varphi(q) \in H_q \cap T_qN$.

(b)$\Longrightarrow$(a) Take a point $q \in N$. By assumption there exists a $v \in (H_q \cap T_qN) \cap (H_q \cap T_qN)^g$, $v \neq 0$. Of course $g(v, v) = 0$ for such a v. Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow N$ be such that $\gamma(0) = q$, $\dot{\gamma}(0) = v$. Differentiating the equality $\varphi(\gamma(t)) = 0$ we get $g(v, \nabla_H\varphi(q)) = 0$. Using elementary linear Lorentzian geometry we deduce that $\nabla_H\varphi(q)$ is either null or spacelike.

Suppose $\nabla_H\varphi(q)$ is spacelike. Assume that following formula

$$H_q \cap T_qN = (Span\{\nabla_H\varphi(q)\})^g \tag{12}$$

true. Then $(H_q \cap T_qN)^g = \nabla_H\varphi(q)$, and since $\nabla_H\varphi(q)$ is spacelike, $T_qN \cap (H_q \cap T_qN)^g = \{0\}$ which is a contradiction with (b).

Thus, to end the proof, it is enough to check (12) under assumption that $\nabla_H\varphi(q)$ is spacelike. Let $X_0, ..., X_n$ be a frame for $TM_{|U}$ such that $X_0, ..., X_k$ is an orthonormal basis of $H_{|U}$ with a time orientation X_0, where U is a suitably small neighbourhood of q_0. Let $\tilde{g}$ be a Lorentzian metric on U defined by assuming the basis $X_0, ..., X_n$ to be orthonormal with respect to $\tilde{g}$ with a time orientation X_0. Let $\tilde{\nabla}\varphi$ denote the gradient of φ with respect to $\tilde{g}$. Evidently

$$g(v, \nabla_H\varphi(q)) = d_q\varphi(v) = \tilde{g}(v, \tilde{\nabla}\varphi(q))$$

for every $v \in H_q$. Next it is clear that since $\nabla_H\varphi(q)$ is spacelike, $\tilde{\nabla}\varphi(q)$ is spacelike too. It implies that $T_qN = \left(Span\{\tilde{\nabla}\varphi(q)\}\right)^{\tilde{g}}$, and to finish the proof of (12) we observe that $H_q \cap \left(Span\{\tilde{\nabla}\varphi(q)\}\right)^{\tilde{g}} = (Span\{\nabla_H\varphi(q)\})^g$.
□

Lemma 5.2. *Suppose that $\varphi : U \longrightarrow \mathbf{R}$ is such a smooth function defined on an open set $U \subset M$, that the horizontal gradient $\nabla_H\varphi$ is everywhere null past directed on U. Denote by $\gamma : [0, T] \longrightarrow U$ an arbitrary trajectory of $-\nabla_H\varphi$. Then $\gamma([0, T]) \subset \tilde{\partial}J^+(\gamma(0), U)$ (i.e. γ is geometrically optimal) and γ is a unique U-maximizer.*

Proof. Take any nspc.f.d. curve $\eta : [\alpha, \beta] \longrightarrow U$ such that $\eta(\alpha) = \gamma(0)$, $\eta(\beta) = \gamma(T)$. Then

$$0 = \varphi(\gamma(T)) - \varphi(\gamma(0)) = \varphi(\eta(\beta)) - \varphi(\eta(\alpha)) = \int_\alpha^\beta (\varphi(\eta(t))^{\dot{}} dt \tag{13}$$

$$= \int_\alpha^\beta g(\dot{\eta}(t), \nabla_H\varphi(\eta(t)))dt \geq 0. \tag{14}$$

(13) means that $\dot{\eta}(t)$ and $-\nabla_H\varphi(\eta(t))$ are parallel a.e. on $[\alpha, \beta]$. Thus η is a reparameterization of γ and the assertion is proven by Theorems 3.1 and 3.2. □

Theorem 5.1. *Suppose that $\gamma : [0,T] \longrightarrow M$, $T > 0$, is a null f.d. Hamiltonian geodesic starting from $\gamma(0) = q_0$, and let U be a neighbourhood of q_0. Then, provided T and U are sufficiently small, $\gamma([0,T]) \subset \tilde{\partial} J^+(q_0, U)$ and γ is a unique U-maximizer between its endpoints.*

Proof. The proof below is modelled, to some extent, on the proof in [16] of local optimality of sub-Riemannian Hamiltonian geodesics. During the whole proof we assume that U is a neighbourhood of q_0 which is as small as we need.

Let $\Gamma(t) = (\gamma(t), \lambda(t))$, $0 \leq t \leq T$, be a Hamiltonian lift of γ, i.e. $\dot{\Gamma}(t) = \overrightarrow{\mathcal{H}}(\Gamma(t))$. Assume that X_0 is a unit t.f.d. vector field on U. Let moreover $Y_1, ..., Y_{n-1}$ be an involutive family of vector fields (Y_j's are not supposed to be horizontal) such that $\dot{\gamma}(0), Y_1(q_0), ..., Y_{n-1}(q_0), X_0(q_0)$ form a basis of $T_{q_0}M$, and $Y_1(q_0), ..., Y_{n-1}(q_0) \in \ker \lambda(0)$. Now we will construct a 1-form $\bar{\lambda}$ on U with the following properties:
(1) $\bar{\lambda}(q_0) = \lambda(0)$;
(2) $\left\langle \bar{\lambda}(q), Y_j(q) \right\rangle = 0$ on U, $j = 1, ..., n-1$;
(3) $\left\langle \bar{\lambda}(q), X_0(q) \right\rangle \neq 0$ on U;
(4) $\mathcal{H}(q, \bar{\lambda}(q)) = 0$ on U.
This can be done, for instance, by use of the implicit function theorem. Let $c = \langle \lambda(0), X_0(q_0) \rangle$; obviously $c \neq 0$. Introduce some local Darboux coordinates $(q, \omega) = (q^0, ..., q^n, \omega_0, ..., \omega_n)$ on $T^*M_{|U}$ and write

$$F(q,\omega) = (\mathcal{H}(q,\omega), \langle \omega, Y_1(q) \rangle, ..., \langle \omega, Y_{n-1}(q) \rangle, \langle \omega, X_0(q) \rangle - c).$$

Direct computation shows that $F(q_0, \lambda(0)) = 0$ and

$$\det \frac{\partial(F^0, ..., F^n)}{\partial(\omega_0, ..., \omega_n)}(q_0, \lambda(0)) = \det[\dot{\gamma}(0), Y_1(q_0), ..., Y_{n-1}(q_0), X_0(q_0)] \neq 0.$$

Thus, if $\omega_\alpha = \omega_\alpha(q)$, $\alpha = 0, ..., n$, is a solution of $F(q, \omega) = 0$ satisfying $\omega_\alpha(q_0) = \lambda_\alpha(0)$ for every α, then $\bar{\lambda} = \sum_{\alpha=0}^{n} \omega_\alpha dq^\alpha$ has properties (1), (2), (3), (4).

Next, let $g^t : U \longrightarrow M$, $|t| < \varepsilon$, $\varepsilon > 0$, be the flow of X_0. By L_t let us denote the integral manifold of the family $\{Y_1, ..., Y_{n-1}\}$ passing through $g^t q_0$, and set $L = \bigcup_{|t|<\varepsilon} L_t$; L is a smooth hypersurface (X_0 is transverse to each L_t) and $q_0 \in L$. For any $q \in L$ let $\Gamma_q = (\gamma_q, \lambda_q) : (-\varepsilon, \varepsilon) \longrightarrow T^*M$ be a curve defined by $\Gamma_q(0) = (q, \bar{\lambda}(q))$, $\dot{\Gamma}_q(t) = \vec{\mathcal{H}}(\Gamma_q(t))$. In particular $\Gamma = \Gamma_{q_0}$. Introduce further a smooth null f.d. vector field X defined by condition

$$X(\gamma_q(t)) = d_{\Gamma_q(t)}\pi\vec{\mathcal{H}}(\Gamma_q(t)) = \dot{\gamma}_q(t).$$

Similarly as in [16] one shows that $\vec{\mathcal{H}}_X(\Gamma_q(t)) = \vec{\mathcal{H}}(\Gamma_q(t))$ for every $q \in L$ and t, $|t| < \varepsilon$, where by $\mathcal{H}_X$ we mean the function $\mathcal{H}_X : T^*M \longrightarrow \mathbf{R}$, $\mathcal{H}_X(q,p) = \langle p, X(q)\rangle$. Denote by h^s the flow of X, again defined for $|s| < \varepsilon$. It is well-known that the flow of $\vec{\mathcal{H}}_X$ has the form

$$(q, \lambda) \longrightarrow (h^s q, \left((d_q h^s)^{-1}\right)^T \lambda). \tag{15}$$

As the next step we define a function $\varphi : U \longrightarrow \mathbf{R}$ by formula $\varphi(h^s q) = t$ whenever $q \in L_t$. Clearly, φ is smooth and $(\partial_{X_0}\varphi)(q_0) \neq 0$ on U which means that $\nabla_H\varphi(q_0) \neq 0$ on U.

To finish the proof let $N_t = \{\varphi = t\}$. Let $q = h^s\bar{q} \in N_t$, $\bar{q} \in L_t$; we have

$$T_q N_t = d_{\bar{q}}h^s(T_{\bar{q}}L_t) \oplus Span\{X(q)\}.$$

Let $w \in d_{\bar{q}}h^s(T_{\bar{q}}L_t) \cap H_q$. Then $w = d_{\bar{q}}h^s(v)$, $v \in T_{\bar{q}}L_t$. Now

$$g(X(q), w) = g(\dot{\gamma}_{\bar{q}}(s), w) = \langle \lambda_{\bar{q}}(s), d_{\bar{q}}h^s(v)\rangle = \left\langle \bar{\lambda}(\bar{q}), v\right\rangle = 0$$

by (15) and property (2) above. We have just proved that

$$X(q) \in T_q N_t \cap (T_q N_t \cap H_q)^g$$

which by Lemma 5.1 shows that $\nabla_H\varphi(q)$ is a null vector. However $q \in U$ was arbitrary, therefore $\nabla_H\varphi$ is a null field on U. Since $(\partial_{X_0}\varphi)(q_0) = 1$, so $\partial_{X_0}\varphi > 0$ on U, and $\nabla_H\varphi$ is past directed. Finally, since $g(\nabla_H\varphi, X) = 0$ on U, the fields $\nabla_H\varphi$ and X must be colinear. We conclude that, after a change of parameterization, γ is a trajectory of $-\nabla_H\varphi$, and the result follows from Lemma 5.2. □

In [6], Proposition 4.1, we proved that t.f.d. Hamiltonian geodesics are locally maximizing. As a corollary of Theorem 5.1 we state a stronger version of this proposition.

Proposition 5.1. *Let $\gamma : [a,b] \longrightarrow M$ be a nspc.f.d. Hamiltonian geodesic. Then for every $t \in (a,b)$ there exists an open set $U \ni \gamma(t)$ such that $\gamma \cap U$ is a unique U-maximizer.*

6 The boundary $\tilde{\partial}J^+(q_0, U)$

Consider a time-oriented sub-Lorentzian manifold (M, H, g). In this section we attempt to describe nspc.f.d. curves that start from q_0 and are contained (at least to some moment of time) in the boundary $\tilde{\partial}I^+(q_0, U) = \tilde{\partial}J^+(q_0, U)$, q_0 being a point in M and U its sufficiently small normal neighbourhood. For instance in the Lorentzian geometry, i.e. for $H = TM$, it is known that $\tilde{\partial}J^+(q_0, U)$ is formed by null f.d. geodesics emanating from q_0, and these are the only nspc.f.d. curves starting from q_0 and contained in $\tilde{\partial}J^+(q_0, U)$; in particular $\tilde{\partial}J^+(q_0, U)$ contains no timelike curves, and the local Lorentzian distance from q_0 vanishes on $\tilde{\partial}J^+(q_0, U)$.

6.1 Preliminary remarks

Everywhere in this section q_0 is a fixed point in M and U denotes its normal neighbourhood. Let $X_0, X_1, ..., X_k$ be an orthonormal frame for (H, g) defined on U with a time orientation X_0.

Recall that all nspc.f.d. curves in U starting from q_0 can be recovered via the control system (5) with the set of control parameters equal to C. Let us observe, however, that instead of (5) it is sometimes more convenient to work with the control affine system

$$\dot{q}(t) = X_0(q(t)) + \sum_{j=1}^{k} u_j(t) X_j(q(t)) \tag{16}$$

with square integrable controls $u : [0, T(u)] \longrightarrow B_k(0, 1)$, where final time $T(u) > 0$ depends on a control and $B_k(0, 1) = \{(u_1, ..., u_k) \in \mathbf{R}^k : \sum_{i=1}^{k} u_i^2 \leq$ $\leq 1\}$ is the unit closed ball centered at zero.

Indeed, every trajectory of the system (16) is at the same time a trajectory of (5). Take a trajectory $\gamma : [0, T] \longrightarrow U$ of (5). Then $\dot{q}(t) = \sum_{\alpha=0}^{k} u_\alpha(t) X_\alpha(q(t))$, where

$$\sum_{j=1}^{k} u_j^2(t) \leq u_0^2(t), \quad u_0(t) > 0 \tag{17}$$

a.e. on $[0, T]$, and $u_0, ..., u_k$ are square integrable. Let us define a real number T_1 and a function $\sigma : [0, T] \longrightarrow [0, T_1]$ by $T_1 = \int_0^T u_0(s) ds$ and

$\sigma(t) = \int_0^t u_0(s)ds$. Since $\dot\sigma(t) > 0$ a.e. and σ is absolutely continuous, σ is increasing. Let $\tau : [0, T_1] \longrightarrow [0, T]$ be the inverse function. Then $\dot\tau(t) = \frac{1}{u_0(\tau(t))}$ a.e. Now let $q_1 : [0, T_1] \longrightarrow U$, $q_1(t) = q(\tau(t))$. Clearly

$$\dot q_1(t) = X_0(q_1(t)) + \sum_{j=1}^{k} \frac{u_j(\tau(t))}{u_0(\tau(t))} X_j(q_1(t))$$

which by (17) implies that $q_1(t)$ is a trajectory of (16). Also, by remark at the end of Section 4, both systems (5) and (16) have the same (up to a change of parameter) geometrically optimal extremals. In this way the dimension of the space of control parameters drops by one, while a drift term appears.

Now we will prove two lemmas which we will use below. To avoid possible misunderstandings let us emphasize that by a smooth curve we mean a 1-dimensional embedded submanifold. Such a notion of smoothness of a curve is invariant with respect to changes of parameter.

Lemma 6.1. *Let $\gamma : [0, T] \longrightarrow U$, $\gamma(0) = q_0$, be a nspc.f.d. geometrically optimal curve, i.e. $\gamma([0, T]) \subset \tilde\partial J^+(q_0, U)$. Suppose that there exists a biextremal $\lambda(t) = (\gamma(t), p(t))$ such that*

$$(\langle p(t), X_0(\gamma(t))\rangle, ..., \langle p(t), X_k(\gamma(t))\rangle) \neq (0, ..., 0), \quad t \in [0, T]. \tag{18}$$

Then γ is null f.d. and smooth.

Proof. Without loss of generality we may assume that γ is parameterized as in (16), so let

$$\dot\gamma(t) = X_0(\gamma(t)) + \sum_{i=1}^{k} u_i(t) X_i(\gamma(t)).$$

Suppose that $k = 1$. By PMP (Theorem 4.1)

$$\langle p(t), X_0(\gamma(t))\rangle + u(t) \langle p(t), X_1(\gamma(t))\rangle = \langle p(t), X_0(\gamma(t))\rangle +$$
$$+ \max_{|u|\le 1} u \langle p(t), X_1(\gamma(t))\rangle = 0 \tag{19}$$

(the first equality in (19) holding a.e.). Therefore, by (18) and (19) $\langle p(t), X_1(\gamma(t))\rangle$ vanishes nowhere. Thus $u(t) = sgn\,\langle p(t), X_1(\gamma(t))\rangle$ which ends the proof for $k = 1$.

Suppose now that $k \geq 2$, $u(\cdot) \in L^2([0,T], B_k(0,1))$. The PMP Hamiltonian is

$$h_u(p,q) = \langle p, X_0(q)\rangle + \sum_{i=1}^{k} u_i \langle p, X_i(q)\rangle ,$$

and the maximum condition of PMP may be rewritten as

$$\begin{cases} \sum_{i=1}^{k} u_i(t) \langle p(t), X_i(\gamma(t))\rangle = \max_{|u|\leq 1} \sum_{i=1}^{k} u_i \langle p(t), X_i(\gamma(t))\rangle \ \text{ a.e.}, \\ \langle p(t), X_0(\gamma(t))\rangle + \sum_{i=1}^{k} u_i(t) \langle p(t), X_i(\gamma(t))\rangle = 0 \quad \text{on } [0,T]. \end{cases} \tag{20}$$

By assumption (18) the maximum in the first equation in (20) is, for almost every t, attained at $u(t) \in \partial B_k(0,1)$. Using (18) and (20) it follows that there exists a neighbourhood Ω of $\lambda([0,T])$ in T^*M such that for $(q,p) \in \Omega$ the function $B_k(0,1) \ni (u_1, ..., u_k) \longrightarrow \langle p, X_0(q)\rangle + \sum_{i=1}^{k} u_i \langle p, X_i(q)\rangle$ attains its maximum at a point $u = u(q,p) \in \partial B_k(0,1)$. Applying Lagrange's multipliers rule we find a function $a = a(q,p)$ such that

$$\langle p, X_i(q)\rangle = a(q,p)u_i(q,p),\ i = 1, ..., k. \tag{21}$$

(21) gives

$$a^2(q,p) = \sum_{i=1}^{k} \langle p, X_i(q)\rangle^2 ,$$

hence $a(q,p) \neq 0$ on Ω. Now, the maximized Hamiltonian (cf. [1])

$$h(q,p) = h_{u(q,p)}(q,p) = \langle p, X_0(q)\rangle + \sqrt{\sum_{i=1}^{k} \langle p, X_i(q)\rangle^2}$$

is smooth on Ω. Evidently $\dot{\lambda}(t) = \overrightarrow{h}(\lambda(t))$, and the proof is over. □

Having proved Lemma 6.1, we can draw one more conclusion from (18).

Lemma 6.2. *Let $\gamma : [0,T] \longrightarrow U$, $\gamma(0) = q_0$, be a nspc.f.d. geometrically optimal curve admitting a biextremal lift $\lambda(t) = (\gamma(t), p(t))$ satisfying the condition (18). Then, up to a change of parameterization, γ is a null f.d. Hamiltonian geodesic.*

Proof. The PMP Hamiltonian applied to the system (5) is equal to

$$h_u(q,p) = \sum_{\alpha=0}^{k} u_\alpha \langle p, X_\alpha(q)\rangle\,, \tag{22}$$

and the maximum condition reads

$$\sum_{\alpha=0}^{k} u_\alpha(t) \langle p(t), X_\alpha(\gamma(t))\rangle = \max_{u\in C} \sum_{\alpha=0}^{k} u_\alpha \langle p(t), X_\alpha(\gamma(t))\rangle = 0, \tag{23}$$

where by Lemma 6.1 we can suppose that the control $u(t)$ generating γ is smooth. By assumption (18) the maximum in (23) is attained on $\partial C \cap \{u_0 > 0\}$. Using Lagrange's multipliers rule there exists a function $a(t)$ such that

$$-a(t)u_0(t) = \langle p(t), X_0(\gamma(t))\rangle\,, \quad a(t)u_j(t) = \langle p(t), X_j(\gamma(t))\rangle\,, \; j = 1, ..., k.$$

$a(t)$ is smooth and does not vanish on $[0,T]$. Let $A(q,p)$ be a smooth non-vanishing function defined on a neighbourhood of $\lambda([0,T])$, such that $A(\lambda(t)) = a(t)$. Let

$$\tilde{u}_0(q,p) = -\frac{\langle p, X_0(q)\rangle}{A(q,p)}, \quad \tilde{u}_j(q,p) = \frac{\langle p, X_j(q)\rangle}{A(q,p)}, \; j = 1, ..., k.$$

We find that in a neighbourhood of $\lambda([0,T])$

$$h(q,p) = h_{\tilde{u}(q,p)}(q,p) = \frac{2}{A(q,p)}\mathcal{H}(q,p),$$

where $\mathcal{H}$ is the geodesic Hamiltonian defined in Section 2.6. Because of (23) $\mathcal{H}(\gamma(t), p(t)) = 0$ for all t. Evidently $\dot{(\gamma(t), p(t))} = \overrightarrow{h}(\gamma(t), p(t))$. Rewriting this in some local Darboux coordinates $(q^0, ..., q^n, p_0, ..., p_n)$ we have

$$\begin{cases} \dot{q}^j = \frac{\partial}{\partial p_j}(\frac{2}{A})\mathcal{H} + \frac{2}{A}\frac{\partial \mathcal{H}}{\partial p_j} = \frac{2}{A}\frac{\partial \mathcal{H}}{\partial p_j} \\ \dot{p}_j = -\frac{\partial}{\partial q_j}(\frac{2}{A})\mathcal{H} - \frac{2}{A}\frac{\partial \mathcal{H}}{\partial q_j} = -\frac{2}{A}\frac{\partial \mathcal{H}}{\partial q_j} \end{cases},$$

and the proof is finished. □

6.2 Generic rank ≥ 3 case

We start with the following lemma.

Lemma 6.3. *Let $\gamma : [0,T] \longrightarrow M$ be a nspc.f.d. curve such that $\gamma(0) = q_0$ and $\gamma([0,T]) \subset \tilde{\partial} J^+(q_0, U)$. Suppose in addition that $\gamma_{|[t_1,t_2]}$, $[t_1, t_2] \subset [0,T]$, is a timelike curve. Then $\gamma_{|[t_1,t_2]}$ is a Goh curve.*

Proof. Again the PMP Hamiltonian is of the form

$$h_u(q,p) = \sum_{\alpha=0}^{k} u_\alpha \langle p, X_\alpha(q) \rangle, \quad q \in U,\, p \in T_q^* M,\, u \in C.$$

Let

$$\dot{\gamma}(t) = \sum_{\alpha=0}^{k} u_\alpha(t) X_\alpha(\gamma(t)),$$

and let $\lambda = (\gamma, p) : [0,T] \longrightarrow T^*M$ be a biextremal covering γ. Clearly $(\lambda_{|[t_1,t_2]}, u_{|[t_1,t_2]})$ enters the maximum condition of PMP

$$\sum_{\alpha=0}^{k} u(t)_\alpha \langle p(t), X_\alpha(\gamma(t)) \rangle = \max_{v \in C} \sum_{\alpha=0}^{k} v_\alpha \langle p(t), X_\alpha(\gamma(t)) \rangle$$

a.e. on $[t_1, t_2]$. Since $u(t) \in C_0$ for almost every $t \in [t_1, t_2]$, this maximum condition reads

$$\langle p(t), X_\alpha(\gamma(t)) \rangle = 0, \quad t \in [t_1, t_2],\, \alpha = 0, ..., k, \tag{24}$$

which is equivalent to saying that $\gamma_{|[t_1,t_2]}$ is abnormal (cf. remark at the end of Section 2.1).

Remark moreover that γ is geometrically optimal also relative to the set C_0 of control parameters. Since C_0 is open and $\lambda_{|[t_1,t_2]}$ is totally singular, $\lambda_{|[t_1,t_2]}$ enters the Goh condition (cf. [1])

$$\langle p(t), [X_\alpha, X_\beta] (\gamma(t)) \rangle = 0, \quad \alpha, \beta = 0, ..., k,\, t \in [t_1, t_2].$$

□

Now we suppose that (M, H, g) is a sub-Lorentzian manifold, where rank $H \geq 3$. It turns out that if H is generic then nontrivial Goh curves do not exist - see [5]. Using this and Lemma 6.3 one obtains the following proposition.

Proposition 6.1. *Let H be a generic (in sense of [5]) distribution of rank ≥ 3, and let g be a Lorentzian metric on H. Then for every $q_0 \in M$ and every normal neighbourhood U of q_0 the set $I^+(q_0, U)$ is open.*

Proof. Suppose $q \in \tilde{\partial} J^+(q_0, U) \cap I^+(q_0, U)$. Then there exists a t.f.d. curve γ joining q_0 to q. By Section 4 γ is contained in $\tilde{\partial} J^+(q_0, U)$, so by Lemma 6.3 γ is a Goh curve. In this way we obtain a contradiction. □

6.3 Further assumptions

Here we are interested in conditions guaranteeing that $f[U]_{|\tilde{\partial} J^+(q_0,U)} = 0$, $f[U]$ being the sub-Lorentzian distance from q_0. Unfortunately the absence of Goh curves, as in the previous subsection, does not exclude possibility of the existence of points $q \in \tilde{\partial} J^+(q_0, U)$ such that $f[U](q) > 0$. This is because there exist nspc.f.d. curves that have no timelike pieces but have positive length. It is easy to construct such curves. Take an interval $[0, T]$, $T > 0$. Let $A \subset [0, T]$ be an arbitrary nowhere dense subset of positive measure. Now let $u_0(t) = 1$ for $t \in [0, T]$, and

$$u_1(t) = ... = u_k(t) = \begin{cases} \frac{1}{\sqrt{k}} : \ t \in [0, T] \backslash A \\ 0 : \ t \in A \end{cases} .$$

Clearly, if $\dot{\gamma}(t) = \sum_{\alpha=0}^{k} u_\alpha(t) X_\alpha(\gamma(t))$, $\gamma(0) = q_0$, then the restriction $\gamma_{|[t_1,t_2]}$ is not a timelike curve for any subinterval $[t_1, t_2] \subset [0, T]$, nevertheless $L(\gamma) > 0$. Our aim is to find a condition that excludes such curves from among the extremals.

So in this subsection we make the following assumption (cf. [4]):

(i) if rank H is even, we suppose that for any $q \in U$ and any non-zero covector $p \in T_q^* M$, the matrix $(\langle p, [X_\alpha, X_\beta](q) \rangle)_{\alpha,\beta=0,\dots,k}$ is invertible;

(ii) if rank H is odd, we suppose H is a 2-generating distribution.

Theorem 6.1. *Let $q_0 \in M$ and let U be a normal neighbourhood of q_0. Under the above assumptions made on H, the set $\tilde{\partial} J^+(q_0, U)$ is made up of null f.d. curves starting from q_0. Consequently, $I^+(q_0, U)$ is open. Moreover, $f[U]_{|\tilde{\partial} J^+(q_0,U)} = 0$ and $f[U]$ is continuous at points of $\tilde{\partial} J^+(q_0, U)$.*

Proof. Suppose that $\gamma : [0, T] \longrightarrow U$, $\gamma(0) = q_0$, is an arbitrary geometrically optimal nspc.f.d. curve, $u : [0, T] \longrightarrow C$ is a control generating γ, and $\lambda = (\gamma, p) : [0, T] \longrightarrow T^*U$ is a biextremal covering γ. Let

$$A = \{t \in [0, T] : \ g(\dot\gamma(t), \dot\gamma(t)) < 0\}.$$

We will show that A is of measure zero which will mean that γ is a null curve. For almost every $t \in A$, the maximum condition of PMP gives

$$\langle p(t), X_0(\gamma(t))\rangle = ... = \langle p(t), X_k(\gamma(t))\rangle = 0.$$

By absolute continuity of the mapping $t \longrightarrow \langle p(t), X_\alpha(\gamma(t))\rangle$, the set

$$\left\{t \in [0, T] : \ \langle p(t), X_\alpha(\gamma(t))\rangle = 0, \ \frac{d}{dt}\langle p(t), X_\alpha(\gamma(t))\rangle \neq 0\right\}$$

has measure zero for every $\alpha = 0, ..., k$, so it suffices to show that

$$A_0 =$$

$$\left\{t \in [0, T] : \langle p(t), X_\alpha(\gamma(t))\rangle = 0, \frac{d}{dt}\langle p(t), X_\alpha(\gamma(t))\rangle = 0, \alpha = 0, .., k\right\}$$

has measure zero. In view of our assumptions made on H, this last statement becomes clear if we recall that

$$\frac{d}{dt}\langle p(t), X_\alpha(\gamma(t))\rangle = \sum_{\beta=0}^{k} u_\beta(t)\langle p(t), [X_\alpha, X_\beta](\gamma(t))\rangle, \alpha = 0, ..., k.$$

It remains to show that $f[U]$ is continuous at points of $\tilde\partial J^+(q_0, U)$. To this end take a $q \in \tilde\partial J^+(q_0, U)$. As we know there exists a U-maximizer γ joining q_0 to q. By Section 4 such a γ is contained in $\tilde\partial J^+(q_0, U)$. Consequently, by the first part of the proof, γ is null f.d. and $f[U](q) = 0$. Take any sequence $\{q_\nu\} \subset J^+(q_0, U)$ such that $q_\nu \longrightarrow q$, and let γ_ν be a U-maximizer connecting q_0 to q_ν. After passing to a subsequence, $\{\gamma_\nu\}$ converges in the C^0 topology on curves to a nspc.f.d. curve $\tilde\gamma$ joining q_0 to q. Again, $\tilde\gamma$ must be contained in $\tilde\partial J^+(q_0, U)$, thus $\tilde\gamma$ is null. By upper semicontinuity of sub-Lorentzian arc length

$$0 \leq \limsup f[U](q_\nu) = \limsup L(\gamma_\nu) \leq L(\tilde\gamma) = 0$$

as $\nu \longrightarrow \infty$, which finally gives $\lim_{\nu\longrightarrow\infty} f[U](q_\nu) = f[U](q)$. □

Unfortunately, we are not able to state any general theorem concerning regularity properties of geometrically optimal curves described by Theorem 6.1. Remark also that the example in 3.2.3 shows that $f[U]$ needs not be continuous at points of $\tilde\partial J^+(q_0, U)$ in the general case.

6.4 Contact case

Let M be a $(2n+1)$-dimensional manifold and suppose that (H, g) is a contact sub-Lorentzian structure on M, i.e. H is a contact distribution. Again U will stand for a normal neighbourhood of a point q_0. We will prove a theorem which generalizes in some sense results obtained in [17] and [15].

Theorem 6.2. *Let $\gamma : [0, T] \longrightarrow U$, $\gamma(0) = q_0$, be a geometrically optimal curve. Then, if U is a normal neighbourhood of q_0, γ is a null f.d. curve with a finite number of non-smooth points. Smooth pieces of γ are null Hamiltonian geodesics.*

Proof. Let $\lambda(t) = (\gamma(t), p(t))$ be a corresponding biextremal, and let $u(t)$ be a geometrically optimal control generating γ. After modification of u on the set of measure zero we can assume that $u(t)$ is defined and non-zero everywhere. Define a set Z by

$$\{t \in [0, T] : (\langle p(t), X_0(\gamma(t))\rangle, ..., \langle p(t), X_{2n-1}(\gamma(t))\rangle) = 0, \gamma(t) \in U\},$$

where $X_0, ..., X_{2n-1}$ is an orthonormal basis of H defined on U. By Lemmas 6.1, 6.2 it is enough to prove that Z is finite.

Denote by X_t the non-autonomous vector field

$$X_t(q) = \sum_{\alpha=0}^{k} u_\alpha(t) X_\alpha(q).$$

As is known (see for instance [16]) $X_t(q)$, $q \in U$, is a so-called *strong bracket generator*. In our particular situation it means that

$$T_qM = Span\{X_0(q), ..., X_{2n-1}(q), [X_t, X_0](q), ..., [X_t, X_{2n-1}](q)\}$$

for every $q \in U$. Suppose now that s_1, $s_1 + s_2 \in Z$, $s_2 > 0$. Let g^t be the flow of X_t computed from time s_1, that is to say $g^s(q) = \eta(s)$, where $\dot{\eta}(t) = X_t(\eta(t))$, $\eta(s_1) = q$. From the proof of PMP (cf. [1]) it is known that $p(s_1 + s_2) = \left(\left(d_{\gamma(s_1)} g^{s_1+s_2}\right)^{-1}\right)^T p(s_1)$. Finally we have

$$0 = \langle p(s_1 + s_2), X_m(\gamma(s_1 + s_2))\rangle =$$
$$\left\langle p(s_1), \left(d_{\gamma(s_1)} g^{s_1+s_2}\right)^{-1} X_m(\gamma(t+s))\right\rangle =$$
$$s_2 \langle p(s_1), [X_{s_1}, X_m](\gamma(s_1))\rangle + O(s_2^2)$$

for every $m = 0, ..., k$, which is impossible for arbitrarily small s_2. □

6.5 The case rank$H = 2$

In this case (16) becomes a control system

$$\dot{q}(t) = X_0(q(t)) + u(t)X_1(q(t)),\ q(0) = q_0,$$

with a scalar input u, $|u| \leq 1$. Evidently, there are exactly two (up to a change of parameter) smooth null f.d. curves γ_+, γ_- starting from q_0; first corresponding to $u(t) \equiv 1$, and the second corresponding to $u(t) \equiv -1$. At the same time there are exactly two null f.d. Hamiltonian geodesics initiating in q_0, so they coincide with γ_+, γ_-. In particular, in rank 2 case, every null f.d. smooth curve is geometrically optimal.

Using above considerations and [21] one can obtain many partial results concerning the boundary of reachable sets for sub-Lorentzian metrics on rank 2 distributions. Here are two examples. The first one is a strengthened version of Theorem 6.2.

Proposition 6.2. *Let H be a generic germ at the origin of a rank 2 distribution on $\mathbf{R}^3$, and let g be a Lorentzian metric on H. Then, for every q_0 sufficiently close to the origin, and for every sufficiently small normal neighbourhood U of q_0, the set $\tilde{\partial} J^+(q_0, U)$ is made up of null f.d. curves starting from q_0. If γ_+ and γ_- stand for the two null f.d. Hamiltonian geodesics starting from q_0, then for every $q \in \tilde{\partial} J^+(q_0, U) \backslash \{\gamma_+, \gamma_-\}$ a unique nspc.f.d. curve joining q_0 to q is a null curve with exactly one non-smooth point. Moreover $N^+(q_0, U) = J^+(q_0, U)$.*

Proof. In fact, only uniqueness of geometrically optimal curves and a number of non-smooth points need to be clarified. We will use results from [17] and [15], where generic control affine systems on $\mathbf{R}^3$ were studied (we can also use Theorem 6.2).

Take a point q_0 and its normal neighbourhood U. We assume that U is so small that the theorem on normal forms from [7] can be applied to it. So suppose that there are coordinates x, y, z on U such that $x(q_0) = y(q_0) = z(q_0) = 0$ and (H, g) possesses an orthonormal frame on U in the normal form

$$\begin{aligned} X &= (1 + y^2\varphi)\tfrac{\partial}{\partial x} + xy\varphi\tfrac{\partial}{\partial y} + \tfrac{1}{2}y(1 + \psi)\tfrac{\partial}{\partial z} \\ Y &= -xy\varphi\tfrac{\partial}{\partial x} + (1 - x^2\varphi)\tfrac{\partial}{\partial x} - \tfrac{1}{2}x(1 + \psi)\tfrac{\partial}{\partial z} \end{aligned} \tag{25}$$

with a time orientation X; here φ, ψ are smooth functions on U satifying $\varphi(0, 0, z) = \psi(0, 0, z) = \frac{\partial \psi}{\partial x}(0, 0, z) = \frac{\partial \psi}{\partial y}(0, 0, z) = 0$. Now, for U sufficiently small, the horizontal gradient of the function $(x, y, z) \longrightarrow x$ is timelike past directed. It follows that for every nspc.f.d. curve $\gamma : [a, b] \longrightarrow U$, the function $t \longrightarrow x(\gamma(t))$ is increasing on $[a, b]$.

Consider now the control affine system on U

$$\dot{q} = X + uY, \quad |u| \leq 1. \tag{26}$$

Denote by $\mathcal{A}(q_0, t, U)$ its accesible set from q_0 at time t in U, and let $\mathcal{A}(q_0, \leq T, U) = \bigcup_{0 \leq t \leq T} \mathcal{A}(q_0, t, U)$. If $\gamma : [0, T] \longrightarrow U$ is an arbitrary trajectory of (26) starting from $\gamma(0) = q_0$, then x-coordinate of γ satisfies

$$x(t) = t + \int_0^t (y - ux) y \beta ds, \tag{27}$$

$0 \leq t \leq T$. Now $|x| \leq d$, $|\beta| \leq \varepsilon$, $|y| \leq |x|$ in $J^+(q_0, U)$, where $d = d(U)$ and $\varepsilon = \varepsilon(U)$ are positive constants that can be taken as small as we wish by shrinking U. Take U so as to have $2d^2\varepsilon \leq \frac{1}{2}$. For such a U, $J^+(q_0, U) \subset \mathcal{A}(q_0, \leq T, U)$, where by (27) $0 < T \leq \frac{d}{1-2d^2\varepsilon}$. Again shrinking U we obtain $T > 0$ small enough to apply [15] or [17]. □

Proposition 6.3. *Let H be a germ at the origin of a rank 2 distribution on $\mathbf{R}^3$. Suppose that H is generic in the class of all distributions admitting abnormal curves, and let g be a Lorentzian metric on H. Then there exists a germ (at the origin) of a hypersurface S such that for every q_0 sufficiently close to the origin, and for every sufficiently small normal neighbourhood U of q_0 :*
(i) if $q_0 \notin S$ then $\tilde{\partial} J^+(q_0, U)$ is described by Proposition 6.2;
(ii) if $q_0 \in S$ then $\tilde{\partial} J^+(q_0, U)$ may additionally contain curves of positive length.

As suggested in [15] we apply [20] to prove one more result, this time in $\mathbf{R}^n$, $n \geq 3$. Introduce the following notation: $(ad^0)Y = Y$, $(ad^{k+1}X)Y = [X, (ad^k X)Y]$, $k = 1, 2, ...$

Proposition 6.4. *Let H be an analytic rank 2 distribution on $\mathbf{R}^n$ defined in a neighbourhood U of 0. Suppose that g is such a Lorentzian metric on H that there exists an analytic orthonormal frame X, Y on U, X is a time orientation, with the following property:*

for every positive integer m there exist analytic functions $\alpha_0^{(m)},\cdots,\alpha_m^{(m)}$, $\beta^{(m)}$ *defined on* U, *such that* $|\beta^{(m)}|<1$ *and*

$$[Y,(ad^mX)Y]=\sum_{i=0}^{m}\alpha_i^{(m)}(ad^iX)Y+\beta^{(m)}(ad^{m+1}X)Y. \tag{28}$$

Then, possibly shrinking U, *for every* $q\in\tilde{\partial}J^+(q_0,U)$ *a nspc.f.d. curve joining* 0 *to* q *is null and has a finite number of non-smooth points.*

Proof. Let us notice that under the condition (28) the Lie algebra generated by X and Y is equal to $L=Span\{X,(ad^iX)Y,\, i=0,1,...\}$. Indeed, to see this it is enough to prove that for every positive integer k the following condition is fulfilled

$$[(ad^kX)Y,(ad^mX)Y]\in L,\; m=0,1,... \tag{29}$$

A proof is by induction. For $k=0$ it is just the condition (28). Suppose (29) true for positive integers $\leq k$. Then

$$[(ad^{k+1}X)Y,(ad^mX)Y]=$$

$$=(adX)[(ad^kX)Y,(ad^mX)Y]-[(ad^kX)Y,(ad^{m+1}X)Y]$$

and the inductive hypothesis gives $[(ad^{k+1}X)Y,(ad^mX)Y]\in L$.

Now suppose that $\gamma:[0,T]\longrightarrow U$ is geometrically optimal. We reparameterize γ as in (16), i.e. $\dot{\gamma}(t)=X(\gamma(t))+u(t)Y(\gamma(t))$, where $u(\cdot)$ is a corresponding geometrically optimal control. Let $\lambda(t)=(\gamma(t),p(t))$ satisfy PMP.

Suppose that $|u(t)|<1$ for $t\in\Delta$, Δ being an interval contained in $[0,T]$. By maximum condition of PMP - the PMP Hamiltonian is $h_u(q,p)=\langle p,X(q)\rangle+u\,\langle p,Y(q)\rangle$ - we have

$$\langle p(t),X(\gamma(t))\rangle=\langle p(t),Y(\gamma(t))\rangle=0,\quad t\in\Delta. \tag{30}$$

Differentiation of (30) with respect to t gives $\langle p(t),(adX)Y(\gamma(t))\rangle=0$ on Δ. Now let us assume that

$$\left\langle p(t),(ad^iX)Y(\gamma(t))\right\rangle=0,\; i=1,...,k,\; t\in\Delta. \tag{31}$$

Differentiating (31) with respect to t for $i = k$ gives

$$\left\langle p(t), (ad^{k+1}X)Y(\gamma(t))\right\rangle + u(t)\left\langle p(t), \left[Y, (ad^k X)Y\right](\gamma(t))\right\rangle = 0$$

which, using (28), (31), reduces to

$$\left(1 + u(t)\beta^{(k)}(t)\right)\left\langle p(t), (ad^{k+1}X)Y(\gamma(t))\right\rangle = 0.$$

Recalling that $|\beta^{(k)}| < 1$ we see that $\left\langle p(t), (ad^k X)Y(\gamma(t))\right\rangle = 0$ on Δ for every $k = 1, 2, \ldots$. But then $p(t) = 0$ by the first part of the proof which contradicts PMP. Thus the fuction $t \longrightarrow \langle p(t), Y(\gamma(t))\rangle$ cannot vanish on any interval, so by [20], Lemma 3, it has only a finite number of zeros. In this way γ is a null f.d. curve with a finite number of switching times. □

6.6 One remark about the image under exponential mapping

Let us mention here that for purely dimensional reasons formulas of type (2) do not hold in the sub-Lorentzian geometry. More precisely

$$\tilde{\partial} J^+(q_0, U) \neq \exp_{q_0}\left(\left\{\lambda \in T^*_{q_0}M : \quad \mathcal{H}(q_0, \lambda) = 0, \quad \langle \lambda, X(q_0)\rangle < 0\right\}\right) \cap U$$

($\langle \lambda, X(q_0)\rangle = g(G\lambda, X(q_0))$, so this expression must be negative (cf. Section 2.6)). At the same time formulas of type (1) do hold, at least in some cases - cf. [9].

7 Appendix. Reachable sets in the Martinet flat case

Let X, Y, H, ω and g be defined as in Section 3.2.3. The structure (H, g) will be referred to as *the sub-Lorentzian Martinet flat structure*. To simplify the notation we will write $I^+(0)$ for $I^+(0, \mathbf{R}^3)$ and $J^+(0)$ for $J^+(0, \mathbf{R}^3)$. We are going to prove the following

Proposition 7.1. *If $I^+(0)$, $J^+(0)$ are reachable sets determined by the sub-Lorentzian Martinet flat structure (H, g), then*

$$I^+(0) = \left\{\frac{1}{4}(-xy^2 + |y|^3) < z < \frac{1}{16}(x^2 - y^2)(x + 3|y|),\ x > 0\right\}$$

$$\cup\{(x,0,0)\colon x>0\}$$

and

$$J^+(0)=\left\{\frac{1}{4}(-xy^2+|y|^3)\le z\le\frac{1}{16}(x^2-y^2)(x+3|y|),\ x\ge 0\right\}. \qquad (32)$$

Moreover, if U is a normal neighbourhood of the zero, then

$$I^+(0,U)=I^+(0)\cap U,\quad J^+(0,U)=J^+(0)\cap U. \qquad (33)$$

Proof. Let us start from the observation that

$$J^+(0)\cap\{y=0\}\cap\{z<0\}=\emptyset, \qquad (34)$$

which follows from Section 3.2.3. Next, for $\Gamma=\{|y|<x,x>0\}$, we have

$$I^+(0)\cap\{z=0\}=\{|y|<x,\,x>0\}\cap\{z=0\}\,. \qquad (35)$$

To see (35) it is enough to notice that for every $a\in(-1,1)$ the curve $t\longrightarrow(t,at,0)$ is t.f.d. It is also obvious that

$$I^+(0)\subset\Gamma, \qquad (36)$$

which is easy when we look at the formulas defining the fields X, Y.

In order to prove Proposition 7.1 we need to consider two families of functions, namely

$$\varphi_a(x,y,z)=-xy^2+\alpha|y|^3-4z$$

and

$$\Phi_\alpha(x,y,z)=-x^3-3\alpha x^2|y|+(1+2\alpha-2\alpha^2)xy^2+\alpha(2\alpha+1)|y|^3+4(1+\alpha)^2z,$$

$0\le\alpha\le 1$. One easily verifies that

$$\nabla_H\varphi_a=3y^2X+3\alpha(sign\ y)y^2Y$$

and

$$\nabla_H\Phi_\alpha=$$
$$-3(x-|y|)(x+(2\alpha+1)|y|)X-3\alpha(sign\ y)(x-|y|)(x+(2\alpha+1)|y|)Y$$

from which it follows that on the set $\Gamma\cap\{y\ne 0\}$ the gradient $\nabla_H\varphi_a$ (resp. $\nabla_H\Phi_a$) is t.f.d for $0\le\alpha<1$, and is null f.d. for $\alpha=1$. We will prove three lemmas.

Lemma 7.1.

$$I^+(0) \cap \{z \neq 0\} \subset \{(x,y,z) : \varphi_1(x,y,z) < 0, \ \Phi_1(x,y,z) < 0, \ x > 0\} .$$

Proof. Let $q_0 = (x_0, y_0, z_0) \in I^+(0) \cap \{z \neq 0\}$. There exists a t.f.d. curve $\gamma : [0,1] \longrightarrow \mathbf{R}^3$ such that $\gamma(0) = 0$, $\gamma(1) = q_0$. The two functions $t \longrightarrow \varphi_1(\gamma(t)), t \longrightarrow \Phi_1(\gamma(t))$ are increasing on every connected component of the set $\{t \in [0,1] : \gamma(t) \in \{y \neq 0\}\}$. On the other hand, if $\gamma(t) \in \{y = 0\}$ for $t \in [t_1, t_2] \subset [0,1]$ then, using (10), $\gamma_{|[t_1,t_2]}$ satisfies $\dot{z} = \frac{1}{2} y(y\dot{x} - x\dot{y}) = 0$, thus is of the form $t \longrightarrow (x(t), 0, z(t_1))$, $t_1 \leq t \leq t_2$. Since $x(t)$ increases, $t \longrightarrow \Phi_1(\gamma(t))$ increases and $t \longrightarrow \varphi_1(\gamma(t))$ is constant. Recalling that $z_0 \neq 0$, the proof is finished in view of (34). □

Lemma 7.2.

$$\{(x,y,z) : \varphi_1(x,y,z) < 0, \ x > 0, \ z < 0\} \subset I^+(0) \cap \{z < 0\} .$$

Proof. Take a $q_0 = (x_0, y_0, z_0) \in \{\varphi_1 < 0, \ x > 0, \ z < 0\}$ with, say, $y_0 > 0$; the case $y_0 < 0$ is treated analogously. Since $\varphi_1(q_0) < 0$, we can find an $\alpha \in (0,1)$ so as to have $\varphi_{1/\alpha}(q_0) < 0$. Fix such an α; in particular $\alpha x_0 - y_0 > 0$. Write out the equations for trajectories of $\nabla_H \varphi_a$:

$$\begin{cases} \dot{x} = 3y^2 \\ \dot{y} = 3\alpha y^2 \\ \dot{z} = \frac{3}{2} y^3 (y - \alpha x) \end{cases} . \qquad (37)$$

In the set $\{y > 0\}$ we can reparameterize (37) to obtain

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha \\ \dot{z} = \frac{1}{2} y (y - \alpha x) \end{cases} .$$

The solution curve γ starting from $(x_0, y_0 z_0)$ at $t = 0$ has the form

$$\begin{cases} x(t) = x_0 + t \\ y(t) = y_0 + \alpha t \\ z(t) = z_0 + \frac{1}{2} y_0 (y_0 - \alpha x_0) t + \frac{1}{4} \alpha (y_0 - \alpha x_0) t^2 \end{cases} .$$

Let $\hat{t} = -\frac{y_0}{\alpha} < 0$. One easily checks that $x(t) > 0$, $y(t) > 0$, and $x(t) - |y(t)| > 0$ for $t \in (\hat{t}, 0)$; by the way we know that γ does not leave Γ for

$t \in (\hat{t}, 0)$, so $\gamma_{|(\hat{t},0]}$ is t.f.d. Finally $z(\hat{t}) = -\frac{1}{4}\varphi_{1/\alpha}(q_0) > 0$. This means that there exists a $t_0 \in (\hat{t}, 0)$ with $z(t_0) = 0$, i.e. $\gamma(t_0) \in I^+(0)$ by (35). Thus also $q_0 \in I^+(0)$ and the proof is finished. $\square$

Lemma 7.3.

$$\{(x, y, z) :\ \Phi_1(x, y, z) < 0,\ x > 0,\ z > 0\} \subset I^+(0) \cap \{z > 0\}\,.$$

Proof. Let $q_0 = (x_0, y_0, z_0) \in \{\Phi_1 < 0,\ x > 0,\ z > 0\}$. At first we investigate the case $y_0 = 0$. Fix a number $u \in (-1, 1)$ and let $\dot{\sigma}(t) = -(X + uY)(\sigma(t))$, $\sigma(0) = q_0$. Clearly, σ is timelike past directed and if $t > 0$ is sufficiently small then $\sigma(t) \in \{\Phi_1 < 0,\ x > 0,\ z > 0\} \cap \{y \neq 0\}$. Thus it suffices to consider the case $y_0 \neq 0$.

So suppose $y_0 > 0$ (the case $y_0 < 0$ is treated similarly). Take an $\alpha \in (0, 1)$ such that $\Phi_{1/\alpha}(q_0) < 0$. For such an α, and after reparameterization in the set $\{y > 0\}$, equations for trajectories of $\nabla_H \Phi_\alpha$ take the form

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = -\alpha \\ \dot{z} = \frac{1}{2}y(y\dot{x} - x\dot{y}) \end{cases}.$$

Integrating with the initial condition (x_0, y_0, z_0) we obtain

$$\begin{cases} x(t) = x_0 + t \\ y(t) = y_0 - \alpha t \\ z(t) = z_0 + \frac{1}{2}y_0(y_0 + \alpha x_0)t - \frac{1}{4}\alpha(y_0 + \alpha x_0)t^2 \end{cases}.$$

Let $\hat{t} = \frac{y_0 - x_0}{1+\alpha} < 0$. It is easily seen that $x(t) > 0$, $y(t) > 0$, $x(t) - y(t) > 0$ for $t \in (\hat{t}, 0)$. We also obtain $z(\hat{t}) = \frac{\alpha^2}{4(1+\alpha)^2}\Phi_{1/\alpha}(q_0) < 0$ from which it follows that there exists a $t_0 \in (\hat{t}, 0)$ with $z(t_0) = 0$. As in the proof of the previous lemma it means that $\gamma(t_0) \in I^+(0)$. $\square$

(35) and (36) together with Lemmas 7.1, 7.2, 7.3 prove (7.1) and (32). Finally, to justify (33), see [9] for the proof of analogous statement in the Heisenberg case. $\square$

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Multidimensional formal Takens normal form

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Abstract

We present a multidimensional analogue of the classical Takens normal form for a nilpotent singularity of a vector field.

Recall the result of F. Takens.

Theorem 1 ([Ta]) *Given an analytic germ of planar vector field of the form* $V = x_2\partial_{x_1} + h.o.t.$ *there exists a formal change of the coordinates* x_1, x_2 *reducing it to the form*

$$V^{Takens} = (x_2 + a(x_1))\partial_{x_1} + b(x_1)\partial_{x_2}$$

where $a(x_1) = a_2x_1^2 + ...$ *and* $b(x_1) = b_2x_1^2 + ...$ *are formal power series.*

The Takens normal form is obtained by solving the homological equation

$$[x_2\partial_{x_1}, Z] = W$$

which is a linear approximation to the condition

$$\left(g_Z^1\right)^* V = V^{Takens},$$

where g_Z^t is the formal flow generated by a formal vector field Z and $V = V^{Takens} + W$. It means that the space $x_1^2\mathbf{C}[[x_1]]\partial_{x_1} + x_1^2\mathbf{C}_2[[x_1]]\partial_{x_2}$ is complementary to the space
$\mathrm{ad}_{x_2\partial_{x_1}}\{\mathbf{C}[[x_1,x_2]]_{\geq 2}\partial_{x_1} + \mathbf{C}[[x_1,x_2]]_{\geq 2}\partial_{x_2}\}$, where $\mathbf{C}[[x_1]]_{\geq 2}$ is the space of series with second order zero at $x_1 = x_2 = 0$. This is the definition of the Takens normal form.

Remark 1 The Takens normal form is not complete. A. Baider and J. Sanders [BS], A. Algaba, E. Freire and E. Gamaro [AFG] and H. Kokubu,

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H. Oka and D. Wang [KOW] showed that some terms in the power series $a(x_1)$ and $b(x_1)$ can be cancelled. In some cases a complete normal form was obtained, but many cases still remain unsolved.

Consider now germs of analytic vector fields in $(\mathbf{C}^n, 0)$ with nilpotent linear part at the singular point $x = 0$. Assume firstly that there is only one Jordan cell. Therefore we take

$$V = X + h.o.t. \tag{1}$$

where

$$X = (n-1)x_2\partial_{x_1} + (n-2)x_3\partial_{x_2} + \ldots + x_n\partial_{x_{n-1}}. \tag{2}$$

(The coefficients before $x_{i+1}\partial_{x_i}$ can be chosen arbitrarily). Define the following additional vector fields

$$\begin{array}{rcl} Y & = & x_1\partial_{x_2} + 2x_2\partial_{x_3} + \ldots + (n-1)x_{n-1}\partial_{x_n}, \\ H & = & -(n-1)x_1\partial_{x_1} - (n-3)x_2\partial_{x_2} + \ldots + (n-1)x_n\partial_{x_n}. \end{array} \tag{3}$$

Lemma 1 *The vector fields X, Y, H define an irreducible representation σ of the Lie algebra $sl\,(2, \mathbf{C})$ such that*

$$\sigma(A) = X, \ \ \sigma(B) = Y, \ \ \sigma(C) = H,$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *generate* $sl(2, \mathbf{C})$.

Proof. See the book of J.-P. Serre [Ser] and the papers [CS1], [CS2]. □

The vector field Y, treated as a differentiation of the ring

$$\mathbf{C}\,[x] = \mathbf{C}[x_1, \ldots, x_n],$$

is a so-called *locally nilpotent derivation* (see [Now]). It means that for any polynomial $f\,(x) \in \mathbf{C}[x]$ we have

$$Y^N(f) \equiv 0$$

for some $N > 0$. (Of course, X is also a locally nilpotent derivation). With any locally nilpotent derivation one associates its ring of constants, i.e.

$$\mathbf{C}[x]^Y = \{g \in \mathbf{C}[x] : Yg = 0\}.$$

Lemma 2 *We have*

$$\mathbf{C}[x]^Y = \mathbf{C}[G_1, G_2, \ldots, G_{n-1}][x_1^{-1}] \cap \mathbf{C}[x]$$

where $G_1 = C_1 = x_1$ *and* G_j *are homogeneous polynomials of degree* j *defined inductively by*

$$\begin{aligned} G_j &= C_j \cdot x_1{}^{j-1}, \\ C_j &= x_{j+1} - \binom{j}{1} C_{j-1} \left(\frac{x_2}{x_1}\right)^1 - \ldots - \binom{j}{j-2} C_2 \left(\frac{x_2}{x_1}\right)^{j-2} \\ &\quad - \binom{j}{j} C_1 \left(\frac{x_2}{x_1}\right)^j. \end{aligned}$$

Proof. The system of equations defining the vector field Y is following

$$\dot{x}_1 = 0,\ \dot{x}_2 = x_1,\ \dot{x}_3 = 2x_2, \ldots$$

Since $x_1(t) \equiv C_1 = \text{const}$ and since we can shift the time t, we can assume that $x_2(t) = x_1 t$, or

$$t = x_2/x_1.$$

The other equations are solved in the form

$$x_{j+1}(t) = C_j + j \int_0^t x_j(s) ds.$$

From this the formulas from the lemma follow. Also the homogeneity of the polynomials G_j follows from this.

On the other hand, the space of solutions is parametrized by the constants of motion C_j. Each C_j, $j \geq 2$, depends linearly on x_{j+1}, with coefficient being a power of x_1; the same is true for G_j, $j \geq 2$. Since any polynomial first integral depends polynomially on $x_3, \ldots, x_n$, we can replace the latter variables by functions of $G_2, \dots, G_{n-1}$ and of x_1 and x_2; moreover, the dependence on x_2 is polynomial. Thus our first integral becomes a polynomial in x_2 with coefficients depending on elementary first integrals $G_1, \dots, G_{n-1}$.

As the latter polynomial represents a first integral of Y, it cannot contain positive powers of x_2. □

Remark 2 For $n = 2$ we get $\mathbf{C}[x]^Y = \mathbf{C}[x_1]$. It is easy to prove that for $n = 3$ we have $\mathbf{C}[x]^Y = \mathbf{C}[G_1, G_2]$.

But for $n = 4$ the ring of constants of the derivation Y is not equal to the polynomial ring of our three polynomials. We have $G_2 = x_1x_3 - x_2^2$, $G_3 = x_1^2x_4 - 3x_1x_2x_3 + 2x_2^3$. However the following first integral $\widetilde{G}_4 = 3x_2^2x_3^2 - 4x_2^3x_4 + 6x_1x_2x_3x_4 - 4x_1x_3^3 - x_1^2x_4^2$ cannot be expressed as a polynomial in G_1, G_2, G_3. In fact, for $n = 4$ the ring $\mathbf{C}[x]^Y$ is a ring of regular functions on the algebraic hypersurface in $\mathbf{C}^4$ defined by $8x^2u - y^3 + 8z^2 = 0$ (see [Now]). Also for greater dimensions the ring $\mathbf{C}[x]^Y$ is not equal to $\mathbf{C}[\mathbf{C}^{n-1}]$.

By a theorem of Weitzenböck [Wei] the ring $\mathbf{C}[x]^Y$ is finitely generated, but its structure for general n is not known. There exist examples of locally nilpotent derivations such that their rings of constants are not finitely generated.

For more information we refer the reader to the habilitation thesis of A. Nowicki [Now] and to the book of Freudenburg [Fre].

Among the first integrals for the vector field Y we distinguish those which are also first integrals for the vector field X. It is easy to see that they are altogether first integrals for the vector field H.

From the examples in Remark 2 we find that $G_2 = x_1x_3 - x_2^2$ is also a first integral for X when $n = 3$; it is invariant with respect to the change $(x_1, x_2, x_3) \to (x_3, x_2, x_1)$. Similarly, the integral $\tilde{G}_4$ is a first integral for $sl(2, \mathbf{C})$ when $n = 4$.

The vector field H defines a quasi-homogeneous gradation $\deg_H$ in the ring $\mathbf{C}[x]$, such that

$$\deg_H x_j = 2j - n - 1.$$

It follows that the first integrals F for Y which are first integrals for $sl(2, \mathbf{C})$ can be characterized by the property

$$\deg_H F = 0,$$

i.e. that it contains only monomials of quasi-homogeneous degree 0.

Note that the first integrals G_j defined in Lemma 2 have $\deg_H G_j < 0$. Generally any first integral of Y contains only terms of $\deg_H \leq 0$. Denote by $\mathbf{C}\,[x]^{Y,0} = \ker Y \cap \ker X$, respectively by $\mathbf{C}\,[x]^{Y,<0} = \ker Y \ominus \ker X$, the ring of polynomial first integrals for $sl(2,\mathbf{C})$, respectively the ring of polynomial first integrals for Y which contain only terms of nonzero quasi-homogeneous degree $\deg_H$.

Remark 3 The three vector fields X, Y, H define a distribution $\mathcal{D} \subset T\mathbf{C}^n$, i.e. a (singular) subbundle such that the fiber $\mathcal{D}_x$ at a point x is spanned by the vectors $X(x), Y(x), H(x)$. If $n \geq 4$ then at a general point the dimension of the space $\mathcal{D}_x$ equals 3, but at some points this dimension falls down. If $n = 2, 3$ then typically $\dim \mathcal{D}_x = 2$.

Since the vector fields generate a Lie algebra, the distribution is integrable. By the Frobenius theorem there exists a foliation $\mathcal{F}$ with typical leaves L of dimension 3 (for $n \geq 4$) or of dimension 2 ($n = 3$). In fact, the leaves are orbits of the action of the Lie group $SL(2,\mathbf{C})$. Since the phase flows g_X^t and g_Y^t are polynomial (as X and Y are locally nilpotent derivations) and since $(g_H^t)^* x_j = e^{t \cdot \deg_H x_j} x_j$ arises from an algebraic action of $\mathbf{C}^*$, the leaves L are algebraic varietes. So there should exist algebraic first integrals for the foliation $\mathcal{F}$.

Existence of polynomial first integrals for $\mathcal{F}$ follows also from the Clebsch–Gordan formula.

We can now formulate the main result of this work. Denote by $\mathbf{C}[x]_k$ and $\mathbf{C}[[x]]_{\geq k}$ (respectively $\mathbf{C}[x]_k^Y$, $\mathbf{C}[[x]]_{\geq k}^Y$, $\mathbf{C}[x]_k^{Y,<0}$, $\mathbf{C}[[x]]_{\geq k}^{Y,<0}$) the subspaces of $\mathbf{C}[[x]]$ (respectively of $\mathbf{C}[[x]]^Y$, $\mathbf{C}[[x]]^{Y,<0}$) consisting of homogeneous polynomials of degree k and of series which have zero of order $\geq k$ at the origin.

Theorem 2 *Any germ of the form* (1) *can be reduced by means of a formal change of variables $x_1, ..., x_n$ to the following*

$$V^{Takens} = X + F_1\,(G)\,\partial_{x_1} + \ldots + F_n\,(G)\,\partial_{x_n}, \tag{4}$$

where $F_j\,(G) = F_j\,(G_1, \ldots, G_{n-1})$ are formal power series in $G_2, \ldots,$ G_{n-1} with coefficients being Laurent polynomials in $G_1 = x_1$ and such that $F_j \circ G\,(x) \in \mathbf{C}[[x]]_{\geq 2}$ and $F_j \in \mathbf{C}[x]^{Y,<0}$ for $j = 1, \ldots, n-1$. Moreover, the form (4) is unique in a sense that the space

$$\mathbf{C}[[x]]_{\geq 2}^{Y,<0} \cdot \partial_{x_1} + \ldots + \mathbf{C}[[x]]_{\geq 2}^{Y,<0} \cdot \partial_{x_{n-1}} + \mathbf{C}[x]_{\geq 2}^{Y} \cdot \partial_{x_n}$$

is complementary to the space

$$\mathrm{ad}_X \left\{ \mathbf{C}[[x]]_{\geq 2} \cdot \partial_{x_1} + \ldots + \mathbf{C}[[x]]_{\geq 2} \cdot \partial_{x_n} \right\}.$$

Example 1 For $n = 3$ the Takens normal form is following

$$\dot{x}_1 = 2x_2 + x_1 \Phi_1(x_1, G_2),\ \dot{x}_2 = x_3 + x_1 \Phi_2(x_1, G_2),\ \dot{x}_3 = \Phi_3(x_1, G_2).$$

For $n = 4$ we have

$$F_j = \sum_{a,b,c,d \geq 0} f^{(j)}_{a,b,c,d} G_1^a G_2^b G_3^c \tilde{G}_4^d,$$

where $a + 2b + 3c + 4d \geq 2$, $a = 0, 1$ if $d > 0$, and $3a + 2b + 3c > 0$ for $j = 1, 2, 3$.

Proof of Theorem 2. Let $Z = Z_1(x)\,\partial_{x_1} + \ldots + Z_n(x)\,\partial_{x_n}$ be a homogeneous vector field of degree k. We have

$$\begin{aligned} \mathrm{ad}_X Z &= X(Z_n)\,\partial_n \\ &\quad + (X(Z_{n-1}) - (n-1)Z_n)\,\partial_{x_{n-1}} \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad + (X(Z_1) - Z_2)\,\partial_{x_1}. \end{aligned}$$

Theorem 2 follows from the following two lemmas.

Lemma 3 *In the space* $\mathbf{C}[x]_k$ *of homogeneous polynomials we have*

$$\begin{aligned} \ker Y \oplus \mathrm{Im} X &= \mathbf{C}[x]_k, \\ \ker X \ominus \ker Y &\subset \mathrm{Im} X, \end{aligned}$$

where $\ker X \ominus \ker Y = \mathbf{C}[x]_k^{X,>0}$ *denotes the space of first integrals for* X *which contain only terms with nonzero quasi-homogeneous degree* $\deg_H$.

Proof. The vector fields X, Y, H define a representation of the Lie algebra $sl(2, \mathbf{C})$ in the space $\mathbf{C}[x]_k$ of homogeneous polynomials. It is known that any finite dimensional representation is split into irreducible representations, so-called *highest weight representations* (see [Ser]). Therefore

$$\mathbf{C}[x]_k = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m,$$

and any $\mathcal{H}_j$ has a basis $\{e_1, \ldots, e_d\}$ such that

$$Xe_1 = 0,\ Xe_2 = (d-1)e_1, \ldots, Xe_d = e_{d-1},$$

$$Ye_1 = e_2, \ldots, Ye_{d-1} = (d-1)e_d,\ Ye_d = 0,$$

$$H\left(e_j\right) = (2j - d - 1)e_j.$$

We see that $\mathrm{Im}X = \mathrm{span}(e_1, \ldots, e_{d-1}),\ \ker X = \mathrm{span}(e_1),\ \ker Y = \mathrm{span}(e_d)$. Hence $\ker Y \oplus \mathrm{Im}X = \mathcal{H}_j$.

If $d > 1$ then we see that $\ker X \subset \mathrm{Im}X$. If $d = 1$ then $X = Y = H = 0$ and $\ker X \ominus \ker Y = 0 \subset \mathrm{Im}X$.

Now the equalities from Lemma 3 hold when restricted to any subspace $\mathcal{H}_j$. Therefore they hold also in $\mathbf{C}[x]_k$. □

Lemma 4 *The space* $\ker Y \ominus \ker X \cdot \partial_{x_1} + \ldots + \ker Y \ominus \ker X \cdot \partial_{x_{n-1}} + \ker Y \cdot \partial_{x_n}$ *is complementary to the space* $\mathrm{ad}_X \mathcal{X}_k$ *in the space* $\mathcal{X}_k$ *of homogeneous vector fields of degree* k.

Proof. From Lemma 3 we see that the last component of the action of ad_X on Z equals $X\left(Z_n\right)$, i.e. lies in the image of X in $\mathbf{C}\left[x\right]_k$. So the n–th component of the normal form should be the kernel of $Y|_{\mathbf{C}[x]_k}$. Note that the Z_n is not unique, when killing a suitable part in ∂_{x_n}; we can add some $\tilde{Z}_n \in \ker X$ to Z_n.

The $(n-1)$–th component of the action ad_X equals $X\left(Z_{n-1}\right) - \lambda_{n-1} Z_n$. So all polynomials from $\mathrm{Im}X$ can be killed.

We can hope to make an additional cancellation using $\tilde{Z}_n$ from $\ker X$. Lemma 3 says that we can write $\tilde{Z}_n = \tilde{Z}_n^{<0} + \tilde{Z}_n^0$, where

— $\tilde{Z}_n^{<0}$ lies in $\mathrm{Im}X$ (and we gain nothing);

— $\tilde{Z}_n^0$ belongs to $\ker Y \cap \ker X$ (here we cancel terms from $\mathbf{C}\left[x\right]_k^{Y,0}$).

So, the $(n-1)$–th component in the normal form is in $\ker Y \ominus \ker X$.

Analogously we consider successively other components. □

Remark 4 We can generalize Theorem 2 to the case when X, the linear part of V, has several nilpotent Jordan cells. For example, when X is given by the matrix

$$\begin{pmatrix} \begin{bmatrix} 0 & n-1 & \dots & 0 \\ & 0 & \dots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & m-1 & \dots & 0 \\ & 0 & \dots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \end{pmatrix}$$

Then X and the vector field Y, which is given by the matrix

$$\begin{pmatrix} \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \dots & 0 & \\ & & n-1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \dots & 0 & \\ & & m-1 & 0 \end{bmatrix} \end{pmatrix},$$

define a representation of the Lie algebra $sl(2, \mathbf{C})$. The normal form is

$$V^{Takens} = X + \sum_{j=1}^{m+n} F_j \left(G \right) \partial_{x_j}$$

where $F_j(G_1, \dots, G_{n-1}, G'_1, \dots, G'_{m-1})$ are formal series of polynomials G_2 , $\dots$, G_{n-1} , G'_2 , $\dots$, G'_{m-1} with coefficients being Laurent polynomials in $G_1 = x_1$ and $G'_1 = x_{n+1}$. The polynomials $G'_1, G'_2, \dots, G'_{m-1}$ generate the field of constants of the part of Y associated with the variables $x_{n+1}, \dots, x_{n+m}$. The polynomials F_j, $j \neq n, n+m$, do not contain terms with zero quasi-homogeneous degree.

Remark 5 Another question is whether the Takens form is analytic (provided that the initial vector field is analytic near the origin). In the two-dimensional case the analyticity was proved in [SZ] and [Lo]. Some partial results in this direction were obtained also by V. Basov [Ba1, Ba2].

We began to study this problem for $n \geq 3$, but it looks very difficult. We think that when $n \geq 3$ the above normal form is not analytic in general. We plan to continue investigations.

Remark 6 R. Cushman and J. Sanders [CS1, CS2] also studied the normal form for the nilpotent singularities and also used the representation theory of the Lie algebra $sl(2, \mathbf{C})$. However their normal form is more complicated than ours. In fact, they applied the representation of this Lie algebra directly in the space $\mathcal{X}_k$ of homogeneous vector fields using the operator ad_X, ad_Y and ad_H, while we are working in the space $\mathbf{C}[x]_k$ of homogeneous polynomials. Moreover, they seem not to explore the property $\ker X \ominus \ker Y \subset \mathrm{Im} X$ from Lemma 3.

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