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# Singularities and Symplectic Geometry

Part VII

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# Preface

This volume contains papers written on the occasion of the 25th anniversary of the Singularity Theory Seminar acting in Warsaw University of Technology. Around eight hundred talks by leading experts in singularities, mathematical physics, symplectic geometry, differential geometry, algebraic geometry, local algebra and related topics were delivered in form of minicourses, research lectures and reports. The Seminar's activity culminated in the condensed school-conferences organized every year. Since 1998 these conferences were dominated by very productive collaboration teams: the "Geometry and Topology of Caustics" team and the "Polish-Japanese Singularity Theory Working Days" team, both oriented towards modern implementations of singularity theory.

In this volume we collect papers on the part of singularity theory related to its geometrical representations:  $\mathcal{A}$ -equivalence of maps according to the volume preserving or symplectic diffeomorphisms. Integrability of implicit Hamiltonian systems around singularities. Reachable sets for the Heisenberg sub-Lorentzian structure. The complex symplectic moduli spaces of unimodal parametric plane curve singularities. Admissible weights for weighted Bergman spaces. Exotic moduli of Goursat distributions. The Euler characteristic of a link of a set defined by a Noetherian family of analytic functions. Drapeau Theorem for differential systems. Formal orbital normal forms for the nilpotent singularity. Surfaces which contain many circles. The Euler number of the normalization of a certain singular hypersurface. Projective invariants associated to the special parabolic points of surfaces and to swallowtails. Centre symmetry sets and other invariants of algebraic sets.

The geometric ideas initiated by considerations of singularity theory have been tremendously successful, and we believe that their interaction with the central mathematical disciplines and other branches of science will bring a substantial feedback and new interesting mathematical methods for applications.

*Stanisław Janeczko*

# Volume-preserving diffeomorphisms on varieties and $\mathcal{A}_\Omega$ -equivalence of maps

Wojciech Domitrz<sup>1</sup> and Joachim Rieger<sup>2</sup>

## Abstract

We describe a cohomological criterion for a pair of diffeomorphic variety-germs (or, more generally, set-germs) to be volume-preserving diffeomorphic. We also show that for the class of varieties that are quasi-homogeneous with respect to a smooth submanifold this cohomology vanishes, so that volume-preserving diffeomorphic is equivalent to diffeomorphic (over  $\mathbb{C}$ ) or orientation-preserving diffeomorphic (over  $\mathbb{R}$ ). Likewise, we give a criterion for a pair of left-right equivalent map-germs to be  $\mathcal{A}_\Omega$ -equivalent (i.e. left-right equivalent by volume-preserving diffeomorphisms on the left) and we show that for the class of weakly quasi-homogeneous map-germs each left-right orbit corresponds to one  $\mathcal{A}_\Omega$ -orbit (over  $\mathbb{C}$ ) or to one or two  $\mathcal{A}_\Omega$ -orbits (over  $\mathbb{R}$ ).

## 1 Introduction

We study the local classification of singular varieties and maps under the action of the group of volume-preserving diffeomorphisms, which is not a geometric subgroup in the sense of Damon [11] (the tangent space fails to have the structure of a finitely generated module). The definition of a geometric subgroup  $\mathcal{G}$  of the Mather groups  $\mathcal{A}$  and  $\mathcal{K}$  collects the properties that are required in the proofs of the finite determinacy and versal unfolding theorems for any such  $\mathcal{G}$ , the central theorems for any classification. For non-geometric subgroups of  $\mathcal{A}$  and  $\mathcal{K}$  moduli (and even functional moduli) often appear rather quickly already in low codimension (even in codimension 0), see e.g. [14]. On the other hand, Martinet wrote 30 years ago in his book (see p. 50 of the English translation [29]) on the classification problem in unimodular geometry – i.e. the classification of map-germs under the subgroup  $\mathcal{A}_\Omega$  of  $\mathcal{A}$

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in which the target diffeomorphisms are volume preserving – that the groups involved “are big enough that there is still some hope of finding a reasonable classification theorem”. It turns out that Martinet was right – the results of the present paper imply, for example, that the classifications of stable maps for  $\mathcal{A}_\Omega$  and  $\mathcal{A}$  agree (at least over  $\mathbb{C}$ ).

Apart from the  $\mathcal{A}_\Omega$ -equivalence of map-germs, we will study volume-preserving diffeomorphisms on set-germs  $V \subset (\mathbb{K}^n, 0)$ . Despite the title of our paper we will only require that  $I(V) \neq \{0\}$  (with  $I(V)$  the vanishing ideal) and use the terms set-germ and germ of a variety to mean the same thing (in many applications of our results  $V$  will indeed be a variety, defined by a collection of analytic or  $C^\infty$  function-germs, with possibly non-isolated singularities, e.g. when  $V$  is the discriminant or the image of a mapping).

The following earlier work is related to our results on volume-preserving diffeomorphisms on set-germs. In [39], [10] the local classification problem for functions under the action of volume-preserving diffeomorphisms was first considered (leading to the so-called isochore Morse Lemma), and more recently isochore versal deformations were studied in [9] and [21]. The results of [39], [10] were generalized in another direction in [16], [17] and [18]. The starting point of the present paper are the articles [2], [4], [38] and [23]. In [2] powers of  $\mathbb{C}$ -analytic volume forms were studied. In particular V. I. Arnol’d proved that two germs of quasi-homogeneous  $\mathbb{C}$ -analytic hypersurfaces with isolated singularity are volume-preserving diffeomorphic if and only if they are diffeomorphic. The generalization of this result was obtained in [38] (see also [25],[26],[27],[28]). In that paper A. N. Varchenko proved that the dimension of the moduli space of germs of  $\mathbb{C}$ -analytic volume forms under the action of diffeomorphism-germs preserving a fixed hypersurface with an isolated singularity is equal to the difference of the Milnor number and the Tjurina number of this hypersurface. (By a result of K. Saito [37] this difference is zero if and only if the hypersurface is quasi-homogeneous.)

The results on  $\mathcal{A}_\Omega$ -equivalence of map-germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  in the present paper are related to the following earlier work. In [4] V. I. Arnol’d introduced the local symplectic algebra and classified singular parameterized curves of the type  $A_{2k}$  on a symplectic manifold. G. Ishikawa and S. Janeczko studied the local classification problem of parameterized curves on 2-dimensional symplectic manifold. (In dimension 2 a symplectic form is a volume form.) All simple germs in this classification are quasi-homogeneous and the orbit of an  $\mathcal{A}$ -simple  $\mathbb{C}$ -analytic quasi-homogeneous germ under the action of  $\mathbb{C}$ -analytic symplectomorphisms is the same as the  $\mathcal{A}$ -orbit of this

germ. As a consequence of our results on  $\mathcal{A}_\Omega$ -equivalence of map-germs we obtain the following generalization of this result on parameterized curve-germs in symplectic 2-manifolds: an  $\mathcal{A}$ -simple map-germ from an  $n$ -manifold into a symplectic 2-manifold is quasi-homogeneous if and only if its  $\mathcal{A}$ -orbit and its  $\mathcal{A}_\Omega$ -orbit coincide (in target dimension 2 the volume preserving diffeomorphisms are symplectomorphisms).

Here is an outline of the content of the present paper. In Section 2 we describe the main equivalence relations for germs of volume forms and map-germs and we summarize our main results.

In Section 3 we define the notion of a weak algebraic restriction of the germ of a  $k$ -form to a germ of a singular variety  $V$ , which is a modification of the notion of algebraic restriction defined by M. Zhitomirskii ([41], see also [12]). The  $k$ -forms with zero weak algebraic restriction to  $V$  form a subcomplex  $\mathcal{W}_0^*(V)$  of the de Rham complex on  $\mathbb{K}^n$ , and we will see that  $H^n(\mathcal{W}_0^*(V))$  vanishes for germs  $V$  that are quasi-homogeneous with respect to a smooth submanifold (as defined in [13]: the quasi-homogeneity with respect to a smooth submanifold may be understood as the quasi-homogeneity with positive and zero weights).

In Section 4 we use the cohomology of the complex  $\mathcal{W}_0^*(V)$  introduced in the previous section to present necessary and sufficient conditions for a pair of diffeomorphic variety-germs – or, more generally, set-germs – to be volume-preserving diffeomorphic ( $\mathcal{R}_\Omega$ -equivalent for some given volume form  $\Omega$ ) and for a pair of volume forms to be  $\mathcal{R}_V$ -equivalent (i.e. to be related by pull-back by a diffeomorphism preserving a given variety-germ  $V$ ). Furthermore, we show that  $\dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V))$  is the dimension of the  $\mathcal{R}_V$ -moduli space and that  $H^n(\mathcal{W}_0^*(V))$  vanishes for those  $V$  that are quasi-homogeneous with respect to a smooth submanifold.

In Sections 5 and 6 we study  $\mathcal{A}_\Omega$ - and  $\mathcal{A}_f$ -equivalence of map-germs and volume forms (in the same way as before  $\mathcal{R}_\Omega$ - and  $\mathcal{R}_V$ -equivalence of varieties and volume forms): for  $\mathcal{A}_\Omega$ -equivalence we fix a volume form  $\Omega$  on the target and require that the target (left) diffeomorphisms in a left-right equivalence of a pair of map-germs preserve  $\Omega$ , whereas for  $\mathcal{A}_f$  equivalence the map-germ  $f$  is given and a pair of volume forms on the target of  $f$  is related by pull-back by some  $\Phi$ , where  $(\Phi, \Psi)$  lies in the isotropy subgroup of  $f$  in the group  $\mathcal{A}$  of left-right equivalences. We present necessary and sufficient conditions for  $\mathcal{A}_\Omega$ -equivalence of map-germs and  $\mathcal{A}_f$ -equivalence of volume forms and we give formulas for the dimensions of the  $\mathcal{A}_f$ - and  $\mathcal{A}_\Omega$ -moduli spaces. Furthermore, we show that for the class of weakly quasi-homogeneous

map-germs each left-right orbit corresponds to one  $\mathcal{A}_\Omega$ -orbit (over  $\mathbb{C}$ ) or to one or two  $\mathcal{A}_\Omega$ -orbits (over  $\mathbb{R}$ ). The study of  $\mathcal{A}_\Omega$ - and  $\mathcal{A}_f$ -equivalence in Section 5 is in the style of that of  $\mathcal{R}_\Omega$ - and  $\mathcal{R}_V$ -equivalence in Section 4. In Section 6 we study the  $\mathcal{A}_\Omega$  classification of map-germs using infinitesimal methods (in the style of Mather, looking at the “tangent spaces” of  $\mathcal{A}_\Omega$ -orbits). Our results give the classification of  $\mathcal{A}_\Omega$ -orbits of maps  $\mathbb{C}^n \rightarrow \mathbb{C}^p$  from the corresponding lists of  $\mathcal{A}$ -simple germs  $f$  as follows:  $f$  is  $\mathcal{A}_\Omega$ -simple if and only if it is weakly quasi-homogeneous and not  $\mathcal{A}$ -adjacent to any non-weakly quasi-homogeneous germ. In dimensions  $(n, p)$  with  $p < 2n$  it is sufficient to work with the smaller class of quasi-homogeneous map-germs, but for  $p \geq 2n$  the wider class of weakly quasi-homogeneous germs is required for obtaining the  $\mathcal{A}_\Omega$ -simple orbits – all our examples of  $\mathcal{A}$ -simple weakly quasi-homogeneous map-germs, which are not quasi-homogeneous, occur for  $p \geq 2n$ .

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## 2 Summary of the main equivalence relations and results

Our results on the equivalence of varieties  $V_0$  and  $V_1$  under volume preserving diffeomorphisms (i.e. under  $\mathcal{R}_\Omega$ -equivalence, where  $\Omega$  is a fixed volume form on the ambient space) will be deduced from corresponding statements about the equivalence of pairs of volume forms  $\Omega_0$  and  $\Omega_1$  under pull-back by diffeomorphisms  $\mathcal{R}_V$ -preserving a fixed variety  $V$ . Likewise, our results about  $\mathcal{A}_\Omega$  equivalence of map-germs  $f, g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  (where  $\Omega$  is a fixed volume form on  $\mathbb{K}^p$ ) will follow from corresponding statements about what we call  $\mathcal{A}_f$  equivalence (with  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  fixed) of volume forms  $\Omega_0$  and  $\Omega_1$  on  $\mathbb{K}^p$  below.



**Convention:** throughout this paper we will be considering germs of maps, coordinate changes, differential forms, vector fields etc. at the origin in  $\mathbb{K}^n$  or  $\mathbb{K}^p$ , and all objects are real-analytic or smooth ( $C^\infty$ ) when  $\mathbb{K} = \mathbb{R}$  and complex-analytic when  $\mathbb{K} = \mathbb{C}$ . Also recall that we do not distinguish germs of a variety  $V$  and set-germs: we only assume that the ideal of (smooth or analytic) function-germs vanishing on  $V$  is different from  $\{0\}$ . With this understood we consider the following equivalences.

(i)  **$\mathcal{R}_V$  and  $\mathcal{IR}_V$  equivalence.** Given a set-germ  $V \subset (\mathbb{K}^n, 0)$  and a pair of volume forms  $\Omega_i, i = 0, 1$  we have:

$$\Omega_0 \sim_{\mathcal{R}_V} \Omega_1 \iff \exists \Phi \in \text{Diff}(\mathbb{K}^n, 0) \text{ such that } \Phi(V) = V, \quad \Phi^* \Omega_1 = \Omega_0.$$

And if, in the above situation,  $\Omega_0$  and  $\Omega_1$  can be joined by a smooth path contained in a single  $\mathcal{R}_V$ -orbit then we write  $\Omega_0 \sim_{\mathcal{IR}_V} \Omega_1$

(ii)  **$\mathcal{R}_\Omega$  equivalence.** Given a volume form  $\Omega$  on  $\mathbb{K}^n$  and a pair of set germs  $V_i \subset \mathbb{K}^n, i = 0, 1$  we have:

$$V_0 \sim_{\mathcal{R}_\Omega} V_1 \iff \exists \Phi \in \text{Diff}(\mathbb{K}^n, 0) \text{ such that } \Phi(V_0) = V_1, \quad \Phi^* \Omega = \Omega.$$

Notice that we obtain an equivalence  $V_0 \sim_{\mathcal{R}_\Omega} V_1$  by first applying an arbitrary diffeomorphism  $\Phi$  with  $\Phi(V_0) = V_1$  that changes the volume form to  $\Omega_1 = \Phi^* \Omega$ , and then mapping  $\Omega_1$  back to  $\Omega$  by pulling back by some  $\Psi \in \mathcal{R}_{V_0}$ .

For map-germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  we consider the following equivalences.

(iii)  **$\mathcal{A}_f$  and  $\mathcal{IA}_f$  equivalence.** For the isotropy subgroup  $\text{Iso}(f) := \{(\Phi, \Psi) \in \mathcal{A} : \Phi \circ f \circ \Psi^{-1} = f\}$  of a map-germ  $f$  and a pair of volume forms  $\Omega_i, i = 0, 1$ , in  $\mathbb{K}^p$  we have:

$$\Omega_0 \sim_{\mathcal{A}_f} \Omega_1 \iff \exists (\Phi, \Psi) \in \text{Iso}(f) \text{ such that } \Phi^* \Omega_1 = \Omega_0.$$

And again if, in the above situation,  $\Omega_0$  and  $\Omega_1$  can be joined by a smooth path contained in a single  $\mathcal{A}_f$ -orbit then we write  $\Omega_0 \sim_{\mathcal{IA}_f} \Omega_1$ .

(iv)  **$\mathcal{A}_\Omega$  equivalence.** Given a volume form  $\Omega$  on the target  $\mathbb{K}^p$  and a pair of map-germs  $f, g$  then we have:

$$f \sim_{\mathcal{A}_\Omega} g \iff \exists (\Phi, \Psi) \in \mathcal{A} \text{ such that } \Phi^* \Omega = \Omega, \quad f = \Phi \circ g \circ \Psi^{-1}.$$

We can obtain an  $\mathcal{A}_\Omega$ -equivalence of  $f$  and  $g$  by first applying an arbitrary  $\mathcal{A}$ -equivalence mapping  $g$  to  $f$  and  $\Omega$  to some  $\Omega_1 = (\Phi^{-1})^* \Omega$  and then pulling

back  $\Omega$  to  $\Omega_1$  using  $\mathcal{A}_f$ -equivalence, as in the diagram below:

$$\begin{array}{ccc}
(\mathbb{K}^n, 0) & \xrightarrow{g} & (\mathbb{K}^p, \Omega, 0) \\
\downarrow \Psi & & \downarrow \Phi \\
(\mathbb{K}^n, 0) & \xrightarrow{f} & (\mathbb{K}^p, \Omega_1, 0) \\
\downarrow \tilde{\Psi} & & \downarrow \tilde{\Phi} \\
(\mathbb{K}^n, 0) & \xrightarrow{f} & (\mathbb{K}^p, \Omega, 0)
\end{array}$$

In fact, given  $f$ ,  $g$  and  $\Omega$  as above, the following statements are equivalent: (i)  $f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent and (ii) there exist  $(\Phi, \Psi) \in \mathcal{A}$  and  $(\tilde{\Phi}, \tilde{\Psi}) \in \text{Iso}(f)$  such that the above diagram commutes.

The following theorem summarizes the main results on  $\mathcal{R}_V$  and  $\mathcal{R}_\Omega$  equivalence (it combines the statements of Theorems 4.3 and 4.9 and Proposition 4.5 below). Given a germ of a variety  $V \subset (\mathbb{K}^n, 0)$ , we say that a  $k$ -form has zero weak algebraic restriction to  $V$  if it is equal to  $d\alpha + \beta$ , where  $\alpha$  is a  $(k-1)$ -form and  $\beta$  a  $k$ -form such that  $df \wedge \alpha|_V = 0$  and  $df \wedge \beta|_V = 0$  for any  $f \in I(V)$ . And we denote by  $[\Omega]_V$  the class of  $\Omega$  in the cohomology group  $H^n(\mathcal{W}_0^*(V))$ , where  $\mathcal{W}_0^*(V)$  is the subcomplex of the de Rham complex of differential forms with zero weak algebraic restriction to  $V$  (see Section 3 below for precise definitions). Furthermore, we say that two cohomology classes  $[\Omega_0]_{V_0}$  and  $[\Omega_1]_{V_1}$  are diffeomorphic if there is a diffeomorphism  $\Phi$  of  $(\mathbb{K}^n, 0)$  that maps  $V_0$  to  $V_1$  and  $[\Phi^*\Omega_1]_{V_0} = [\Omega_0]_{V_0}$  (see 4.8 below). Finally let  $\text{div}(\text{Derlog}(V)) \subset C_n$  be the  $\mathbb{K}$ -subspace of  $C_n$  ( $C_n$ —the local ring of smooth or analytic function-germs on  $(\mathbb{K}^n, 0)$ ) consisting of the divergences of the logarithmic vector fields on  $V$ . Then we have the following

**Theorem 2.1.** *For germs of varieties  $V, V_0, V_1 \subset (\mathbb{K}^n, 0)$  and of volume forms  $\Omega, \Omega_0, \Omega_1$  we have the following.*

- (i) *Given  $V$ :  $\Omega_0 \sim_{\mathcal{R}_V} \Omega_1 \iff [\Omega_0]_V = [\Omega_1]_V$ .*
- (ii) *Fixing  $\Omega$ :  $V_0 \sim_{\mathcal{R}_\Omega} V_1 \iff [\Omega]_{V_0}$  and  $[\Omega]_{V_1}$  are diffeomorphic.*
- (iii) *Fixing  $V$ , the number of  $\mathcal{R}_V$ -moduli of germs of volume forms on  $(\mathbb{K}^n, 0)$  is equal to*

$$\dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V)) = \dim_{\mathbb{K}} C_n / \text{div}(\text{Derlog}(V)).$$

The next result summarizes the statements in Theorems 4.4 and 4.10 (varieties  $V$  that are quasi-homogeneous with respect to a smooth submanifold were introduced in [13], in Section 3 we will see that for such  $V$  we have  $H^n(\mathcal{W}_0^*(V)) = 0$ ).

**Theorem 2.2.** *Consider germs of varieties  $V, V_1 \subset (\mathbb{K}^n, 0)$  and of volume forms  $\Omega, \Omega_0, \Omega_1$ . If  $V$  is quasi-homogeneous with respect to a smooth sub-manifold then the following statements hold.*

(i)  $\Omega_0 \sim_{\mathcal{R}_V} \Omega_1$ , for any pair  $\Omega_0, \Omega_1$  (over  $\mathbb{K} = \mathbb{C}$ ) or for any pair  $\Omega_0, \Omega_1$  such that  $\Omega_0|_0, \Omega_1|_0$  define the same orientation (over  $\mathbb{K} = \mathbb{R}$ ).

(ii) Fixing  $\Omega$ :  $V_1 \sim_{\mathcal{R}_\Omega} V \iff \Phi(V) = V_1$  for some germ  $\Phi$ , of a diffeomorphism (over  $\mathbb{K} = \mathbb{C}$ ) or an orientation-preserving diffeomorphism (over  $\mathbb{K} = \mathbb{R}$ ).

For map-germs  $f$  the following theorem summarizes the main results on  $\mathcal{A}_f$  and  $\mathcal{A}_\Omega$  equivalence (it combines the results of Theorems 5.7 and 5.19, Corollary 5.8 and Proposition 6.7 below). Here  $Lift(f)$  denotes the  $C_p$ -module ( $C_p$  – local ring of functions on the target of  $f$ ) of vector fields  $Y$  liftable over  $f$ ,  $Lift_0(f)$  the submodule of  $Lift(f)$  consisting of those  $Y$  that lift to source vector fields vanishing at 0 and let  $\text{div}(Lift_0(f)) := \{\text{div}(Y) : Y \in Lift_0(f)\} \subset C_p$ . The symbol  $[\Omega]_f$  denotes the class of the volume form  $\Omega$  under the following equivalence ( $\text{div}Lift_0(f)$ -equivalence see Def. 5.5 below):  $\Omega_0$  and  $\Omega_1$  are equivalent if  $\Omega_1 - \Omega_0 = \text{div}(Y)\Omega$  for some  $Y \in Lift_0(f)$  and some (or, in fact, any) volume form  $\Omega$ . For  $\mathcal{A}$ -equivalent map-germs  $f$  and  $g$ , i.e.  $g = \Phi \circ f \circ \Psi^{-1}$ , the classes  $[\Omega]_f$  and  $[\Omega]_g$  are diffeomorphic if  $[\Phi^*\Omega]_g = [\Omega]_f$ . Then we have the following

**Theorem 2.3.** *For map-germs  $f, g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  and germs of volume forms  $\Omega, \Omega_0, \Omega_1$  in  $(\mathbb{K}^p, 0)$  we have the following.*

(i) Given  $f$ :  $\Omega_0 \sim_{\mathcal{I}\mathcal{A}_f} \Omega_1 \iff [\Omega_0]_f = [\Omega_1]_f$ .

(ii) Fixing  $\Omega$ :  $f \sim_{\mathcal{A}_\Omega} g \iff [\Omega]_f$  and  $[\Omega]_g$  are diffeomorphic.

(iii) Fixing  $f$ , the number of  $\mathcal{A}_f$ -moduli of germs of volume forms on  $(\mathbb{K}^p, 0)$  is equal to

$$\dim_{\mathbb{K}} C_p / \text{div}(Lift_0(f)) = \dim_{\mathbb{K}} T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f,$$

where in the second term  $\Omega$  is some fixed volume form and  $T\mathcal{A}_\Omega \cdot f$  denotes the tangent space to the  $\mathcal{A}_\Omega$ -orbit through  $f$  at  $f$ .

A map-germ  $f$  is weakly quasi-homogeneous if it is quasi-homogeneous for integer weights, non-negative weighted degrees and positive total degree (for some choice of coordinates in source and target). The following is a summary of Theorems 5.14 and 5.20.

**Theorem 2.4.** *Consider map-germs  $f, g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  and germs of volume forms  $\Omega, \Omega_0, \Omega_1$  in  $(\mathbb{K}^p, 0)$ . If  $f$  is weakly quasi-homogeneous then the following statements hold.*

(i)  $\Omega_0 \sim_{\mathcal{A}_f} \Omega_1$  for any pair  $\Omega_0, \Omega_1$  (over  $\mathbb{K} = \mathbb{C}$ ) or for any pair  $\Omega_0, \Omega_1$  such that  $\Omega_0|_0, \Omega_1|_0$  define the same orientation (over  $\mathbb{K} = \mathbb{R}$ ).

(ii) Fixing  $\Omega$ :  $g \sim_{\mathcal{A}_\Omega} f \iff g \sim_{\mathcal{A}} f$ , where the target diffeomorphisms (in the elements of  $\mathcal{A}$ ) are arbitrary (over  $\mathbb{K} = \mathbb{C}$ ) or are orientation-preserving (over  $\mathbb{K} = \mathbb{R}$ ).

**Remark 2.5.** (1) *One could deduce Theorems 2.3 and 2.4 on map-germs  $f$  from Theorems 2.1 and 2.2 by considering the set-germ  $\Delta(f) \subset (\mathbb{K}^p, 0)$ , which denotes the discriminant of  $f$  (for  $n \geq p$ ) or the image (for  $n < p$ ), and using the results in [32], [33], [8] on critical normalizations. However, we shall give direct, self-contained proofs of these theorems that apply to a larger class of map-germs  $f$  than the ones considered in [32], [33], [8]. Also notice that one could try to obtain Theorem 2.4 from its “infinitesimal version” in Proposition 6.1, which says that for weakly quasi-homogeneous maps  $f$  we have the equality  $T\mathcal{A} \cdot f = T\mathcal{A}_\Omega \cdot f$  of tangent spaces, using arguments similar to the ones in [19].*

(2) *Our results on mono-germs  $f$  can be generalized to multi-germs, this just complicates the notation without really changing the proofs.*

### 3 Cohomology of forms with zero weak algebraic restriction to a singular set

Let  $V$  be a germ of a subset of  $\mathbb{K}^n$  at 0. Let  $I(V)$  be the ideal of smooth ( $\mathbb{K}$ -analytic) function-germs vanishing on  $V$ . We assume that  $I(V) \neq \{0\}$ .

**Definition 3.1.** *We say that germs of  $k$ -forms  $\omega_0$  and  $\omega_1$  have the same weak algebraic restriction to  $V$  if there exists a germ of a  $(k-1)$ -form  $\alpha$  and a germ of a  $k$ -form  $\beta$  such that  $\omega_1 - \omega_0 = d\alpha + \beta$  and  $df \wedge \alpha|_V = 0$ ,  $df \wedge \beta|_V = 0$  for any  $f \in I(V)$ .*

*We say that the germ of a  $k$ -form  $\omega$  has zero weak algebraic restriction to  $V$  if  $\omega$  and a germ of the zero  $k$ -form have the same weak algebraic restriction to  $V$ .*

This definition is a modification of the definition of an algebraic restriction to a singular set  $V$  introduced by M. Zhitomirskii for 1-forms in [41] and

generalized to arbitrary  $k$ -forms in [12]. Germs of  $k$ -forms  $\omega_0$  and  $\omega_1$  have the same *algebraic restriction to  $V$*  if there exists a germ of a  $(k-1)$ -form  $\alpha$  and a germ of a  $k$ -form  $\beta$  such that  $\omega_1 - \omega_0 = d\alpha + \beta$  and  $\alpha|_V = 0$ ,  $\beta|_V = 0$ .

It is obvious that if  $\omega_0$  and  $\omega_1$  have the same algebraic restriction to  $V$  then they have the same weak algebraic restriction to  $V$ . But it is easy to see that the weak algebraic restriction of any  $n$ -form on  $\mathbb{K}^n$  to a subset  $V \subset \mathbb{K}^n$  such that  $I(V) \neq \{0\}$  is zero (see Proposition 3.5) and that this is not true for algebraic restrictions of volume forms to singular hypersurfaces.

If  $V$  is a germ of a smooth submanifold then  $\omega_0$  and  $\omega_1$  have the same algebraic restriction to  $V$  if and only if  $\omega_0$  and  $\omega_1$  have the same pullback to  $V$ , i.e. they have the same geometric restriction to  $V$  (see [12]). The same is true for the weak algebraic restriction to a smooth submanifold. On smooth submanifolds the notions of algebraic restriction, weak algebraic restriction and geometric restriction coincide. This is not true on singular sets (see Example 3.4)

**Proposition 3.2.** *If  $V$  is a smooth submanifold then  $\omega_0$  and  $\omega_1$  have the same weak algebraic restriction to  $V$  if and only if  $\omega_0$  and  $\omega_1$  have the same pullback to  $V$ .*

*Proof:* It is enough to show that a germ of a  $k$ -form  $\omega$  has zero weak algebraic restriction if and only if the pullback of  $\omega$  to  $V$  vanishes. We may assume that  $k < n$ . Let  $V$  be a germ of a  $p$ -dimensional submanifold. Then there exists a local coordinate system  $(x_1, \dots, x_n)$  on  $\mathbb{K}^n$  such that  $V = \{x_1 = \dots = x_{n-p} = 0\}$ . The pullback of  $\omega$  to  $V$  vanishes if and only if  $\omega = \sum_{i=1}^{n-p} x_i \alpha_i + dx_i \wedge \beta_i$ , where  $\alpha_i$  is a germ of a  $k$ -form and  $\beta_i$  is a germ of a  $(k-1)$ -form for  $i = 1, \dots, n-p$ . But then

$$\omega = \sum_{i=1}^{n-p} x_i (\alpha_i - d\beta_i) + d\left(\sum_{i=1}^{n-p} x_i \beta_i\right).$$

Thus  $\omega$  has zero weak algebraic restriction to  $V$ .

Now assume that  $\omega$  has zero weak algebraic restriction to  $V$ . Thus

$$\omega = \alpha + d\beta,$$

where  $\alpha$  is a germ of a  $k$ -form and  $\beta$  is a germ of a  $(k-1)$ -form such that  $df \wedge \alpha|_V = 0$ ,  $df \wedge \beta|_V = 0$  for any  $f \in I(V)$ .

We may write  $\alpha$  in the following way

$$\alpha = \sum_{i=1}^{n-p} x_i \gamma_i + dx_i \wedge \delta_i + \pi_V^* \sigma,$$

where  $\gamma_i$  is a germ of a  $k$ -form on  $\mathbb{K}^n$ ,  $\delta_i$  is a germ of a  $(k-1)$ -form on  $\mathbb{K}^n$  ( $i = 1, \dots, n-p$ ),  $\sigma$  is a germ of a  $k$ -form on  $V$  and  $\pi_V : \mathbb{K}^n \ni (x_1, \dots, x_n) \mapsto (x_{n-p+1}, \dots, x_n) \in V$  is the germ of the standard projection on  $V$ . But  $dx_i \wedge \alpha|_V = 0$ . Thus we obtain that  $\sigma = 0$  (for  $k < n$ ). Therefore the pullback of  $\alpha$  to  $V$  vanishes. In the same way we can show that the pullback of  $\beta$  to  $V$  vanishes.  $\square$

If  $V$  is a hypersurface then we have the following proposition.

**Proposition 3.3.** *If  $I(V) = \langle f \rangle$ , where  $f$  is a smooth ( $\mathbb{K}$ -analytic) function-germ, then a  $k$ -form  $\omega$  has zero weak algebraic restriction to  $V$  if and only if the pullback of  $\omega$  to the regular part of  $V$  vanishes.*

*Proof:* The pullback of  $\omega$  to the regular part of  $V$  vanishes if and only if  $df \wedge \omega|_V = 0$ . But this means that  $\omega$  has zero weak algebraic restriction to  $V$ .  $\square$

Let  $V$  be a germ of subset of  $\mathbb{K}^n$ . If  $\omega$  has weak zero algebraic restriction to  $V$  then the pullback of  $\omega$  to the regular part of  $V$  vanishes. If  $V$  is the germ of a singular curve in  $\mathbb{K}^3$  then the pullback of the germ of any 2-form to the regular part of  $V$  vanishes. But it is easy to see that there are 2-forms which do not have zero weak algebraic restriction to  $V$ .

**Example 3.4.** *Let  $V = \{x_1^2 - x_2^2 - x_3^2 = x_2x_3 = 0\}$  be a germ of a subset of  $\mathbb{K}^3$  at 0. Then pullback of any 2-form to the regular part of  $V$  vanishes, because the regular part of  $V$  is 1-dimensional, but the germ of a 2-form  $dx_1 \wedge dx_2$  does not have zero weak algebraic restriction to  $V$ .*

*The germ of the form  $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$  has zero weak algebraic restriction to  $V$ , because  $d(x_1^2 - x_2^2 - x_3^2) \wedge \omega = 2(x_1^2 - x_2^2 - x_3^2) dx_1 \wedge dx_2 \wedge dx_3$  and  $d(x_2x_3) \wedge \omega = 2x_2x_3 dx_1 \wedge dx_2 \wedge dx_3$ . But  $\omega$  does not have zero algebraic restriction to  $V$  (see [12]).*

We denote by  $\mathcal{W}_0^k(V)$  the set of  $k$ -forms with zero weak algebraic restriction to  $V$ . One can show that

**Proposition 3.5.** *Let  $\omega$  be a germ of an  $n$ -form on  $\mathbb{K}^n$ . If  $I(V) \neq \{0\}$  then  $\omega \in \mathcal{W}_0^n(V)$ .*

*Proof:* If  $f \neq 0$  then  $df \wedge \omega$  is the germ of an  $(n + 1)$ -form on  $\mathbb{K}^n$ . Thus  $df \wedge \omega = 0$  for any  $f \in I(V)$ .  $\square$

It is easy to see that  $\mathcal{W}_0^*(V)$  is a differential subcomplex of the complex of differential forms on  $\mathbb{K}^n$ . Thus we can define the cohomology groups of this subcomplex

$$H^k(\mathcal{W}_0^*(V)) = \frac{\{\omega \in \mathcal{W}_0^k(V) : d\omega = 0\}}{\{d\alpha : \alpha \in \mathcal{W}_0^{k-1}(V)\}}$$

By Proposition 3.3, if  $H$  is a hypersurface then  $H^*(\mathcal{W}_0^*(H))$  is the cohomology of the subcomplex of forms with vanishing pullback to the regular part of  $H$ , which we denote by  $H_{Giv}^*(H)$ . These were considered in [F] and [22] in the  $\mathbb{C}$ -analytic category (see also [13]). In [22] (see also [F]) it was proved that if  $V$  is the germ of a quasi-homogeneous  $\mathbb{C}$ -analytic variety then  $H_{Giv}^k(V) = \{0\}$  for  $k > 0$ . This result was generalized in [13].

We are interested in  $H^n(\mathcal{W}_0^*(V))$ . We define a class of subsets of  $\mathbb{K}^n$  for which this cohomology group vanishes. These are the subsets which are quasi-homogeneous with respect of a smooth submanifold, as defined in ([13])

**Definition 3.6.** *A germ of a subset  $V \subset \mathbb{K}^n$  is called quasi-homogeneous with respect to a smooth submanifold  $S$  if there exists a local coordinate system*

$$(x_1, \dots, x_k, y_1, \dots, y_{n-k})$$

*and positive numbers*

$$w_1, \dots, w_k$$

*(called weights) such that  $S$  is given by equations  $x_1 = 0, \dots, x_k = 0$  and, for all  $t$ , the map*

$$F_t : (x_1, \dots, x_k, y_1, \dots, y_{n-k}) \mapsto (t^{w_1}x_1, \dots, t^{w_k}x_k, y_1, \dots, y_{n-k})$$

*sends any point  $p \in V$  to a point  $F_t(p) \in V$ , provided that  $p$  and  $F_t(p)$  are sufficiently close to 0.*

This is a generalization of the classical quasi-homogeneity. The classical quasi-homogeneity is quasi-homogeneity with respect to  $S = \{0\}$ . The quasi-homogeneity with respect to a smooth submanifold is the classical quasi-homogeneity with some of the weights allowed to be 0.

**Example 3.7.** *Let*

$$N = \{(x_1, x_2, y) \in \mathbb{R}^3 : (x_1^2 - x_2^2)^2 + yx_1^2x_2^2 = 0\}.$$

*The set  $N$  is quasi-homogeneous with respect to the curve  $S : \{x_1 = x_2 = 0\}$  with weights  $(1, 1)$ : if  $(x_1, x_2, y) \in N$  then  $(tx_1, tx_2, y) \in N$ . In ([13]) it was shown that  $N$  is not quasi-homogeneous (in any coordinate system) in the classical sense.*

**Example 3.8.** *Let  $V$  be a germ of a subset of a smooth hypersurface  $H$ . Then  $V$  is quasi-homogeneous with respect to  $H$ . Indeed, there exists a coordinate system  $(x_1, x_2, \dots, x_n)$  on  $\mathbb{K}^n$  such that  $H = \{x_1 = 0\}$ .*

*Thus  $F_t(x_1, x_2, \dots, x_n) = (tx_1, x_2, \dots, x_n)$  maps  $V$  to  $V$ .*

The vector field

$$E_w = \sum_{i=1}^k w_i x_i \frac{\partial}{\partial x_i}.$$

is called the Euler vector field for a coordinate system  $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$  with weights  $w = (w_1, \dots, w_k)$ . If  $V$  is quasi-homogeneous with respect to a submanifold  $\{x_1 = \dots = x_k = 0\}$  with positive weights  $(w_1, \dots, w_k)$  then  $E_w$  is tangent to  $V$ . In fact it was proved in [13] that

**Theorem 3.9.** *Let  $S$  be a smooth submanifold of  $\mathbb{R}^n$ . Let  $N = \{H_1 = \dots = H_p = 0\}$ , where  $H_1, \dots, H_p$  generate the ideal of smooth (or  $\mathbb{K}$ -analytic) function-germs vanishing on  $N$ , and suppose that the set of non-singular points of  $N$  (the points near which  $N$  has the structure of a smooth submanifold of  $\mathbb{R}^n$ ) is dense in  $N$ .*

*Then  $N$  is quasi-homogeneous with respect to  $S$  if and only if there exists a vector field which is tangent to  $N$ , vanishes at any point  $x \in S$ , and whose eigenvalues at  $x \in S$  corresponding to directions transversal to  $S$  do not depend on  $x$  and are real positive numbers.*

**Proposition 3.10.** *Let  $V$  be the germ of a subset of  $\mathbb{K}^n$ . If  $V$  is quasi-homogeneous with respect to a smooth  $k$ -dimensional submanifold ( $k < n$ ) then  $H^n(\mathcal{W}_0^*(V)) = 0$ .*

The proof the above proposition is based on the following lemma.

**Lemma 3.11.** *Let  $E_w$  be a germ of the Euler vector field for a coordinate system  $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$  with positive weights  $w = (w_1, \dots, w_k)$  and*



let  $\Omega_0$  be the germ of the volume-form  $dx_1 \wedge \dots \wedge dx_k \wedge dy_1 \wedge \dots \wedge dy_{n-k}$ . If  $\omega$  is the germ of a smooth ( $\mathbb{K}$ -analytic)  $n$ -form on  $\mathbb{K}^n$  then there exists a smooth ( $\mathbb{K}$ -analytic) function-germ  $g$  on  $\mathbb{K}^n$  such that  $\omega = d(g(E_w]\Omega_0))$ .

*Proof:* Let  $G_t(x_1, \dots, x_k, y_1, \dots, y_{n-k}) = (e^{w_1 t} x_1, \dots, e^{w_k t} x_k, y_1, \dots, y_{n-k})$  for  $t \leq 0$  and  $x = (x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in \mathbb{K}^n$ . It is easy to see that

$$(G_t)' := \frac{d}{dt} G_t = E_w \circ G_t, \quad G_0 = Id_{\mathbb{K}^n}, \quad \lim_{t \rightarrow -\infty} G_t(x, y) = (0, y)$$

for any  $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in \mathbb{K}^n$ . Thus

$$\omega = G_0^* \omega - \lim_{t \rightarrow -\infty} G_t^* \omega = \int_{-\infty}^0 (G_t^* \omega)' dt. \quad (3.1)$$

But  $\omega = f\Omega_0$ , where  $f$  is a smooth ( $\mathbb{K}$ -analytic) function-germ and

$$(G_t^* \omega)' = G_t^* L_{E_w} \omega = G_t^* d(E_w] \omega) = d(G_t^*(E_w] \omega)).$$

Thus

$$(G_t^* \omega)' = d(G_t^*(E_w] f\Omega_0)) = d((f \circ G_t) G_t^*(E_w] \Omega_0)).$$

It is easy to check by direct calculation that  $G_t^*(E_w] \Omega_0) = e^{t \sum_{i=1}^k w_i} (E_w] \Omega_0)$ . Therefore  $(G_t^* \omega)' = d((f \circ G_t) e^{t \sum_{i=1}^k w_i} (E_w] \Omega_0))$ . Combining this with (3.1) we obtain

$$\omega = d\left(\int_{-\infty}^0 ((f \circ G_t) e^{t \sum_{i=1}^k w_i}) dt (E_w] \Omega_0)\right) = d(g(E_w] \Omega_0)),$$

where the function-germ  $g$  on  $\mathbb{K}^n$  is defined in the following way:

$$g(x, y) = \int_{-\infty}^0 e^{t \sum_{i=1}^k w_i} (f(G_t(x, y))) dt.$$

It is easy to see that  $g$  is smooth ( $\mathbb{K}$ -analytic), because

$$\int_{-\infty}^0 e^{t \sum_{i=1}^k w_i} (f(G_t(x, y))) dt = \int_0^1 s^\alpha f(F_s(x, y)) ds$$

where  $\alpha = (\sum_{i=1}^k w_i) - 1$  and

$$F_s(x_1, \dots, x_k, y_1, \dots, y_{n-k}) = (s^{w_1} x_1, \dots, s^{w_k} x_k, y_1, \dots, y_{n-k})$$

for any  $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_{n-k})$  and  $s \in [0, 1]$ . Multiplying weights by a sufficiently large constant we may assume that  $\alpha > 1$ .  $\square$

*Proof:* [Proof of Proposition 3.10] Let  $\omega$  be a germ of an  $n$ -form on  $\mathbb{K}^n$ . It is obvious that  $df \wedge \omega = 0$  for any  $f \in I(V)$ . By Lemma 3.1 there exists an  $n$ -form  $\beta$  such that  $\omega = d(E_w \lrcorner \beta)$  where  $E_w$  is the Euler vector field for  $V$ . But for any  $f \in I(V)$   $df \wedge (E_w \lrcorner \beta) = (E_w \lrcorner df)\beta$  and  $E_w \lrcorner df \in I(V)$ . Thus  $df \wedge (E_w \lrcorner \beta)|_V = 0$ .  $\square$

**Proposition 3.12.** *Let  $V$  be a germ of a subset of  $\mathbb{K}^n$ . If there exists a germ of a vector field  $Y$  tangent to  $V$ , such that  $Y|_0 \neq 0$ , then  $H^n(\mathcal{W}_0^*(V)) = 0$ .*

*Proof:* There exists a coordinate system  $(x_1, x_2, \dots, x_n)$  on  $\mathbb{K}^n$  such that  $Y = \frac{\partial}{\partial x_1}$ . If  $\omega$  is a germ of an  $n$ -form then

$$\omega = g dx_1 \wedge dx_2 \wedge \dots \wedge dx_n = d\left(\int_0^{x_1} g(t, x_2, \dots, x_n) dt \frac{\partial}{\partial x_1} \lrcorner dx_1 \wedge dx_2 \wedge \dots \wedge dx_n\right),$$

where  $g$  is a function-germ. It is easy to see that

$$df \wedge \left(\int_0^{x_1} g(t, x_2, \dots, x_n) dt \frac{\partial}{\partial x_1} \lrcorner dx_1 \wedge dx_2 \wedge \dots \wedge dx_n\right)|_V = 0$$

for any  $f \in I(V)$ . Thus  $H^n(\mathcal{W}_0^*(V)) = 0$ .  $\square$

**Remark 3.13.** *We can also prove Proposition 3.12 in a different way. If  $\frac{\partial}{\partial x_1}$  is tangent to  $V$  then  $x_1 \frac{\partial}{\partial x_1}$  is tangent to  $V$  too. By Lemma 3.1, using the same argument as in the proof of Proposition 3.10, we obtain  $H^n(\mathcal{W}_0^*(V)) = 0$ . If  $V$  satisfies the assumptions of Theorem 3.9 then  $V$  is quasi-homogeneous with respect to  $\{x_1 = 0\}$ .*

Let  $C_n$  be the ring of smooth ( $\mathbb{K}$ -analytic) function-germs at 0 on  $\mathbb{K}^n$ . Let  $I(V)$  be the ideal in  $C_n$  of function-germs vanishing on  $V$ .

Let  $Derlog(V)$  denote the module of germs at 0 of vector fields tangent to  $V$ , i.e.  $Y \in Derlog(V)$  if  $Y(I(V)) \subset I(V)$ .

Let  $\text{div}(Y)$  denote the divergence of a vector field  $Y = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$  with respect to the volume form  $\Omega_0 = dx_1 \wedge \dots \wedge dx_n$ , i.e.

$$\text{div}(Y) = \frac{\mathcal{L}_Y \Omega_0}{\Omega_0} = \frac{d(Y \lrcorner \Omega_0)}{\Omega_0} = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}.$$

Then  $\text{div}(\text{Derlog}(V)) := \{\text{div}(Y) \in C_n : Y \in \text{Derlog}(V)\}$  is a  $\mathbb{K}$ -vector subspace of  $C_n$ .

**Proposition 3.14.**  $H^n(\mathcal{W}_0^*(V)) \cong \frac{C_n}{\text{div}(\text{Derlog}(V))}$ .

*Proof:* Let  $\omega_0$  and  $\omega_1$  be germs of  $n$ -forms on  $\mathbb{K}^n$ . Let  $\Omega_0$  be the germ of a volume form on  $\mathbb{K}^n$ . Then  $\omega_0 = g_0\Omega_0$  and  $\omega_1 = g_1\Omega_0$ , where  $g_0$  and  $g_1$  are function-germs on  $\mathbb{K}^n$ , and  $[\omega_0]_V = [\omega_1]_V$  in  $H^k(\mathcal{W}_0^*(V))$  if and only if  $\omega_0 - \omega_1 = d\alpha$ , where  $\alpha$  is the germ of an  $(n-1)$ -form on  $\mathbb{K}^n$  such that  $df \wedge \alpha|_V = 0$  for any  $f \in I(V)$ . Let  $X$  be a germ of a vector field such that  $\alpha = X \lrcorner \Omega_0$  then

$$df \wedge \alpha = df \wedge (X \lrcorner \Omega_0) = (X \lrcorner df) \Omega_0.$$

Thus  $df \wedge \alpha|_V = 0$  if and only if  $X \lrcorner df \in I(V)$ .

We obtain that  $[\omega_0]_V = [\omega_1]_V$  in  $H^n(\mathcal{W}_0^*(V))$  if and only if  $g_0 - g_1 = \text{div}(X)$ , where  $X \in \text{Derlog}(V)$ .  $\square$

**Remark 3.15.** *Proposition 3.10 can be proved using Proposition 3.14. It is easy to show that if  $V$  is quasi-homogeneous with respect to the smooth submanifold then  $\text{div}(\text{Derlog}(V))$  is  $C_n$ .*

**Definition 3.16.** *We say that a cohomology class  $a \in H^n(\mathcal{W}_0^*(V))$  is realizable by a volume form if there exists a germ  $\Omega$  of a volume form such that  $[\Omega]_V = a$ . We denote by  $[\text{Vol}]_V$  the set of all cohomology classes realizable by volume forms.*

**Proposition 3.17.** *If there exists a germ of a volume form with zero cohomology class in  $H^n(\mathcal{W}_0^*(V))$  then  $[\text{Vol}]_V = H^n(\mathcal{W}_0^*(V))$ .*

*Proof:* Let  $\Omega$  be a germ of a volume form such that  $[\Omega]_V = [0]_V \in H^n(\mathcal{W}_0^*(V))$ . Let  $\alpha$  be a germ of an  $n$ -form on  $\mathbb{K}^n$ . If  $\alpha|_0 = 0$  then  $\beta = \alpha + \Omega$  is a germ of a volume form and  $[\beta]_V = [\alpha]_V \in H^n(\mathcal{W}_0^*(V))$ .  $\square$

**Proposition 3.18.** *If  $H^n(\mathcal{W}_0^*(V)) \setminus [\text{Vol}]_V \neq \emptyset$  then  $H^n(\mathcal{W}_0^*(V)) \setminus [\text{Vol}]_V$  is a  $\mathbb{K}$ -linear subspace of  $H^n(\mathcal{W}_0^*(V))$ .*

*If  $L$  is a finite dimensional linear subspace of  $H^n(\mathcal{W}_0^*(V))$  such that  $L \cap [\text{Vol}]_V \neq \emptyset$  then there exists a base  $a_1, \dots, a_{\dim_{\mathbb{K}} L}$  of  $L$  such that  $a_1, \dots, a_{\dim_{\mathbb{K}} L}$  are realizable by volume forms.*

*Proof:* By Proposition 3.17,  $[0]_V$  is not realizable by a volume form. If  $a \in H^n(\mathcal{W}_0^*(V))$  is not realizable by a volume form then neither is  $\lambda a$  for any  $\lambda \in \mathbb{K}$ , because if  $\lambda \neq 0$  and there exists a germ  $\Omega$  of a volume form such that  $\lambda a = [\Omega]_V$  then  $a = [\frac{1}{\lambda}\Omega]_V$ . In the same way we show that if  $a, b \in H^n(\mathcal{W}_0^*(V))$  are not realizable by a volume form then neither is  $a + b$ .

Let us assume that  $H^n(\mathcal{W}_0^*(V)) \setminus [Vol]_V \neq \emptyset$ . Then by Proposition 3.17 any germ of a volume form has non-zero cohomology class in  $H^n(\mathcal{W}_0^*(V))$ . Let  $\Omega$  be a germ of a volume form such that  $[\Omega]_V \in L$ . Then  $[\Omega]_V \neq [0]_V$  and let  $[\Omega]_V, [\beta_2]_V, \dots, [\beta_{\dim_{\mathbb{K}} L}]_V$  be a base of  $L$ . Let  $a_1 = [\Omega]_V$  and  $a_i = [\beta_i]_V$  if  $\beta_i|_0 \neq 0$  and  $a_i = [\beta_i + \Omega]_V$  if  $\beta_i|_0 = 0$  for  $i = 2, \dots, \dim_{\mathbb{K}} L$ . It is easy to see that  $a_1, \dots, a_{\dim_{\mathbb{K}} L}$  is a base of  $L$  and  $a_1, \dots, a_{\dim_{\mathbb{K}} L}$  are realizable by volume forms.  $\square$

By Propositions 3.17 and 3.18 we obtain the following corollary.

**Corollary 3.19.**  $\dim_{\mathbb{K}}[Vol]_V = \dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V))$ .

*Proof:* If there exists a germ of a volume form with zero cohomology class in  $H^n(\mathcal{W}_0^*(V))$  then  $[Vol]_V = H^n(\mathcal{W}_0^*(V))$  by Proposition 3.17. But if  $H^n(\mathcal{W}_0^*(V)) \setminus [Vol]_V \neq \emptyset$  then  $[Vol]_V = H^n(\mathcal{W}_0^*(V)) \setminus T$ , where  $T$  is a subspace of codimension at least 1 by Proposition 3.18.  $\square$

## 4 Volume preserving diffeomorphisms on varieties

Let  $\Omega_0, \Omega_1$  be germs at 0 of smooth ( $\mathbb{K}$ -analytic) volume-forms on  $\mathbb{K}^n$ . Let  $V$  be the germ at 0 of a subvariety of  $\mathbb{K}^n$ . We define two equivalence relations on the space of volume forms on  $\mathbb{K}^n$ .

**Definition 4.1.** We say that  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{R}_V$ -equivalent if there exists a germ of a diffeomorphism  $\Phi : (\mathbb{K}^n, V, 0) \rightarrow (\mathbb{K}^n, V, 0)$  (preserving  $V$ ) such that  $\Phi^*\Omega_1 = \Omega_0$ .

**Definition 4.2.** We say that  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{IR}_V$ -equivalent if there exists a smooth family of germs of diffeomorphisms  $\Phi_t : (\mathbb{K}^n, V, 0) \rightarrow (\mathbb{K}^n, V, 0)$  (preserving  $V$ ) for  $t \in [0, 1]$  such that  $\Phi_0 = Id$  and  $\Phi_1^*\Omega_0 = \Omega_1$ .

It is obvious that  $\mathcal{IR}_V$ -equivalence implies  $\mathcal{R}_V$ -equivalence. We now prove one of the main results of the paper.

**Theorem 4.3.** *Let  $V$  be a germ of a subset of  $\mathbb{K}^n$ . Let  $\Omega_0$  and  $\Omega_1$  be germs of smooth (or  $\mathbb{K}$ -analytic) volume forms on  $\mathbb{K}^n$ . If  $\mathbb{K} = \mathbb{R}$  we assume that  $\Omega_0$  and  $\Omega_1$  define the same orientation of  $\mathbb{K}^n$ .*

*Then  $[\Omega_0]_V = [\Omega_1]_V$  in  $H^n(\mathcal{W}_0^*(V))$  if and only if  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{IR}_V$ -equivalent.*

*Proof:* First we assume that any germ of a vector field tangent to  $V$  vanishes at 0. If  $[\Omega_0]_V = [\Omega_1]_V$  in  $H^n(\mathcal{W}_0^*(V))$  then there exists a germ of an  $(n-1)$ -form  $\alpha$  on  $\mathbb{K}^n$  such that

$$\Omega_0 - \Omega_1 = d\alpha, \quad df \wedge \alpha|_V = 0 \quad \forall f \in I(V). \quad (4.1)$$

We use Moser's homotopy method ([31]). Let  $\Omega_t = \Omega_0 + t(\Omega_1 - \Omega_0)$  for  $t \in [0, 1]$ . It is easy to see that if  $\Omega_0$  and  $\Omega_1$  define the same orientation of  $\mathbb{R}^n$  then  $\Omega_t$  is a germ of a volume-form at 0 for any  $t \in [0, 1]$ . We are looking for diffeomorphisms  $\Phi_t : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  for  $t \in [0, 1]$  such that

$$\Phi_t^* \Omega_t = \Omega_0 \quad (4.2)$$

and  $\Phi_0 = Id_{\mathbb{K}^n}$ ,  $\Phi_t(V) \subset V$ . Differentiating (4.2) we obtain

$$\Phi_t^*(L_{U_t} \Omega_t + \Omega_1 - \Omega_0) = 0,$$

where  $U_t \circ \Phi_t = \frac{d}{dt} \Phi_t$ .  $\Phi_t(V) \subset V$  if and only if  $U_t$  is tangent to  $V$ . Thus by (4.1) we have

$$d(U_t \rfloor \Omega_t) = \Omega_0 - \Omega_1 = d\alpha. \quad (4.3)$$

Thus we want to find  $U_t$  tangent to  $V$  such that

$$U_t \rfloor \Omega_t = \alpha. \quad (4.4)$$

Let  $X$  be a germ of a vector field such that  $X \rfloor \Omega_0 = \alpha$ . Then  $df \wedge \alpha = df \wedge (X \rfloor \Omega_0) = (X \rfloor df) \Omega_0$ . We have that  $df \wedge \alpha|_V = 0$  for any  $f \in I(V)$ . Thus  $X \rfloor df \in I(V)$  for any  $f \in I(V)$ , which means that  $X$  is tangent to  $V$ . By the above assumption  $X|_0 = 0$ . But

$$\Omega_t = g_t \Omega_0$$

where  $g_t$  is a non-vanishing function-germ at 0. Hence  $U_t = \frac{1}{g_t}X$  is a solution of (4.4), which is tangent to  $V$  and vanishes at 0. By integration of  $U_t$  we obtain a smooth family of germs of diffeomorphism  $\Phi_t : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  for  $t \in [0, 1]$  such that  $\Phi_t(V) \subset V$ ,  $\Phi_0 = Id$  and  $\Phi_1^*\Omega_1 = \Omega_0$ .

To conclude the proof of the forward direction, assume that there exists a germ of vector field  $Y$  tangent to  $V$  such that  $Y|_0 \neq 0$ . Then there exists a coordinate system  $(x_1, \dots, x_n)$  on  $\mathbb{K}^n$  such that  $Y = \frac{\partial}{\partial x_1}$ . If  $\omega$  is a germ of an  $n$ -form then

$$\begin{aligned} \Omega_0 - \Omega_1 &= g dx_1 \wedge dx_2 \wedge \dots \wedge dx_n = \\ &= d\left(\int_0^{x_1} g(t, x_2, \dots, x_n) dt \frac{\partial}{\partial x_1}\right) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n, \end{aligned}$$

where  $g$  is some function-germ and  $X = \int_0^{x_1} g(t, x_2, \dots, x_n) dt \frac{\partial}{\partial x_1}$  vanishes at 0.

But  $Y|_0 \neq 0$  also implies that  $H^n(\mathcal{W}_0^*) = 0$ , by Proposition 3.12. Now we can continue in the same way as in the first part of the proof.

For the converse, assume that  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{IR}_V$ -equivalent. Then there exists a smooth family of germs of diffeomorphisms  $\Phi_t : (\mathbb{K}^n, V, 0) \rightarrow (\mathbb{K}^n, V, 0)$  (preserving  $V$ ) for all  $t \in [0, 1]$  such that  $\Phi_0 = Id$  and  $\Phi_1^*\Omega_0 = \Omega_1$ . Let  $(\Phi_t)' = \frac{d}{dt}\Phi_t = X_t \circ \Phi_t$ . Then

$$\Omega_1 - \Omega_0 = \Phi_1^*\Omega_0 - \Omega_0 = \int_0^1 (\Phi_t^*\Omega_0)' dt = \int_0^1 (\Phi_t^*\mathcal{L}_{X_t}\Omega_0) dt = \int_0^1 \Phi_t^*d(X_t \lrcorner \Omega_0) dt.$$

Thus

$$\Omega_1 - \Omega_0 = d\left(\int_0^1 \Phi_t^*(X_t \lrcorner \Omega_0) dt\right).$$

$X_t$  is tangent to  $V$ . Thus for any  $f \in I(V)$

$$df \wedge (X_t \lrcorner \Omega_0)|_V = (X_t \lrcorner df) \wedge \Omega_0|_V = 0.$$

But

$$df \wedge \Phi_t^*(X_t \lrcorner \Omega_0) = \Phi_t^*(d((\Phi_t^{-1})^*f) \wedge (X_t \lrcorner \Omega_0))$$

and  $\Phi_t(V) \subset V$ . Hence  $df \wedge \Phi_t^*(X_t \lrcorner \Omega_0)|_V = 0$ , which implies that

$$df \wedge \int_0^1 \Phi_t^*d(X_t \lrcorner \Omega_0) dt|_V = 0.$$

Thus  $[\Omega_0]_V = [\Omega_1]_V$  in  $H^n(\mathcal{W}_0^*(V))$ . □

In [2] (see also [38], [28]) it was shown that if  $V$  is the germ of a  $\mathbb{C}$ -analytic quasi-homogeneous hypersurface with an isolated singularity then any two holomorphic volume forms on  $\mathbb{C}^n$  are  $\mathcal{R}_V$ -equivalent. More generally we have (for set-germs  $V$  of any codimension and with possibly non-isolated singularities) the following

**Theorem 4.4.** *Let  $V$  be a germ of a subset of  $\mathbb{K}^n$ .*

1. *Let  $\mathbb{K} = \mathbb{R}$  and let  $\Omega_0$  and  $\Omega_1$  be germs of smooth or  $\mathbb{R}$ -analytic volume-forms at 0. If  $V$  is quasi-homogeneous with respect to a smooth submanifold of  $\mathbb{R}^n$  and  $\Omega_0|_0, \Omega_1|_0$  define the same orientation of  $T_0\mathbb{R}^n$  then  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{R}_V$ -equivalent.*
2. *Let  $\mathbb{K} = \mathbb{C}$  and let  $\Omega_0$  and  $\Omega_1$  be germs of  $\mathbb{C}$ -analytic volume-forms at 0. If  $V$  is quasi-homogeneous with respect to a  $\mathbb{C}$ -analytic submanifold of  $\mathbb{C}^n$  then  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{R}_V$ -equivalent.*

In [38] it was shown that if  $H = \{f = 0\}$  is a germ at 0 of a  $\mathbb{C}$ -analytic hypersurface in  $\mathbb{C}^n$  with isolated singularity at 0 then the number of  $\mathcal{IR}_H$ -moduli of volume forms on  $\mathbb{C}^n$  is equal to  $\mu - \tau$ , where  $\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle}$  is the Milnor number of  $H$  and  $\tau = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle}$  is the Tjurina number of  $H$ .

Let  $V$  be the germ of a subset of  $\mathbb{K}^n$  and let  $\Omega$  be the germ of a volume form on  $\mathbb{K}^n$ . The  $\mathcal{IR}_V$ -orbit of  $\Omega$  is the identity component of the  $\mathcal{R}_V$ -orbit of  $\Omega$ . Thus the number of  $\mathcal{R}_V$ -moduli of germs of volume forms is equal to the number of  $\mathcal{IR}_V$ -moduli of germs of volume forms. By Theorem 4.3 and Proposition 3.14 we obtain the following result.

**Proposition 4.5.** *Let  $V$  be a germ of a subset of  $\mathbb{K}^n$ . Then the number of  $\mathcal{R}_V$ -moduli of germs of volume forms is equal to*

$$\dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V)) = \dim_{\mathbb{K}} \frac{C_n}{\operatorname{div}(\operatorname{Derlog}(V))}.$$

*Proof:* Let  $\Omega_0$  and  $\Omega_1$  be germs of volume forms on  $\mathbb{K}^n$ . If  $[\Omega_0]_V = [\Omega_1]_V$  in  $H^n(\mathcal{W}_0^*(V))$  then  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{R}_V$ -equivalent by Theorem 4.3. By Corollary 3.19 and Proposition 3.14 we have  $\dim_{\mathbb{K}} [Vol]_V = \dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V)) = \dim_{\mathbb{K}} \frac{C_n}{\operatorname{div}(\operatorname{Derlog}(V))}$ . Now assume that  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{R}_V$ -equivalent. We also

may assume that they are in the same component of an  $\mathcal{R}_V$ -orbit. But then they are  $\mathcal{IR}_V$ -equivalent. Hence  $[\Omega_1]_V = [\Omega_0]_V$  in  $H^n(\mathcal{W}_0^*(V))$  by Theorem 4.3.  $\square$

Let  $\Omega$  be the germ of a fixed smooth ( $\mathbb{K}$ -analytic) volume-form on  $\mathbb{K}^n$  at 0, and let  $V_1, V_2$  be germs of subsets of  $\mathbb{K}^n$  at  $0 \in \mathbb{K}^n$ .

**Definition 4.6.**  $V_1$  and  $V_2$  are  $\mathcal{R}_\Omega$ -equivalent if there exists a volume-preserving diffeomorphism-germ  $\Phi : (\mathbb{K}^n, \Omega, 0) \rightarrow (\mathbb{K}^n, \Omega, 0)$  (i.e.  $\Phi^*\Omega = \Omega$ ) such that  $\Phi(V_1) = V_2$ .

It is easy to prove that

**Proposition 4.7.** Let  $\Phi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  be a germ of a diffeomorphism such that  $\Phi(V_1) = V_2$ .

If  $\omega$  is the germ of a  $k$ -form which has zero weak algebraic restriction to  $V_2$  then  $\Phi^*\omega$  has zero weak algebraic restriction to  $V_1$ .

If  $\omega$  is the germ of a  $k$ -form which has zero weak algebraic restriction to  $V_2$  and  $[\omega]_{V_2} = 0$  in  $H^k(\mathcal{W}_0^*(V_2))$  then  $[\Phi^*\omega]_{V_1} = 0$  in  $H^k(\mathcal{W}_0^*(V_1))$ .

*Proof:* If  $\omega$  has zero weak algebraic restriction to  $V_2$  then  $\omega = \alpha + d\beta$ , where  $\alpha$  is the germ of a  $k$ -form and  $\beta$  is the germ of a  $(k-1)$ -form such that  $df \wedge \alpha|_{V_2} = 0$  and  $df \wedge \beta|_{V_2} = 0$  for any  $f \in I(V_2)$ . Then  $\Phi^*\omega = \Phi^*\alpha + d\Phi^*\beta$ . Let  $g$  be a function-germ vanishing on  $V_1$ . Then  $(\Phi^{-1})^*g \in I(V_2)$ , because  $\Phi(V_1) = V_2$ . Thus  $d(\Phi^{-1})^*g \wedge \alpha|_{V_2} = 0$ . Hence

$$dg \wedge \Phi^*\alpha|_{V_1} = \Phi^*(d((\Phi^{-1})^*g) \wedge \alpha)|_{V_1} = 0.$$

In the same way we show that  $dg \wedge \Phi^*\beta|_{V_1} = 0$ . The proof of the second statement is similar.  $\square$

Thus we can define diffeomorphic cohomology classes.

**Definition 4.8.** We say that cohomology classes  $[\omega_1]_{V_1} \in H^k(\mathcal{W}_0^*(V_1))$  and  $[\omega_2]_{V_2} \in H^k(\mathcal{W}_0^*(V_2))$  are diffeomorphic if there exists a germ of a diffeomorphism  $\Phi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $\Phi(V_1) = V_2$  and  $[\Phi^*\omega_2]_{V_1} = [\omega_1]_{V_1}$  in  $H^k(\mathcal{W}_0^*(V_1))$ .

Now, as a corollary of Theorem 4.3, we obtain the following theorem.

**Theorem 4.9.**  $V_1$  and  $V_2$  are  $\mathcal{R}_\Omega$ -equivalent if and only if the classes  $[\Omega]_{V_1} \in H^n(\mathcal{W}_0^*(V_1))$  and  $[\Omega]_{V_2} \in H^n(\mathcal{W}_0^*(V_2))$  are diffeomorphic.



*Proof:* It is obvious that if  $V_1$  and  $V_2$  are  $\mathcal{R}_\Omega$ -equivalent then the classes  $[\Omega]_{V_1} \in H^n(\mathcal{W}_0^*(V_1))$  and  $[\Omega]_{V_2} \in H^n(\mathcal{W}_0^*(V_2))$  are diffeomorphic.

If  $[\Omega]_{V_1} \in H^n(\mathcal{W}_0^*(V_1))$  and  $[\Omega]_{V_2} \in H^n(\mathcal{W}_0^*(V_2))$  are diffeomorphic then there exists a germ of a diffeomorphism  $\Phi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $\Phi(V_1) = V_2$  and  $[\Phi^*\Omega]_{V_1} = [\Omega]_{V_1}$  in  $H^n(\mathcal{W}_0^*(V_1))$ . Theorem 4.3 implies that there exists a germ of a diffeomorphism  $\Psi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $\Psi(V_1) = V_1$  and  $\Psi^*\Phi^*\Omega = \Omega$ . Thus the germ of the diffeomorphism  $\Phi \circ \Psi$  preserves the volume form  $\Omega$  and maps  $V_1$  to  $V_2$ .  $\square$

Theorem 4.4 can be formulated in the following way.

**Theorem 4.10.** *Let  $V, V_1$  be germs of subsets of  $\mathbb{K}^n$  at  $p$ .*

1. *Let  $\mathbb{K} = \mathbb{R}$  and fix a germ  $\Omega$  of a smooth (or  $\mathbb{R}$ -analytic) volume-form at  $p$ . If  $V$  is quasi-homogeneous with respect to a smooth submanifold then  $V$  and  $V_1$  are  $\mathcal{R}_\Omega$ -equivalent if and only if there exists a germ of smooth ( $\mathbb{R}$ -analytic) orientation-preserving diffeomorphism  $\Phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  such that  $\Phi(V) = V_1$ .*
2. *Let  $\mathbb{K} = \mathbb{C}$  and fix a germ  $\Omega$  of a  $\mathbb{C}$ -analytic volume-form at  $p$ . If  $V$  is quasi-homogeneous with respect to a  $\mathbb{C}$ -analytic submanifold then  $V$  and  $V_1$  are  $\mathcal{R}_\Omega$ -equivalent if and only if  $V$  and  $V_1$  are  $\mathbb{C}$ -analytically diffeomorphic.*

Here is an example of a germ of a singular curve on  $\mathbb{R}^2$  which determines an orientation on  $\mathbb{R}^2$ .

**Example 4.11.** *Let  $V_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 - x_1^5 = 0\}$  and  $V_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 + x_1^5 = 0\}$ .  $V_1$  and  $V_2$  are quasi-homogeneous with weights  $(3, 5)$  and they are diffeomorphic. For example take  $\Phi(x_1, x_2) = (-x_1, x_2)$ . But they are not  $\mathcal{R}_\Omega$ -equivalent, because any diffeomorphism mapping  $V_1$  to  $V_2$  changes the orientation of  $\mathbb{R}^2$ .*

## 5 $\mathcal{A}_\Omega$ -equivalence of maps

We now study  $\mathcal{A}_\Omega$ -equivalence of map-germs  $f$ , where  $\Omega$  is a fixed volume form on the target of  $f$ . The results about  $\mathcal{A}_\Omega$ -equivalence will follow from those about  $\mathcal{A}_f$ -equivalence of volume forms, where the map-germ  $f$  is fixed,

in the same way as the results about  $\mathcal{R}_\Omega$  were obtained in the previous section from those about  $\mathcal{R}_V$ .

Furthermore, as mentioned already in Remark 2.5, a slightly weaker versions of the results about maps  $f$  can be deduced from those about set-germs  $V$ , by considering vector fields tangent to the discriminant (or the image)  $\Delta(f)$  of  $f$ . More precisely, using the results in [32], [33], [8] – and in particular the necessary and sufficient conditions in Theorem 2 of [8] for a vector field tangent to  $\Delta(f)$  to lift over  $f$  – we could deduce our results about  $\mathcal{A}_\Omega$  and  $\mathcal{A}_f$  from those about  $\mathcal{R}_\Omega$  and  $\mathcal{R}_V$  (taking  $V = \Delta(f)$ ) for the class of maps  $f$  satisfying the conditions in Theorem 2 of [8]. This class would, for example over  $\mathbb{C}$ , include all  $\mathcal{A}$ -finite map-germs  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  for all pairs  $(n, p)$  different from  $(n, 2)$ ,  $n \geq 2$ . Nevertheless, working directly with liftable vector fields, we can avoid these restrictions – see Example 5.13 below for a weakly quasi-homogeneous map-germ  $f$  (to which our Theorem 2.4 applies) which does not satisfy the conditions in [8].

First, we recall the definition of a liftable vector field ([1], [8]).

**Definition 5.1.** *The germ of a vector field  $Y$  on  $\mathbb{K}^p$  is liftable or lifts over a map-germ  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  to a germ of a vector field  $X$  at 0 on  $\mathbb{K}^n$  if*

$$df(X) = Y \circ f.$$

(Considering homomorphisms  $tf$  from the  $C_n$ -module of source vector fields to the  $C_n$ -module  $\theta_f$  of smooth sections of  $f^*T\mathbb{K}^p$  and  $\omega f$  from the  $C_p$ -module of target vector fields to  $\theta_f$ , given by  $X \mapsto df(X)$  and  $Y \mapsto Y \circ f$  respectively, the above equation becomes  $tf(X) = \omega f(Y)$  – this notation will be used in the next section when we study the  $\mathcal{A}_\Omega$ -tangent space of  $f$ .) We denote the  $C_p$ -module of germs of vector fields liftable over  $f$  by  $Lift(f)$  and the  $C_p$ -submodule of  $Lift(f)$  of germs of vector fields liftable over  $f$  to germs of vector fields **vanishing at 0** by  $Lift_0(f)$ .

Let  $\Omega_0, \Omega_1$  be germs at 0 of smooth ( $\mathbb{K}$ -analytic) volume-forms on  $\mathbb{K}^p$  and let  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  be a smooth (or  $\mathbb{K}$ -analytic) map-germ.

**Definition 5.2.** *We say that  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{A}_f$ -equivalent if there exist germs of diffeomorphisms  $\Phi : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $\Phi^*\Omega_1 = \Omega_0$  and  $\Phi \circ f = f \circ \Psi$ .*

**Remark 5.3.** *The condition  $\Phi \circ f = f \circ \Psi$  is equivalent to  $\Phi \circ f \circ \Psi^{-1} = f$ . It means that  $f$  is preserved by  $\mathcal{A}$ -action of  $(\Psi^{-1}, \Phi)$ .*

**Definition 5.4.** We say that  $\Omega_0, \Omega_1$  are  $\mathcal{IA}_f$ -equivalent if there exist smooth families of germs of diffeomorphisms  $\Phi_t : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi_t : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  for  $t \in [0, 1]$  such that  $\Phi_0 = Id_{\mathbb{K}^p}$ ,  $\Psi_0 = Id_{\mathbb{K}^n}$ ,  $\Phi_1^* \Omega_0 = \Omega_1$  and  $\Phi_t \circ f = f \circ \Psi_t$  for  $t \in [0, 1]$ .

Obviously  $\mathcal{IA}_f$ -equivalence implies  $\mathcal{A}_f$ -equivalence.

Let  $\text{div}(Y)$  denote the divergence of a germ of a vector field  $Y$  with respect to the germ of the volume form  $\Omega$ , i.e.  $\text{div}(Y)\Omega = d(Y \lrcorner \Omega) = \mathcal{L}_Y \Omega$ .

**Definition 5.5.** We say that germs of volume forms  $\Omega_0$  and  $\Omega_1$  are  $\text{divLift}_0(f)$ -equivalent if there exists a germ of a vector field  $Y \in \text{Lift}_0(f)$  such that  $\Omega_1 - \Omega_0 = \text{div}(Y)\Omega$ .

**Remark 5.6.**  $\text{divLift}_0(f)$  is an equivalence relation.

We denote the  $\text{divLift}_0(f)$ -equivalence class of a germ of a volume form  $\Omega_0$  by  $[\Omega_0]_f$ . The definition of  $\text{divLift}_0(f)$  does not depend of the choice of germ of a volume form  $\Omega$ . If  $\tilde{\Omega}$  is the germ of another volume form then  $\Omega = g\tilde{\Omega}$ . If  $\Omega_1 - \Omega_0 = d(Y \lrcorner \Omega)$  for  $Y \in \text{Lift}_0(f)$  then  $\Omega_1 - \Omega_0 = d(gY \lrcorner \tilde{\Omega})$  and  $gY \in \text{Lift}_0(f)$ .

Now we state one of the main results of this section.

**Theorem 5.7.** Let  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  be a smooth ( $\mathbb{K}$ -analytic) map-germ. Let  $\Omega_0$  and  $\Omega_1$  be germs of smooth ( $\mathbb{K}$ -analytic) volume forms on  $\mathbb{C}^p$ . If  $\mathbb{K} = \mathbb{R}$  we assume that  $\Omega_0$  and  $\Omega_1$  define the same orientation of  $\mathbb{R}^p$ .

Then  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{IA}_f$ -equivalent if and only if  $\Omega_0$  and  $\Omega_1$  are  $\text{divLift}_0(f)$ -equivalent.

*Proof:* First we assume that  $\Omega_1 - \Omega_0 = \text{div}(Y)\Omega_0$  and  $df(X) = Y \circ f$ , where  $X$  is the germ of a vector field at 0 on  $\mathbb{K}^n$  such that  $X|_0 = 0$ . We use Moser's homotopy method ([31]). Let  $\Omega_t = \Omega_0 + t(\Omega_1 - \Omega_0)$  for  $t \in [0, 1]$ . It is easy to see that  $\Omega_t$  is the germ of a volume-form at 0 for any  $t \in [0, 1]$ . We are looking for diffeomorphisms  $\Phi_t : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$ ,  $\Psi_t : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  for  $t \in [0, 1]$  such that

$$\Phi_t^* \Omega_t = \Omega_0 \tag{5.1}$$

and  $\Phi_0 = Id_{\mathbb{K}^p}$ ,  $\Psi_0 = Id_{\mathbb{K}^n}$ ,  $\Phi_t \circ f = f \circ \Psi_t$ . Differentiating (5.1) we obtain

$$\Phi_t^*(L_{Y_t} \Omega_t + \Omega_1 - \Omega_0) = 0,$$

where  $Y_t \circ \Phi_t = \frac{d}{dt}\Phi_t$ . We have  $\Phi_t \circ f = f \circ \Psi_t$  if and only if  $Y_t \circ f = df(X_t)$ , where  $X_t \circ \Psi_t = \frac{d}{dt}\Psi_t$ . By the assumptions

$$d(Y_t \rfloor \Omega_t) = \Omega_0 - \Omega_1 = \operatorname{div}(Y)\Omega_0 = d(Y \rfloor \Omega_0). \quad (5.2)$$

But

$$\Omega_t = h_t \Omega_0,$$

where  $h_t$  is a non-vanishing function-germ at 0. Hence  $Y_t = \frac{1}{h_t}Y$  is a solution of (5.2). Multiplying  $df(X) = Y \circ f$  by  $1/h_t \in C_p$  (using the  $C_p$ -module structure of  $Lift(f)$ ) we have

$$Y_t \circ f = \left(\frac{1}{h_t}Y\right) \circ f = df\left(\left(\frac{1}{h_t} \circ f\right)X\right) = df(X_t),$$

where  $X_t := \left(\frac{1}{h_t} \circ f\right)X$  vanishes at 0, because  $X$  vanishes at 0. And  $Y_t$  vanishes at 0 too, because  $Y_t|_0 = Y_t|_{f(0)} = df|_0(X_t|_0) = df|_0(0) = 0$ .

Integrating  $Y_t$  and  $X_t$  we obtain diffeomorphism-germs  $\Phi_t : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi_t : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $\Phi_1^*\Omega_1 = \Omega_0$  and  $\Phi_t \circ f = f \circ \Psi_t$ .

Now assume that  $\Omega_1$  and  $\Omega_0$  are  $\mathcal{I}\mathcal{A}_f$ -equivalent. Then there exist smooth families of germs of diffeomorphisms  $\Phi_t : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi_t : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $\Phi_0 = Id_{\mathbb{K}^p}$ ,  $\Psi_0 = Id_{\mathbb{K}^n}$ ,  $\Phi_1^*\Omega_0 = \Omega_1$  and  $\Phi_t \circ f = f \circ \Psi_t$  for all  $t \in [0, 1]$ . Thus

$$\Omega_1 - \Omega_0 = \int_0^1 \frac{d}{dt}\Phi_t^*\Omega_0 dt = \int_0^1 \Phi_t^*d(Y_t \rfloor \Omega_0) dt = d \int_0^1 \Phi_t^*(Y_t \rfloor \Omega_0) dt,$$

where  $Y_t$  is a germ of a vector field on  $\mathbb{K}^p$  such that  $Y_t \circ \Phi_t = \frac{d}{dt}\Phi_t$ .

But

$$\int_0^1 \Phi_t^*(Y_t \rfloor \Omega_0) dt = \int_0^1 (d(\Phi_t^{-1})(Y_t \circ \Phi_t)) \rfloor \Phi_t^*\Omega_0 dt = \int_0^1 Jac(\Phi_t)\tilde{Y}_t dt \rfloor \Omega_0,$$

where  $\tilde{Y}_t = d(\Phi_t^{-1})(Y_t \circ \Phi_t)$  is a germ of a vector field on  $\mathbb{K}^p$  and  $Jac(\Phi_t) = \det d\Phi_t$ .

Differentiating  $\Phi_t \circ f = f \circ \Psi_t$  we obtain  $Y_t \circ f = df(X_t)$ , where  $X_t$  is a germ of a vector field at 0 such that  $X_t \circ \Psi_t = \frac{d}{dt}\Psi_t$ .

Then  $\tilde{Y}_t \circ f = d(\Phi_t^{-1})(Y_t \circ \Phi_t \circ f) = d(\Phi_t^{-1})(Y_t \circ f \circ \Psi_t) = d(\Phi_t^{-1})(df(X_t \circ \Psi_t)) = d(\Phi_t^{-1} \circ f)(X_t \circ \Psi_t) = d(f \circ \Psi_t^{-1})(X_t \circ \Psi_t) = df(d(\Psi_t^{-1})(X_t \circ \Psi_t))$ .

Thus

$$\tilde{Y}_t \circ f = df(\tilde{X}_t), \quad t \in [0, 1],$$

where  $\tilde{X}_t = d(\Psi_t^{-1})(X_t \circ \Psi_t)$  is a germ of a vector field on  $\mathbb{K}^n$ .

Hence

$$\int_0^1 \text{Jac}(\Phi_t) \tilde{Y}_t dt \circ f = df \left( \int_0^1 (\text{Jac}(\Phi_t) \circ f) \tilde{X}_t dt \right).$$

Thus a germ of a vector field  $Y = \int_0^1 \text{Jac}(\Phi_t) \tilde{Y}_t dt$  lifts over  $f$  to a germ of a vector field  $X = \int_0^1 (\text{Jac}(\Phi_t) \circ f) \tilde{X}_t dt$  and  $\Omega_1 - \Omega_0 = d(Y \rfloor \Omega_0) = \text{div}(Y) \Omega_0$ . It is easy to see that  $X$  vanishes at 0, because  $X_t|_0 = 0$ .  $\square$

From Theorem 5.7 we get the following corollary, using the fact the  $\mathcal{IA}_f$ -orbit of  $\Omega$  is the identity component of the  $\mathcal{A}_f$ -orbit of  $\Omega$ .

**Corollary 5.8.** *The number of  $\mathcal{A}_f$ -moduli of germs of volume forms on  $\mathbb{K}^n$  is equal to  $\dim_{\mathbb{K}} \frac{C_p}{\text{div}(\text{Lift}_0(f))}$ .*

We now define a class of map-germs  $f$  for which  $\text{div}(\text{Lift}_0(f))$  is  $C_p$ .

**Definition 5.9.** *We say that a smooth ( $\mathbb{K}$ -analytic) map-germ  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  is weakly quasi-homogeneous if it is quasi-homogeneous with **non-negative** degrees  $\delta_1, \dots, \delta_p$  and a **positive** total degree  $\delta_1 + \dots + \delta_p$  for **integer** weights  $w_1, \dots, w_n$  (fixing coordinates  $(x_1, \dots, x_n) \in \mathbb{K}^n$  and  $(X_1, \dots, X_p) \in \mathbb{K}^p$ ).*

*For convenience, we assume that the degree of the zero function-germ is 1 (see Example 5.12 below).*

Define Euler-like vector fields on  $\mathbb{K}^n$  (with  $w_i \in \mathbb{Z}$ )

$$E_w = \sum_{i=1}^n w_i x_i \frac{\partial}{\partial x_i}.$$

and on  $\mathbb{K}^p$  (with  $\delta_i \geq 0$ ,  $\sum_i \delta_i > 0$ )

$$E_\delta = \sum_{i=1}^p \delta_i X_i \frac{\partial}{\partial X_i}.$$

The following easy proposition gives one more equivalent definition of weak quasi-homogeneity of a map.

**Proposition 5.10.** *The following conditions on a smooth ( $\mathbb{K}$ -analytic) map-germ  $f$  are equivalent:*

(i)  $f$  is weakly quasi-homogeneous with degrees  $\delta = (\delta_1, \dots, \delta_p)$  for weights  $w = (w_1, \dots, w_n)$  in coordinate systems  $x = (x_1, \dots, x_n)$  on  $\mathbb{K}^n$  and  $X = (X_1, \dots, X_p)$  on  $\mathbb{K}^p$ ;

(ii)  $E_\delta \circ f = df(E_w)$ .

All quasi-homogeneous map-germs are weakly quasi-homogeneous. But there are weakly quasi-homogeneous map-germs which are not quasi-homogeneous.

**Example 5.11.** Let  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  be a map-germ  $\mathcal{A}$ -equivalent to

$$\tilde{f}(u, x_1, \dots, x_{n-1}) = (u, g(x_1, \dots, x_{n-1}))$$

for some map-germ  $g : (\mathbb{K}^{n-1}, 0) \rightarrow (\mathbb{K}^{p-1}, 0)$ . Then  $f$  is weakly quasi-homogeneous. For  $\tilde{f}$  we take weights  $(1, 0, \dots, 0)$ .

**Example 5.12.** Let  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  be a map-germ  $\mathcal{A}$ -equivalent to

$$\tilde{f}(x_1, \dots, x_n) = (g(x_1, \dots, x_n), 0)$$

for some map-germ  $g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^{p-1}, 0)$ . Then  $f$  is weakly quasi-homogeneous. For  $\tilde{f}$  we take weights  $(0, \dots, 0)$ . The total degree of  $f$  is 1, because the degree of the zero component function-germ is 1.

**Example 5.13.** In [8]

$$f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^2, 0), \quad f(u, x, y) = (u, x^4 + y^4 + ux^2y^2)$$

is presented as an example of a map-germ which is not generically a trivial unfolding of a quasi-homogeneous germ. This  $f$  is also an example of a weakly quasi-homogeneous map-germ (for the weights  $(0, 1, 1)$ ).

Notice that the map-germ  $f$  in the third example fails to be  $\mathcal{A}$ -finite and that in the first example  $f$  is either stable (if  $g$  is stable) or it fails to be  $\mathcal{A}$ -finite. In the next section we describe a more subtle example of a weakly quasi-homogeneous (but not quasi-homogeneous)  $\mathcal{A}$ -simple map-germ without zero component functions (see Example 6.4).

Using Lemma 3.1 we prove the following theorem.

**Theorem 5.14.** 1. Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a smooth ( $\mathbb{R}$ -analytic) weakly quasi-homogeneous map-germ and let  $\Omega_0$  and  $\Omega_1$  be germs of smooth ( $\mathbb{R}$ -analytic) volume-forms at 0. If  $\Omega_0|_0$  and  $\Omega_1|_0$  define the same orientation of  $T_0\mathbb{R}^p$  then  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{A}_f$ -equivalent.

2. Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  be a  $\mathbb{C}$ -analytic weakly quasi-homogeneous map-germ and let  $\Omega_0$  and  $\Omega_1$  be germs  $\mathbb{C}$ -analytic volume-forms at 0. Then  $\Omega_0$  and  $\Omega_1$  are  $\mathcal{A}_f$ -equivalent.

*Proof:* Let  $E_w$  be the Euler-like vector field on  $\mathbb{K}^n$  for a coordinate system  $(x_1, \dots, x_n)$  with weights  $w = (w_1, \dots, w_n)$  of weak quasi-homogeneity for  $f$  and let  $E_\delta$  be the Euler vector field on  $\mathbb{K}^p$  for a coordinate system  $(X_1, \dots, X_p)$  with non-negative weights  $\delta = (\delta_1, \dots, \delta_p)$ , which are degrees of weak quasi-homogeneity of  $f$ . We may assume that  $\delta_1, \dots, \delta_k$  are positive integers and  $\delta_{k+1} = \dots = \delta_p = 0$ . Therefore  $E_\delta$  satisfies the conditions of Lemma 3.1. Let  $\Omega_1$  be a germ of volume form on  $\mathbb{K}^p$  and let  $\Omega_0 = \pm dX_1 \wedge \dots \wedge dX_p$  (we choose the sign of  $\Omega_0$  in such way that  $\Omega_0|_0$  and  $\Omega_1|_0$  define the same orientation of  $T_0\mathbb{R}^p$  for  $\mathbb{K} = \mathbb{R}$ , and  $\Omega_0 = dX_1 \wedge \dots \wedge dX_p$  for  $\mathbb{K} = \mathbb{C}$ ).

We show that  $\Omega_1$  and  $\Omega_0$  are  $\mathcal{A}_f$ -equivalent.

By Lemma 3.1 we have

$$\Omega_0 - \Omega_1 = d(h(E_\delta|_0)\Omega_0), \quad (5.3)$$

where  $h$  is a function-germ. From Proposition 5.10 we have  $E_\delta \circ f = df(E_w)$  and multiplying by  $h \circ f$  we get

$$(hE_\delta) \circ f = df((h \circ f)E_w),$$

and  $(h \circ f)E_w|_0 = 0$ , because  $E_w|_0 = 0$ .

Hence, by Theorem 5.7,  $\Omega_1$  and  $\Omega_0$  are  $\mathcal{I}\mathcal{A}_f$ -equivalent.  $\square$

**Remark 5.15.** *Theorem 5.14 can also be proved using Proposition 5.8. It is easy to show that if  $f$  is weakly quasi-homogeneous then  $\text{div}(\text{Lift}_0(f))$  is  $C_p$ .*

Let  $\Omega$  be a germ of a volume form on  $\mathbb{K}^p$  and let  $f, g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  be map-germs.

**Definition 5.16.**  $f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent if there exists a germ of a volume-preserving diffeomorphism  $\Phi : (\mathbb{K}^p, \Omega, 0) \rightarrow (\mathbb{K}^p, \Omega, 0)$  (i.e.  $\Phi^*\Omega = \Omega$ ) and a germ of a diffeomorphism  $\Psi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ , such that

$$f \circ \Psi = \Phi \circ g.$$

**Proposition 5.17.** *Let  $\Phi : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  be germs of diffeomorphisms such that*

$$f \circ \Psi = \Phi \circ g.$$

*If the germs of volume forms  $\Omega_0$  and  $\Omega_1$  are  $\text{divLift}_0(f)$ -equivalent then  $\Phi^*\Omega_0$  and  $\Phi^*\Omega_1$  are  $\text{divLift}_0(g)$ -equivalent.*

*Proof:* There exist germs of vector fields  $X$  on  $\mathbb{K}^n$  and  $Y$  on  $\mathbb{K}^p$  such that  $\Omega_1 - \Omega_0 = d(Y \rfloor \Omega)$ ,  $df(X) = Y \circ f$  and  $X|_0 = 0$ , because  $\Omega_0$  and  $\Omega_1$  are  $\text{divLift}_0(f)$ -equivalent.

$\Phi^*\Omega_1 - \Phi^*\Omega_0 = d(\Phi^*(Y \rfloor \Omega)) = d(\tilde{Y} \rfloor \Phi^*\Omega)$ , where  $\tilde{Y} = d(\Phi^{-1})(Y \circ \Phi)$  is a germ of a vector field on  $\mathbb{K}^p$ .

But  $\tilde{Y} \circ g = d(\Phi^{-1})(Y \circ \Phi \circ g) = d(\Phi^{-1})(Y \circ f \circ \Psi) = d(\Phi^{-1})(df(X \circ \Psi)) = d(\Phi^{-1} \circ f)(X \circ \Psi) = d(g \circ \Psi^{-1})(X \circ \Psi) = dg(d(\Psi^{-1})(X \circ \Psi))$ .

Thus  $\tilde{Y} \circ g = dg(\tilde{X})$ , where  $\tilde{X} = d(\Psi^{-1})(X \circ \Psi)$  is a germ of a vector field on  $\mathbb{K}^n$  vanishing at 0, because  $X|_0 = 0$ .

Hence  $\Phi^*\Omega_0$  and  $\Phi^*\Omega_1$  are  $\text{divLift}_0(g)$ -equivalent.  $\square$

Next we can define diffeomorphic  $\text{divLift}_0$ -equivalence classes.

**Definition 5.18.**  $[\Omega]_f$  and  $[\Omega]_g$  are diffeomorphic if there exist germs of diffeomorphisms  $\Phi : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that

$$f \circ \Psi = \Phi \circ g, \quad [\Phi^*\Omega]_g = [\Omega]_g.$$

As a corollary of Theorem 5.7 we obtain the following theorem.

**Theorem 5.19.**  *$f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent if and only if  $[\Omega]_f$  and  $[\Omega]_g$  are diffeomorphic.*

*Proof:* If  $f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent then it is obvious that  $[\Omega]_f$  and  $[\Omega]_g$  are diffeomorphic.

Now assume that  $[\Omega]_f$  and  $[\Omega]_g$  are diffeomorphic. Then there exist germs of diffeomorphisms  $\Phi : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $f \circ \Psi = \Phi \circ g$  and  $[\Phi^*\Omega]_g = [\Omega]_g$ . By Theorem 5.7 there exist germs of diffeomorphisms  $\Phi_1 : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$  and  $\Psi_1 : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $g \circ \Psi_1 = \Phi_1 \circ g$  and  $\Phi_1^*(\Phi^*\Omega) = \Omega$ . Then  $f \circ \Psi \circ \Psi_1 = \Phi \circ g \circ \Psi_1 = \Phi \circ \Phi_1 \circ g$  and  $(\Phi \circ \Phi_1)^*\Omega = \Omega$ . Hence  $f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent.  $\square$

From Theorem 5.14 we obtain the following corollary.



**Theorem 5.20.** 1. Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a smooth ( $\mathbb{R}$ -analytic) weakly quasi-homogeneous map-germ and let  $\Omega$  be the germ of a smooth ( $\mathbb{R}$ -analytic) volume-form at 0. Let  $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be another smooth ( $\mathbb{R}$ -analytic) map-germ, then  $f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent if and only if there exist a germ of a smooth ( $\mathbb{R}$ -analytic) orientation-preserving diffeomorphism  $\Phi : (\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^p, 0)$  and a germ of a smooth ( $\mathbb{R}$ -analytic) diffeomorphism  $\Psi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  such that  $\Phi \circ g \circ \Psi = f$ .

2. Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  be a  $\mathbb{C}$ -analytic weakly quasi-homogeneous map-germ and let  $\Omega$  be the germ of a  $\mathbb{C}$ -analytic volume-form at 0. Let  $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  be another  $\mathbb{C}$ -analytic map-germ, then  $f$  and  $g$  are  $\mathcal{A}_\Omega$ -equivalent if and only if  $f$  and  $g$  are  $\mathcal{A}$ -equivalent.

The following example explains the role of the orientation in the real case.

**Example 5.21.** Let  $f_1 : \mathbb{R} \ni t \mapsto (t^3, t^5) \in \mathbb{R}^2$  and  $f_2 : \mathbb{R} \ni t \mapsto (t^3, -t^5) \in \mathbb{R}^2$  (see [23]). It is easy to see that  $f_1$  and  $f_2$  are quasi-homogeneous and  $\mathcal{A}$ -equivalent. But they are not  $\mathcal{A}_\Omega$ -equivalent, because their images  $f_1(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 - x_1^5 = 0\}$  and  $f_2(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 + x_1^5 = 0\}$  fail to be  $\mathcal{R}_\Omega$ -equivalent as set-germs at  $0 \in \mathbb{R}^2$  (see Example 4.11).

## 6 $\mathcal{A}_\Omega$ -classification of maps

From the work of Mather it is clear that the  $\mathcal{A}$ -stable map-germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  are all quasi-homogeneous. The classification of the  $\mathcal{A}_\Omega$ -stable germs therefore coincides with the corresponding  $\mathcal{A}$ -classification for  $\mathbb{K} = \mathbb{C}$ , for  $\mathbb{K} = \mathbb{R}$  a given  $\mathcal{A}$ -orbit corresponds to one or two  $\mathcal{A}_\Omega$ -orbits. Also, Mather's nice pairs of dimensions  $(n, p)$  are the same for  $\mathcal{A}$  and  $\mathcal{A}_\Omega$ . In the present section we describe the classification of the  $\mathcal{A}_\Omega$ -simple orbits in those dimensions  $(n, p)$  for which the  $\mathcal{A}$ -simple orbits are known and we study the foliation of  $\mathcal{A}$ -orbits of modality one and two by  $\mathcal{A}_\Omega$ -orbits for  $(n, 2)$ ,  $n \geq 2$ . We also relate the number of  $\mathcal{A}_f$ -moduli to the dimension of  $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$ , which requires some notation for the tangent spaces to orbits.

Given a map-germ  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  we use coordinates  $x = (x_1, \dots, x_n)$  in the source and  $(X_1, \dots, X_p)$  in the target, and we also denote the vector fields  $\partial/\partial x_i$  and  $\partial/\partial X_i$  by  $e_i$ . Recall the usual definition of the  $\mathcal{A}$ -tangent space  $T\mathcal{A} \cdot f = tf(\mathcal{M}_n\theta_n) + \omega f(\mathcal{M}_p\theta_p)$ , where  $\mathcal{M}_n$  and  $\mathcal{M}_p$  denote the maximal ideals in the local rings  $C_n$  and  $C_p$  of function-germs in the source and target of  $f$ , respectively, and where  $tf : \theta_n \rightarrow \theta_f$ ,  $a \mapsto df(a)$

and  $\omega f : \theta_p \rightarrow \theta_f$ ,  $b \mapsto b \circ f$  are homomorphisms from the  $C_n$ -module of source vector fields  $\theta_n$  and the  $C_p$ -module of target vector fields  $\theta_p$  into the  $C_n$ -module  $\theta_f$  of smooth sections of  $f^*T\mathbb{K}^p$ . For the subgroup  $\mathcal{A}_\Omega = \mathcal{R} \times \mathcal{L}_\Omega$  of  $\mathcal{A}$  we have to restrict  $\theta_p$  to divergence-free vector fields, the left tangent space  $T\mathcal{L}_\Omega \cdot f$  is no longer a  $C_p$ -module. Let  $\Lambda_d$  denote the  $\mathbb{K}$ -vector space of homogeneous divergence-free vector fields in  $\mathbb{K}^p$  of degree  $d$ . Notice that  $\Lambda_d$  is the kernel of the epimorphism

$$\text{div} : (\theta_p)_{(d)} := \frac{\mathcal{M}_p^d \cdot \theta_p}{\mathcal{M}_p^{d+1} \cdot \theta_p} \rightarrow H_{(d-1)} := \frac{\mathcal{M}_p^{d-1}}{\mathcal{M}_p^d},$$

which maps a vector field on  $\mathbb{K}^p$  of degree  $d$  to its divergence. Hence

$$\dim \Lambda_d = \dim(\theta_p)_{(d)} - \dim H_{(d-1)} = (p-1) \binom{p+d-1}{d} + \binom{p+d-2}{d}.$$

The  $\dim \Lambda_d$  vector fields

$$\prod_{l \neq i} X_l^{\alpha_l} e_i, \quad \sum_l \alpha_l = d, \quad i = 1, \dots, p$$

and (setting  $h_{X_i} := \partial h / \partial X_i$ )

$$-h_{X_j} e_1 + h_{X_1} e_j, \quad h = \prod_l X_l^{\alpha_l}, \quad \alpha_1, \alpha_j \geq 1, \quad \sum_l \alpha_l = d+1, \quad j = 2, \dots, p$$

are clearly linearly independent and hence form a basis for  $\Lambda_d$ . The tangent space to the  $\mathcal{L}_\Omega$ -orbit at  $f$  is then given by  $T\mathcal{L}_\Omega \cdot f = f^* \oplus_{d \geq 1} \Lambda_d$ . As usual, for any group of equivalences  $\mathcal{G}$  we denote the extended pseudogroup of non-origin preserving diffeomorphisms by  $\mathcal{G}_e$ .

The following result is the infinitesimal version (and, of course, a direct consequence) of Theorem 5.20, but we give a short direct proof.

**Proposition 6.1.** *Let  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  be a weakly quasi-homogeneous map-germ. Then we have an inclusion*

$$T\mathcal{L} \cdot f \subset T\mathcal{A}_\Omega \cdot f,$$

and hence  $T\mathcal{A}_\Omega \cdot f = T\mathcal{A} \cdot f$ . One also has  $T\mathcal{A}_{\Omega,e} \cdot f = T\mathcal{A}_e \cdot f$ .

*Proof:* Fixing coordinates, we suppose that  $f = (f_1, \dots, f_p)$  is weighted homogeneous with non-negative degrees and positive total degree  $\delta_1 + \dots + \delta_p$  for integer weights  $w_1, \dots, w_n$ . Let  $X^\alpha = \prod_l X_l^{\alpha_l}$  and  $|\alpha| \geq 0$ . The following elements of  $T\mathcal{A}_\Omega \cdot f$  yield  $\omega f(X^\alpha X_i \cdot \partial / \partial X_i) \in T\mathcal{L} \cdot f$ ,  $i = 1, \dots, p$ :

$$\omega f(-(1 + \alpha_j)X_1 X^\alpha \cdot \partial / \partial X_1 + (1 + \alpha_1)X_j X^\alpha \cdot \partial / \partial X_j), \quad j = 2, \dots, p$$

together with

$$\begin{aligned} tf(f^*(X^\alpha) \sum_{i=1}^n w_i x_i \cdot \partial / \partial x_i) - \sum_{j=2}^p \delta_j \cdot \omega f\left(-\frac{1 + \alpha_j}{1 + \alpha_1} X^\alpha X_1 \cdot \partial / \partial X_1 + X^\alpha X_j \cdot \partial / \partial X_j\right) \\ = (1 + \alpha_1)^{-1} \sum_{j=1}^p (1 + \alpha_j) \delta_j \cdot \omega f(X^\alpha X_1 \cdot \partial / \partial X_1) \end{aligned}$$

(notice that  $\sum_j (1 + \alpha_j) \delta_j \neq 0$ , for any exponent vector  $\alpha$ , is equivalent to the weak quasi-homogeneity of  $f$ ). And the remaining elements of  $T\mathcal{L} \cdot f$  are of the form

$$\omega f\left(\prod_{l \neq i} X_l^{\alpha_l} \cdot \partial / \partial X_i\right) \in T\mathcal{L}_\Omega \cdot f.$$

The desired inclusion now follows, and the last statement is trivial, because the constant target vector fields are divergence-free.  $\square$

**Remark 6.2.** Notice that the above proof for  $\mathcal{A}_\Omega = \mathcal{R} \times \mathcal{L}_\Omega$  cannot be adapted to  $\mathcal{A}'_\Omega := \mathcal{R}_\Omega \times \mathcal{L}$ , but we can obtain a corresponding result for the subgroup  $\mathcal{K}'_\Omega$  of the contact group  $\mathcal{K}$  for which the diffeomorphisms on the right are volume-preserving (simply exchange the roles of source and target vector fields in the above proof, and use multiplication by  $x^\alpha \in C_n$ ). For  $f$  quasi-homogeneous for non-negative weights  $w_i$  and positive total weight  $w_1 + \dots + w_n$  we obtain that  $\mathcal{K}'_\Omega \cdot f = TK \cdot f$ , and for general  $f$  one obtains a formula for  $\dim_{\mathbb{K}} TK \cdot f / TK'_\Omega \cdot f$ . Notice that considering volume-preserving diffeomorphisms on the right is more closely related to earlier work on the isochore Morse lemma [39, 10] (and can be considered as an extension of this work on functions to the case of mappings). But, in fact, the  $\mathcal{K}'_\Omega$ -classification of map-germs  $f$  corresponds to the classification of varieties  $V = f^{-1}(0)$  up to volume-preserving diffeomorphisms in Sections 3 and 4 in the same way as ordinary  $\mathcal{K}$ -equivalence corresponds to the equivalence of varieties up to ambient diffeomorphisms (at least when  $C_n / f^* \mathcal{M}_p$  is reduced).

The criterion in the next easy lemma is sufficient for detecting in the existing classifications of  $\mathcal{A}$ -simple orbits those which are foliated by an  $r$ -parameter family,  $r \geq 1$ , of  $\mathcal{A}_\Omega$ -orbits.

**Lemma 6.3.** *Consider a germ  $f_u : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$  of the form  $f_u = f + u \cdot M$ , where  $f$  is a quasi-homogeneous germ,  $u \in \mathbb{K}$  and  $M = X^\alpha \cdot \partial / \partial X_j \notin T\mathcal{A} \cdot f = T\mathcal{A}_\Omega \cdot f$  is a monomial vector of positive weighted degree (with respect to the weights of  $f$ ). Then we have the following:*

(i) *The coefficient  $u$  is not a modulus for  $\mathcal{A}$ -equivalence.*

(ii) *For a set of weights for which  $f$  is weighted homogeneous, let  $(\theta_n)_0$ ,  $(\theta_p)_0$  and  $(\theta_f)_0$  denote the filtration-0 parts of the modules of source and target vector fields and vector fields along  $f$ , respectively. If the kernel of the linear map*

$$\gamma(f) : (\theta_n)_0 \oplus (\theta_p)_0 \rightarrow (\theta_f)_0, \quad (a, b) \mapsto tf(a) - \omega f(b),$$

*of  $\mathbb{K}$ -vector spaces is 1-dimensional then  $u$  is an  $\mathcal{A}_\Omega$ -modulus of  $f_u$ .*

*Proof:* Let  $f$  be weighted homogeneous for the weights  $w_1, \dots, w_n$ , and associate to the target variables the weights  $\delta_1, \dots, \delta_p$ . Then the weighted degree of  $\partial / \partial X_i$  is  $-\delta_i$ , so that  $f$  has filtration 0 and  $M$  has filtration  $r > 0$ .

For  $\mathcal{A}$ -equivalence we consider the following element of  $T\mathcal{A} \cdot f_u$ :

$$tf_u\left(\sum_{i=1}^n w_i x_i \cdot \partial / \partial x_i\right) - \omega f_u\left(\sum_{j=1}^p \delta_j X_j \cdot \partial / \partial X_j\right) = ruM.$$

From Mather's lemma (Lemma 3.1 in [30]) we conclude that the connected components of  $\mathbb{K} \setminus \{0\}$  of the parameter axis lie in a single  $\mathcal{A}$ -orbit, hence  $u$  is not a modulus for  $\mathcal{A}$ .

For the second statement we observe that  $\dim \ker \gamma(f) = 1$  implies that this kernel is spanned by the pair of Euler fields  $(E_w, E_\delta)$  (which is unique up to a multiplication by an element of  $\mathbb{K}^*$ ). For  $M \notin T\mathcal{A} \cdot f$  implies that the only generator of  $M$  in  $T\mathcal{A} \cdot f_u$  must be of the form  $tf_u(a) - \omega f_u(b)$  with  $(a, b)$  a non-zero multiple of  $(E_w, E_\delta)$ , but  $E_\delta$  has non-zero divergence, hence this generator does not belong to  $T\mathcal{A}_\Omega \cdot f_u$ . Now Mather's lemma implies that  $u$  is a modulus for  $\mathcal{A}_\Omega$ .  $\square$

We can distinguish three types of weakly quasi-homogeneous map-germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ : (a) quasi-homogeneous, (b) not quasi-homogeneous

and having an  $\mathcal{A}$ -representative with a zero component function (their image lies in a smooth hypersurface in  $\mathbb{K}^p$ ) and (c) the remaining ones. Looking at the existing classifications of  $\mathcal{A}$ -simple germs we see that all non-weakly quasi-homogeneous map-germs satisfy condition (ii) of the lemma above and hence have an  $\mathcal{A}_\Omega$ -modulus along the positive filtration term. Then for dimensions  $(n, p)$  with  $p < 2n$  the weakly quasi-homogeneous  $\mathcal{A}$ -simple germs are all in fact quasi-homogeneous (i.e. of type (a)). For  $p = 2n$  and  $n \geq 2$  the weakly quasi-homogeneous germs are of type (c) or are quasi-homogeneous, see the classification in [24]. For  $p > 2n$  and  $n \geq 2$  we have all three types of weakly quasi-homogeneous germs. Finally, for curves in  $\mathbb{K}^p$  the weakly quasi-homogeneous germs for  $p = 2$  are all quasi-homogeneous, for  $p \geq 3$  they are quasi-homogeneous or of type (b) (see [6],[3]).

**Example 6.4.** Consider the series  $22_k$  of map germs  $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ ,  $n \geq 3$ , from the classification of  $\mathcal{A}$ -simple germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^{2n}, 0)$ ,  $n \geq 2$ , in [24] given by:

$$g_k = (x_1, \dots, x_{n-1}, x_1y + y^3, x_2y, \dots, x_{n-1}y, x_1y^2 + y^{2k+1}, x_2y^2 + y^4), k \geq 2.$$

The germs  $22_k$  are not semi-quasi-homogeneous: if we write  $g_k = f + y^{2k+1} \cdot e_{2n-1}$  then the weighted homogeneous initial part  $f$  is not  $\mathcal{A}$ -finite. For  $n = 3$  all the germs  $22_k$  are  $\mathcal{A}$ -simple, for  $n \geq 4$  only  $22_2$  is  $\mathcal{A}$ -simple (the germs  $22_{\geq 3}$  do not have an  $\mathcal{A}$ -modulus, but they lie in the closure of non-simple  $\mathcal{A}$ -orbits).

Now consider  $\mathcal{A}_\Omega$ -equivalence. Writing  $f_u = f + u \cdot y^{2k+1} \cdot e_{2n-1}$  we see that  $\dim \ker \gamma(f) = n - 2$ . For  $n = 3$  part (ii) of Lemma 6.3 implies that the coefficient  $u$  is an  $\mathcal{A}_\Omega$ -modulus. For  $n \geq 4$  the germs  $f_u$  are weakly quasi-homogeneous of type (c) (take  $w(x_1) = w(x_2) = w(y) = 0$  and  $w(x_i) = 1$ ,  $i \geq 3$ ) and  $\mathcal{A}_\Omega$ -equivalent to  $g_k$  (for  $u \neq 0$ ).

Next, one obtains the classification of  $\mathcal{A}_\Omega$ -simple orbits from the existing  $\mathcal{A}$ -classification using the following

**Remark 6.5.** An  $\mathcal{A}$ -simple germ in the existing classifications is  $\mathcal{A}_\Omega$ -simple if and only if it does not lie in the closure of the orbit of any non-weakly quasi-homogeneous germ.

As an example, let us consider map-germs with target dimension  $p = 2$ . The classification of  $\mathcal{A}_\Omega$ -simple map-germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^2, 0)$  in the curve case  $n = 1$  has been described in [23] (for target dimension 2 the symplectic

forms are volume forms). For any  $n \geq 2$  we have the following classification over  $\mathbb{K} = \mathbb{C}$ .

**Proposition 6.6.** (i) *The following map-germs  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^2, 0)$  represent the  $\mathcal{A}_\Omega$ -orbits within the  $\mathcal{A}$ -simple orbits (here “type  $r$ ” refers to the notation in [34, 36] for the  $\mathcal{A}$ -orbits).*

$\mathcal{A}_\Omega$ -modality 0:  $(x, y)$  (type 1);  $(x, y^2 + \sum_{i=1}^{n-2} z_i^2)$  (type 2);  $(x, xy + y^3 + \sum_{i=1}^{n-2} z_i^2)$  (type 3);  $(x, y^3 + x^k y + \sum_{i=1}^{n-2} z_i^2)$ ,  $k > 1$  (type  $4_k$ );  $(x, xy + y^4 + \sum_{i=1}^{n-2} z_i^2)$  (type 5).

$\mathcal{A}_\Omega$ -modality  $\geq 1$ :  $(x, xy + y^5 + ay^7 + \sum_{i=1}^{n-2} z_i^2)$  (types 6, 7);  $(*) (x, xy^2 + y^4 + \sum_{k \geq 2} a_k y^{2k+1} + \sum_{i=1}^{n-2} z_i^2)$  (types  $11_{2k+1}$ );  $(x, xy^2 + y^5 + ay^6 + by^9 + \sum_{i=1}^{n-2} z_i^2)$  (types 12, 13, 14);  $(x, x^2 y + y^4 + ay^5 + \sum_{i=1}^{n-2} z_i^2)$  (types 16, 17);  $(*) (x^2 + ay^{2l+1}, y^2 + x^{2m+1})$ ,  $l \geq m \geq 1$  (type  $\text{II}_{2,2}^{l,m}$ ).

(ii) *The  $\mathcal{A}$ -unimodal orbits of map-germs  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^2, 0)$  of lowest codimension in their  $\mathcal{K}$ -orbit, containing the remaining  $\mathcal{A}$ -unimodal orbits in their closures, have the following representatives (see [35]):*

$$(x, y^4 + x^3 y + ax^2 y^2 + x^3 y^2 + \sum_{i=1}^{n-2} z_i^2), \quad a \neq -3/2 \quad (\text{type 19})$$

$$(x, xy + y^6 + y^8 + ay^9 + \sum_{i=1}^{n-2} z_i^2) \quad (\text{type 8})$$

$$(x, xy + y^3 + ay^2 z + z^3 + z^5) \quad (\text{type I}).$$

For  $\mathcal{A}_\Omega$ -equivalence the corresponding normal forms are:

$$(x, y^4 + x^3 y + ax^2 y^2 + bx^3 y^2 + \sum_{i=1}^{n-2} z_i^2)$$

$$(x, xy + y^6 + ay^8 + by^9 + cy^{14} + \sum_{i=1}^{n-2} z_i^2)$$

$$(x, xy + y^3 + ay^2 z + z^3 + bz^5).$$

All  $\mathcal{A}$ -unimodal germs therefore have  $\mathcal{A}_\Omega$ -modality at least two.

*Proof:* The classification of the orbits of  $\mathcal{A}_\Omega$ -modality 0 follows from the remark preceding the statement of the proposition.

For the semi-quasi-homogeneous map-germs  $f = f_0 + f_+$  the  $\mathcal{A}$ -determinacy degree, say  $r$ , of the initial part  $f_0$  is an upper bound for the  $\mathcal{A}_\Omega$ -determinacy degree of  $f$ . The normal forms of such  $f$  can then be obtained with the usual techniques (Mather's lemma) applied to the Lie group  $\mathcal{A}_\Omega^r$  of  $r$ -jets of elements of  $\mathcal{A}_\Omega$  (we omit the details of these calculations).

Finally, the map-germs  $f = f_0 + f_+$  marked by an  $(*)$  are not semi-quasi-homogeneous. As a consequence we cannot use the  $\mathcal{A}$ -determinacy degree of the weighted homogeneous initial parts  $f_0$  as an upper bound for the  $\mathcal{A}_\Omega$ -determinacy degree of  $f$ . For the germs  $\Pi_{2,2}^{l,m}$  we can show that these are formally  $(2l+1)$ - $\mathcal{A}_\Omega$ -determined (for  $a \neq 0$ ). For the germs  $11_{2k+1}$  we only know that  $a_2$  is an  $\mathcal{A}_\Omega$ -modulus (and we can set  $a_3$  and  $a_4$  to zero, provided  $a_2$  is non-zero).

See also Remark 6.8 (below) on the problem of  $\mathcal{A}_\Omega$ -determinacy.  $\square$

Denote the number of  $\mathcal{G}$ -moduli ( $\mathcal{G} = \mathcal{A}$  or  $\mathcal{A}_\Omega$ ) of a map-germ  $f$  by  $m_{\mathcal{G}}(f)$ , and let  $m_{\mathcal{A}/\mathcal{A}_\Omega}(f) := \dim_{\mathbb{K}}(T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f)$  be the dimension of the  $\mathcal{A}_\Omega$ -moduli space. Then  $m_{\mathcal{A}_\Omega}(f) = m_{\mathcal{A}}(f) + m_{\mathcal{A}/\mathcal{A}_\Omega}(f)$ , where  $m_{\mathcal{A}/\mathcal{A}_\Omega}(f)$  is equal to the number of  $\mathcal{A}_f$ -moduli by Corollary 5.8 and the following result.

**Proposition 6.7.** *There is an isomorphism*

$$\frac{T\mathcal{A} \cdot f}{T\mathcal{A}_\Omega \cdot f} \cong \frac{C_p}{\text{div}(\text{Lift}_0(f))}.$$

*Proof:* We use the notation  $T_{id}\mathcal{A} = T_{id}\mathcal{R} + T_{id}\mathcal{L}$  for the tangent space to the  $\mathcal{A}$ -group at the identity, rather than the usual notation  $T_{id}\mathcal{R} = \mathcal{M}_n \cdot \theta_n$  and  $T_{id}\mathcal{L} = \mathcal{M}_p \cdot \theta_p$ .

The map  $\beta : T_{id}\mathcal{A} \rightarrow C_p$ ,  $(a, b) \mapsto \text{div}b$  is surjective with kernel  $T_{id}\mathcal{A}_\Omega$ , hence there is an isomorphism  $\bar{\beta} : T_{id}\mathcal{A}/T_{id}\mathcal{A}_\Omega \rightarrow C_p$  and the map

$$\bar{\gamma} : \frac{T_{id}\mathcal{A}}{T_{id}\mathcal{A}_\Omega} \rightarrow \frac{\mathcal{M}_n \cdot \theta_f}{T\mathcal{A}_\Omega \cdot f}, \quad [(a, b)] \mapsto [tf(a) - wf(b)]$$

is independent of the choice of  $(\tilde{a}, \tilde{b}) \in [(a, b)]$  and hence well-defined. Its image is  $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$  and

$$\ker \bar{\gamma} = \{[(a, b)] : tf(a) - wf(b) \in T\mathcal{A}_\Omega \cdot f\}.$$

Hence  $C_p / \bar{\beta}(\ker \bar{\gamma})$  is isomorphic to  $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$ . And it is easy to see that  $\bar{\beta}(\ker \bar{\gamma}) = \text{div}(\text{Lift}_0(f))$ .  $\square$

**Remark 6.8.** For  $\mathcal{A}$ -finite map-germs  $f$ , that are not semi-quasi-homogeneous, two questions arise: (i) is such an  $f$  always  $\mathcal{A}_\Omega$ -finite? (ii) Is there an estimate for the  $\mathcal{A}_\Omega$ -determinacy degree?

(i) Composing the above isomorphism

$$\sigma : C_p / \text{div}(\text{Lift}_0(f)) \rightarrow T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$$

with an epimorphism

$$g : \mathcal{M}_n \cdot \theta_f / T\mathcal{A}_\Omega \cdot f \rightarrow \mathcal{M}_n \cdot \theta_f / T\mathcal{A} \cdot f$$

with kernel  $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$  (coming from the second isomorphism theorem for the vector spaces  $T\mathcal{A}_\Omega \cdot f \subset T\mathcal{A} \cdot f \subset \mathcal{M}_n \cdot \theta_f$ ) we get an exact sequence

$$0 \longrightarrow \frac{C_p}{\text{div}(\text{Lift}_0(f))} \xrightarrow{\sigma} \frac{\mathcal{M}_n \cdot \theta_f}{T\mathcal{A}_\Omega \cdot f} \xrightarrow{g} \frac{\mathcal{M}_n \cdot \theta_f}{T\mathcal{A} \cdot f} \longrightarrow 0.$$

Hence we have:

$$\text{cod}(\mathcal{A}_\Omega, f) = \text{cod}(\mathcal{A}, f) + m_{\mathcal{A}/\mathcal{A}_\Omega}(f),$$

and for  $\mathcal{A}$ -finite  $f$  the germ  $\tilde{f}$  of  $f$  at any point of a punctured neighborhood of the origin in the source is  $\mathcal{A}$ -stable, hence weighted homogeneous, so that  $m_{\mathcal{A}/\mathcal{A}_\Omega}(\tilde{f}) = 0$ . Unfortunately,  $\text{div}(\text{Lift}_0(f))$  is not an ideal in  $C_p$ , so we cannot conclude (from the Nullstellensatz) that  $m_{\mathcal{A}/\mathcal{A}_\Omega}(f)$  is finite. For plane curves, Ishikawa and Janeczko [23] have shown that  $\text{cod}(\mathcal{A}_{\Omega,e}, f)$  is equal to the delta invariant  $\delta(f)$ , and it is known that  $\delta(f) < \infty$  is equivalent to the  $\mathcal{A}$ -finiteness of  $f$ . The analogue of the delta invariant for map-germs  $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^{2n}, 0)$ ,  $n \geq 2$ , is the double-point number  $d(f)$ , whose finiteness is again equivalent to the  $\mathcal{A}$ -finiteness of  $f$  (see [24]), but the equation  $\text{cod}(\mathcal{A}_{\Omega,e}, f) = d(f)$  does not hold for  $n \geq 2$ .

(ii) Suppose that for an  $r$ - $\mathcal{A}$ -determined map-germ  $f$  the following holds:  $\mathcal{M}_n^l \theta_f \subset T\mathcal{K}_e \cdot f$  ( $\mathcal{K}_e$  is the extended contact group) and  $\mathcal{M}_p^k \subset \text{div}(\text{Lift}_0(f))$  (so that  $f$  is  $\mathcal{A}_\Omega$ -finite). Then  $f$  is  $(r + l(k + 2) - 1)$ - $\mathcal{A}_\Omega$ -determined. To see this, let  $\mathcal{G}_s$  denote the subgroup of  $\mathcal{G}$  consisting of elements of  $\mathcal{G}$  with identity  $s$ -jet and apply the inverse  $\bar{\beta}^{-1}$  of the isomorphism  $\bar{\beta} : T_{\text{id}}\mathcal{A}/T_{\text{id}}\mathcal{A}_\Omega \rightarrow C_p$  in the proof of the above proposition to  $\mathcal{M}_p^k$ . It follows that  $T\mathcal{A}_{\Omega, k+1} \cdot f = T\mathcal{A}_{k+1} \cdot f$ , and 2.7.3 of [7] then implies that  $f$  is  $(r + l(k + 2) - 1)$ -determined for  $\mathcal{A}_{k+1}$  and hence for  $\mathcal{A}_\Omega$ .



## References

- [1] V. I. Arnol'd, Wave front evolution and equivariant Morse lemma, *Comm. Pure Appl. Math.* 29 (1976), 557–582.
- [2] V. I. Arnol'd, Remarks on Poisson structures on a plane and on other powers of volume elements, *J. Soviet Math.* 47 (1989), 2509–2516.
- [3] V. I. Arnol'd, Simple singularities of curves, *Proc. Steklov Inst. Math.* 226 (1999), 20–28.
- [4] V. I. Arnol'd, First steps of local symplectic algebra, *Adv. in Soviet Math.*, D. Fuchs birthday volume, Providence, Amer. Math. Soc., 1999.
- [5] V. I. Arnol'd, A. B. Givental', Symplectic geometry, in *Dynamical systems IV*, 1–138, Encyclopedia of Mathematical Sciences, vol. 4, Springer, Berlin, (2001).
- [6] J. W. Bruce, T. Gaffney, Simple singularities of maps  $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ , *J. London Math. Soc.* 26 (1982), 464–474.
- [7] J.W. Bruce, A.A. du Plessis and C.T.C. Wall, Determinacy and unipotency, *Inventiones Math.* 88 (1987), 521–554.
- [8] J.W. Bruce, A.A. du Plessis and L.C. Wilson, Discriminants and liftable vector fields, *J. Algebraic Geometry* 3 (1994), 725–753.
- [9] Y. Colin de Verdiere, Singular Lagrangian manifolds and semi-classical analysis, *Duke Math. J.* 116 (2003), 263–298.
- [10] Y. Colin de Verdiere and J. Vey, Le lemme de Morse isochore, *Topology* 18 (1979), 283–293.
- [11] J. Damon, The unfolding and determinacy results for subgroups of  $\mathcal{A}$  and  $\mathcal{K}$ , in *Proc. of Symposia in Pure Mathematics* 40, Part 1 (American Mathematical Society, Providence, R.I., 1983), 233–254
- [12] W. Domitrz, S. Janeczko, M. Zhitomirskii, Relative Darboux theorem for singular submanifolds of a symplectic manifold, preprint.

- [13] W. Domitrz, S. Janeczko, M. Zhitomirskii, Relative Poincare lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety, *Illinois J. of Math.* 48 (2004), 803–835.
- [14] J.P. Dufour, Sur la stabilité des diagrammes d’applications différentiables, *Ann. scient. Éc. Norm. Sup.* 10 (1977), 153–174.
- [15] A. Ferrari, Cohomology and holomorphic differential forms on complex analytic spaces, *Ann. Scuola Norm. Sup. Pisa* 24 (1970), 65–77.
- [16] J-P. Francoise, Modèle local simultané d’une fonction et d’une forme de volume, *Astérisque S. M. F.* 59-60 (1978), 119–130.
- [17] J-P. Francoise, Réduction simultanée d’un croisement normal et d’un volume, *Bol. Soc. Bras. Mat.* 13 (1982), 79–83.
- [18] A. Fronville, Théorème de la singularité résiduelle isochore, *C. R. Acad. Sci. Paris*, t. 322, Série I, 1996, 857-860.
- [19] T. Gaffney and A. du Plessis, More on the determinacy of smooth map-germs, *Inventiones Math.* 66 (1982), 137–163.
- [20] T. Gaffney and L. Wilson, Equivalence of generic mappings and  $C^\infty$  normalization, *Compositio Math.* 49 (1983), 291–308.
- [21] M.D. Garay, An isochore versal deformation theorem, *Topology* 43 (2004), 1081–1088.
- [22] A.B. Givental’, Singular Lagrangian manifolds and their Lagrangian mappings, *J. Soviet Math.* 52 (1990), 3246–3278.
- [23] G. Ishikawa and S. Janeczko, Symplectic bifurcations of plane curves and isotropic liftings, *Quart. J. Math. Oxford Ser. (2)* 54 (2003), 73–102.
- [24] C. Klotz, O. Pop and J.H. Rieger, Real double points of deformations of  $\mathcal{A}$ -simple map-germs from  $\mathbb{R}^n$  to  $\mathbb{R}^{2n}$ , preprint.
- [25] V. P. Kostov and S. K. Lando, Versal deformations of powers of volume forms, *Computational algebraic geometry (Nice, 1992)*, 143–162, Progr. Math., 109, Birkhäuser Boston, Boston, MA, 1993.

- [26] S. K. Lando, Deformations of differential forms, *Trudy Mat. Inst. Steklov.* 209 (1995), Osob. Gladkikh Otobrazh. s Dop. Strukt., 167–199 (Russian).
- [27] S. K. Lando, Singularities of the differential forms of the complex degree and their deformations, *Singularity theory (Trieste, 1991)*, 289–305, World Sci. Publishing, River Edge, NJ, 1995.
- [28] S. K. Lando, Normal forms of powers of volume forms, *Funktsional. Anal. i Prilozhen* 19 (1985), 78–79 (Russian).
- [29] J. Martinet, *Singularities of smooth functions and maps*, London Math. Soc., Lect. Note 58, Cambridge Univ. Press 1982.
- [30] J.N. Mather, Stability of  $C^\infty$  mappings, IV: classification of stable germs by  $\mathbb{R}$ -algebras, *Publ. Math. I.H.E.S.* 37 (1969), 223–248.
- [31] J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.* 120 (1965), 280–296.
- [32] A. du Plessis and L. Wilson, On right-equivalence, *Math. Z.* 190, 163–205 (1985).
- [33] A. du Plessis, T. Gaffney and L. Wilson, Maps-germs determined by their discriminants, *Proceedings of Hawaii-Provence Singularities Conferences*, 1990, Travaux en Cours, Hermann, Paris, 1–40.
- [34] J.H. Rieger, Families of maps from the plane to the plane, *J. London Math. Soc.* 36 (1987), 351–369.
- [35] J.H. Rieger,  $\mathcal{A}$ -unimodal map-germs into the plane, *Hokkaido Math. J.* 33 (2004), 47–64.
- [36] J.H. Rieger and M.A.S. Ruas, Classification of  $\mathcal{A}$ -simple germs from  $k^n$  to  $k^2$ , *Compositio Math.* 79 (1991), 99–108.
- [37] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen, *Invent. Math.* 14 (1971), 123–142.
- [38] A. N. Varchenko, Local classification of volume forms in the presence of a hypersurface, *Funktsional. Anal. i Prilozhen* 19 (1985), no. 4, 23–31 (Russian).

- [39] J. Vey, Sur le lemme de Morse, *Inventiones Math.* 40 (1977), 1–10.
- [40] C.T.C. Wall, Finite determinacy of smooth map-germs, *Bull. London Math. Soc.* 13 (1981), 481–539.
- [41] M. Zhitomirskii, Relative Darboux theorem for singular manifolds and local contact algebra, *Canadian J. of Math.* 57 (2005), 1314–1340.

# Implicit Hamiltonian systems with fold singularities on the plane

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## 1 Introduction

Let  $\mathbf{R}^2$  be the symplectic plane endowed with the symplectic structure  $\omega = dy \wedge dx$ , where  $(x, y)$  are the standard coordinates on  $\mathbf{R}^2$  (cf. [6]). Then the tangent bundle  $T\mathbf{R}^2$  of  $\mathbf{R}^2$  is also a symplectic manifold with the natural symplectic structure

$$\bar{\omega} = dy \wedge dx - dx \wedge dy,$$

where  $(p, q) = ((x, y), (\dot{x}, \dot{y}))$  are coordinates on  $T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$ . Let

$$\pi : T\mathbf{R}^2 \rightarrow \mathbf{R}^2 \quad \pi(p, q) = p$$

denote the projection of the tangent bundle onto the plane.

A Lagrangean submanifold of the symplectic manifold  $(T\mathbf{R}^2, \bar{\omega})$  is called an *implicit Hamiltonian system* on the symplectic plane  $(\mathbf{R}^2, \omega)$  (see [1], [3], [2]).

Let  $M \subset T\mathbf{R}^2$  be an implicit Hamiltonian system. Singular points of the restricted map

$$\pi|_M : M \rightarrow \mathbf{R}^2$$

are called singular points of the implicit Hamiltonian system  $M$  (see [4], [5]).

A symplectomorphism  $\phi : P \rightarrow P'$  between open subsets of  $\mathbf{R}^2$  induces a symplectomorphism between their tangent bundles

$$\bar{\phi} : TP \rightarrow TP' \quad \text{defined by} \quad \bar{\phi}(p, q) = (\phi(p), d\phi_p(q)).$$

In this note, we will classify integrable fold singularities of implicit Hamiltonian systems on  $\mathbf{R}^2$  up to germs of symplectomorphisms of the tangent bundle  $T\mathbf{R}^2$  induced by germs of symplectomorphisms of  $\mathbf{R}^2$ .

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**Definition 1.1.** Let  $M, N \subset T\mathbf{R}^2$  be implicit Hamiltonian systems and let  $(p, q) \in M$  and  $(p', q') \in N$  be their singular points. We say that the germs  $(M, (p, q))$  and  $(N, (p', q'))$  are symplectomorphic if there exists a germ of symplectomorphism  $\phi : (\mathbf{R}^2, p) \rightarrow (\mathbf{R}^2, p')$  such that  $\bar{\phi}((M, (p, q))) = (N, (p', q'))$ .

## 2 Main Theorem

Now we formulate our main theorem, which will be proved in the last section.

**Theorem 2.1.** 1) If  $((0, 0), q) \in M \subset T\mathbf{R}^2$  is a fold singular point of an implicit Hamiltonian system, then the germ  $(M, (p, q))$  of the implicit Hamiltonian system is symplectomorphic to the germ  $(M_F, ((0, 0), q))$  of the implicit Hamiltonian system  $M_F$  generated by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y),$$

$$\text{where } q = \left( \frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0) \right).$$

2) The implicit Hamiltonian system generated by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y)$$

is smoothly integrable if and only if

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{for some function-germ } a(x, y).$$

3) Let  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  be two germs of integrable implicit Hamiltonian systems generated by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y),$$

$$\text{where } q = \left( \frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0) \right), \quad q' = \left( \frac{\partial G}{\partial y}(0, 0, 0), -\frac{\partial G}{\partial x}(0, 0, 0) \right).$$

Then  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  are symplectomorphic if and only if  $a(x, y)$  and  $b(x, y)$  are symplectomorphic under a symplectomorphism of the form  $\phi(x, y) = (x + \alpha(y), y)$ :

$$b(x, y) = a(x + \alpha(y), y).$$

### 3 A classification of fold singularities: polynomial case

In this section, as corollaries of Theorem 2.1, we give a complete classification of fold singularities of implicit Hamiltonian systems generated by functions of the form

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y)$$

where functions  $a(x, y)$  are polynomials in variable  $x$  with coefficients in  $\mathcal{E}_y$ :

$$a(x, y) = a_n(y)x^n + a_{n-1}(y)x^{n-1} + \cdots + a_0(y), \quad a_i \in \mathcal{E}_y.$$

**Corollary 3.1.** *Let  $a(x, y)$  be a polynomial in variable  $x$  with coefficients in  $\mathcal{E}_y$ :*

$$a(x, y) = a_n(y)x^n + a_{n-1}(y)x^{n-1} + \cdots + a_0(y).$$

*Then coefficient  $a_n(y)$  of the highest degree term  $x^n$  is a symplectic invariant of the implicit Hamiltonian system germ  $(M_F, ((0, 0), q))$  generated by the function*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y).$$

*In other words, if  $(M_F, ((0, 0), q))$  is symplectomorphic to another implicit Hamiltonian system germ  $(M_G, ((0, 0), q))$  generated by the function*

$$G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y),$$

*with*

$$b(x, y) = b_m(y)x^m + b_{m-1}(y)x^{m-1} + \cdots + b_0(y),$$

*then  $m = n$  and  $b_n(y) = a_n(y)$ .*

**Proof** For a symplectomorphism  $\phi$  of the form  $\phi(x, y) = (x + \alpha(y), y)$ , we have

$$\begin{aligned} a \circ \phi(x, y) &= a_n(y)(x + \alpha(y))^n + a_{n-1}(y)(x + \alpha(y))^{n-1} + \cdots + a_0(y) \\ &= a_n(y)x^n + (na_n(y)\alpha(y) + a_{n-1}(y))x^{n-1} + \cdots \end{aligned}$$

Thus symplectomorphisms  $\phi$  of the form  $\phi(x, y) = (x + \alpha(y), y)$  do not change the coefficient  $a_n(y)$  of the highest degree term  $x^n$ .  $\square$

**Theorem 3.2.** *Let  $a(x, y)$  be a polynomial in variable  $x$  with coefficients in  $\mathcal{E}_y$ :*

$$a(x, y) = a_n(y)x^n + a_{n-1}(y)x^{n-1} + \cdots + a_0(y),$$

and let

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y).$$

1) *If*

$$\text{the degree of } a_n(y) < \text{the degree of } a_{n-1}(y),$$

*then the implicit Hamiltonian system germ  $(M_F, ((0, 0), q))$  is symplectomorphic to the germ of the implicit Hamiltonian system generated by*

$$G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y)$$

*for some polynomial  $b(x, y)$  in variable  $x$*

$$b(x, y) = a_n(y)x^n + b_{n-2}(y)x^{n-2} + \cdots + b_0(y).$$

*of the same degree  $n$  as  $a(x, y)$  with the same coefficient  $a_n(y)$  of the highest degree term and null coefficient of the term  $x^{n-1}$ .*

2) *Suppose that polynomials  $a(x, y)$  and  $b(x, y)$  in  $x$  have the same coefficients of the highest degree term  $x^n$  and null coefficients of  $x^{n-1}$ :*

$$a(x, y) = a_n(y)x^n + a_{n-2}(y)x^{n-2} + \cdots + a_0(y),$$

$$b(x, y) = a_n(y)x^n + b_{n-2}(y)x^{n-2} + \cdots + b_0(y).$$

*Then the implicit Hamiltonian system germs  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  generated respectively by*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y)$$

*are symplectomorphic to each other if and only if  $a(x, y) = b(x, y)$ .*

**Proof** 1) Set

$$\alpha(y) = -\frac{a_{n-1}(y)}{na_n(y)} \quad \text{and}$$

$$\phi(x, y) = \left(x - \frac{a_{n-1}(y)}{na_n(y)}, y\right).$$



Then we have

$$\begin{aligned}
a \circ \phi(x, y) &= a_n(y)(x + \alpha(y))^n + a_{n-1}(y)(x + \alpha(y))^{n-1} + \cdots + a_0(y) \\
&= a_n(y)x^n + (na_n(y)\alpha(y) + a_{n-1}(y))x^{n-1} + \cdots \\
&= a_n(y)x^n + (na_n(y)\left(-\frac{a_{n-1}(y)}{na_n(y)}\right) + a_{n-1}(y))x^{n-1} + \cdots \\
&= a_n(y)x^n + 0 \times x^{n-1} + \cdots \quad .
\end{aligned}$$

2) Of course, if  $a(x, y) = b(x, y)$ , then we have  $(M_F, ((0, 0), q)) = (M_G, ((0, 0), q'))$ .

Conversely, suppose that  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  are symplectomorphic to each other. Then from Theorem 2.1, 3),

$$\begin{aligned}
b(x, y) &= a(x + \alpha(y), y) \quad \text{for some } \alpha(y) \in m_y \\
&= a_n(y)(x + \alpha(y))^n + a_{n-1}(y)(x + \alpha(y))^{n-1} + \cdots \\
&= a_n(y)x^n + na_n(y)\alpha(y)x^{n-1} + \cdots \quad .
\end{aligned}$$

If  $\alpha(y) \neq 0$ , then the coefficient  $na_n(y)\alpha(y)$  of  $x^{n-1}$  is not 0, which contradicts the assumption that the coefficient of  $x^{n-1}$  in  $b(x, y)$  is 0. Thus  $\alpha(y)$  must be 0 and we have  $b(x, y) = a(x, y)$ .  $\square$

**Theorem 3.3.** *Let  $a(x, y)$  be a polynomial in variable  $x$  with coefficients in  $\mathcal{E}_y$ :*

$$a(x, y) = a_n(y)x^n + a_{n-1}(y)x^{n-1} + \cdots + a_0(y),$$

and let

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y).$$

1) If

$$\text{the degree of } a_n(y) \geq \text{the degree of } a_{n-1}(y),$$

then the implicit Hamiltonian system germ  $(M_F, ((0, 0), q))$  is symplectomorphic to the germ of the implicit Hamiltonian system generated by

$$G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y)$$

for some polynomial  $b(x, y)$  in  $x$  with the same coefficient  $a_n(y)$  of the highest degree term  $x^n$  as  $a(x, y)$  and of the form

$$b(x, y) = a_n(y)x^n + b_{n-1}(y)x^{n-1} + \cdots + b_0(y)$$

such that  $b_{n-1}(y)$  is a polynomial in  $y$  with

$$\text{the degree of } b_{n-1}(y) \leq \text{the degree of } a_n(y).$$

2) Suppose that

$$a(x, y) = a_n(y)x^n + a_{n-1}(y)x^{n-1} + \cdots + a_0(y)$$

and

$$b(x, y) = a_n(y)x^n + b_{n-1}(y)x^{n-1} + \cdots + b_0(y)$$

have the same coefficient  $a_n(y)$  of the highest degree term  $x^n$  and that  $a_{n-1}(y)$  and  $b_{n-1}(y)$  are polynomials in  $y$  such that

$$\text{the degree of } a_{n-1}(y) \leq \text{the degree of } a_n(y),$$

$$\text{the degree of } b_{n-1}(y) \leq \text{the degree of } a_n(y).$$

Then the implicit Hamiltonian system germs  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  generated respectively by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y).$$

are symplectomorphic to each other if and only if  $a(x, y) = b(x, y)$ .

**Proof** 1) Let  $r$  be the degree of  $a_n(y)$ . Represent the coefficient  $a_{n-1}(y)$  in a sum of Taylor polynomial of  $a_{n-1}(y)$  of degree  $r$  and the higher term:

$$a_{n-1}(y) = b_{n-1}(y) + c_{n-1}(y), \quad b_{n-1}(y) = \sum_{i=0}^r \frac{1}{i!} \frac{d^i a_{n-1}}{dy^i}(0) y^i.$$

Then we have degree of  $c_{n-1}(y) > \text{degree of } a_n(y)$ . Set

$$\phi(x, y) = \left( x - \frac{c_{n-1}(y)}{na_n(y)}, y \right).$$

Then we have

$$a \circ \phi(x, y) = a_n(y)x^n + b_{n-1}(y)x^{n-1} + \cdots,$$

and

$$\text{the degree of } b_{n-1}(y) \leq \text{the degree of } a_n(y).$$

2) Suppose that

$$b(x, y) = a(x + \alpha(y), y) \quad \text{for some } \alpha(y) \in m_y$$

Then

$$\begin{aligned} b(x, y) &= a_n(y)(x + \alpha(y))^n + a_{n-1}(y)(x + \alpha(y))^{n-1} + \dots \\ &= a_n(y)x^n + (na_n(y)\alpha(y) + a_{n-1}(y))x^{n-1} + \dots, \\ \text{and } b_{n-1}(y) &= na_n(y)\alpha(y) + a_{n-1}(y). \end{aligned}$$

On one hand  $b_{n-1}(y)$  is a polynomial of degree less than the order of  $a_n(y)$ . On the other hand, if  $\alpha(y) \neq 0$ , then  $na_n(y)\alpha(y) + a_{n-1}(y)$  is not a polynomial of degree less than the degree of  $a_n(y)$ . Thus  $\alpha(y)$  must be 0 and  $b(x, y) = a(x, y)$ .  $\square$

The above two theorems give a complete classification of fold singularities of integrable implicit Hamiltonian systems generated by functions of the form

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y),$$

$a(x, y)$  being a polynomial in  $x$  with coefficients in  $\mathcal{E}_y$ .

## 4 Non-polynomial case: tangent spaces of symplectic orbits

We have no such a complete classification for the case where  $a(x, y)$ 's are not polynomials in  $x$  with coefficients in  $\mathcal{E}_y$ . However we can deduce some symplectic invariants for such systems from the main theorem. For example we have the following

**Corollary 4.1.** *Let  $a(x, y)$  be of the form*

$$a(x, y) = a_0(x) + ya_1(x, y).$$

*Then the function  $a_0(x)$  is a symplectic invariant of the germ  $(M_F, ((0, 0), q))$  of the implicit Hamiltonian system  $M_F$  defined by*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y);$$

i.e. if another germ  $(M_G, ((0, 0), q'))$  of the implicit Hamiltonian system  $M_G$  defined by

$$G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y) \quad \text{with} \quad b(x, y) = b_0(x) + yb_1(x, y)$$

is symplectomorphic to  $(M_F, ((0, 0), q))$ , then we have  $a_0(x) = b_0(x)$ .

**Proof** Since  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  are symplectomorphic, from 3) of the theorem, we have

$$b(x, y) = a(x + \alpha(y), y) \quad \text{for some function} \quad \alpha(y)$$

and

$$\begin{aligned} b_0(x) + yb_1(x, y) &= a_0(x + \alpha(y)) + ya_1(x + \alpha(y), y) \\ &= a_0(x) + \alpha(y)\bar{a}_0(x, y) + ya_1(x + \alpha(y), y) \\ &\quad \text{for some function} \quad \bar{a}_0(x, y). \end{aligned}$$

Thus we have  $b_0(x) = a_0(x)$ . □

More interesting symplectic invariants are derived from the tangent spaces of symplectic orbits. From 1) of Theorem 2.1, every fold singularity of implicit Hamiltonian system is generated by a generating function of the form

$$\lambda^3 + y\lambda + c(x, y), \quad c(x, y) \in m_{x,y}.$$

From 3) of Theorem 2.1, two generating families

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y),$$

generate symplectomorphic germs of integrable implicit Hamiltonian systems if and only if  $a(x, y)$  and  $b(x, y)$  are symplectomorphic under a symplectomorphism of the form  $\phi(x, y) = (x + \alpha(y), y)$ :

$$b(x, y) = a(x + \alpha(y), y).$$

Therefore in order to classify fold singularities of integrable implicit Hamiltonian systems, it is enough to classify functions  $a(x, y) \in \mathcal{E}_{x,y}$  up to symplectomorphism  $\phi(x, y)$  of the form

$$\phi(x, y) = (x + \alpha(y), y).$$

Let  $\mathcal{S}_0$  denote the set of symplectomorphisms of the form  $\phi(x, y) = (x + \alpha(y), y)$ :

$$\mathcal{S}_0 = \{\phi(x, y) \mid \phi(x, y) = (x + \alpha(y), y) \text{ for some } \alpha \in m_y\}.$$

The composition of  $\phi(x, y) = (x + \alpha(y), y)$  and  $\psi(x, y) = (x + \beta(y), y)$  is

$$\psi \circ \phi(x, y) = \psi(x + \alpha(y), y) = (x + \alpha(y) + \beta(y), y).$$

Thus we have

**Lemma 4.2.**  *$\mathcal{S}_0$  is an abelian group with composition of maps. Moreover it is a real vector space isomorphic to the vector space  $m_y$  with the correspondence*

$$\phi(x, y) = (x + \alpha(y), y) \in \mathcal{S}_0 \quad \mapsto \quad \alpha(y) \in m_y.$$

The orbit  $\mathcal{S}_0(a(x, y))$  of a function  $a(x, y) \in \mathcal{E}_{xy}$  is given by

$$\mathcal{S}_0(a(x, y)) = \{a(x + \alpha(y), y) \mid \alpha(y) \in m_y\}.$$

Since  $\mathcal{S}_0$  is a real vector space isomorphic to the vector space  $m_y$  with the above correspondence, for  $a \in \mathcal{E}_{xy}$ , a vector tangent to  $\mathcal{S}_0(a)$  at  $a$  is given by

$$\left. \frac{da(x + t\alpha(y), y)}{dt} \right|_{t=0} \quad \text{for some } \alpha(y) \in m_y.$$

Therefore vectors tangent to  $\mathcal{S}_0(a)$  at  $a$  have the form

$$v = \frac{\partial a}{\partial x} \alpha.$$

Thus we call the set

$$\left\{ \frac{\partial a}{\partial x} \alpha \mid \alpha \in m_y \right\} = \frac{\partial a}{\partial x} m_y$$

the *tangent space* of the orbit  $\mathcal{S}_0(a)$  at  $a$ .

Let  $b(x, y)$  be symplectomorphic to  $a(x, y)$ :  $b(x, y) = a(x + \alpha(y), y)$ . Then

$$\frac{\partial b}{\partial x}(x, y) = \frac{\partial a}{\partial x}(x + \alpha(y), y).$$

Now consider the isomorphism

$$\phi^* : m_{xy} \rightarrow m_{xy} \quad \text{defined by} \quad \phi^*(c) = c \circ \phi.$$

Then we have

$$\phi^*\left(\frac{\partial a}{\partial x}\right) = \frac{\partial(a \circ \phi)}{\partial x} \quad \text{and} \quad \phi^*\left(\left(\frac{\partial a}{\partial x}\right)m_y\right) = \left(\frac{\partial a \circ \phi}{\partial x}\right)m_y = \left(\frac{\partial b}{\partial x}\right)m_y.$$

Since  $\phi^* : m_{xy} \rightarrow m_{xy}$  is an isomorphism, we have the isomorphism between the quotient spaces

$$\phi^* : \frac{m_{xy}}{\frac{\partial a}{\partial x}m_y} \cong \frac{m_{xy}}{\frac{\partial b}{\partial x}m_y}.$$

And for each positive integer  $k$ , we have also an isomorphism

$$\phi^* : \frac{m_{xy}}{\frac{\partial a}{\partial x}m_y + m_{xy}^{k+1}} \cong \frac{m_{xy}}{\frac{\partial b}{\partial x}m_y + m_{xy}^{k+1}}.$$

Thus we have

**Theorem 4.3.** *The numbers*

$$c_k(a) = \dim_{\mathbf{R}} \frac{m_{xy}}{\frac{\partial a}{\partial x}m_y + m_{xy}^{k+1}}, \quad k = 1, 2, \dots,$$

*are symplectic invariants.*

## 5 Preliminaries to the proof of the main theorem

Here we check elementary properties of generating functions.

### 5.1 Generating functions of implicit Hamiltonian systems

Let  $M \subset T\mathbf{R}^2$  be an implicit Hamiltonian system and let  $(p, q) \in M$  be a fold or cusp singularity of  $M$ , i.e. a fold or cusp singular point of  $\pi|_M : M \rightarrow \mathbf{R}^2$ . Then the corank  $d(\pi|_M)_{(p,q)} = 1$ . Therefore, by Hörmander -Arnold-Weinstein generating family representation of  $M$ , there exists a smooth function germ  $F : (\mathbf{R}^2 \times \mathbf{R}, (p, 0)) \rightarrow (\mathbf{R}, 0)$  such that

$(M, (p, q)) =$  the germ at  $(p, q)$  of the set

$$\left\{ (x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)) \mid \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0, (x, y, \lambda) \in \mathbf{R}^2 \times \mathbf{R} \right\},$$

and such that

$$(p, q) = \left(p, \frac{\partial F}{\partial y}(p, 0), -\frac{\partial F}{\partial x}(p, 0)\right),$$

$$\text{rank}\left(\frac{\partial^2 F}{\partial x \partial \lambda}, \frac{\partial^2 F}{\partial y \partial \lambda}\right)(p, 0) = 1, \quad \text{and} \quad \frac{\partial^2 F}{\partial \lambda^2}(p, 0) = 0.$$

In what follows, we assume that  $M$  is generated by some generating family-germ  $F$  as above and denote it by  $M_F$ . Without loss of generality, we may assume that germs of implicit Hamiltonian systems we consider are those at  $((0, 0), q) \in T\mathbf{R}^2$ , i.e.  $(M_F, ((0, 0), q))$ .

Our purpose in this note is to classify integrable fold and cusp singularities of implicit Hamiltonian systems on  $\mathbf{R}^2$  up to germs of symplectomorphisms of the tangent bundle  $T\mathbf{R}^2$  induced by germs of symplectomorphisms of  $\mathbf{R}^2$ . In order to do so, as we will see below, it is enough to classify their generating families up to germs of diffeomorphisms of  $(\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0))$  of the form

$$\tilde{\phi}(x, y, \lambda) = (\phi(x, y), \Lambda(x, y, \lambda)),$$

$$\phi : (\mathbf{R}^2, (0, 0)) \rightarrow (\mathbf{R}^2, (0, 0)), \quad \text{being a symplectomorphism-germ.}$$

**Definition 5.1.** *A diffeomorphism germ*

$$\tilde{\phi} : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0))$$

$$\text{of the form} \quad \tilde{\phi}(x, y, \lambda) = (\phi(x, y), \Lambda(x, y, \lambda)),$$

$\phi : (\mathbf{R}^2, (0, 0)) \rightarrow (\mathbf{R}^2, (0, 0))$  being a symplectomorphism, is called a *generalized symplectomorphism*.

**Definition 5.2.** *Two generating family germs  $F$  and  $G : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}, 0)$  are said to be symplectomorphic if there exists a generalized symplectomorphism germ*

$$\tilde{\phi} : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)), \quad \tilde{\phi}(x, y, \lambda) = (\phi(x, y), \Lambda(x, y, \lambda))$$

such that

$$G(x, y, \lambda) = F(\phi(x, y), \Lambda(x, y, \lambda)).$$

**Lemma 5.3.** *Let  $F$  and  $G : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}, 0)$  be symplectomorphic generating family germs. Then the implicit Hamiltonian system germs  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  are symplectomorphic.*

We prove the lemma by proving the following two lemmas.

**Lemma 5.4.** *Let  $F : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}, 0)$  be a symplectomorphic generating family germ. Let  $\phi = (\phi_1, \phi_2) : (\mathbf{R}^2, (0, 0)) \rightarrow (\mathbf{R}^2, (0, 0))$  be a symplectomorphism germ and let  $G : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}, 0)$  be the generating family germ defined by*

$$G(x, y, \lambda) = F(\phi(x, y), \lambda).$$

Then we have

$$\bar{\phi}(M_G, ((0, 0), q')) = (M_F, ((0, 0), q));$$

the germs  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  of the implicit Hamiltonian systems generated by  $F$  and  $G$ , respectively are symplectomorphic, where  $q = (\partial F/\partial y(0, 0), -\partial F/\partial x(0, 0))$  and  $q' = (\partial G/\partial y(0, 0), -\partial G/\partial x(0, 0))$ .

**Proof** Since

$$\frac{\partial G}{\partial x}(x, y, \lambda) = \frac{\partial F}{\partial x}(\phi(x, y), \lambda) \frac{\partial \phi_1}{\partial x}(x, y) + \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \frac{\partial \phi_2}{\partial x}(x, y),$$

$$\frac{\partial G}{\partial y}(x, y, \lambda) = \frac{\partial F}{\partial x}(\phi(x, y), \lambda) \frac{\partial \phi_1}{\partial y}(x, y) + \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \frac{\partial \phi_2}{\partial y}(x, y),$$

we have

$$\begin{aligned} \begin{pmatrix} \frac{\partial G}{\partial y}(x, y, \lambda) \\ -\frac{\partial G}{\partial x}(x, y, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial G}{\partial x}(x, y, \lambda) \\ \frac{\partial G}{\partial y}(x, y, \lambda) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \phi_1}{\partial x}(x, y) & \frac{\partial \phi_2}{\partial x}(x, y) \\ \frac{\partial \phi_1}{\partial y}(x, y) & \frac{\partial \phi_2}{\partial y}(x, y) \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x}(\phi(x, y), \lambda) \\ \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \phi_1}{\partial x}(x, y) & \frac{\partial \phi_1}{\partial y}(x, y) \\ \frac{\partial \phi_2}{\partial x}(x, y) & \frac{\partial \phi_2}{\partial y}(x, y) \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \\ -\frac{\partial F}{\partial x}(\phi(x, y), \lambda) \end{pmatrix}. \end{aligned}$$

Thus we have

$$\begin{aligned} J\phi(x, y) \begin{pmatrix} \frac{\partial G}{\partial y}(x, y, \lambda) \\ -\frac{\partial G}{\partial x}(x, y, \lambda) \end{pmatrix} &= \\ J\phi(x, y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} J\phi(x, y)^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \\ -\frac{\partial F}{\partial x}(\phi(x, y), \lambda) \end{pmatrix} &= \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \\ -\frac{\partial F}{\partial x}(\phi(x, y), \lambda) \end{pmatrix} &= \begin{pmatrix} \frac{\partial F}{\partial y}(\phi(x, y), \lambda) \\ -\frac{\partial F}{\partial x}(\phi(x, y), \lambda) \end{pmatrix}. \end{aligned}$$



Therefore we have

$$\begin{aligned} \bar{\phi}(x, y, \frac{\partial G}{\partial y}(x, y, \lambda), -\frac{\partial G}{\partial x}(x, y, \lambda)) = \\ (\phi(x, y), \frac{\partial F}{\partial y}(\phi(x, y), \lambda), -\frac{\partial F}{\partial x}(\phi(x, y), \lambda)) \end{aligned}$$

and

$$\bar{\phi}(M_G, ((0, 0), q')) = (M_F, ((0, 0), q)).$$

□

**Lemma 5.5.** *Let  $(M_F, ((0, 0), q)) \subset (T\mathbf{R}^2, ((0, 0), q))$  be an implicit Hamiltonian system generated by  $F(x, y, \lambda)$ . Let  $\phi : (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0)) \rightarrow (\mathbf{R}^2 \times \mathbf{R}, (0, 0, 0))$  be a generalized symplectomorphism of the form*

$$\tilde{\phi}(x, y, \lambda) = (x, y, \Lambda(x, y, \lambda)).$$

Set

$$G(x, y, \lambda) = F(x, y, \Lambda(x, y, \lambda)).$$

Then we have

$$(M_G, ((0, 0), q)) = (M_F, ((0, 0), q)).$$

**Proof** Set

$$\widetilde{M}_F = \{(x, y, \lambda) \mid \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0\},$$

$$\widetilde{M}_G = \{(x, y, \lambda) \mid \frac{\partial G}{\partial \lambda}(x, y, \lambda) = 0\},$$

$$\Psi_F : \mathbf{R}^2 \times \mathbf{R} \rightarrow T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2,$$

$$\Psi_F(x, y, \lambda) = (x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)),$$

$$\Psi_G : \mathbf{R}^2 \times \mathbf{R} \rightarrow T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2,$$

$$\Psi_G(x, y, \lambda) = (x, y, \frac{\partial G}{\partial y}(x, y, \lambda), -\frac{\partial G}{\partial x}(x, y, \lambda)).$$

Since

$$\frac{\partial G}{\partial \lambda}(x, y, \lambda) = \frac{\partial F}{\partial \lambda}(x, y, \Lambda(x, y, \lambda)) \frac{\partial \Lambda}{\partial \lambda}(x, y, \lambda) \quad \text{and} \quad \frac{\partial \Lambda}{\partial \lambda}(x, y, \lambda) \neq 0,$$

we have  $\tilde{\phi}(\widetilde{M}_G) = \widetilde{M}_F$ .

Moreover, since

$$\frac{\partial G}{\partial x}(x, y, \lambda) = \frac{\partial F}{\partial x}(x, y, \Lambda(x, y, \lambda)) \quad \text{on } \widetilde{M}_G,$$

$$\frac{\partial G}{\partial y}(x, y, \lambda) = \frac{\partial F}{\partial y}(x, y, \Lambda(x, y, \lambda)) \quad \text{on } \widetilde{M}_G,$$

we have

$$\Psi_G|_{\widetilde{M}_G} = (\Psi_F \circ \tilde{\phi})|_{\widetilde{M}_G},$$

and

$$M_F = \Psi_F(\widetilde{M}_F) = \Psi_F(\tilde{\phi}(\widetilde{M}_G)) = (\Psi_F \circ \tilde{\phi})(\widetilde{M}_G) = (\Psi_G)(\widetilde{M}_G) = M_G.$$

□

## 5.2 Generating functions of fold and cusp singularities

Let  $(M_F, ((0, 0), q)) \subset (T\mathbf{R}^2, ((0, 0), q))$  be a fold or cusp singularity of an implicit Hamiltonian system generated by  $F(x, y, \lambda)$ . Let

$$\tilde{\pi} : \mathbf{R}^2 \times \mathbf{R} \rightarrow \mathbf{R}^2 \quad \text{be the projection,} \quad \tilde{\pi}(x, y, \lambda) = (x, y),$$

and set

$$\widetilde{M}_F = \{(x, y, \lambda) \mid \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0\}.$$

From the definition of generating functions,

$$((0, 0), q) = ((0, 0), \frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0)), \quad \text{and}$$

$$\text{rank}\left(\frac{\partial^2 F}{\partial x \partial \lambda}, \frac{\partial^2 F}{\partial y \partial \lambda}\right)(0, 0, 0) = 1.$$

Therefore, without loss of generality, we may assume that

$$\frac{\partial^2 F}{\partial y \partial \lambda}(0, 0, 0) \neq 0.$$

Then, by the implicit function theorem, there exists a function  $\eta(x, \lambda)$  such that  $\widetilde{M}_F$  has the form

$$\widetilde{M}_F = \{(x, \eta(x, \lambda), \lambda) \mid (x, \lambda) \in \mathbf{R}^2\},$$

and that the restriction of the projection  $\tilde{\pi}$  to  $\widetilde{M}_F$  can be expressed in terms of the coordinates  $(x, \lambda)$  of  $\widetilde{M}_F$ :

$$\tilde{\pi}|_{\widetilde{M}_F}(x, \lambda) = (x, \eta(x, \lambda)).$$

Now, suppose that  $((0, 0), q) \in M_F$  is a fold singular point. Then we have

$$\frac{\partial \eta}{\partial \lambda}(0, 0) = 0, \quad \frac{\partial^2 \eta}{\partial \lambda^2}(0, 0) \neq 0,$$

and, by the relation of derivatives of  $\eta(x, \lambda)$  and  $\frac{\partial F}{\partial \lambda}(x, y, \lambda)$ , we have

$$\frac{\partial F}{\partial \lambda}(0, 0, 0) = \frac{\partial^2 F}{\partial \lambda^2}(0, 0, 0) = 0 \quad \text{and} \quad \frac{\partial^3 F}{\partial \lambda^3}(0, 0, 0) \neq 0.$$

In the case where  $((0, 0), q) \in M_F$  is a cusp singular point, we have

$$\frac{\partial \eta}{\partial \lambda}(0, 0) = \frac{\partial^2 \eta}{\partial \lambda^2}(0, 0) = 0, \quad \frac{\partial^3 \eta}{\partial \lambda^3}(0, 0) \neq 0,$$

and

$$\frac{\partial^k F}{\partial \lambda^k}(0, 0, 0) = 0, \quad (k = 1, 2, 3), \quad \frac{\partial^4 F}{\partial \lambda^4}(0, 0, 0) \neq 0.$$

Thus we have

**Lemma 5.6.** 1) *If  $((0, 0), q) \in M_F$  is a fold singular point, then we have*

$$\frac{\partial F}{\partial \lambda}(0, 0, 0) = \frac{\partial^2 F}{\partial \lambda^2}(0, 0, 0) = 0 \quad \text{and} \quad \frac{\partial^3 F}{\partial \lambda^3}(0, 0, 0) \neq 0.$$

2) *If  $((0, 0), q) \in M_F$  is a cusp singular point, then we have*

$$\frac{\partial^k F}{\partial \lambda^k}(0, 0, 0) = 0, \quad (k = 1, 2, 3), \quad \frac{\partial^4 F}{\partial \lambda^4}(0, 0, 0) \neq 0.$$

Now, from Lemma 5.6, after a coordinate change of the variable  $\lambda$  in the form of  $\Lambda(x, y, \lambda)$ , we have

**Lemma 5.7.** *We may suppose that*

1) *If  $((0, 0), q) \in M_F$  is a fold singular point, then  $F(x, y, \lambda)$  is an unfolding of  $\lambda^3$ , and*

2) *if  $((0, 0), q) \in M_F$  is a cusp singular point, then  $F(x, y, \lambda)$  is an unfolding of  $\lambda^4$ .*

Now a versal unfolding of  $\lambda^3$  is given by

$$G(u, \lambda) = \lambda^3 + u\lambda.$$

From the versality of  $G(u, \lambda)$ , for unfolding  $F(x, y, \lambda)$  of  $\lambda^3$ , there is a coordinate change of  $\lambda$  of the form  $\Lambda(x, y, \lambda)$  and functions  $a(x, y)$  and  $b(x, y)$  such that

$$F(x, y, \lambda) = \Lambda(x, y, \lambda)^3 + a(x, y)\Lambda(x, y, \lambda) + c(x, y).$$

Again, from Lemma 5.6, we may assume that  $F(x, y, \lambda)$  has the form

$$F(x, y, \lambda) = \lambda^3 + a(x, y)\lambda + c(x, y).$$

At the beginning of this subsection we assumed that

$$\frac{\partial^2 F}{\partial y \partial \lambda}(0, 0, 0) \neq 0.$$

Therefore we have

$$\frac{\partial a}{\partial y}(0, 0) \neq 0.$$

**Lemma 5.8.** *For  $a(x, y)$  with*

$$\frac{\partial a}{\partial y}(0, 0) \neq 0.$$

*there is always a function  $b(x, y)$  such that the mapping  $\phi$  defined by*

$$\phi(x, y) = (a(x, y), b(x, y))$$

*is a symplectomorphism.*

**Proof** A diffeomorphism  $\phi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is a symplectomorphism if and only if  $\det J\phi(x, y) \equiv 1$ . Since

$$\frac{\partial a}{\partial y}(0, 0) \neq 0,$$

the Hamiltonian vector field

$$X_a(x, y) = \frac{\partial a}{\partial y}(x, y) \frac{\partial}{\partial x} - \frac{\partial a}{\partial x}(x, y) \frac{\partial}{\partial y}$$

is regular at  $(0, 0)$  and so is  $-X_a(x, y)$ . Then, from the linearization theorem, there exists a diffeomorphism-germ  $h = (h_1, h_2) : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$  such that

$$Jh(x, y)(-X_a(x, y)) = \frac{\partial}{\partial x},$$

$$i.e. \quad \begin{pmatrix} \frac{\partial h_1}{\partial x}(x, y) & \frac{\partial h_1}{\partial y}(x, y) \\ \frac{\partial h_2}{\partial x}(x, y) & \frac{\partial h_2}{\partial y}(x, y) \end{pmatrix} \begin{pmatrix} -\frac{\partial a}{\partial y}(x, y) \\ \frac{\partial a}{\partial x}(x, y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus we have

$$\det J(a, h_2)(x, y) = \det \begin{pmatrix} \frac{\partial a}{\partial x}(x, y) & \frac{\partial a}{\partial y}(x, y) \\ \frac{\partial h_1}{\partial x}(x, y) & \frac{\partial h_1}{\partial y}(x, y) \end{pmatrix} = 1$$

and we may choose  $h_1(x, y)$  as a desired function.  $\square$

Since our purpose is to classify singularities of implicit Hamiltonian systems up to symplectomorphisms, by Lemma 5.8 we may assume that  $a(x, y) = y$ . Thus we have

**Lemma 5.9.** *1) If  $((0, 0), q) \in M \subset T\mathbf{R}^2$  is a fold singular point of an implicit Hamiltonian system, then the germ  $(M, (p, q))$  of the implicit Hamiltonian system is symplectomorphic to the germ  $(M_F, ((0, 0), q))$  of the implicit Hamiltonian system  $M_F$  generated by a function  $F(x, y, \lambda)$  of the form*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y),$$

$$where \quad q = \left( \frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0) \right).$$

*2) In the same way, if  $((0, 0), q) \in M \subset T\mathbf{R}^2$  is a cusp singular point of an implicit Hamiltonian system, then the germ  $(M, ((0, 0), q))$  of the implicit Hamiltonian system is symplectomorphic to the germ  $(M_F, ((0, 0), q))$  of the implicit Hamiltonian system  $M_F$  generated by the function*

$$F(x, y, \lambda) = \lambda^4 + b(x, y)\lambda^2 + y\lambda + c(x, y),$$

$$where \quad q = \left( \frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0) \right).$$

## 6 Proof of the main theorem

In this section we prove our main theorem. Let us recall it first.

**Theorem 2.1** 1) *If  $((0,0), q) \in M \subset T\mathbb{R}^2$  is a fold singular point of an implicit Hamiltonian system, then the germ  $(M, (p, q))$  of the implicit Hamiltonian system is symplectomorphic to the germ  $(M_F, ((0,0), q))$  of the implicit Hamiltonian system  $M_F$  generated by a function  $F(x, y, \lambda)$  of the form*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y),$$

where  $q = \left(\frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0)\right)$ .

2) *The implicit Hamiltonian system generated by*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y)$$

*is smoothly integrable if and only if*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{for some function } a(x, y).$$

3) *Let  $(M_F, ((0,0), q))$  and  $(M_G, ((0,0), q'))$  be the germs of integrable implicit Hamiltonian systems generated by*

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y),$$

where  $q = \left(\frac{\partial F}{\partial y}(0, 0, 0), -\frac{\partial F}{\partial x}(0, 0, 0)\right)$ ,  $q' = \left(\frac{\partial G}{\partial y}(0, 0, 0), -\frac{\partial G}{\partial x}(0, 0, 0)\right)$ .

*Then  $(M_F, ((0,0), q))$  and  $(M_G, ((0,0), q'))$  are symplectomorphic if and only if  $a(x, y)$  and  $b(x, y)$  are symplectomorphic under a symplectomorphism of the form  $\phi(x, y) = (x + \alpha(y), y)$ :*

$$b(x, y) = a(x + \alpha(y), y).$$

### 6.1 Proof of the main theorem

**Proof of 1)** 1) is the first part of Lemma 5.9.

**Proof of 2)** The implicit Hamiltonian system generated by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y)$$

is smoothly integrable if and only if the linear equation

$$\frac{\partial^2 F}{\partial \lambda^2} \mu = \left\{ \frac{\partial F}{\partial \lambda}, F \right\} \quad (\text{mod } \frac{\partial F}{\partial \lambda})$$

has a smooth solution  $\mu(x, y, \lambda)$ . Since

$$\frac{\partial F}{\partial \lambda} = 3\lambda^2 + y, \quad \frac{\partial^2 F}{\partial \lambda^2} = 6\lambda \quad \text{and} \quad \left\{ \frac{\partial F}{\partial \lambda}, F \right\} = \frac{\partial c}{\partial x}(x, y),$$

the linear equation is of the form

$$6\lambda\mu(x, y, \lambda) = \frac{\partial c}{\partial x}(x, y) \quad (\text{mod } \langle 3\lambda^2 + y \rangle_{\mathcal{E}_{xy\lambda}}).$$

Therefore the linear equation has a smooth solution if and only if

$$\frac{\partial c}{\partial x}(x, y) \in \langle 3\lambda^2 + y, 6\lambda \rangle_{\mathcal{E}_{xy\lambda}} = \langle y, \lambda \rangle_{\mathcal{E}_{xy\lambda}}.$$

Since  $c(x, y)$  is a function of  $x$  and  $y$ , the equation has a smooth solution if and only if

$$\frac{\partial c}{\partial x}(x, y) \in \langle y \rangle_{\mathcal{E}_{xy}}, \quad \text{i.e.} \quad c(x, y) = ya(x, y) \quad \text{for some } a(x, y) \in \mathcal{E}_{xy}.$$

Thus the implicit Hamiltonian system generated by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + c(x, y)$$

is smoothly integrable if and only if  $F(x, y, \lambda)$  is of the form

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{for some function } a(x, y).$$

**Proof of 3)** Suppose that the germs of implicit Hamiltonian systems  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  generated respectively by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y),$$

are symplectomorphic.

**Lemma 6.1.** *If  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  are symplectomorphic, then there exists a symplectomorphism*

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y))$$

such that

$$\begin{aligned} \lambda^3 + y\lambda + yb(x, y) &= G(x, y, \lambda) = \\ F(\phi(x, y), \lambda) &= \lambda^3 + \phi_2(x, y)\lambda + \phi_2(x, y)a(\phi_1(x, y), \phi_2(x, y)). \end{aligned}$$

We postpone the proof of the lemma and continue the proof of the main theorem. By Lemma 6.1,

$$\lambda^3 + y\lambda + yb(x, y) = \lambda^3 + \phi_2(x, y)\lambda + \phi_2(x, y)a(\phi_1(x, y), \phi_2(x, y)),$$

for some symplectomorphism

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y)).$$

Comparing the coefficients of  $\lambda$ , we have

$$\phi_2(x, y) = y.$$

Since  $\phi(x, y) = (\phi_1(x, y), y)$  is a symplectomorphism,

$$\det J\phi(x, y) = \det \begin{pmatrix} \frac{\partial \phi_1}{\partial x}(x, y) & \frac{\partial \phi_1}{\partial y}(x, y) \\ 0 & 1 \end{pmatrix} = 1.$$

Thus we have

$$\frac{\partial \phi_1}{\partial x}(x, y) \equiv 1 \quad \text{and} \quad \phi_1(x, y) = x + \alpha(y) \quad \text{for some} \quad \alpha \in m_y$$

$$\text{and} \quad b(x, y) = a(x + \alpha(y), y).$$

Conversely, since  $F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y)$  and  $G(x, y, \lambda) = \lambda^3 + y\lambda + ya(x + \alpha(y), y)$  are symplectomorphic, by Lemma 5.9, they generate the symplectomorphic implicit Hamiltonian systems.  $\square$

## 6.2 Proof of Lemma 6.1

Suppose that the germs of implicit Hamiltonian systems  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  generated respectively by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + ya(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + yb(x, y),$$

are symplectomorphic. Then we will prove that there exists a symplectomorphism

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y))$$

such that  $G(x, y, \lambda) = F(\phi(x, y), \lambda)$ .



**Lemma 6.2.**

If  $(M_F, ((0, 0), q)) = (M_G, ((0, 0), q'))$ , then  $F(x, y, \lambda) = G(x, y, \lambda)$ .

**Proof of Lemma 6.2.**

Since  $(M_F, ((0, 0), q)) = (M_G, ((0, 0), q'))$ , we have

$$\frac{\partial F}{\partial x}(x, y, \lambda) = \frac{\partial G}{\partial x}(x, y, \lambda) \quad \text{for } -3\lambda^2 = y.$$

Therefore we have

$$\frac{\partial a}{\partial x}(x, y) = \frac{\partial b}{\partial x}(x, y) \quad \text{and hence } b(x, y) = a(x, y) + c(y).$$

Moreover, since  $(M_F, ((0, 0), q)) = (M_G, ((0, 0), q'))$ , we have

$$\frac{\partial F}{\partial y}(x, y, \lambda) = \frac{\partial G}{\partial y}(x, y, \lambda) \quad \text{for } -3\lambda^2 = y.$$

Therefore we have

$$a(x, y) + y \frac{\partial a}{\partial y}(x, y) = b(x, y) + y \frac{\partial b}{\partial y}(x, y),$$

and, since  $b(x, y) = a(x, y) + c(y)$ , we have

$$a(x, y) + y \frac{\partial a}{\partial y}(x, y) = a(x, y) + c(y) + y \left( \frac{\partial a}{\partial y}(x, y) + \frac{\partial c}{\partial y}(y) \right).$$

Thus

$$c(y) + y \frac{\partial c}{\partial y}(y) = 0 \quad \text{and} \quad c(y) \equiv 0.$$

Hence

$$a(x, y) = b(x, y) \quad \text{and} \quad F(x, y, \lambda) = G(x, y, \lambda).$$

□

Now let us prove Lemma 6.1. Suppose that the germs of implicit Hamiltonian systems  $(M_F, ((0, 0), q))$  and  $(M_G, ((0, 0), q'))$  generated respectively by

$$F(x, y, \lambda) = \lambda^3 + y\lambda + a(x, y) \quad \text{and} \quad G(x, y, \lambda) = \lambda^3 + y\lambda + b(x, y),$$

are symplectomorphic; *i.e.* there exists a symplectomorphism

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y))$$

such that

$$\bar{\phi}((M_G, ((0, 0), q'))) = (M_F, ((0, 0), q)),$$

$$\text{where } \bar{\phi}(x, y, \dot{x}, \dot{y}) = (\phi(x, y), d\phi_{(x,y)}(\dot{x}, \dot{y})).$$

Setting  $H(x, y, \lambda) = F(\phi(x, y), \lambda)$ , from Lemma 5.8, we have

$$\bar{\phi}((M_H, ((0, 0), q'))) = (M_F, ((0, 0), q)),$$

which was proved to be equal to  $\bar{\phi}((M_G, ((0, 0), q')))$ .

Thus we have  $(M_H, ((0, 0), q')) = (M_G, ((0, 0), q))$ , which implies, from Lemma 6.2, that

$$G(x, y, \lambda) = H(x, y, \lambda) = F(\phi(x, y), \lambda).$$

□

of Lemma 6.1

## References

- [1] P.A.M. Dirac, *Generalized Hamiltonian Dynamics*, Canadian J. Math. **2** (1950), 129–148.
- [2] T. Fukuda, S. Janeczko, *Singularities of implicit differential systems and their integrability*, Banach Center Publications, "Geometric Singularity Theory", Vol. **65** (2004), 23–47.
- [3] S. Janeczko, *On implicit Lagrangian differential systems*, Annales Polonici Mathematici, **LXXIV** (2000), 133–141
- [4] J. Martinet, *Singularities of Smooth Functions and Maps*, Cambridge Univ. Press, Cambridge, 1982.
- [5] J.N. Mather, *Solutions of generic linear equations*, Dynamical Systems, Proc. Sympos. Univ. Bahia, Salvador 1971, 185–193 (1973).
- [6] A. Weinstein, *Lectures on Symplectic Manifolds*, CBMS Regional Conf. Ser. in Math., 29, AMS Providence, R.I. 1977.

# Reachable sets from a point for the Heisenberg sub-Lorentzian structure on $\mathbb{R}^3$ . An estimate for the distance function

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## Abstract

In this paper we compute reachable sets from a point for the Heisenberg sub-Lorentzian metric on  $\mathbb{R}^3$ , and give one estimate (from below) for the distance function. We also give examples of maximizing non-Hamiltonian and non-smooth geodesics which are regular curves.

## 1 Introduction

### 1.1 Statement of the results

Suppose that  $(M, H, g)$  is a sub-Riemannian manifold. For a fixed point  $p_0$  let  $f(p)$  stand for the sub-Riemannian distance from  $p_0$ . It can be proved (cf. [2]; see also [6]) that one can choose local coordinates  $x_1, \dots, x_n$  around  $p_0$  (the so-called *privileged coordinates* centered at  $p_0$ ) such that  $x_i(p_0) = 0$  and the following estimate holds true

$$c(|x_1|^{1/w_1} + \dots + |x_n|^{1/w_n}) \leq f(x_1, \dots, x_n) \leq C(|x_1|^{1/w_1} + \dots + |x_n|^{1/w_n}),$$

or equivalently

$$\tilde{c}\sqrt{|x_1|^{2/w_1} + \dots + |x_n|^{2/w_n}} \leq f(x_1, \dots, x_n) \leq \tilde{C}\sqrt{|x_1|^{2/w_1} + \dots + |x_n|^{2/w_n}},$$

where  $c, C$  (resp.  $\tilde{c}, \tilde{C}$ ) are positive constants and  $w_1, \dots, w_n$  are positive integers determined by the sub-Riemannian structure under consideration ( $w_i$  is called a *weight* of  $x_i$ ). In particular, if  $f$  is the Heisenberg sub-Riemannian distance from the origin in  $\mathbb{R}^3$ , and  $x, y, z$  are standard coordinates, then

$$c\sqrt{x^2 + y^2 + |z|} \leq f(x, y, z) \leq C\sqrt{x^2 + y^2 + |z|}, \quad (1.1)$$

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$C, c > 0$  (compare it with the Riemannian case where the distance from a point, in suitable exponential coordinates, is given by  $\sqrt{x_1^2 + \dots + x_n^2}$ ).

The aim of this paper is to prove an estimate similar to this in (1.1) for the distance function  $f[U]$  determined by the Heisenberg sub-Lorentzian structure on  $\mathbb{R}^3$ ; unfortunately, only estimate from below is true. Since the Heisenberg sub-Lorentzian structure, up to a change of coordinates, is a nilpotent approximation of any contact sub-Lorentzian structure on  $\mathbb{R}^3$  (cf. normal forms in [4]), this paper is a first step toward computations in a general contact case.

For all the details concerning the sub-Lorentzian geometry the reader is referred to [3]. A review of basic notions and facts is presented in Section 1.2 (note that Proposition 1.2 and Examples 1.3, 1.4 are new in comparison with the previous papers by the author).

In Section 2 we compute reachable sets  $I^+(p_0, U)$  and  $J^+(p_0, U)$  for the Heisenberg metric, where  $U$  is a normal neighbourhood of  $p_0$  or  $U = \mathbb{R}^3$  (Theorem 2.1).

In Section 3 we prove that the image of the set of "timelike" covectors under exponential mapping is equal to  $I^+(0)$  (Proposition 3.1).

In Section 4 we prove Theorem 4.3 which states that for every  $(x, y, z) \in J^+(0, U)$

$$\sqrt{x^2 - y^2 - 4|z|} \leq f[U](x, y, z),$$

where  $U$  is a normal neighbourhood of the origin or  $U = \mathbb{R}^3$ . (Note here that the distance from the origin in the Minkowski  $\mathbb{R}^3$  space with standard coordinates  $x, y, z$  is given on the set  $\{-x^2 + y^2 + z^2 < 0, x > 0\}$  by formula  $f(x, y, z) = \sqrt{x^2 - y^2 - z^2}$ ).

In Section 5 we construct null non-Hamiltonian and non-smooth maximizing geodesics which are regular curves. By the way, we study the problem of uniqueness of null maximizers (Proposition 5.2).

## 1.2 Basic definitions and facts on sub-Lorentzian geometry

A *sub-Lorentzian manifold* is a triple  $(M, H, h)$ , where  $M$  is a smooth  $(n+1)$ -dimensional manifold,  $H$  is a smooth bracket generating distribution on  $M$  of constant rank  $k+1$ , and  $h$  is a smooth Lorentzian metric on  $H$ .

Fix a sub-Lorentzian manifold  $(M, H, h)$ . For each point  $p \in M$  a vector  $v \in H_p$  is called *horizontal*. An absolutely continuous curve which is tangent

to  $H$  a.e. and has square integrable derivative is called a *horizontal* (or *admissible*) *curve*.

Unless otherwise specified, all vectors and curves are supposed to be horizontal.

Fix a point  $p \in M$ , and denote by  $\Omega_p$  the set of all (horizontal) curves  $\gamma : [0, 1] \rightarrow M$  starting from  $\gamma(0) = p$ . The *endpoint map*  $end_p : \Omega_p \rightarrow M$  is a mapping that assigns to each curve  $\gamma \in \Omega_p$  its end  $\gamma(1)$ .  $end_p$  is of class  $C^\infty$  with respect to the structure of Hilbert manifold on  $\Omega_p$ . Now, a curve  $\gamma \in \Omega_p$  is said to be *regular* (resp. *singular*) if it is a regular (resp. critical) point of  $end_p$ . It can be proved that in a contact case only constant curves are singular.

A vector  $v \in H_p$  is called *timelike* if  $h(v, v) < 0$ , *spacelike* if  $h(v, v) > 0$  or  $v = 0$ , *null* if  $h(v, v) = 0$  and  $v \neq 0$ , *nonspacelike* if  $h(v, v) \leq 0$ . A curve is called *timelike* if its tangent is timelike a.e.; similarly for *spacelike*, *null* and *nonspacelike curves*.

By a *time orientation* of  $(M, H, h)$  we mean a continuous timelike vector field on  $M$ . From now on we suppose our  $(M, H, h)$  to be time-oriented.

If  $X$  is a time orientation, then a nonspacelike  $v \in H_p$  is called *future directed* if  $h(v, X(p)) < 0$ , and is called *past directed* if  $h(v, X(p)) > 0$ .

Throughout this paper *f.d.* stands for "future directed", *t.* for "timelike", and *nspc.* for "nonspacelike". So, for instance, a *t.f.d. curve* is a curve which is horizontal and timelike future directed.

By  $\mathcal{H}$  we will denote the Hamiltonian associated with our sub-Lorentzian metric. Locally it can be defined as follows: let  $X_0, X_1, \dots, X_k$  be an orthonormal frame for  $H$  defined on an open  $U$  with  $X_0$  timelike; then

$$\mathcal{H}(x, \lambda) = -\frac{1}{2} \langle \lambda, X_0 \rangle^2 + \frac{1}{2} \sum_{j=1}^k \langle \lambda, X_j \rangle^2$$

on  $T^*M|U$ . By  $\vec{\mathcal{H}}$  we denote the corresponding Hamiltonian vector field on  $T^*M$ , and  $\Phi_s$  stands for its flow. Notice that  $\pi \circ \Phi_s(\lambda) = \pi \circ \Phi_1(s\lambda)$  for any covector  $\lambda$ , where  $\pi : T^*M \rightarrow M$  is the canonical projection.

A curve  $\gamma : [\alpha, \beta] \rightarrow U$  is called a *Hamiltonian geodesic* if it is of the form  $\gamma(s) = \pi \circ \Phi_s(\lambda)$ ; in such a case  $[\alpha, \beta] \ni s \rightarrow \Phi_s(\lambda)$  is called a *Hamiltonian lift* of  $\gamma$ . Note that a Hamiltonian geodesic which is a regular curve has a unique Hamiltonian lift.

Each Hamiltonian geodesic is either timelike, spacelike or null (i.e. it does not change its causal character).

For a point  $p \in M$  denote by  $\exp_p$  the exponential mapping with the pole at  $p$ , which is defined as follows. Let  $D_p$  stand for the set of all covectors  $\lambda \in T_p^*M$  such that the curve  $s \rightarrow \pi \circ \Phi_s(\lambda)$  is defined on the interval  $[0, 1]$ . The set  $D_p$  is open and  $\exp_p : D_p \rightarrow M$  acts by formula  $\exp_p(\lambda) = \pi \circ \Phi_1(\lambda)$ .

For a nspc. curve  $\gamma : [\alpha, \beta] \rightarrow M$  we define its *length* to be

$$L(\gamma) = \int_{\alpha}^{\beta} |h(\dot{\gamma}, \dot{\gamma})|^{1/2} dt.$$

Fix an open set  $U$ ; a nspc.f.d. curve  $\gamma : [\alpha, \beta] \rightarrow U$  is called a *U-maximizer*, if it is the longest curve from  $\gamma(\alpha)$  to  $\gamma(\beta)$  among all nspc.f.d. curves contained in  $U$  and joining  $\gamma(\alpha)$  to  $\gamma(\beta)$ . We also use a name *U-geodesic* for a curve in  $U$  whose each suitably short sub-arc is a *U-maximizer* (note that in [3] only timelike curves were used).

By a *unique U-maximizer* (or a unique maximizing *U-geodesic*) we mean a (nspc.f.d.) curve  $\gamma : [\alpha, \beta] \rightarrow U$  such that for each  $t_1, t_2 \in [\alpha, \beta]$  with  $t_1 < t_2$ , the restriction  $\gamma|_{[t_1, t_2]}$  is the only *U-maximizer* between  $\gamma(t_1)$  and  $\gamma(t_2)$ . It can be proved that if  $\gamma : [\alpha, \beta] \rightarrow M$  is a t.f.d. Hamiltonian geodesic then for each  $t \in (\alpha, \beta)$  there is a neighbourhood  $U$  of  $\gamma(t)$  such that  $\gamma \cap U$  is a unique *U-maximizer*.

If  $\varphi : U \rightarrow \mathbb{R}$  is a smooth function on an open  $U$ , then its *horizontal gradient*  $\nabla_H \varphi$  is, by definition, a vector field on  $U$  such that  $(\partial_v \varphi)(p) = h(v, \nabla_H \varphi(p))$  for any  $v \in H_p$  and  $p \in U$ . If  $\nabla_H \varphi$  is unit timelike past directed on  $U$  then the trajectories of  $-\nabla_H \varphi$  are unique *U-maximizers*.

$d[U](\cdot, \cdot)$  will denote the *sub-Lorentzian distance function relative to a set*  $U \subset M$ , which is defined as follows. For  $p, q \in U$  let  $\Omega_{p,q}^-(U)$  be the set of all nspc.f.d. curves contained in  $U$  and joining  $p$  to  $q$ ; then

$$d[U](p, q) = \begin{cases} \sup \{L(\gamma) : \gamma \in \Omega_{p,q}^-(U)\} & \text{in case } \Omega_{p,q}^-(U) \neq \emptyset \\ 0 & \text{in case } \Omega_{p,q}^-(U) = \emptyset. \end{cases}$$

For a general  $U$  very little can be said about  $d[U]$ . However for a fixed point  $p_0$  one can construct a family of certain special neighbourhoods of  $p_0$ . Their construction goes as follows. We take any neighbourhood  $U$  of  $p_0$ . Shrinking it we assume  $\bar{U}$  compact, and that there is a basis  $X_0, \dots, X_n$  of  $TM$  over  $\bar{U}$  such that  $X_0, \dots, X_k$  is an orthonormal basis of  $H$  with  $X_0$  being a time orientation. Now we define a Lorentzian metric  $\tilde{h}$  on  $U$  by declaring the basis  $X_0, \dots, X_n$  to be orthonormal with respect to  $\tilde{h}$ , with the same time orientation  $X_0$ . Finally, again shrinking  $U$  we can assume that  $U$  is a convex

neighbourhood of  $p_0$  with respect to  $\tilde{h}$ , and that  $\bar{U}$  is contained in some other convex neighbourhood of  $p_0$ . A set  $U$  obtained in this way is called a *normal neighbourhood* of  $p_0$ . Remark that the Lorentzian manifold  $(U, \tilde{h})$  is *strongly causal* [10].

For a fixed  $p_0$  and  $U \ni p_0$ , let  $f[U](p) = d[U](p_0, p)$ .

**Proposition 1.1 ([3]).** *If  $U$  is a normal neighbourhood of  $p_0$  then  $f[U]$  is finite and upper semicontinuous on  $U$ . Moreover, if  $\gamma$  is a  $U$ -maximizer joining  $p_0$  to a point  $p \in U$ , such that  $\gamma$  is a t.f.d. Hamiltonian geodesic which is a regular curve, then  $f[U]$  is continuous at each point of  $\gamma$ .*

We need some notion of convergence of sequences of curves. Suppose that  $\gamma_\nu, \gamma : [a, b] \rightarrow M$ ,  $\nu = 1, 2, \dots$ , are curves in  $M$ ; we say that  $\{\gamma_\nu\}$  converges to  $\gamma$  in the  $C^0$  topology on curves if  $\gamma_\nu(a) \rightarrow \gamma(a)$ ,  $\gamma_\nu(b) \rightarrow \gamma(b)$ , and for each open  $V$  containing  $\gamma$  there is an integer  $\Lambda$  such that  $\gamma_\nu \subset V$  for all  $\nu > \Lambda$ . Now, let  $U$  be a normal neighbourhood of  $p_0$  and take a sequence  $\gamma_\nu : [0, l] \rightarrow U$  of nspc.f.d. curves such that  $\gamma_\nu(0) = p_0$  and their endpoints  $\gamma_\nu(l)$  tend to a point  $p \in U$ . Then it can be proved that there exists a subsequence  $\{\gamma_{\nu_i}\}$  convergent in the  $C^0$  topology to a nspc.f.d. curve joining  $p_0$  to  $p$  and contained in  $U$ .

For an open set  $U$  and fixed  $p_0 \in U$ , we define two *reachable sets*:  $I^+(p_0, U)$  (resp.  $J^+(p_0, U)$ ) is the set of all points  $p \in U$  that can be reached from  $p_0$  along a t.f.d. (resp. nspc.f.d.) curve contained in  $U$ . In terms of control theory (cf. [7])  $I^+(p_0, U)$  is just the set reachable from  $p_0$  for the family of all smooth t.f.d. vector fields on  $U$ ; in the Lorentzian geometry it is called the *chronological future* of  $p_0$  (with respect to  $U$ ). Similarly,  $J^+(p_0, U)$  is the set reachable from  $p_0$  for the family of all smooth nspc.f.d. vector fields on  $U$ , and in the Lorentzian geometry it is called the *causal future* of  $p_0$  (with respect to  $U$ ).

**Proposition 1.2.** *For any normal neighbourhood  $U$  of a point  $p_0$  (i)  $J^+(p_0, U)$  is a closed subset in  $U$ ; (ii)  $cl(I^+(p_0, U)) = J^+(p_0, U)$ , where  $cl$  stands for the closure with respect to  $U$ .*

*Proof:* Take any sequence  $\{p_\nu\}$ ,  $p_\nu \in J^+(p_0, U)$ , and  $p_\nu \rightarrow p$  with  $p \in U$ . Let  $\gamma_\nu$  be a nspc.f.d. curve in  $U$  which connects  $p_0$  to  $p_\nu$ . Passing to a subsequence we can assume that  $\gamma_\nu \rightarrow \gamma$  in the  $C^0$  topology on curves. As we already know  $\gamma$  is nspc.f.d., joins  $p_0$  to  $p$ , and  $\gamma \subset U$  which proves (i).

Using the same argument as in (i) one shows the inclusion  $cl(I^+(p_0, U)) \subset J^+(p_0, U)$ . To prove the reverse inclusion take a point  $p \in J^+(p_0, U)$  and

an open set  $V$ ,  $p \in V \subset U$ . Let  $\gamma : [0, T] \rightarrow U$  be a nspc.f.d. curve such that  $\gamma(0) = p_0$ ,  $\gamma(T) = p$ . Let  $\eta : [0, T] \rightarrow U$  be a smooth nspc.f.d. approximation of  $\gamma$  such that  $\eta(0) = p_0$  and  $\eta(T) \in V$ . Extending the domain of  $\eta$  we can assume that  $\eta : (-\varepsilon, T + \varepsilon) \rightarrow U$  is nspc.f.d. and smooth. Next let  $Z$  be a smooth nspc.f.d. vector field defined on a neighbourhood, say  $G$ , of the set  $\eta((-\varepsilon, T + \varepsilon))$  such that  $\dot{\eta}(t) = Z(\eta(t))$ ,  $t \in (-\varepsilon, T + \varepsilon)$ . Denote by  $X_0, X_1, \dots, X_k$  an orthonormal basis of  $H$  over  $U$  (which exists by definition of normal neighbourhoods); then

$$Z = \varphi_0 X_0 + \sum_{j=1}^k \varphi_j X_j$$

where  $\varphi_0 > 0$  and  $-\varphi_0^2 + \sum_{j=1}^k \varphi_j^2 \leq 0$ . Finally choose a sequence of real numbers  $\{a_\nu\}$  such that  $0 < a_\nu \nearrow 1$ , and define

$$Z_\nu = \varphi_0 X_0 + a_\nu \sum_{j=1}^k \varphi_j X_j.$$

Of course, for every  $\nu$ ,  $Z_\nu$  is a smooth t.f.d. vector field on  $G$ . Moreover,  $Z_\nu \rightrightarrows Z$  on any compact  $K$ , where  $\eta([0, T]) \subset K \subset G$ . Now for each  $\nu$  consider a curve  $\eta_\nu$  which is (the unique) solution to the following problem:  $\dot{\eta}_\nu(t) = Z_\nu(\eta_\nu(t))$ ,  $\eta_\nu(0) = p_0$ . For every sufficiently large  $\nu$  the curve  $\eta_\nu$  is defined on  $[0, T]$  and  $\eta_\nu \rightrightarrows \eta$ . It follows that there is a  $\nu$  such that  $\eta_\nu(T) \in V$ , from which  $I^+(p_0, U) \cap V \neq \emptyset$  and the proof is over.  $\square$

In the Lorentzian geometry the above proposition is also true; one can prove even more, namely  $I^+(p_0, U)$  is always open. The two examples below show that  $I^+(p_0, U)$  need not be open in the sub-Lorentzian case, and that the sub-Lorentzian distance function from  $p_0$  need not be continuous.

**Example 1.3.** (Cf. [8]) Let  $\omega = x^2 dy - (1 - x) dz$  be a 1-form on  $\mathbb{R}^3$  and set  $H = \ker \omega$ . Let  $X = (1 - x)\partial/\partial y + x^2\partial/\partial z$ ,  $Y = \partial/\partial x$ . Define a sub-Lorentzian metric  $h$  with formulae

$$h(X, X) = -1, \quad h(Y, Y) = 1, \quad h(X, Y) = 0, \quad (1.2)$$

and take  $X$  as a time orientation of  $(\mathbb{R}^3, H, h)$ . Consider a curve  $\gamma : [0, b] \rightarrow \mathbb{R}^3$ ,  $\gamma(t) = (0, t, 0)$ ,  $b > 0$ . Of course  $\gamma$  is t.f.d.; it can also be proved, in the same way as in the sub-Riemannian geometry, that  $\gamma$  is a singular



curve which is not a Hamiltonian geodesic. We will show that  $\Omega_{(0,0,0),(0,b,0)}^- = \Omega_{(0,0,0),(0,b,0)}^-(\mathbb{R}^3)$  consists, up to a change of parameterization, of a single curve, namely  $\gamma$ . This will imply, in particular, that  $\gamma$  is a  $U$ -maximizer for any normal neighbourhood  $U$  of the origin. Suppose that  $\eta \in \Omega_{(0,0,0),(0,b,0)}^-$ ,  $\eta(t) = (x(t), y(t), z(t))$ ,  $0 \leq t \leq T$ . Then

$$\dot{\eta}(t) = \alpha(t)X(\eta(t)) + \beta(t)Y(\eta(t))$$

which gives

$$\dot{\eta}(t) = (\beta(t), \alpha(t)(1 - x(t)), \alpha(t)x^2(t)).$$

Since  $\eta$  is nsc.p.f.d.,  $\alpha(t) > 0$  a.e. Now

$$z(T) = \int_0^T \alpha(t)x^2(t)dt = 0 \tag{1.3}$$

from which  $x(t)$  vanishes a.e. It means that  $z(t) = 0$  everywhere and  $\eta$  coincides with  $\gamma$  up to a change of parameterization. Using the same argument one shows that a point of the form  $(0, b, c)$  with  $c < 0$  does not belong to  $I^+(0, U)$ ,  $U$  being any normal neighbourhood of the zero containing  $(0, b, c)$ . Therefore  $I^+(0, U)$  is not open and  $f[U]$ , the sub-Lorentzian distance from the origin, is not continuous at points  $(0, b, 0)$ ,  $b > 0$ .

**Example 1.4.** (Cf. [9]) Let  $\omega = dz - x^2dy$  be (the Martinet) 1-form on  $\mathbb{R}^3$ . Again set  $H = \ker \omega$ . Let  $X = \partial/\partial y + x^2\partial/\partial z$ , and  $Y = \partial/\partial x$ . Define a sub-Lorentzian metric  $h$  using (1.2) and take  $X$  as a time orientation. Again consider a curve  $\gamma : [0, b] \rightarrow \mathbb{R}^3$ ,  $\gamma(t) = (0, t, 0)$ ,  $b > 0$ . One sees that  $\gamma$  is t.f.d., it is a singular curve, but this time  $\gamma$  is a Hamiltonian geodesic. Now, if  $\eta \in \Omega_{(0,0,0),(0,b,0)}^-$ ,  $\eta(t) = (x(t), y(t), z(t))$ ,  $0 \leq t \leq T$ , then again (1.3) holds with  $\alpha(t) > 0$  a.e. This shows that  $\Omega_{(0,0,0),(0,b,0)}^-$  contains, up to a change of parameterization, only one curve  $\gamma$ ,  $I^+(0, U)$  is not open, and the corresponding sub-Lorentzian distance  $f[U]$  is not continuous at points of  $\gamma \setminus \{0\}$ .

These two examples show that the statement of Proposition 1.1 cannot be strengthened. Let us also note here that a t.f.d. geodesic which is a regular curve cannot be contained in the boundary  $\partial I^+(0, U)$ .

At the end let us extract one more property of normal neighbourhoods. Once more we fix a point  $p_0$  and its normal neighbourhood  $U$ . Let  $\tilde{h}$  be a Lorentzian metric on  $U$  arising from the definition of normal neighbourhoods.

Next, let  $\tilde{f}[U]$  be the Lorentzian distance from  $p_0$  relative to  $U$  determined by  $\tilde{h}$  and let  $\tilde{I}^+(p_0, U)$  be the set of all points from  $U$  that can be reached from  $p_0$  by a (not necessarily horizontal) curve in  $U$  which is t.f.d. with respect to  $\tilde{h}$ . It is well known ([1]) that  $\tilde{f}[U]$  is smooth on  $\tilde{I}^+(p_0, U)$  and the gradient  $\nabla\tilde{f}[U]$  with respect to  $\tilde{h}$  is timelike past directed. This last statement implies that  $\tilde{f}[U]$  is non-decreasing along nspc.f.d. curves in  $U$ . Obviously  $\partial I^+(p_0, U) \cap \partial U \subset \partial\tilde{I}^+(p_0, U) \cap \partial U$  and the latter set is a smooth hypersurface. Therefore:

**Proposition 1.5.** *There exists a (non-horizontal) vector field  $W = \nabla\tilde{f}[U]$  which is transverse to  $\partial\tilde{I}^+(p_0, U) \cap \partial U$ , and which "pushes" all nspc.f.d. curves which are in  $\tilde{I}^+(p_0, M) \setminus \tilde{I}^+(p_0, U)$  away from  $\partial\tilde{I}^+(p_0, U) \cap \partial U$ . In particular, no nspc.f.d. curve can enter the set  $I^+(p_0, U)$  through the set  $\partial I^+(p_0, U) \cap \partial U$ .*

## 2 Reachable sets for the Heisenberg metric

Consider  $\mathbb{R}^3$  equipped with the Heisenberg sub-Lorentzian structure  $(H, h)$ , i.e.  $H = \ker \omega$ ,  $\omega = dz - \frac{1}{2}(ydx - xdy)$ , and  $h$  is defined by  $h(X, X) = -1$ ,  $h(Y, Y) = 1$ ,  $h(X, Y) = 0$ , where

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial z}; \quad (2.1)$$

we take  $X$  to be a time orientation. The aim of this section is to compute the sets  $I^+(p_0, U)$  and  $J^+(p_0, U)$ , where  $p_0 \in \mathbb{R}^3$  and  $U$  is its arbitrary normal neighbourhood.

We start from the case  $p_0 = 0$  and  $I^+(0) = I^+(0, \mathbb{R}^3)$ . To this end let us consider a family of functions  $\eta_\alpha = -x^2 + y^2 + \alpha|z|$ ,  $|\alpha| \leq 4$ . Then

$$\nabla_H \eta_\alpha = (2x - \frac{1}{2}\alpha y)X + (2y - \frac{1}{2}\alpha x)Y, \quad z > 0$$

and

$$\nabla_H \eta_\alpha = (2x + \frac{1}{2}\alpha y)X + (2y + \frac{1}{2}\alpha x)Y, \quad z < 0.$$

Let

$$\Gamma_\alpha = \{\eta_\alpha < 0, x > 0\}.$$

In both cases above the field  $\nabla_H \eta_\alpha$  is t.f.d. for  $|\alpha| < 4$  and is null f.d. for  $|\alpha| = 4$  on the set  $\Gamma_0 \cap \{z \neq 0\}$ .

Consider at first  $\eta_0$ . Looking at the behaviour of the fields  $X$  and  $Y$  on  $\partial\Gamma_0$  we have

$$J^+(0) \subset \bar{\Gamma}_0 = \{\eta_0 \leq 0, x \geq 0\},$$

where we set  $J^+(0) = J^+(0, \mathbb{R}^3)$  (cf. [4]). Now, since  $\nabla_H \eta_0$  is nspc.f.d. on  $\bar{\Gamma}_0 \setminus \{0\}$ , and since each nspc.f.d. curve which projects on the set

$$\{y = \pm x, x > 0, z = 0\}$$

must be a null curve, we see that for any t.f.d. curve  $\gamma : [0, T] \rightarrow \mathbb{R}^3$ ,  $\gamma(0) = 0$ , the function  $t \rightarrow \eta_0(\gamma(t))$  is decreasing a.e. It follows that  $I^+(0) \subset \Gamma_0$ .

We will show that

$$I^+(0) = \Gamma_4. \quad (2.2)$$

First of all let us notice that

$$\Gamma_0 \cap \{z = 0\} \subset I^+(0);$$

this is because each line  $x = t \cosh \varphi$ ,  $y = t \sinh \varphi$ ,  $z = 0$ ,  $t > 0$ , is a t.f.d. curve.

Take any  $p \in I^+(0)$ , i.e.  $p = \gamma(T)$ , where  $\gamma : [0, T] \rightarrow \mathbb{R}^3$  is t.f.d. and  $\gamma(0) = 0$ . Since, as we already know,  $\gamma((0, T]) \subset \Gamma_0$  and  $\nabla_H \eta_4$  is null f.d. on  $\Gamma_0$ , the function  $t \rightarrow \eta_4(\gamma(t))$  is decreasing a.e. It means, however, that  $\eta_4(p) < 0$  and "  $\subset$  " is true in (2.2).

To prove the reverse inclusion take a  $p \in \Gamma_4$ ,  $p = (x_0, y_0, z_0)$ . Fix an  $\varepsilon > 0$  this small that

$$-x_0^2 + y_0^2 + \frac{16}{4-\varepsilon}|z_0| < 0, \quad 4 - \varepsilon > 0. \quad (2.3)$$

Put  $\alpha = 4 - \varepsilon$  and write equations for  $\nabla_H \eta_\alpha$ . We distinguish two cases:  $z_0 > 0$

$$\begin{cases} \dot{x} = 2x - \frac{1}{2}\alpha y \\ \dot{y} = -\frac{1}{2}\alpha x + 2y \\ \dot{z} = \frac{1}{4}\alpha(x^2 - y^2) \end{cases} \quad (2.4)$$

and  $z_0 < 0$

$$\begin{cases} \dot{x} = 2x + \frac{1}{2}\alpha y \\ \dot{y} = \frac{1}{2}\alpha x + 2y \\ \dot{z} = \frac{1}{4}\alpha(y^2 - x^2) \end{cases} \quad (2.5)$$

(if  $z_0 = 0$  then there is nothing to do, see remark above).

Consider the case  $z_0 > 0$ . Solving (2.4) with initial condition  $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$  we get

$$z(t) = z_0 + \frac{\alpha}{16}(x_0^2 - y_0^2)(e^{4t} - 1).$$

Because of (2.3) the equation  $z(t) = 0$  has a negative solution with respect to  $t$ :

$$t = \frac{1}{4} \ln \frac{x_0^2 - y_0^2 - (16/\alpha)|z_0|}{x_0^2 - y_0^2}.$$

It shows that there is a trajectory of  $-\nabla_H \eta_\alpha$  joining  $p$  with a point  $\tilde{p} \in \Gamma_0 \cap \{z = 0\} \subset I^+(0)$ . After time reversal we obtain a trajectory, say,  $\sigma$  of the field  $\nabla_H \eta_\alpha$  that joins  $\tilde{p}$  to  $p$ . Since such a  $\sigma$  is t.f.d.,  $p \in I^+(0)$ . Similar argument works for  $z_0 < 0$ , thus "⊃" in (2.2) is proved.

Next, let  $U$  be a normal neighbourhood of the origin; we will show that

$$I^+(0, U) = I^+(0) \cap U. \quad (2.6)$$

Evidently  $I^+(0, U) \subset I^+(0) \cap U$ . Take a point  $p \in I^+(0) \cap U$  and let  $\gamma : [0, T] \rightarrow \mathbb{R}^3$  be a t.f.d. curve joining 0 to  $p$ . Suppose  $\gamma$  leaves  $U$ . Since it cannot leave the set  $I^+(0)$  it falls into the set  $I^+(0) \setminus U$ . Now  $\partial(I^+(0) \setminus U) = B_1 \cup B_2$ , where  $B_1 = \{\eta_4 = 0, x > 0\} \setminus U$  and  $B_2 = \{\eta_4 \leq 0, x > 0\} \cap \partial U$ . Consider the vector field  $\nabla_H \eta_4$  in a neighbourhood of  $B_1$  and the field  $W$  from Proposition 1.5 in a neighbourhood of  $B_2$ . Both fields taken together prevent all nspc.f.d. curves which are in  $I^+(0) \setminus U$  from leaving  $I^+(0) \setminus U$ . Thus we get a contradiction since  $\gamma$  cannot reenter  $U$  and reach  $p$ . This ends the proof of (2.6).

Finally, take any  $p_0 = (x_0, y_0, z_0)$ . It is easy to see that the mapping

$$\Phi(x, y, z) = (x - x_0, y - y_0, \frac{1}{2}(yx_0 - xy_0) + z - z_0), \quad (2.7)$$

which carries  $p_0$  to 0, preserves the Heisenberg sub-Lorentzian structure:  $\Phi_* X = X$ ,  $\Phi_* Y = Y$ ; in particular, if  $\gamma$  is a maximizer joining  $p_0$  to  $p_1$  then  $\Phi \circ \gamma$  is a maximizer joining 0 to  $\Phi(p_1)$ .

In this way, and with the aid of Proposition 1.2, we have proved the following

**Theorem 2.1.** *Consider the Heisenberg sub-Lorentzian metric on  $\mathbb{R}^3$ . Then*

$$I^+(0, \mathbb{R}^3) = \{-x^2 + y^2 + 4|z| < 0, x > 0\}$$

and

$$J^+(0, \mathbb{R}^3) = \{-x^2 + y^2 + 4|z| \leq 0, x \geq 0\}.$$

Also, for any normal neighbourhood  $U$  of zero,

$$I^+(0, U) = I^+(0, \mathbb{R}^3) \cap U$$

and

$$J^+(0, U) = J^+(0, \mathbb{R}^3) \cap U.$$

Next, if  $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$  is arbitrary and  $U$  is its normal neighbourhood or  $U = \mathbb{R}^3$ , then

$$I^+(p_0, U) = \Phi^{-1}(I^+(0, U))$$

and

$$J^+(p_0, U) = \Phi^{-1}(J^+(0, U)),$$

where  $\Phi$  is defined by (2.7).

**Remark 2.2.** The equality  $J^+(0, \mathbb{R}^3) = \{\eta_4 \leq 0, x \geq 0\}$  requires some explanations. Observe that a function  $\varphi(x, y, z) = x$  satisfies  $\nabla_H \varphi = -X$  on the whole  $\mathbb{R}^3$ . Such a  $\varphi$  is called a "time function", and its existence insures that Lemma 3.1 of [3], and hence Proposition 1.2 above, applies to  $U = \mathbb{R}^3$  in case of the Heisenberg sub-Lorentzian metric.

From now on by  $f[U]$  we will denote the sub-Lorentzian distance from the origin induced by the Heisenberg structure; we also set  $f = f[\mathbb{R}^3]$ . As a consequence of above considerations we obtain two corollaries.

**Corollary 2.3.** For any normal neighbourhood  $U$  of zero  $f[U] = f|_{J^+(0, U)}$ .

*Proof:* We use two fields:  $\nabla_H \eta_4$  and the one from Proposition 1.5 □

**Corollary 2.4.** For any normal neighbourhood  $U$  of the origin  $f[U]$  vanishes on  $\{\eta_4 = 0\} \cap U$  and is continuous at points of the set  $\{\eta_4 = 0\} \cap U$ .

*Proof:* The first part follows from the fact that  $\nabla_H \eta_4$  keeps all nspc.f.d. curves which are in  $I^+(0, U)$  from reaching the set  $\{\eta_4 = 0\}$ . Then any nspc.f.d. curve  $\gamma$  which joins zero to a point of  $\{\eta_4 = 0\}$  must be entirely contained in  $\{\eta_4 = 0\}$ . Obviously, for such a  $\gamma$ ,  $\eta_4(\gamma(t)) = 0$  for all  $t$ , from which  $h(\dot{\gamma}, \nabla_H \eta_4) = 0$ . However  $\nabla_H \eta_4$  is a null field, so  $\gamma$  must be a null curve.

To prove the second part let us fix a  $p \in \{\eta_4 = 0\} \cap U$  and take a sequence  $I^+(0, U) \ni p_\nu \rightarrow p$ . Let  $\gamma_\nu$  be a  $U$ -maximizer connecting zero to  $p_\nu$ . Passing to a subsequence,  $\gamma_\nu \rightarrow \gamma$  in the  $C^0$  topology, where  $\gamma$  is nspc.f.d. and joins the zero to  $p$ . Thus  $\gamma$  is a null curve and  $\limsup f[U](p_\nu) \leq f[U](p) = 0$ , i.e.  $\lim f[U](p_\nu) = f[U](p)$ , which ends the proof.  $\square$

### 3 Image under exponential mapping

For any  $p \in \mathbb{R}^3$  and  $\lambda \in T_p^* \mathbb{R}^3$  let

$$\mathcal{H}(p, \lambda) = -\frac{1}{2} \langle \lambda, X \rangle + \frac{1}{2} \langle \lambda, Y \rangle$$

be the Hamiltonian associated with the Heisenberg sub-Lorentzian structure. Next, let

$$E_{p_0} = \left\{ \lambda \in T_{p_0}^* \mathbb{R}^3 : \mathcal{H}(p_0, \lambda) < 0, \left\langle \lambda, \frac{\partial}{\partial x} \right\rangle < 0 \right\}.$$

In this section we prove the following

**Proposition 3.1.**  $\exp_{p_0}(E_{p_0}) = I^+(p_0, \mathbb{R}^3)$ .

Take a point  $p_0 = (x_0, y_0, z_0)$ , and let  $\Phi$  be the mapping as in (2.7).

**Lemma 3.2.** *If  $\gamma$  is a t.f.d. Hamiltonian geodesic then so is  $\Phi \circ \gamma$ .*

*Proof:* Let  $\gamma : [a, b] \rightarrow \mathbb{R}^3$  be a t.f.d. Hamiltonian geodesic. Extend it at both ends to a t.f.d. Hamiltonian geodesics  $\gamma$ , now defined on  $[a - \varepsilon, b + \varepsilon]$ ,  $\varepsilon > 0$ . Suppose that  $[a, b] \subset \bigcup_{i=1}^m \Delta_i$ , where  $\Delta_1, \dots, \Delta_m$  are intervals such that  $\Delta_i \cap \Delta_{i+1} \neq \emptyset$ , and  $\gamma_i = \gamma|_{\Delta_i}$  is a unique  $U_i$ -maximizer for a suitable open  $U_i$ ,  $i = 1, \dots, m - 1$  ([3], prop. 4.1). Now,  $\Phi \circ \gamma_i = \Phi \circ \gamma|_{\Delta_i}$  is a unique  $\Phi(U_i)$ -maximizer for each  $i$ . Using Proposition 5.1 below,  $\Phi \circ \gamma_i$  is a Hamiltonian geodesic, i.e. it possesses a Hamiltonian lift to the cotangent bundle. Uniqueness of Hamiltonian lifts for Hamiltonian geodesics which are regular curves insures existence of the Hamiltonian lift of  $\Phi \circ \gamma$ .  $\square$

Since it is clear that  $\exp_{p_0}(E_{p_0}) = \Phi^{-1}(\exp_0(E_0))$ , it suffices to prove  $\exp_0(E_0) = I^+(0)$ . To this end let us recall (see [4]) equations of t.f.d. Hamiltonian geodesics (parametrized by arc length) starting from the origin:

$$\begin{cases} x(t, \varphi, r_0) = \frac{1}{r_0} \sinh \varphi - \frac{1}{r_0} \sinh(\varphi - r_0 t) \\ y(t, \varphi, r_0) = \frac{1}{r_0} \cosh \varphi - \frac{1}{r_0} \cosh(\varphi - r_0 t) \\ z(t, \varphi, r_0) = -\frac{1}{2r_0^2}(r_0 t - \sinh(r_0 t)) \end{cases} \quad (3.1)$$

for  $r_0 \neq 0$ , and

$$\begin{cases} x(t, \varphi, 0) = t \cosh \varphi \\ y(t, \varphi, 0) = t \sinh \varphi \\ z(t, \varphi, 0) = 0 \end{cases} \quad (3.2)$$

for  $r_0 = 0$ .

Fix a point  $p = (x_0, y_0, z_0) \in I^+(0)$ , i.e.  $|z_0| < \frac{1}{4}(x_0^2 - y_0^2)$ ,  $|y_0| < x_0$ ,  $x_0 > 0$ , and assume  $z_0 \neq 0$ . In order to prove Proposition 3.1 we must show that  $p$  can be reached from 0 by one of the geodesics (3.1). We see that projections on the  $(x, y)$ -plane of these geodesics are suitable branches of hyperbolas

$$\left(y - \frac{\cosh \varphi}{r_0}\right)^2 - \left(x - \frac{\sinh \varphi}{r_0}\right)^2 = \frac{1}{r_0^2}, \quad (3.3)$$

and (3.3) describes the projection of a geodesic joining 0 to  $p$ , if

$$r_0 = 2 \frac{x_0 \sinh \varphi - y_0 \cosh \varphi}{x_0^2 - y_0^2}. \quad (3.4)$$

At first consider the case  $z_0 > 0$ . If a curve in (3.1) is supposed to connect 0 to  $p$  we must have  $r_0 > 0$ , and its projection on the  $(x, y)$ -plane can be described as the graph of the function

$$y(x; \varphi) = \frac{\cosh \varphi}{r_0} - \sqrt{\left(x - \frac{\sinh \varphi}{r_0}\right)^2 + \frac{1}{r_0^2}} \quad (3.5)$$

with  $r_0$  as in (3.4) and  $\varphi_0 < \varphi < \infty$ , where  $\varphi_0 = \frac{1}{2} \ln \frac{x_0 + y_0}{x_0 - y_0}$ . Now we lift (3.5) to a geodesic (3.1) using the formula  $2dz = ydx - xdy$ . This geodesic will be defined by equations  $y = y(x; \varphi)$ ,  $z = z(x; \varphi)$  (it is indeed a Hamiltonian geodesic because for any given  $p = (x_0, y_0, z_0)$  each curve in the  $(x, y)$ -plane passing through  $(x_0, y_0)$  has a unique lift to a horizontal curve in  $\mathbb{R}^3$  passing

through  $p$ ), where

$$z(x; \varphi) = \frac{\cosh \varphi}{2r_0} x - \frac{1}{2r_0^2} \ln \frac{x - \frac{\sinh \varphi}{r_0} + \sqrt{x - \frac{\sinh \varphi}{r_0} + \frac{1}{r_0^2}}}{\frac{1}{r_0} (\cosh \varphi - \sinh \varphi)} + \\ + \frac{\sinh \varphi}{2r_0} \sqrt{\left(x - \frac{\sinh \varphi}{r_0}\right)^2 + \frac{1}{r_0^2}} - \frac{\sinh \varphi \cosh \varphi}{2r_0^2}.$$

Finally, we obtain

$$\lim_{\varphi \rightarrow \varphi_0^+} z(x_0; \varphi) = 0, \quad \lim_{\varphi \rightarrow \infty} z(x_0; \varphi) = \frac{1}{4}(x_0^2 - y_0^2),$$

so there is a  $\varphi$  for which  $z(x_0; \varphi) = z_0$  and the corresponding geodesic (3.1) reaches  $p$ .

The case  $z_0 < 0$  is treated in the similar manner. If a curve in (3.1) is supposed to connect 0 to  $p$ , then we must have  $r_0 < 0$ , and its projection on the  $(x, y)$ -plane can be described as the graph of the function

$$y(x; \varphi) = \frac{\cosh \varphi}{r_0} + \sqrt{\left(x - \frac{\sinh \varphi}{r_0}\right)^2 + \frac{1}{r_0^2}};$$

again  $r_0$  is as in (3.4),  $-\infty < \varphi < \varphi_0$ ,  $\varphi_0$  as above. We lift it to a geodesic

$$z(x; \varphi) = \frac{\cosh \varphi}{2r_0} x - \frac{1}{2r_0^2} \ln \frac{x - \frac{\sinh \varphi}{r_0} - \sqrt{x - \frac{\sinh \varphi}{r_0} + \frac{1}{r_0^2}}}{\frac{1}{r_0} (\cosh \varphi - \sinh \varphi)} + \\ - \frac{\sinh \varphi}{2r_0} \sqrt{\left(x - \frac{\sinh \varphi}{r_0}\right)^2 + \frac{1}{r_0^2}} - \frac{\sinh \varphi \cosh \varphi}{2r_0^2},$$

and finish the proof of this case by checking

$$\lim_{\varphi \rightarrow -\infty} z(x_0; \varphi) = -\frac{1}{4}(x_0^2 - y_0^2), \quad \lim_{\varphi \rightarrow \varphi_0^-} z(x_0; \varphi) = 0.$$

The remaining case  $z_0 = 0$  is trivial using (3.2).

## 4 The Distance Function

### 4.1 Dilations $\delta_\mu$ and their properties

Let  $\delta_\mu : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $\delta_\mu(x, y, z) = (\mu x, \mu y, \mu^2 z)$ ,  $\mu > 0$ . It is a simple matter to verify that the two fields  $X, Y$  in (2.1) defining the Heisenberg sub-Lorentzian



structure satisfy

$$(\delta_\mu)_*X = \mu X, \quad (\delta_\mu)_*Y = \mu Y. \quad (4.1)$$

Thanks to (4.1) the  $\delta_\mu$ 's play a key role in the sequel (see [2] for the sub-Riemannian case).

We start with the observation that each  $\delta_\mu$  preserves causal character of curves. Let  $\gamma : [a, b] \longrightarrow \mathbb{R}^3$  be, for instance, a t.f.d. curve. Then

$$\dot{\gamma}(t) = u_0(t)X(\gamma(t)) + u_1(t)Y(\gamma(t)), \quad -u_0(t)^2 + u_1(t)^2 < 0, \quad u_0(t) > 0 \quad \text{a.e.}$$

Now (4.1) gives

$$(\delta_\mu \circ \gamma)(t) = \mu u_0(t)X((\delta_\mu \circ \gamma)(t)) + \mu u_1(t)Y((\delta_\mu \circ \gamma)(t)),$$

so  $\delta_\mu \circ \gamma : [a, b] \longrightarrow \mathbb{R}^3$  is t.f.d. It follows that for each  $\mu > 0$

$$\delta_\mu(I^+(0)) \subset I^+(0), \quad \delta_\mu(J^+(0)) \subset J^+(0).$$

Next we see that for each nspc.f.d. curve  $\gamma$  and any  $\mu > 0$ ,  $L(\delta_\mu \circ \gamma) = \mu L(\gamma)$ . Applying this to the distance function we have

**Proposition 4.1.** *For each  $\mu > 0$  sufficiently small*

$$f[U](\delta_\mu p) = \mu f[U](p).$$

*Proof:* By the definition of  $f[U]$ , for each positive integer  $\nu$  there is a nspc.f.d. curve  $\gamma_\nu$  joining 0 to  $p$  in  $U$ , such that

$$f[U](p) - \frac{1}{\nu} \leq L(\gamma_\nu). \quad (4.2)$$

At the same time for each  $\nu$  the curve  $t \longrightarrow \delta_\mu(\gamma_\nu(t))$  joins 0 to  $\delta_\mu p$ , so

$$\mu L(\gamma_\nu) = L(\delta_\mu \circ \gamma_\nu) \leq f[U](\delta_\mu p). \quad (4.3)$$

(4.2) and (4.3) together with letting  $\nu \longrightarrow \infty$  give

$$\mu f[U](p) \leq f[U](\delta_\mu p).$$

Using the same argument we get

$$\mu^{-1} f[U](\delta_\mu p) \leq f[U](\delta_{\mu^{-1}} \delta_\mu p) = f[U](p).$$

□

Recall that by  $f$  we mean  $f[\mathbb{R}^3]$ .

**Corollary 4.2.**  *$f$  is finite. It follows that for each  $\mu > 0$  and every  $p \in I^+(0)$ ,  $f(\delta_\mu p) = \mu f(p)$ , and consequently  $f$  is upper semi-continuous.*

*Proof:* Suppose that  $f(p) = \infty$  for a  $p \in I^+(0)$ . Thus, for every positive integer  $\nu$ , there exists a nspc.f.d. curve  $\gamma_\nu$  joining 0 to  $p$  and such that  $L(\gamma_\nu) > \nu$ . Take a  $\mu > 0$  with  $\delta_\mu p \in U$ , where  $U$  is a normal neighbourhood of the origin. Now  $\delta_\mu \circ \gamma_\nu$  is a nspc.f.d. curve in  $U$  that joins 0 to  $\delta_\mu p$ , and  $L(\delta_\mu \circ \gamma_\nu) > \mu\nu$  for every  $\nu$ . This is, however, a contradiction with finiteness of  $f[U]$ .

Next, using the same argument as in the proof of Proposition 4.1 we show that  $f(\delta_\mu p) = \mu f(p)$ ,  $p \in I^+(0)$ .

To prove the last part take a  $p \in I^+(0)$  and such a  $\mu > 0$  that  $\delta_\mu p \in U$ ,  $U$  being a normal neighbourhood of zero. Now it suffices to observe (cf. Corollary 2.3) that  $f = \mu^{-1}f[U] \circ \delta_\mu$  in a neighbourhood of  $p$ .  $\square$

## 4.2 Estimate for the Distance Function

Let  $U$  be a normal neighbourhood of the origin or  $U = \mathbb{R}^3$ . All we can say about  $f[U]$  is that it is upper semi-continuous, it is differentiable on a neighbourhood of the set  $I^+(0, U) \cap \{z = 0\}$  (cf. [4], [5]), and it is continuous at all points of the set  $\partial I^+(0, U) \setminus \partial U$  (see Corollary 2.4). The main reason for our little knowledge about  $f[U]$  is that we do not know if all non-null  $U$ -maximizers are t.f.d. Hamiltonian geodesics (if we knew this the situation would get much simpler). In this section we prove one more thing yet about  $f[U]$  –a certain estimate from below.

Let

$$g(x, y, z) = \sqrt{\eta_4(x, y, z)} = \sqrt{x^2 - y^2 - 4|z|}$$

be defined on  $J^+(0)$ . Evidently,  $g(\delta_\mu p) = \mu g(p)$ ; moreover  $f$  and  $g$  coincide on  $J^+(0) \cap \{z = 0\}$ .

Take an  $\varepsilon > 0$  and define  $M_\varepsilon = \{g = \varepsilon\}$ . Now, consider all t.f.d. Hamiltonian geodesics with  $r_0 \neq 0$ , joining the origin to  $M_\varepsilon$ , i.e. the ones which are solutions of (3.1) and additionally satisfy

$$x^2 - y^2 - 4|z| = \varepsilon^2. \quad (4.4)$$

Suppose at first that  $r_0 > 0$ . The condition (4.4) means

$$\sinh^2\left(\frac{r_0 t}{2}\right) + \frac{r_0 t}{2} - \sinh\left(\frac{r_0 t}{2}\right) \cosh\left(\frac{r_0 t}{2}\right) = \frac{1}{4}r_0^2 \varepsilon^2. \quad (4.5)$$

Setting  $\alpha = r_0 t$  and

$$F(\alpha) = 2 \left( \sinh^2 \frac{\alpha}{2} + \frac{\alpha}{2} - \sinh \frac{\alpha}{2} \cosh \frac{\alpha}{2} \right) = \alpha - 1 + e^{-\alpha},$$

(3.1) can be rewritten as

$$F(\alpha) = \frac{1}{2} r_0^2 \varepsilon^2, \quad (4.6)$$

$\alpha > 0$ . Clearly,  $F$  increases; moreover  $F(\alpha) = \frac{1}{2} \alpha^2 + o(\alpha^2)$ , so  $F$  admits the inverse  $G$  of the form  $G(\beta) = \sqrt{2} \beta^{1/2} + o(\beta^{1/2})$ ,  $\beta > 0$ . Applying  $G$  to both sides of (4.6) we get

$$t(r_0; \varepsilon) = \frac{1}{r_0} G\left(\frac{1}{2} r_0^2 \varepsilon^2\right),$$

where  $t(r_0; \varepsilon)$  is the length of a segment (between 0 and  $M_\varepsilon$ ) of a geodesic (3.1) with the initial covector  $(-\cosh \varphi, \sinh \varphi, r_0) \in T_0^* \mathbb{R}^3$ . Since the function  $\beta \rightarrow \frac{G(\beta)}{\sqrt{\beta}}$ ,  $\beta > 0$ , increases, one easily verifies the following equalities:

$$\inf_{\beta > 0} \frac{G(\beta)}{\sqrt{\beta}} = \lim_{\beta \rightarrow 0^+} \frac{G(\beta)}{\sqrt{\beta}} = \sqrt{2}, \quad \sup_{\beta > 0} \frac{G(\beta)}{\sqrt{\beta}} = \lim_{\beta \rightarrow \infty} \frac{G(\beta)}{\sqrt{\beta}} = \infty.$$

Recall that by Proposition 3.1 each point of  $M_\varepsilon$  can be reached by a Hamiltonian geodesic starting from zero. Now, for any  $p = (x, y, z) \in M_\varepsilon$  with  $z > 0$  we have

$$\frac{f(p)}{g(p)} = \frac{f(p)}{\varepsilon} \geq \inf_{r_0 > 0} \frac{1}{r_0 \varepsilon} G\left(\frac{1}{2} r_0^2 \varepsilon^2\right) = 1 \quad (4.7)$$

and

$$\sup_{p \in M_\varepsilon} \frac{f(p)}{g(p)} = \sup_{p \in M_\varepsilon} \frac{f(p)}{\varepsilon} \geq \sup_{r_0 > 0} \frac{1}{r_0 \varepsilon} G\left(\frac{1}{2} r_0^2 \varepsilon^2\right) = \infty. \quad (4.8)$$

Fix a normal neighbourhood  $U$  of the origin. Take an arbitrary  $q \in I^+(0, U)$ ; then  $q = \delta_\mu p$ , where  $p = \delta_{\varepsilon/g(q)} q \in M_\varepsilon$ ,  $\mu = g(q)/\varepsilon$ . Now it is clear from (4.7) that

$$\frac{f[U](q)}{g(q)} = \frac{f(\delta_\mu p)}{g(\delta_\mu p)} = \frac{f(p)}{g(p)} \geq 1$$

from which  $f[U] \geq g$  on  $J^+(0, U)$ .

At the same time take an  $N > 0$ ; by (4.8) there exists a  $p \in M_\varepsilon$  such that  $f(p)/g(p) > N$ . Now, for a suitable  $\mu > 0$ ,  $q = \delta_\mu p \in I^+(0, U)$  and

$$\frac{f[U](q)}{g(q)} = \frac{f(\delta_\mu p)}{g(\delta_\mu p)} = \frac{f(p)}{g(p)} > N.$$

Similar arguments work for  $r_0 < 0$ , and therefore  $z < 0$ , so in this way we have completed the proof of the following

**Theorem 4.3.** *For every  $(x, y, z) \in J^+(0, U)$ , where  $U$  is a normal neighbourhood of the origin or  $U = \mathbb{R}^3$ , the following estimate is true:*

$$f[U](x, y, z) \geq \sqrt{x^2 - y^2 - 4|z|}.$$

*On the other hand, the reverse inequality, i.e.  $f[U] \leq Cg$ , does not hold for any constant  $C$ .*

## 5 Non-Hamiltonian geodesics which are regular curves

Fix a time-oriented sub-Lorentzian manifold  $(M, H, h)$ ,  $\dim M = n + 1$ ,  $\text{rank } H = k + 1$ . Let an open set  $U \subset M$  be a domain of local coordinates, and suppose that  $p, q \in U$  are such that  $q \in I^+(p, U)$ . We want to apply the Maximum Principle of Pontryagin (PMP) to the following problem: (TFD) *among all t.f.d. curves joining  $p$  to  $q$  and contained in  $U$  find the longest one.* To this end let  $X_0, \dots, X_k$  be an orthonormal frame for  $H$  on  $U$  with a time orientation  $X_0$ . It is natural to take

$$K_1 = \left\{ (u_0, \dots, u_k) \in \mathbb{R}^{k+1} : -u_0^2 + \sum_{j=0}^k u_j^2 < 0, u_0 > 0 \right\}$$

as the set of control parameters. Consider the system of equations

$$\dot{x}(t) = \sum_{\alpha=0}^k u_\alpha(t) X_\alpha(x(t)),$$

where  $u = (u_0, \dots, u_k) \in L^2([0, T], K_1)$ . Further, for any  $u = (u_0, \dots, u_k) \in K_1$  let  $\|u\| = |-u_0^2 + u_1^2 + \dots + u_k^2|^{1/2}$  and define the cost function to be

$$J(x, u) = - \int_0^T \|u(t)\| dt.$$

Finally, introduce the usual Hamiltonian for PMP:  $\mathcal{H}_{\lambda_0} : T^*U \times K_1 \longrightarrow \mathbb{R}$ ,

$$\mathcal{H}_{\lambda_0}(x, \lambda, u) = \sum_{\alpha=0}^k u_\alpha \langle \lambda, X_\alpha \rangle + \lambda_0 \|u\|,$$

where  $\lambda_0 \in \mathbb{R}$ ,  $u \in K_1$ ,  $\lambda \in T_x^*U$ . By  $\vec{\mathcal{H}}_{\lambda_0}$  denote the Hamiltonian vector field on  $T^*U$  corresponding to the function  $\mathcal{H}_{\lambda_0}$  of variables  $x$  and  $\lambda$ , where  $u$  is a parameter. Now the PMP applied to our problem says: if  $(x(t), u(t))$ ,  $0 \leq t \leq T$ , is an optimal trajectory for (TFD), then there exists a  $\lambda_0 \leq 0$  and an absolutely continuous lift  $(x, \lambda, u) : [0, T] \rightarrow T^*U \times K_1$  of  $x$  such that  $\lambda(t) \neq 0$ ,  $(x, \dot{\lambda}) = \vec{\mathcal{H}}_{\lambda_0}$ , and

$$\mathcal{H}_{\lambda_0}(x(t), \lambda(t), u(t)) = \sup \{ \mathcal{H}_{\lambda_0}(x(t), \lambda(t), v) : v \in K_1 \} = 0. \quad (5.1)$$

It turns out that, similarly to in the sub-Riemannian case, there are only two types of solutions to (TFD), as is described in the proposition below.

**Proposition 5.1.** *Each extremal of the PMP with the set  $K_1$  of control parameters is either a Hamiltonian geodesic (then it can be a regular or a singular curve) or a strictly abnormal extremal (which is a singular curve). Moreover, each abnormal extremal is a singular curve.*

*Proof:* Since  $K_1$  is open, we can replace the condition (5.1) with

$$\frac{\partial \mathcal{H}_{\lambda_0}}{\partial u_a}(x(t), \lambda(t), u(t)) = 0,$$

$\alpha = 0, 1, \dots, k$  and proceed analogously as in the sub-Riemannian case.  $\square$

Unfortunately, the set  $K_1$  of control parameters is not appropriate for studying  $U$ -maximizers. Instead we must consider the problem: (NSPCFD) *among all nspc.f.d. curves joining  $p$  to  $q$  and contained in  $U$  find the longest one.* This time the suitable set of control parameters is

$$K_2 = \left\{ (u_0, \dots, u_k) \in \mathbb{R}^{k+1} : -u_0^2 + \sum_{j=0}^k u_j^2 \leq 0, u_0 > 0 \right\}.$$

Now, if we apply PMP to the problem (NSPCFD), Proposition 5.1 is no longer true – all we can say is that a t.f.d.  $U$ -maximizer being a smooth curve is a Hamiltonian geodesic. In particular, one can find non-Hamiltonian  $U$ -maximizers which are regular curves (a phenomenon which does not occur in the sub-Riemannian situation); every such maximizer is either null, non-smooth or it changes its causal character (i.e. it has null and timelike pieces).

In our case of the Heisenberg sub-Lorentzian metric, it is very easy to construct an example of a null non-Hamiltonian and non-smooth maximizer.

First of all observe that the results of Section 2 imply that each nspc.f.d. curve starting from 0 and contained in  $\partial J^+(0)$  is a null maximizer, and these are the only null maximizers. Moreover, since there are only two null f.d. Hamiltonian geodesics (namely half-lines  $\{y = \pm x, x \geq 0, z = 0\}$  – see [5]), any maximizer that connects 0 to a  $p = (x_0, y_0, z_0) \in \partial J^+(0)$ ,  $z_0 \neq 0$ , cannot be Hamiltonian. Suppose that  $\gamma$  is such a non-Hamiltonian and null maximizer. Assume  $\gamma$  smooth. Every smooth and null curve satisfies  $\dot{y} = \dot{x}$  (resp.  $\dot{y} = -\dot{x}$ ), i.e.  $y = \text{const} + x$  (resp.  $y = \text{const} - x$ ). In our case  $\gamma(0) = 0$ , so  $y = x$  (resp.  $y = -x$ ) along  $\gamma$ ; in other words  $\gamma$  coincides with one of the two null f.d. Hamiltonian geodesics, which is not true. Thus null non-Hamiltonian maximizers are not smooth.

Next, for a fixed  $p = (x_0, y_0, z_0) \in \partial J^+(0) = \{\eta_4 = 0\}$  with  $z_0 \neq 0$  we will construct the unique null maximizer joining 0 to  $p$  (see [5] for the proof of existence of such curves). At first let us make two observations. Lift a curve  $\tilde{\xi}(t) = (a + t, b - t, 0)$ ,  $a > 0$ ,  $|b| < a$ , to a (horizontal) curve lying on the surface  $\{\eta_4 = 0, x \geq 0\}$ . It can be done in a unique way:

$$\xi(t) = \begin{cases} x(t) = a + t \\ y(t) = b - t \\ z(t) = \frac{1}{4}(a^2 - b^2) + \frac{1}{2}(a + b)t; \end{cases}$$

the resulting curve is contained in  $\{\eta_4 = 0, x \geq 0, z \geq 0\}$  and is always null. Analogously, a curve  $\tilde{\zeta}(t) = (a + t, b + t, 0)$ ,  $a > 0$ ,  $|b| < a$ , has a unique (horizontal) lift to the surface  $\{\eta_4 = 0, x \geq 0\}$ , and this lift happens to be a null curve

$$\zeta(t) = \begin{cases} x(t) = a + t \\ y(t) = b + t \\ z(t) = \frac{1}{4}(b^2 - a^2) + \frac{1}{2}(b - a)t \end{cases}$$

contained in  $\{\eta_4 = 0, x \geq 0, z \leq 0\}$ . In particular, every null maximizer that leaves the set  $\{y = x, x > 0, z = 0\}$  (resp.  $\{y = -x, x > 0, z = 0\}$ ) stays in  $\{\eta_4 = 0, x > 0, z > 0\}$  (resp.  $\{\eta_4 = 0, x > 0, z < 0\}$ ).

Now, take a  $p = (x_0, y_0, z_0) \in \partial J^+(0)$  with, say,  $z_0 > 0$ . Using the above procedure we lift the curve

$$\tilde{\gamma}(t) = \begin{cases} (t, t, 0) : 0 \leq t \leq \frac{1}{2}(x_0 + y_0) \\ (t, x_0 + y_0 - t, 0) : \frac{1}{2}(x_0 + y_0) < t \leq x_0 \end{cases}$$

to the surface  $\{\eta_4 = 0\}$  and obtain a null maximizer

$$\gamma(t) = \begin{cases} (t, t, 0) : 0 \leq t \leq \frac{1}{2}(x_0 + y_0) \\ (t, x_0 + y_0 - t, \frac{1}{2}(x_0 + y_0)t - \frac{1}{4}(x_0 + y_0)^2) : \frac{1}{2}(x_0 + y_0) < t \leq x_0 \end{cases}$$

joining 0 to  $p$ . Remarks above show that any such maximizer, after leaving the set  $\{y = x, x \geq 0, z = 0\}$ , must satisfy  $\dot{y} = \dot{x}$ ; this proves uniqueness of  $\gamma$ .

Similar construction yields a null and unique maximizer that joins 0 to a  $p = (x_0, y_0, z_0) \in \partial J^+(0)$  with  $z_0 < 0$ .

**Proposition 5.2.** *For each  $p = (x_0, y_0, z_0) \in \partial J^+(0)$  there exists the unique null maximizer  $\gamma_p$  joining the origin to  $p$ . Every such  $\gamma_p$  is contained in  $\partial J^+(0)$ . In case  $z_0 \neq 0$ ,  $\gamma_p$  is not smooth and is not a Hamiltonian geodesic.*

At the end let us remark that away from the set  $\{z = 0\}$  all null maximizers are trajectories of the horizontal gradient  $\nabla_H \eta_4$ .

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## References

- [1] J.K. Beem, P.E. Ehrlich, K.L. Easley, *Global Lorentzian Geometry*, Marcel Dekker, 1996.
- [2] A. Bellaïche, *The tangent space in sub-Riemannian geometry*, in *Sub-Riemannian Geometry*, Birkhauser, 1996.
- [3] M. Grochowski, *Geodesics in the sub-Lorentzian Geometry*, Bulletin of the Polish Academy of Sciences, Vol 50, No. 2, 2002.
- [4] M. Grochowski, *Normal Forms of Germs of Contact sub-Lorentzian Structures on  $\mathbb{R}^3$ . Differentiability of the sub-Lorentzian Distance*, Journal of Dynamical and Control Systems, Vol. 9, No. 4, 2003.
- [5] M. Grochowski, *On the Heisenberg sub-Lorentian Metric on  $\mathbb{R}^3$* , in *Geometric Singularity Theory*, Banach Center Publication, Vol. 6, Warszawa 2004.
- [6] M. Gromov, *Carnot Carathéodory Metrics Seen From Within*, in *Sub-Riemannian Geometry*, Birkhauser, 1996.
- [7] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

- [8] W. Liu, H. Sussman, *Shortest Paths for Sub-Riemannian Metrics on Rank-Two Distributions*, Memoires of the American Mathematical Society, Vol. 118, No. 564, 1995.
- [9] R. Montgomery, *A Survey of Singular Curves in the Sub-Riemannian Geometry*, Journal of Dynamical and Control Systems, Vol. 1, No. 1, 1995.
- [10] R. Penrose, *Techniques of Differential Topology in Relativity*, Regional Conference Series in App. Math., vol. 7, SIAM, Philadelphia, 1972.



# The complex symplectic moduli spaces of uni-modal parametric plane curve singularities

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## Abstract

Classification of zero-modal singularities of parametric plane curves under diffeomorphism equivalence is extended to uni-modal singularities. Both the simple and uni-modal singularities of parametric plane curves are classified further under symplectomorphic equivalence. In particular the corresponding cyclic symplectic moduli spaces are reconstructed as canonical ambient spaces for the diffeomorphism moduli spaces which are no longer Hausdorff spaces.

## 1 Introduction

In [3], Bruce and Gaffney classified the simple (0-modal) singularities of parametric plane curves  $f : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  under diffeomorphism equivalence (right-left equivalence) in the complex analytic category into the classes  $A_{2\ell}$ ,  $E_{6\ell}$ ,  $E_{6\ell+2}$ ,  $W_{12}$ ,  $W_{18}$  and  $W_{1,2\ell-1}^\#$  ( $\ell = 1, 2, 3, \dots$ ); see also [2] and Table 1. In the present paper the classification is extended to the uni-modal singularities as follows:

**Theorem 1.1.** *Under diffeomorphism equivalence the uni-modal singularities of parametric plane curves  $f : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  are classified into the*

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following list:

$$\begin{aligned}
N_{20} : & \quad (t^5, t^6 + t^8 + \lambda t^9) \ (-\lambda \sim \lambda), \ (t^5, t^6 + t^9), \ (t^5, t^6 + t^{14}), \ (t^5, t^6), \\
N_{24} : & \quad (t^5, t^7 + t^8 + \lambda t^{11}), \ (t^5, t^7 + t^{11} + \lambda t^{13}) \ (-\lambda \sim \lambda), \\
& \quad (t^5, t^7 + t^{13}), \ (t^5, t^7 + t^{18}), \ (t^5, t^7), \\
N_{28} : & \quad (t^5, t^8 + t^9 + \lambda t^{12}), \ (t^5, t^8 + t^{12} + \lambda t^{14}) \ (-\lambda \sim \lambda), \\
& \quad (t^5, t^8 + t^{14} + \lambda t^{17}) \ (-\lambda \sim \lambda), \ (t^5, t^8 + t^{17}), \ (t^5, t^8 + t^{22}), \ (t^5, t^8), \\
W_{24} : & \quad (t^4, t^9 + t^{10} + \lambda t^{11}) \ (\lambda \neq \frac{19}{18}), \ (t^4, t^9 + t^{10} + \frac{19}{18}t^{11} + \lambda t^{15}), \\
& \quad (t^4, t^9 + t^{11}), \ (t^4, t^9 + t^{15}), \ (t^4, t^9 + t^{19}), \ (t^4, t^9), \\
W_{30} : & \quad (t^4, t^{11} + t^{13} + \lambda t^{14}) \ (-\lambda \sim \lambda), \ (t^4, t^{11} + t^{14} + \lambda t^{17}) \ (\lambda \neq \frac{25}{22}), \\
& \quad (t^4, t^{11} + t^{14} + \frac{25}{22}t^{17} + \lambda t^{21}) \ (\omega\lambda \sim \lambda, \omega^3 = 1), \\
& \quad (t^4, t^{11} + t^{17}), \ (t^4, t^{11} + t^{21}), \ (t^4, t^{11} + t^{25}), \ (t^4, t^{11}), \\
W_{2,2\ell-1}^\# : & \quad (t^4, t^{10} + t^{2\ell+9} + \lambda t^{2\ell+11}) \ (\omega\lambda \sim \lambda, \omega^{2\ell-1} = 1) \ (\ell = 1, 2, 3, \dots).
\end{aligned}$$

In the list, for instance  $-\lambda \sim \lambda$  means that  $(t^5, t^6 + t^8 + \lambda' t^9)$  is diffeomorphic to  $(t^5, t^6 + t^8 + \lambda t^9)$  if and only if  $\lambda' = \pm\lambda$ .

In [7], motivated by the symplectic bifurcation problem, we gave the symplectic classification of simple singularities of parametric plane curves in the real case. (For the higher dimensional case, see [8]). In this paper we classify symplectically both the simple and uni-modal singularities of parametric plane curves in the complex case.

We call holomorphic parametric curve-germs  $f, g : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  *diffeomorphic* (resp. *symplectomorphic*) if there exist a bi-holomorphic diffeomorphism  $\sigma$  of  $(\mathbf{C}, 0)$  and a bi-holomorphic diffeomorphism  $\tau$  (resp. a bi-holomorphic symplectomorphism  $\tau$ ) of  $(\mathbf{C}^2, 0)$  (for the holomorphic symplectic form  $dx \wedge dy$  on  $\mathbf{C}^2$ ) satisfying  $\tau(g(t)) = f(\sigma(t))$ .

Let  $r$  be a non-negative integer. A curve-germ  $f$  is called *r-modal* if a finite number of  $s$ -parameter families ( $0 \leq s \leq r$ ) of diffeomorphism classes form a neighborhood of  $f$  in the space of curve-germs. Then we have:

**Theorem 1.2.** *A simple or uni-modal singularity  $f : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  is symplectomorphic to a germ which belongs to one of the following families*

(called “symplectic normal forms”):

$$\begin{aligned}
A_{2\ell} &: (t^2, t^{2\ell+1}), \\
E_{6\ell} &: (t^3, t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}), \\
E_{6\ell+2} &: (t^3, t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}), \\
W_{12} &: (t^4, t^5 + \lambda t^7), \\
W_{18} &: (t^4, t^7 + \lambda t^9 + \mu t^{13}), \\
W_{1,2\ell-1}^\# &: (t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9}), \lambda \neq 0, (\ell = 1, 2, \dots) \\
\\
N_{20} &: (t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14}), \\
N_{24} &: (t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18}), \\
N_{28} &: (t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22}), \\
W_{24} &: (t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19}), \\
W_{30} &: (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25}), \\
W_{2,2\ell-1}^\# &: (t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \\
&\lambda_1 \neq 0, (\ell = 1, 2, \dots)
\end{aligned}$$

Moreover we determine their symplectic moduli spaces as listed in Tables 1 and 2:

**Theorem 1.3.** *Let  $f_\lambda(t) = (t^m, t^n + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \dots + \lambda_s t^{r_s})$  be one of the symplectic normal forms of simple or uni-modal singularities. Then two curve-germs  $f_\lambda$  and  $f_{\lambda'}$  belonging to the same family are symplectomorphic if and only if there exists an  $(m+n)$ -th root  $\zeta \in \mathbf{C}$  of unity satisfying*

$$\lambda'_1 = \zeta^{r_1-n} \lambda_1, \lambda'_2 = \zeta^{r_2-n} \lambda_2, \dots, \lambda'_s = \zeta^{r_s-n} \lambda_s.$$

*In particular each symplectic moduli space of a family is a Hausdorff space in the natural topology and it is extended to a cyclic quotient singularity.*

In his lecture notes [18], O. Zariski studied the moduli space of parametric plane curve-germs, under diffeomorphism equivalence, for a given topological type, or the equi-singularity class  $(m, \beta_1, \dots, \beta_g)$ . (See §2). In particular, Zariski determined the moduli spaces for the classes  $(2, 2\ell + 1)$ ,  $(3, 3\ell + 1)$ ,  $(4, 5)$ ,  $(4, 6, 2\ell + 5)$ ,  $(5, 6)$  and  $(6, 7)$ . He did not mention symplectomorphic equivalence at all, but surprisingly, he used, as pre-normal forms, several symplectic normal forms given in Theorem 1.2. For instance, in [18] page 68, he started with

$$x = t^5, y = t^6 + a_8 t^8 + a_9 t^9 + a_{14} t^{14}$$

in the concrete classification of the case (5, 6).

In this paper, clarifying the role of symplectomorphism equivalence, we proceed Zariski's classification via modality: by Bruce-Gaffney's classification and by Theorem 1.1, we determine the moduli spaces for the classes (4, 7), (5, 7), (5, 8), (4, 9), (4, 11) and (4, 10,  $2\ell + 9$ ) beyond Zariski's result, except for the class (6, 7) which is actually bi-modal. Moreover we can treat the case (6, 7) by the same method developed in this paper.

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The classification of plane curve singularities is closely related to the classification of Legendre curve singularities and the classification of Goursat distributions ([12],[19],[13]). Actually P. Mormul has predicted several forms in Theorem 1.1 from his classification results for uni-modal singularities of Goursat distributions (private communication to the first author). Note however, that these classification problems have different features, and therefore, to get the exact classification, we need a detailed analysis in each case.

In the next section we give an outline of the proofs of Theorem 1.2 and Theorem 1.3. In the last section we outline the proof of Theorem 1.1.

## 2 Symplectic normal forms

Let  $f : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  be a germ of parametric holomorphic plane curve. Then the following conditions on  $f$  are known to be equivalent ([16],[7]):

- (i)  $f$  has an injective representative.
- (ii)  $f$  is a normalization onto the image.
- (iii) The diffeomorphism class of  $f$  is determined by a finite jet of  $f$ .
- (iv) The symplectomorphism class of  $f$  is determined by a finite jet of  $f$ .
- (v) The quotient vector space  $\mathcal{O}_1/f^*\mathcal{O}_2$  is finite dimensional.

We assume that  $f$  satisfies one (and therefore all) of the above conditions. Here  $f^* : \mathcal{O}_2 = \mathbf{C}\{x, y\} \rightarrow \mathcal{O}_1 = \mathbf{C}\{t\}$  is defined by composition:  $f^*(h) = h \circ f$ . Recall that *the number of double points*  $\delta(f) = \dim_{\mathbf{C}} \mathcal{O}_1/f^*\mathcal{O}_2$  ([11],[17]) also has the meaning of *the symplectic codimension* of  $f$ , that is the number of parameters needed to produce its versal unfolding via symplectomorphism equivalence ([7]).

	DIFF. NORMAL FORM	SYM. NORMAL FORM	SYM. MODULI SPACE
$A_{2\ell}$	$(t^2, t^{2\ell+1})$	$(t^2, t^{2\ell+1})$	
$E_{6\ell}$ ( $\ell \geq 1$ )	$(t^3, t^{3\ell+1} + t^{3(\ell+p)+2})$ ( $0 \leq p \leq \ell - 2$ ) $(t^3, t^{3\ell+1})$	$(t^3, t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1})$	$\mathbf{C}^{\ell-1}/G$ , $G = \mathbf{Z}/(3\ell+4)\mathbf{Z}$ $(\lambda_1, \dots, \lambda_{\ell-1}) \mapsto$ $(\zeta \lambda_1, \dots, \zeta^{3j-2} \lambda_j, \dots, \zeta^{3\ell-5} \lambda_{\ell-1})$ $(\zeta^{3\ell+4} = 1, \text{primitive})$
$E_{6\ell+2}$ ( $\ell \geq 1$ )	$(t^3, t^{3\ell+2} + t^{3(\ell+p)+4})$ ( $0 \leq p \leq \ell - 2$ ) $(t^3, t^{3\ell+2})$	$(t^3, t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1})$	$\mathbf{C}^{\ell-1}/G$ , $G = \mathbf{Z}/(3\ell+5)\mathbf{Z}$ $(\lambda_1, \dots, \lambda_{\ell-1}) \mapsto$ $(\zeta^2 \lambda_1, \dots, \zeta^{3j-1} \lambda_j, \dots, \zeta^{3\ell-4} \lambda_{\ell-1})$ $(\zeta^{3\ell+5} = 1, \text{primitive})$
$W_{12}$	$(t^4, t^5 + t^7)$ $(t^4, t^5)$	$(t^4, t^5 + \lambda t^7)$	$\mathbf{C}/G$ , $G = \mathbf{Z}/9\mathbf{Z}$ $\lambda \mapsto \zeta \lambda$ , ( $\zeta^9 = 1$ )
$W_{18}$	$(t^4, t^7 + t^9)$ $(t^4, t^7 + t^{13})$ $(t^4, t^7)$	$(t^4, t^7 + \lambda t^9 + \mu t^{13})$	$\mathbf{C}^2/G$ , $G = \mathbf{Z}/11\mathbf{Z}$ $(\lambda, \mu) \mapsto (\zeta \lambda, \zeta^3 \mu)$ , ( $\zeta^{11} = 1$ )
$W_{1,2\ell-1}^\#$ ( $\ell \geq 1$ )	$(t^4, t^6 + t^{2\ell+5})$	$(t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9})$ ( $\lambda \neq 0$ )	$(\mathbf{C}^* \times \mathbf{C})/G$ , $G = \mathbf{Z}/10\mathbf{Z}$ , $(\lambda, \mu) \mapsto (\zeta^{2\ell-1} \lambda, \zeta^{2\ell+3} \mu)$ , $(\zeta^{10} = 1, \text{primitive})$

Table 1: The complex symplectic moduli spaces of simple parametric plane curve singularities.

	SYM. NORMAL FORM	SYM. MODULI SPACE
$N_{20}$	$(t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14})$	$\mathbf{C}^3/G$ , $G = \mathbf{Z}/11\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\zeta^2 \lambda_1, \zeta^3 \lambda_2, \zeta^8 \lambda_3)$ , $(\zeta^{11} = 1)$
$N_{24}$	$(t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18})$	$\mathbf{C}^4/G$ , $G = \mathbf{Z}/12\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto$ $(\zeta \lambda_1, \zeta^4 \lambda_2, \zeta^6 \lambda_3, \zeta^{11} \lambda_4)$ , $(\zeta^{12} = 1, \text{primitive})$
$N_{28}$	$(t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14}$ $+ \lambda_4 t^{17} + \lambda_5 t^{22})$	$\mathbf{C}^5/G$ , $G = \mathbf{Z}/13\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta \lambda_1, \zeta^4 \lambda_2, \zeta^6 \lambda_3, \zeta^9 \lambda_4, \zeta^{14} \lambda_5)$ , $(\zeta^{13} = 1, \text{primitive})$
$W_{24}$	$(t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19})$	$\mathbf{C}^4/G$ , $G = \mathbf{Z}/13\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto$ $(\zeta \lambda_1, \zeta^2 \lambda_2, \zeta^6 \lambda_3, \zeta^{10} \lambda_4)$ , $(\zeta^{13} = 1)$
$W_{30}$	$(t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17}$ $+ \lambda_4 t^{21} + \lambda_5 t^{25})$	$\mathbf{C}^5/G$ , $G = \mathbf{Z}/15\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta^2 \lambda_1, \zeta^5 \lambda_2, \zeta^6 \lambda_3, \zeta^{10} \lambda_4, \zeta^{14} \lambda_5)$ , $(\zeta^{15} = 1, \text{primitive})$
$W_{2,2\ell-1}^\#$	$(t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13}$ $+ \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21})$ , ( $\lambda_1 \neq 0$ ).	$(\mathbf{C}^* \times \mathbf{C}^4)/G$ , $G = \mathbf{Z}/14\mathbf{Z}$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \mapsto$ $(\zeta^{2\ell-1} \lambda_1, \zeta^{2\ell+1} \lambda_2, \zeta^{2\ell+3} \lambda_3,$ $\zeta^{2\ell+7} \lambda_4, \zeta^{2\ell+11} \lambda_5)$ , $(\zeta^{14} = 1, \text{primitive})$

Table 2: The complex symplectic moduli spaces of uni-modal parametric plane curve singularities.

We briefly recall the theory developed in [7] §7: The symplectic codimension of  $f$  is defined by

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} \frac{V_f}{tf(V_1) + wf(VH_2)}$$

as an infinitesimal symplectic invariant of Mather's type. Here  $V_f$  is the space of germs of holomorphic vector fields  $v : (\mathbf{C}, 0) \rightarrow T\mathbf{C}^2$  along  $f$ , which is the space of infinitesimal deformations of  $f$ ;  $V_1$  is the space of germs of holomorphic vector fields over  $(\mathbf{C}, 0)$  and  $VH_2$  is the space of germs of holomorphic Hamiltonian vector fields over  $(\mathbf{C}^2, 0)$ . The homomorphisms  $tf : V_1 \rightarrow V_f$  and  $wf : VH_2 \rightarrow V_f$  are defined by  $tf(\xi) := f_*(\xi)$ ,  $\xi \in V_1$  and  $wf(\eta) := \eta \circ f$  respectively. An unfolding  $F : (\mathbf{C} \times \mathbf{C}^r, (0, 0)) \rightarrow (\mathbf{C}^2 \times \mathbf{C}^r, (0, 0))$  of  $f$ ,  $F(t, u) = (f_u(t), u)$ , is *symplectically versal* if  $\frac{\partial f_u}{\partial u_1}(t, 0), \dots, \frac{\partial f_u}{\partial u_r}(t, 0)$  generate  $V_f$ , over  $\mathbb{R}$ , up to the space  $tf(V_1) + wf(VH_2)$  of deformations which are covered by symplectomorphisms ([7], Proposition 7.1).

Set  $f(t) = (x(t), y(t))$ . For an infinitesimal deformation  $v(t) = a(t)\frac{\partial}{\partial x} + b(t)\frac{\partial}{\partial y} \in V_f$ , we define a *generating function*  $e(t) \in \mathcal{O}_1$  of  $v$  by  $d(e(t)) = b(t)d(x(t)) - a(t)d(y(t))$ , or  $e'(t) = b(t)x'(t) - a(t)y'(t)$  up to the constant term. The generating function of  $tf(\xi) + wf(H_k)$  is equal to  $f^*k$ , where  $k$  is the Hamiltonian function of the Hamiltonian vector field  $H_k$ . Then there exists an exact sequence of vector spaces:

$$0 \rightarrow \frac{V'_f}{tf(V_1)} \rightarrow \frac{V_f}{tf(V_1) + wf(VH_2)} \rightarrow \frac{R_f}{f^*\mathcal{O}_2} \rightarrow 0,$$

where  $R_f = \{e(t) \in \mathcal{O}_1 \mid \text{ord}(e'(t)) \geq \text{ord}(f) - 1\}$  and

$$V'_f = \{v(t) = a(t)\partial/\partial x + b(t)\partial/\partial y \in V_f \mid b(t)x'(t) - a(t)y'(t) = 0\}.$$

Then we see that  $\dim_{\mathbf{C}} V'_f/tf(V_1) = \text{ord}(f) - 1 = \dim_{\mathbf{C}} \mathcal{O}_1/R_f$ . Therefore we have

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} V'_f/tf(V_1) + \dim_{\mathbf{C}} R_f/f^*\mathcal{O}_2 = \dim_{\mathbf{C}} \mathcal{O}_1/f^*\mathcal{O}_2 = \delta(f).$$

Some of parameters of the symplectically versal unfolding correspond to deformations into less singular germs, and the remaining parameters provide

the symplectic normal form within a given *equi-singular class* up to discrete symplectomorphism equivalence classes. We recall a basic fact from the text-book [17] in our context. Set  $m = \text{ord}(f)$ . Then  $f$  is symplectomorphic to a germ of the form  $(t^m, \sum_{k=m}^{\infty} a_k t^k)$ . Suppose  $m \geq 2$ , that is,  $f$  is not an immersion. Set  $\beta_1 = \min\{k \mid a_k \neq 0, m \nmid k\}$  and let  $e_1$  be the greatest common divisor of  $m$  and  $\beta_1$ , and inductively set  $\beta_q = \min\{k \mid a_k \neq 0, e_{q-1} \nmid k\}$ , and let  $e_q$  be the greatest common divisor of  $\beta_q$  and  $e_{q-1}$ ,  $q \geq 2$ . Then  $e_g = 1$  for sufficiently large  $g$ , and we call  $(m = \beta_0, \beta_1, \beta_2, \dots, \beta_g)$  the *Puiseux characteristic* of  $f$ , which is a basic diffeomorphism invariant. Setting  $e_0 = m$ , we have  $\delta(f) = \frac{1}{2} \sum_{q=1}^g (\beta_q - 1)(e_{q-1} - e_q)$  ([11],[17]). Moreover the Puiseux characteristic determines the homeomorphism equivalence class of  $f$  ([10],[18]). We call a deformation of plane curve singularities *equi-singular* if the Puiseux characteristic is preserved. Under an equi-singular deformation of  $f$ , we can take a common monomial basis of  $\mathcal{O}_1/f^*\mathcal{O}_2$ .

First we have:

**Lemma 2.1.**  *$f$  is symplectomorphic to a germ of the form  $(t^m, t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k)$ .*

*Proof:* Set  $\psi(x) = \sum_{k=m}^{\beta_1-1} a_k x^{k/m}$  and  $\tau_1(x, y) = (x, y - \psi(x))$ . Then  $\tau_1(f(t)) = (t^m, \sum_{k=\beta_1}^{\infty} a_k t^k)$  with  $a_{\beta_1} \neq 0$ . Define  $\alpha \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$  by  $\alpha^{m+\beta_1} a_{\beta_1} = 1$ , and set  $\sigma(t) = \alpha t$  and  $\tau_2(x, y) = (\alpha^{-m} x, \alpha^m y)$ . Then  $\tau_1, \tau_2$  are both symplectomorphisms and we see that  $\tau_2(\tau_1(f(\sigma(t))))$  has the required form.  $\square$

To get symplectic normal forms, we first remark the following:

**Lemma 2.2.** *Suppose  $\mathcal{O}_1/f^*\mathcal{O}_2$  has a monomial basis*

$$t, t^2, \dots, t^{m-1}, t^{m+1}, \dots, t^{r_1+m}, \dots, t^{r_s+m}$$

where  $r_1 + m, \dots, r_s + m$  are all exponents greater than  $\beta_1 + m$ . Then the family

$$f_c(t) = (t^m, t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k + \sum_{j=1}^s c_j t^{r_j}),$$

$c = (c_1, \dots, c_s) \in \mathbf{C}^s$ , gives a transversal to the symplectomorphism orbit.

*Proof:* Let  $v = \psi(t) \left( \frac{\partial}{\partial y} \circ f \right)$ ,  $\psi(t) = \sum_{k=\beta_1+1}^{\infty} u_k t^k$ , be an infinitesimal deformation of  $f$  among the forms given in Lemma 2.1. Take the generating

function  $e$  of  $v$  satisfying  $de(t) = \psi(t)d(t^m)$ ,  $e(0) = 0$ . Then there exist  $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{m-1}, \tilde{b}_{m+1}, \dots, \tilde{b}_{r_1+m}, \dots, \tilde{b}_{r_s+m} \in \mathbf{C}$  such that, setting

$$\varphi(t) = \tilde{b}_1 t + \tilde{b}_2 t^2 + \dots + \tilde{b}_{m-1} t^{m-1} + \tilde{b}_{m+1} t^{m+1} + \dots + \tilde{b}_{r_1+m} t^{r_1+m} + \dots + \tilde{b}_{r_s+m} t^{r_s+m},$$

we have  $e - \varphi \in f^* \mathcal{O}_2$ . Set  $e - \varphi = h \circ f$ . Since  $\text{ord}(e) \geq \beta_1 + m$ , we see  $\text{ord}(h) \geq 2$ . On the other hand,  $\varphi(t) = \tilde{b}_{r_1+m} t^{r_1+m} + \dots + \tilde{b}_{r_s+m} t^{r_s+m}$ . Then we have

$$\begin{aligned} d\varphi(t) &= \{(r_1 + m)\tilde{b}_{r_1+m} t^{r_1+m-1} + \dots + (r_s + m)\tilde{b}_{r_s+m} t^{r_s+m-1}\} dt \\ &= \left( \frac{r_1+m}{m} \tilde{b}_{r_1+m} t^{r_1} + \dots + \frac{r_s+m}{m} \tilde{b}_{r_s+m} t^{r_s} \right) d(t^m). \end{aligned}$$

Set  $w = \left( \frac{r_1+m}{m} \tilde{b}_{r_1+m} t^{r_1} + \dots + \frac{r_s+m}{m} \tilde{b}_{r_s+m} t^{r_s} \right) \left( \frac{\partial}{\partial y} \circ f \right)$ . Consider the Hamiltonian vector field  $X_h$ . Then, the field  $(v-w) - wf(X_h)$  has a zero as a generating function, that is,  $(v-w) - wf(X_h) \in V'_f$ . Then there exists  $\xi \in V_1$  with  $\xi(0) = 0$  satisfying  $tf(\xi) = (v-w) - wf(X_h)$ , that is  $v = w + tf(\xi) + wf(X_h)$  (cf. Lemma 8.2 and Theorem 8.7 of [7]). This means that the above family is transversal to the symplectomorphism orbit through  $f$ .  $\square$

A monomial basis of  $\mathcal{O}_1/f^* \mathcal{O}_2$  can be calculated by considering *the semigroup*  $S(f) = \{\text{ord}(h) \mid h \in f^* \mathcal{O}_2\} \subseteq \mathbf{N}$ . In fact  $\{t^r \mid r \in \mathbf{N} \setminus S(f), r > 0\}$  forms a monomial basis of  $\mathcal{O}_1/f^* \mathcal{O}_2$ . Note that a system of generators for the semigroup  $S(f)$  is calculated explicitly from the Puiseux characteristic. Moreover there exists a number  $N$  depending only on the Puiseux characteristic of  $f$  such that if  $\phi \in \mathcal{O}_1$  has order  $\geq N$ , then  $\phi \in f^* \mathcal{O}_2$  ([17]).

**Example 2.3.** (1) ( $W_{30}$ ) Let  $m = 4$ ,  $\beta_1 = 11$ . Then the semigroup  $S(f)$  is generated by 4 and 11. A monomial basis of  $\mathcal{O}_1/f^* \mathcal{O}_2$  is given by  $t, t^2, t^3, t^5, t^6, t^7, t^9, t^{10}, t^{13}, t^{14}, t^{17}, t^{18}, t^{21}, t^{25}, t^{29}$ .

(2) ( $W_{1,2\ell-1}^\#$ ) Let  $m = 4$ ,  $\beta_1 = 6$  and  $\beta_2 = 2\ell + 5$ . Then  $S(f)$  is generated by 4, 6 and  $2\ell + 11$ . The complement  $\mathbf{N} \setminus S(f)$  consists of  $1, 2, 3, 5, 7, 9, 11, \dots, 2\ell + 9, 2\ell + 13$ .

(3) ( $W_{2,2\ell-1}^\#$ ) Let  $m = 4$ ,  $\beta_1 = 10$  and  $\beta_2 = 2\ell + 9$ . Then  $S(f)$  is generated by 4, 10 and  $2\ell + 19$ . The complement  $\mathbf{N} \setminus S(f)$  consists of  $1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, \dots, 2\ell + 13, 2\ell + 15, 2\ell + 17, 2\ell + 21, 2\ell + 25$ .

*Proof of Theorem 1.2 :* Under the notations of Lemma 2.2, consider the infinitesimal deformation  $v = \kappa(t) \left( \frac{\partial}{\partial y} \circ f \right)$ ,  $\kappa(t) = -\sum b_k t^k$ , where the summation runs over  $k$  different from  $r_1, \dots, r_s$ . Then the Puiseux characteristics



are preserved under the deformation

$$f_u = (t^m, t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k - u\kappa(t))$$

( $u \in [0, 1]$ ) corresponding to  $v$ . This is clear when the greatest common divisor  $e_1$  of  $m$  and  $\beta_1$  is equal to 1. From Example 2.3 (2)(3), it also holds for  $W_{1,2\ell-1}^{\#}$  and  $W_{2,2\ell-1}^{\#}$ . Then there exist  $w_u = \sum_{j=1}^s c_{j,u} t^{r_j}$ ,  $c_{j,u} \in \mathbf{C}$ ,  $\xi_u \in V_1$ ,  $\xi_u(0) = 0$ , and  $\eta_u \in VH_2$ ,  $\eta_u(0) = 0$ , smoothly depending on  $u$  and satisfying  $v = w_u + tf_u(\xi_u) + wf_u(\eta_u)$ . By integrating from  $u = 0$  to  $u = 1$  we see that  $f$  is symplectomorphic to

$$f_{\lambda}(t) = (t^m, t^{\beta_1} + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \dots + \lambda_s t^{r_s}),$$

for some  $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbf{C}^s$ . In all cases except  $W_{1,2\ell-1}^{\#}$  and  $W_{2,2\ell-1}^{\#}$ , there are no restrictions on  $\lambda$  and we get the symplectic normal forms given in Theorem 1.2. In the case of  $W_{1,2\ell-1}^{\#}$ ,  $f$  is symplectomorphic to

$$f_c = (t^4, t^6 + c_1 t^7 + \dots + c_{\ell} t^{2\ell+5} + c_{\ell+1} t^{2\ell+7}).$$

Since the Puiseux characteristic of  $f$  is  $(4; 6, 2\ell + 5)$ , we necessarily have  $c_1 = 0, \dots, c_{\ell-1} = 0$  and  $c_{\ell} \neq 0$ . Setting  $\lambda = c_{\ell}$ ,  $\mu = c_{\ell+1}$ , we get the symplectic normal form. In the case of  $W_{2,2\ell-1}^{\#}$ ,  $f$  is symplectomorphic to

$$f_c = (t^4, t^{10} + c_1 t^{11} + c_2 t^{13} + c_3^{15} + \dots + c_{\ell} t^{2\ell+9} + c_{\ell+1} t^{2\ell+11} + c_{\ell+2} t^{2\ell+13} \\ + c_{\ell+3} t^{2\ell+17} + c_{\ell+4} t^{2\ell+21}).$$

Since the Puiseux characteristic of  $f$  is  $(4; 10, 2\ell + 9)$ , we have  $c_1 = 0, \dots, c_{\ell-1} = 0$  and  $c_{\ell} \neq 0$ , which gives the symplectic normal form.  $\square$

In the process of symplectic classification, we observe a kind of rigidity. Let  $f_{\lambda}$  and  $f_{\lambda'}$ , with  $\lambda \neq \lambda'$ , be germs belonging to one of the symplectic normal forms of simple or uni-modal parametric plane curve singularities. Then  $f_{\lambda}$  and  $f_{\lambda'}$  are not isotopic by symplectomorphisms. Moreover we have the following strong rigidity which implies Theorem 1.3 in each case:

**Proposition 2.4.** *Let  $f_{\lambda}$  and  $f_{\lambda'}$  be germs belonging to one of the symplectic normal forms of simple or uni-modal parametric plane curve singularities. If  $f_{\lambda}$  and  $f_{\lambda'}$  are symplectomorphic, then they are linearly symplectomorphic:*

If there exists a symplectomorphism equivalence  $(\sigma, \tau)$  satisfying  $\tau \circ f_{\lambda'} = f_{\lambda} \circ \sigma$ , then there exists a symplectomorphism equivalence  $(\Sigma, T)$  such that  $T \circ f_{\lambda'} = f_{\lambda} \circ \Sigma$ ,  $\Sigma : (\mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$  is a complex linear transformation, and  $T : (\mathbf{C}^2, 0) \rightarrow (\mathbf{C}^2, 0)$  is a complex linear symplectic transformation.

*Proof:* We give the calculation in the case  $W_{30}$ . Other cases can be treated similarly. Set  $f_{\lambda} = (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25})$ , and suppose  $f_{\lambda}$  and  $f_{\lambda'}$  are symplectomorphic for  $\lambda = (\lambda_1, \dots, \lambda_5)$  and  $\lambda' = (\lambda'_1, \dots, \lambda'_5)$ .

Set  $\sigma(t) = a_1 t + a_2 t^2 + \dots$  and, as components of  $\tau(x, y)$ ,

$$\begin{aligned} x \circ \tau(x, y) &= ax + by + h_1 x^2 + h_2 xy + h_3 y^2 + \ell_1 x^3 + \ell_2 x^2 y \\ &\quad + \ell_3 xy^2 + \ell_4 y^3 + \dots, \\ y \circ \tau(x, y) &= cx + ey + k_1 x^2 + k_2 xy + k_3 y^2 + m_1 x^3 + m_2 x^2 y \\ &\quad + m_3 xy^2 + m_4 y^3 + \dots \end{aligned}$$

Consider the equation  $f_{\lambda}(\sigma(t)) = \tau(f_{\lambda'}(t))$ :

$$\begin{aligned} \sigma(t)^m &= x \circ \tau(t^4, t^{11} + \lambda'_1 t^{13} + \lambda'_2 t^{14} + \lambda'_3 t^{17} + \lambda'_4 t^{21} + \lambda'_5 t^{25}) \quad \dots\dots\dots (*), \\ \sigma(t)^{11} + \lambda_1 \sigma(t)^{13} + \lambda_2 \sigma(t)^{14} + \lambda_3 \sigma(t)^{17} + \lambda_4 \sigma(t)^{21} + \lambda_5 \sigma(t)^{25} \\ &= y \circ \tau(t^4, t^{11} + \lambda'_1 t^{13} + \lambda'_2 t^{14} + \lambda'_3 t^{17} + \lambda'_4 t^{21} + \lambda'_5 t^{25}) \quad \dots\dots\dots (**). \end{aligned}$$

Now we are going to determine the coefficients of  $\sigma$  and  $\tau$  of lower degree terms, using the equations (\*) and (\*\*) in a zigzag manner. We denote for comparison of terms in (\*) (resp. (\*\*)) of degree  $i$  by  $(*i)$  (resp.  $(**i)$ ). First by (\*4), we have  $a_1^4 = a$ . By (\*5)(\*6)(\*7), we have  $a_2 = 0$ ,  $a_3 = 0$ ,  $a_4 = 0$ . By (\*\*4),  $c = 0$ . By (\*\*8),  $k_1 = 0$ . By (\*\*11), we have  $a_1^{11} = e$ . Since  $\tau$  is a symplectomorphism, we see that  $ae = 1$ , so we have  $a_1^{15} = 1$ . By (\*\*12),  $m_1 = 0$ . By (\*\*13), we get  $\lambda_1 a^{13} = e \lambda'_1$  and therefore  $\lambda_1 a^2 = \lambda'_1$ . By (\*\*14),  $\lambda_2 a^{14} = e \lambda'_2$  and therefore  $\lambda_1 a^3 = \lambda'_1$ . By (\*\*15),  $11a_1^{10} a_5 = k_2$ . From (\*8), we have  $4a_3 a_5 = h_1$ . Since  $\tau$  is a symplectomorphism, we have  $2h_1 e + a k_2 = 0$ . Thus we see that  $a_5 = 0$ . Then  $k_2 = 0$ ,  $h_1 = 0$ . By (\*9),  $4a_3 a_6 = 0$  so  $a_6 = 0$ . By (\*10),  $a_7 = 0$ . By (\*11), we have  $4a_1^3 a_8 = b$ . Then by (\*\*17), we have  $\lambda_3 a^{17} = e \lambda'_3$ , thus  $\lambda_3 a^6 = \lambda'_3$ . By (\*\*18), we have  $a_8 = 0$ . Therefore we have  $b = 0$ . By (\*12),  $4a_1^3 a_8 = \ell_1$ . By (\*\*19),  $11a_1^{10} a_9 = m_2$ . Since  $\tau$  is a symplectomorphism we have  $6\ell_1 e + 2am_2 = 0$ . Thus we have  $a_9 = 0$ . Then we have  $\ell_1 = 0$ ,  $m_2 = 0$ . By (\*13)(\*14), we have  $a_{10} = 0$ ,  $a_{11} = 0$ . By (\*\*21), we have  $\lambda_4 a^{21} = e \lambda'_4$  so  $\lambda_4 a^{10} = \lambda'_4$ . By (\*\*22),  $11a_1^{10} a_{12} = k_3$ . By (\*15), we have  $4a_1^3 a_{12} = h_2$ . Since  $\tau$  is a symplectomorphism we have  $h_2 e + 2ak_3 = 0$ . Therefore  $a_{12} = 0$ , and  $k_3 = 0$ ,  $h_2 = 0$ . Then, by (\*\*23),

we have  $a_{13} = 0$ , and by (\*17)(\*18),  $a_{14} = 0$ ,  $a_{15} = 0$ . Finally, by (\*\*25), we have  $\lambda_5 a^{25} = e\lambda'_5$ , and  $\lambda_5 a^{13} = \lambda'_5$ . Therefore, setting  $T$  and  $\Sigma$  as the linear parts of  $\tau$  and  $\sigma$  respectively, we have  $T \circ f_{\lambda'} = f_{\lambda} \circ \Sigma$ .  $\square$

**Remark 2.5.** If two curve-germs  $f, g : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  are symplectomorphic, then they are *symplectically isotopic*, that is there exist  $C^\infty$  families of bi-holomorphic diffeomorphisms  $\sigma_s$  and bi-holomorphic symplectomorphisms  $\tau_s$  ( $s \in [0, 1]$ ) on  $(\mathbf{C}, 0)$  and  $(\mathbf{C}^2, 0)$  respectively such that  $\sigma_0(t) = t$ ,  $\tau_0(x, y) = (x, y)$  and  $\tau_1(g(t)) = f(\sigma_1(t))$ . This fact is a feature of the complex case and it is proved by using the fact that  $\mathrm{SL}(2, \mathbf{C})$  is arcwise connected and the group of symplectomorphisms with identity linear part is arcwise connected (cf. [6]). Thus our symplectic moduli space in Tables 1 and 2 are also moduli spaces for the symplectic isotopy equivalence.

### 3 Differential normal forms

The proof of Theorem 1.1 is similar to the one in [3]. We note that the symplectic normal forms (Proposition 2.2) can play the role of an intermediate classification, which also makes the diffeomorphic classification easier and clearer.

First we have

**Lemma 3.1.** *Let  $f : (\mathbf{C}, 0) \rightarrow (\mathbf{C}^2, 0)$  have the Puiseux characteristic  $(m, \beta_1, \dots)$ . If  $m \geq 4$  and  $\beta_1 \geq 13$ , or  $m \geq 5$  and  $\beta_1 \geq 9$ , or  $m \geq 6$  then the modality of  $f$  is at least 2.*

*Proof:* For instance, assume  $m = 4$ ,  $\beta_1 = 13$ . Then, in any neighborhood of  $f$ , there exists a two-parameter family of germs at 0 which are diffeomorphic to  $g_\lambda = (t^4, t^{13} + t^{14} + \lambda_1 t^{15} + \lambda_2 t^{19})$ . We find this family by an infinitesimal calculation. First, for each  $\alpha \in \mathbf{N}$ , we try to find  $\xi \in \mathcal{O}_1$  and  $\eta_1, \eta_2 \in \mathcal{O}_2$  with  $\xi(0) = 0, \eta_1(0, 0) = 0, \eta_2(0, 0) = 0$ , satisfying the equation

$$\begin{pmatrix} 0 \\ t^\alpha \end{pmatrix} \equiv \begin{pmatrix} 4t^3\xi \\ (13t^{12} + \dots)\xi \end{pmatrix} + \begin{pmatrix} \eta_1(t^4, t^{13} + \dots) \\ \eta_2(t^4, t^{13} + \dots) \end{pmatrix} \pmod{\begin{pmatrix} 0 \\ t^{\alpha+1}\mathcal{O}_1 \end{pmatrix}}.$$

Then we see that the equation is not solvable for  $\alpha = 15$  and  $\alpha = 19$ . Second, by a formal calculation, we verify that  $g_\lambda$  and  $g_{\lambda'}$  are diffeomorphic if and only if  $\lambda' = \lambda$ . From this observation we see that, if  $m \geq 4$ ,  $\beta_1 \geq 13$ , then the modality of  $f$  is  $\geq 2$ .

Other cases can be treated in a similar way.  $\square$

Thus Theorem 1.1 will be proved if we check all remaining cases. Here we will treat only the class  $W_{30}$  with the Puiseux characteristic (4, 11).

Consider the symplectic normal form  $f_\lambda(t) = (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25})$ . Suppose  $\lambda_1 \neq 0$ . Consider, for given  $\rho(t)$ , the equation

$$\begin{pmatrix} 0 \\ \rho(t) \end{pmatrix} = \xi \begin{pmatrix} 4t^3 \\ 11t^{10} + 13\lambda_1 t^{12} + \dots \end{pmatrix} + \begin{pmatrix} \eta_1(f_\lambda(t)) \\ \eta_2(f_\lambda(t)) \end{pmatrix},$$

and try to find  $\xi(t), \eta_1(x, y), \eta_2(x, y)$  with  $\xi(0) = 0, \eta_1(0, 0) = 0, \eta_2(0, 0) = 0$ . The equation is solvable for  $\rho(t) = t^{13}$ , up to higher order terms, and solvable for any  $\rho(t)$  with  $\text{ord}\rho(t) \geq 15$ . Then, by the homotopy method, we see that: if  $\lambda_1 \neq 0$  then  $f$  is diffeomorphic to  $(t^4, t^{11} + t^{13} + \lambda t^{14})$  for some  $\lambda \in \mathbf{C}$ ; if  $\lambda_1 = 0, \lambda_2 \neq 0$ , then  $f$  is diffeomorphic to  $(t^4, t^{11} + t^{14} + \lambda t^{17})$  for some  $\lambda \in \mathbf{C}, \lambda \neq \frac{25}{22}$ , or to  $(t^4, t^{11} + t^{14} + \frac{25}{22}t^{17} + \lambda t^{21})$  for some  $\lambda \in \mathbf{C}$ ; if  $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$  (resp.  $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 \neq 0; \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 \neq 0$ ), then  $f$  is diffeomorphic to  $(t^4, t^{11} + t^{17})$  (resp.  $(t^4, t^{11} + t^{21}); (t^4, t^{11} + t^{25})$ ). The exact determination of the moduli space is completed by direct formal calculations. The other cases are classified in a similar way.

**Remark 3.2.** In general, for each equi-singularity class, the symplectic moduli space is mapped canonically onto the differential moduli space, i.e. the ordinary moduli space. The dimension of the fiber over a diffeomorphism class  $[f]$  is called the *symplectic defect* and denoted by  $\text{sd}(f)$  in [7]. It is known that  $\text{sd}(f) = \mu(f) - \tau(f)$ , where  $\mu(f) = 2\delta(f)$  is the Milnor number of  $f$  and  $\tau(f)$  is the Tyurina number of  $f$  ([15],[9],[4]). Let  $s(f)$  (resp.  $c(f)$ ) be the symplectic modality, that is the number of parameters in the symplectic normal form of  $f$  (resp. the codimension of the locus in the parameter space corresponding to germs diffeomorphic to  $f$ ). Then  $s(f) - c(f) = \text{sd}(f)$ . Thus we have the formula for the Tyurina number (by means of Varchenko-Lando's formula) as

$$\tau(f) = 2\delta(f) + c(f) - s(f).$$

For example, for  $f = (t^4, t^{11} + t^{21})$  in the case of  $W_{30}$ , we have  $\delta(f) = 15, c(f) = 3, s(f) = 5$  and in fact  $\tau(f) = 28$ .

Note that the differential moduli space is not a Hausdorff space, while the symplectic moduli space is, at least for 0-modal and 1-modal cases, as

we clearly observe in Theorems 1.1 and 1.3. Therefore the symplectic moduli space can be called a *Hausdorffization* of the differential moduli space.

**Remark 3.3.** The adjacency of simple and uni-modal singularities of parametric plane curves is generated (as an ordering) by  $A_{2\ell} \leftarrow A_{2\ell+2}, E_{6\ell} \leftarrow E_{6\ell+2} \leftarrow E_{6\ell+6}$  ( $\ell = 1, 2, \dots$ ),  $A_{6s-2} \leftarrow E_{12s-6}, A_{6s} \leftarrow E_{12s}, A_{6s-2} \leftarrow E_{12s-4}, A_{6s+2} \leftarrow E_{12s+2}$  ( $s = 1, 2, \dots$ ),  $E_8 \leftarrow W_{12} \leftarrow W_{18}, W_{12} \leftarrow W_{1,1}^\#, E_{12} \leftarrow W_{1,1}^\# \leftarrow W_{18}, W_{1,2\ell-1}^\# \leftarrow W_{1,2\ell+1}^\#$  ( $\ell = 1, 2, \dots$ ),  $W_{1,1}^\# \leftarrow N_{20} \leftarrow N_{24} \leftarrow N_{28}, W_{18} \leftarrow N_{24}, W_{24} \leftarrow N_{28}, W_{18} \leftarrow W_{24} \leftarrow W_{30}, E_{18} \leftarrow W_{24} \leftarrow W_{2,1}^\#, E_{20} \leftarrow W_{30}, W_{2,2\ell-1}^\# \leftarrow W_{2,2\ell+1}^\#$  ( $\ell = 1, 2, \dots$ ).

## References

- [1] V.I. Arnold, *First steps of local symplectic algebra*, Adv. in Soviet Math., D. Fuchs birthday volume, Providence, Amer. Math. Soc., 1999.
- [2] V.I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vasil'ev, *Singularity Theory II: Classification and Applications*, Encyclopaedia of Math. Sci., Vol. 39, Dynamical System VIII, Springer-Verlag (1993).
- [3] J.W. Bruce, T. Gaffney, *Simple singularities of mappings  $\mathbf{C}, 0 \rightarrow \mathbf{C}^2, 0$* , J. London Math. Soc., **26** (1982), 465–474.
- [4] W. Domitrz, S. Janeczko, M. Zhitomirskii, *Relative Poincaré lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety*, Illinois J. Math. **48–3** (2004), 803–835.
- [5] A.B. Givental', *Singular Lagrangian manifolds and their Lagrangian mappings*, Itogi Nauki Tekh., Ser. Sovrem. Prob. Mat., (Contemporary Problems of Mathematics) **33**, VINITI, 1988, pp. 55–112. English transl., J. Soviet Math. **52** (1990), no. 4, 3246–3278.
- [6] G. Ishikawa, *Determinacy, transversality and Lagrange stability*, Banach Center Publ. **50** (1999), 123–135.
- [7] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings*, Quarterly J. Math., **54** (2003), 73–102.

- [8] G. Ishikawa, S. Janeczko, *Symplectic singularities of isotropic mappings*, Geometric Singularity Theory, Banach Center Publications **65**, (2004), 85-106.
- [9] S.K. Lando, *Normal forms of the degrees of a volume form*, Funct. Anal. Appl. **19–2** (1984), 146–148.
- [10] Lê Dũng Tráng, *Sur un critère d'équisingularité*, C. R. Acad. Sci. Paris Sér. A, **272** (1971), 138–140.
- [11] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies, **61**, Princeton Univ. Press, 1986.
- [12] R. Montgomery, M. Zhitomirskii, *Geometric approach to Goursat flags*, Ann. Inst. H. Poincaré, **18–4** (2001), 459–493.
- [13] P. Mormul, *Real moduli in codimension 2 in local classification of Goursat flags*, Preprint.
- [14] P. Orlik, L. Solomon, *Singularities. II. Automorphisms of forms*, Math. Ann. **231–3** (1977/78), 229–240.
- [15] A.N. Varchenko, *Local classification of volume forms in the presence of a hypersurface*, Funct. Anal. Appl. **19–4** (1984), 269–276.
- [16] C.T.C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. **13** (1981), 481–539.
- [17] C.T.C. Wall, *Singular Points of Plane Curves*, Cambridge Univ. Press (2004).
- [18] O. Zariski, *Le problème des modules pour les branches planes*, Cours donné au Centre de mathématiques de l'École Polytechnique, 1973, (ed. F. Kmety, M. Merle, with an appendix of B. Tessier), Hermann, Paris (1987).
- [19] M. Zhitomirskii, *Germes of integral curves in contact 3-space, plane and space curves*, Isaac Newton Inst. Preprint NI00043-SGT, December 2000.

## On the space of admissible weights for weighted Bergman spaces

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### Abstract

The space  $AW(\Omega)$  of admissible weights on an open set in  $\mathbf{C}^n$  and the functional transformation defined on this space by weighted Bergman functions are considered. The new natural topology on  $AW(\Omega)$  is defined and properties of this topology are examined. Suitable examples illustrating the general considerations are given.

**Key words and phrases:** Bergman space, Bergman kernel, weighted Bergman function, functional transform.

**1991 AMS Subject Classification Code.** 32H02, 32H10, 32L05, 53C55, 57R22, 58C10.

## 1 Introduction

In this paper we are concerned with the space of weights of integration, the so called admissible weights, or a-weights for short, for which the weighted Bergman function (the Bergman reproducing kernel) can be defined. The concept of the a-weight was introduced and initially investigated in [12]. It was also considered in [13]. The weighted Bergman functions have been investigated by many authors in different contexts (see [1], [2], [3], [4], [6], [7], [8], [9], and [15]). In particular they have appeared in theoretical physics in the model of quantum theory described in [10] and [11]. The weights of integration considered in this models are assumed to satisfy the following equation of Monge-Ampère type

$$\det \left[ \frac{\partial^2 \log \mu(z)}{\partial z_j \partial \bar{z}_k} \right] = \frac{C}{n!} (-1)^{\frac{n(n+1)}{2}} \mu(z) K_\mu(z, z),$$

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where  $C$  is a constant,  $\mu$  is an  $a$ -weight on an open set  $\Omega \subset \mathbf{C}^n$  and  $K_\mu$  is the weighted Bergman function defined by  $\mu$ . It is clear that for the investigation of this equation one should know the properties of the transformation  $\mu \mapsto \mathcal{K}[\mu] := K_\mu$  and the properties (topological and differential) of the space of all  $a$ -weights. Some of them have been described in [13], [14] and [16]. The main purpose of the presented study is to give a description of a new topological structure on the space of all  $a$ -weights which seems to be the most natural one.

The paper is divided into three sections and this Introduction is the first one. Preliminary notions and results have been collected in Section 2. In Section 3 we describe a new topology on the space of admissible weights and prove that the definition of this topology is correct (Def. 3.1, Th 3.1, Th. 3.2 and Th. 3.3). We find connected component of the space of admissible weights. In the last part of this section we give sufficient conditions for the space of admissible weights to be a Hausdorff space

Without any other explanations we use the following symbols:  $\mathbf{N}$ —the set of natural numbers;  $\mathbf{R}$ —the set of reals;  $\mathbf{C}$ —the complex plane.

## 2 Preliminaries

Any positive, Lebesgue measurable real function on an open set  $\Omega \subset \mathbf{C}^n$  is called a *weight (of integration) on  $\Omega$* . The set of all weights on  $\Omega$  will be denoted by  $W(\Omega)$  (we consider two weights as equivalent if they are equal almost everywhere with respect to the Lebesgue measure on  $\Omega$ ). If  $\mu \in W(\Omega)$  then  $L^2(\Omega, \mu)$  denotes the space of all Lebesgue measurable complex valued functions  $f$  on  $\Omega$  such that  $(\|f\|_\mu)^2 := \int_\Omega |f(z)|^2 \mu(z) d^{2n}z < \infty$ . The space  $L^2(\Omega, \mu)$  (the space of all  $\mu$ -square integrable functions on  $\Omega$ ) equipped with the scalar product

$$\langle f|g \rangle_\mu := \int_\Omega \overline{f(z)}g(z)\mu(z)d^{2n}z,$$

where  $f, g \in L^2(\Omega, \mu)$ , is a separable Hilbert space (see [17]). The space of all holomorphic  $\mu$ -square integrable functions on  $\Omega$  is called *the  $\mu$ -Bergman space* and is denoted by  $L^2H(\Omega, \mu)$ . For any  $z \in \Omega$  we define the evaluation functional  $\mathcal{E}_z$  on  $L^2H(\Omega, \mu)$  as follows:

$$\mathcal{E}_z f := f(z), \quad f \in L^2H(\Omega, \mu).$$



**Definition 2.1.** A weight  $\mu \in W(\Omega)$  is said to be *admissible* if:

- (i)  $L^2H(\Omega, \mu)$  is a closed subspace of  $L^2(\Omega, \mu)$ ;
- (ii) for any  $z \in \Omega$  the evaluation functional  $\mathcal{E}_z$  is continuous on  $L^2H(\Omega, \mu)$ .

The set of all admissible weights on  $\Omega$  will be denoted by  $AW(\Omega)$ .

Let  $\mu \in AW(\Omega)$  and let for any  $z \in \Omega$  the function  $e_{z,\mu}$  represent the evaluation functional  $\mathcal{E}_z$  (in the sense of the Riesz representation theorem).

**Definition 2.2.** The function  $K_\mu : \Omega \times \Omega \rightarrow \mathbf{C}$  given by the formula

$$K_\mu(z, w) := \overline{e_{z,\mu}(w)}, \quad z, w \in \Omega,$$

is called *the  $\mu$ -Bergman function* of the set  $\Omega$  (see [1], [12] or [13]).

The above definition can be interpreted as a definition of the transformation  $\mathcal{K}$  with the domain  $AW(\Omega)$ , i.e.

$$\mathcal{K}[\mu] := K_\mu, \quad \mu \in AW(\Omega).$$

To characterize the range of this transformation we need the following result.

**Theorem 2.1.** *If  $\mu \in AW(\Omega)$  then:*

- (i) *for any  $z, w \in \Omega$*

$$K_\mu(w, z) = \overline{K_\mu(z, w)};$$

- (ii) *the function  $K_\mu(z, w)$  is analytic in the real sense, holomorphic in  $z$  and antiholomorphic in  $w$ ;*

- (iii)  *$K_\mu$  is the integral kernel of the operator  $P_\mu$  of  $\langle \cdot | \cdot \rangle_\mu$ -orthogonal projection of  $L^2(\Omega, \mu)$  onto  $L^2H(\Omega, \mu)$ , i.e. for any  $z \in \Omega$  and any  $f \in L^2(\Omega, \mu)$*

$$[P_\mu f](z) = \int_{\Omega} K_\mu(z, w) f(w) \mu(w) d^{2n}w; \quad (2.1)$$

- (iv) *the family  $\{K_\mu(\cdot, w) : w \in \Omega\}$  is linearly dense in  $L^2H(\Omega, \mu)$  and for any  $z, w \in \Omega$*

$$\langle K_\mu(\cdot, z) | K_\mu(\cdot, w) \rangle_\mu = K_\mu(z, w). \quad (2.2)$$

*Proof.* For the proof of (i), (ii) and (iii) see [12], Th. 2.1 or [13], Th. 2.1. Point (iv) follows immediately from (iii).  $\square$

Let  $HA(\Omega)$  be the real vector space of all complex-valued functions  $F$  on  $\Omega \times \Omega$  which are analytic in the real sense, holomorphic with respect to

the first  $n$  variables, antiholomorphic with respect to the last  $n$  variables and satisfy the equality

$$F(w, z) = \overline{F(z, w)}, \quad w, z \in \Omega.$$

We endow  $HA(\Omega)$  with a Fréchet space topology given by the family of seminorms  $\{\|\cdot\|_X : X \subset \Omega, X \text{ compact}\}$ , where

$$\|F\|_X := \sup_{(z,w) \in X \times X} |F(z, w)|, \quad F \in HA(\Omega).$$

### 3 Topological and analytic structures on $AW(\Omega)$

In [12] the set  $W(\Omega)$  was endowed with the following topological and differential structure. Let us consider the set

$$U(\Omega) := \{g \in L_{\mathbf{R}}^{\infty}(\Omega) : \text{ess inf}_{z \in \Omega} g(z) > 0\}.$$

It is an open subset of the Banach space  $L_{\mathbf{R}}^{\infty}(\Omega)$ . Let for any  $\mu \in W(\Omega)$  the map  $\Phi_{\mu} : U(\Omega) \rightarrow W(\Omega)$  be given by the formula

$$[\Phi_{\mu}(g)](z) := g(z)\mu(z), \quad g \in U(\Omega), \quad z \in \Omega.$$

It is clear that for any  $\mu \in W(\Omega)$  the map  $\Phi_{\mu}$  is a bijection of  $U(\Omega)$  onto the set  $U(\Omega, \mu) := \Phi_{\mu}(U(\Omega))$ .

**Proposition 3.1.** *Let  $\mu \in W(\Omega)$ . Then*

- (i) *for any  $\nu \in W(\Omega)$  if  $U(\Omega, \nu) \cap U(\Omega, \mu) \neq \emptyset$  then  $U(\Omega, \nu) = U(\Omega, \mu)$ ;*
- (ii) *the family  $\{(\Phi_{\mu}(X) : \mu \in W(\Omega), X \text{-an open subset of } U(\Omega))\}$  forms a basis of some topology  $\tau_0$  on  $W(\Omega)$ ;*
- (iii) *the family  $\{(\Phi_{\mu}^{-1}, U(\Omega, \mu)) : \mu \in W(\Omega)\}$  forms an analytic atlas of a Banach manifold on the topological space  $(W(\Omega), \tau_0)$ ;*
- (iv) *for any  $\nu_1, \nu_2 \in U(\Omega, \mu)$  the spaces  $L^2(\Omega, \nu_1)$  and  $L^2(\Omega, \nu_2)$  coincide as vector spaces and the norms  $\|\cdot\|_{\nu_1}$  and  $\|\cdot\|_{\nu_2}$  are equivalent;*
- (v) *if  $\mu \in AW(\Omega)$  then  $U(\Omega, \mu) \subset AW(\Omega)$  i.e.,  $AW(\Omega)$  is an open submanifold of  $W(\Omega)$*

*Proof.* For the proof see [12], Proposition 2.3. □

It is proved in [12] that the transformation  $\mathcal{K}$  is analytic with respect to the analytic (linear) structure on the space  $HA(\Omega)$  described in the previous

section and the analytic structure on  $AW(\Omega)$  considered in Proposition 3.1 ([12], Theorem 5.2). Although this analytic structure on  $AW(\Omega)$  is convenient in the proof of the result, the topology  $\tau_0$  seems to be much too strong. The following example illustrates the situation.

**Example 3.1.** Let  $\Omega := \{z \in \mathbf{C} : |z| < 1\}$  be the unit disc in  $\mathbf{C}$  and let  $\mu_a := |z|^a$ , for  $z \in \Omega$ , where  $a \in \mathbf{R}$ . It is shown in [12], Example 3.2 that for  $a > -2$  the  $\mu_a$ -Bergman function  $\mathcal{K}[\mu_a]$  is given by the formula

$$\mathcal{K}[\mu_a](z, w) = \frac{1}{\pi(1 - z\bar{w})^2} + \frac{a}{2\pi(1 - z\bar{w})}. \quad (3.1)$$

Note that the map  $(-2; +\infty) \ni a \mapsto \mathcal{K}[\mu_a] \in HA(\Omega)$  is analytic on  $(-2; +\infty)$ . On the other hand, if  $a_1 \neq a_2$  then  $U(\Omega, \mu_{a_1}) \cap U(\Omega, \mu_{a_2}) = \emptyset$  and therefore  $(-2; +\infty) \ni a \mapsto \mu_a \in AW(\Omega)$  is not a continuous map. Hence the analyticity of the map  $a \mapsto \mathcal{K}[\mu_a]$  does not follow from the results of [12].

The main purpose of this paper is to introduce the topology and the analytic structure on  $AW(\Omega)$  which is weaker and more natural than the previous one but seems to be strong enough to obtain the analyticity of the map  $a \mapsto \mathcal{K}[\mu_a]$ . At first we will prove the following theorem.

**Theorem 3.1.** *Let  $X$  be a vector space endowed with two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  such that:*

- (i)  $(X, \|\cdot\|_1)$  and  $(X, \|\cdot\|_2)$  are Banach spaces;
- (ii) for any sequence  $(x_n)$  in  $X$  and any  $a, b \in X$ , the condition

$$\lim_{n \rightarrow \infty} \|x_n - a\|_1 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|x_n - b\|_2 = 0$$

*implies  $a = b$ .*

*Then norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.*

*Proof.* Let  $\|\cdot\|$  be the norm in  $X$  given by

$$\|x\| = \|x\|_1 + \|x\|_2, \quad x \in X.$$

For any  $x \in X$  we have  $\|x\|_1 \leq \|x\|$  and  $\|x\|_2 \leq \|x\|$ . Then if  $(x_n)$  is a Cauchy sequence with respect to  $\|\cdot\|$ , then it is a Cauchy sequence with respect to norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ . By (i)  $(x_n)$  converges to an element  $y_1$  in  $(X, \|\cdot\|_1)$  and  $y_2$  in  $(X, \|\cdot\|_2)$ . Then by (ii)  $y_1 = y_2$  and  $(x_n)$  converges to  $y = y_1 = y_2$  in  $(X, \|\cdot\|)$ . This means that  $(X, \|\cdot\|)$  is a Banach space. Let  $A_i : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|_i)$ ,  $i = 1, 2$ , be the identity operator. Then  $A_i$  is

continuous and invertible. By the Banach inverse mapping theorem  $A_i^{-1}$  is continuous for  $i = 1, 2$ . Therefore  $\|\cdot\|$  and  $\|\cdot\|_i$ ,  $i = 1, 2$ , are equivalent, i.e. there exist constants  $C_i$ ,  $i = 1, 2$ , such that

$$\begin{aligned}\|x\|_2 &\leq \|x\| \leq C_1\|x\|_1, \\ \|x\|_1 &\leq \|x\| \leq C_2\|x\|_2.\end{aligned}$$

Consequently  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.  $\square$

As a simple corollary we obtain

**Theorem 3.2.** *Let  $\mu_1, \mu_2 \in AW(\Omega)$  be such that the Bergman spaces  $L^2H(\Omega, \mu_1)$  and  $L^2H(\Omega, \mu_2)$  coincide as vector spaces. Then the norms  $\|\cdot\|_{\mu_1}$  and  $\|\cdot\|_{\mu_2}$  are equivalent.*

*Proof.* Let  $(f_n)$  be a sequence in  $X := L^2H(\Omega, \mu_1) = L^2H(\Omega, \mu_2)$  such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\mu_1} = 0 \text{ and } \lim_{n \rightarrow \infty} \|f_n - g\|_{\mu_2} = 0$$

for some  $f, g \in X$ . Then by the continuity of evaluation functionals (Definition 2.1) for any  $z \in \Omega$

$$\lim_{n \rightarrow \infty} f_n(z) = \lim_{n \rightarrow \infty} \mathcal{E}_z f_n = \mathcal{E}_z f = f(z)$$

and

$$\lim_{n \rightarrow \infty} f_n(z) = \lim_{n \rightarrow \infty} \mathcal{E}_z f_n = \mathcal{E}_z g = g(z)$$

Hence  $f = g$  and by Theorem 3.1 the norms  $\|\cdot\|_{\mu_1}$  and  $\|\cdot\|_{\mu_2}$  are equivalent.  $\square$

For any  $\mu \in AW(\Omega)$  we denote by  $V(\Omega, \mu)$  the set of all  $\nu \in AW(\Omega)$  for which  $L^2H(\Omega, \mu)$  and  $L^2H(\Omega, \nu)$  coincide as vector spaces. We want to endow  $V(\Omega, \mu)$  with the topological and analytic structure. We are going to do it using the map

$$V(\Omega, \mu) \ni \nu \mapsto \langle \cdot | \cdot \rangle_\nu \in \text{Heq}(L^2H(\Omega, \mu)), \quad (3.2)$$

where  $\text{Heq}(L^2H(\Omega, \mu))$  denotes the set of all hermitian products on  $L^2H(\Omega, \mu)$  defining norms equivalent to  $\|\cdot\|_\mu$ . This set is an open cone in the real Banach space  $H(L^2H(\Omega, \mu))$  of all bounded hermitian forms on  $L^2H(\Omega, \mu)$ . Let us recall that for any  $h \in H(L^2H(\Omega, \mu))$  the norm  $\|h\|_H$  is defined as the

infimum of all such constants  $C > 0$  that  $|h(f, g)| \leq C\|f\|_\mu\|g\|_\mu$  for any  $f, g \in L^2H(\Omega, \mu)$ . Since  $\|\cdot\|_\mu$  and  $\|\cdot\|_\nu$  are equivalent it is known that for any  $\nu \in V(\Omega, \mu)$  there exists the unique linear hermitian operator  $A_\nu$  on the space  $L^2H(\Omega, \mu)$  such that  $\langle f|g \rangle_\nu = \langle f|A_\nu g \rangle_\mu$  for all  $f, g \in L^2H(\Omega, \mu)$ . It is positive definite and if  $\|A_\nu\|_\mu$  denotes the operator norm of  $A_\nu$  in  $L^2H(\Omega, \mu)$  then we have the equality  $\|\langle \cdot | \cdot \rangle_\nu\|_H = \|A_\nu\|_\mu$ .

Now we are ready to introduce a new topology on  $AW(\Omega)$ .

**Definition 3.1.** Let  $\mu \in AW(\Omega)$ . By  $\tau_\mu$  we denote the weakest topology on  $V(\Omega, \mu)$  for which the map (3.2) is continuous.

**Theorem 3.3.** *Let  $\mu \in AW(\Omega)$ . Then*

(i) *for any  $\nu \in AW(\Omega)$ , if  $V(\Omega, \nu) \cap V(\Omega, \mu) \neq \emptyset$  then  $V(\Omega, \nu) = V(\Omega, \mu)$ ;*

(ii) *the family*

$$\mathcal{B} := \bigcup_{\mu \in AW(\Omega)} \tau_\mu$$

*form a basis of some topology  $\tau$  on  $AW(\Omega)$ ;*

(iii) *any set  $V(\Omega, \mu)$  is open and it is a connected component in the topological space  $(AW(\Omega), \tau)$ .*

*Proof.* If  $\kappa \in V(\Omega, \nu) \cap V(\Omega, \mu)$  then the following equality of vector spaces holds

$$L^2H(\Omega, \mu) = L^2H(\Omega, \kappa) = L^2H(\Omega, \nu).$$

This implies that for any  $\mu_1 \in V(\Omega, \mu)$  and  $\nu_1 \in V(\Omega, \nu)$

$$L^2H(\Omega, \mu_1) = L^2H(\Omega, \nu_1).$$

Hence  $\nu_1 \in V(\Omega, \mu)$  and  $\mu_1 \in V(\Omega, \nu)$ . Consequently  $V(\Omega, \nu) = V(\Omega, \mu)$ .

For the proof of (ii) note that  $\mathcal{B}$  is a covering of  $AW(\Omega)$ . Moreover, if  $X, Y \in \mathcal{B}$  and  $X \cap Y \neq \emptyset$  then by (i) there exists  $\mu \in AW(\Omega)$  such that  $X, Y \in \tau_\mu$ . Hence  $X \cap Y \in \tau_\mu \subset \mathcal{B}$ . It means that  $\mathcal{B}$  is a basis of some topology on  $AW(\Omega)$ .

For the proof of (iii) it is obvious that  $V(\Omega, \mu)$  is open with respect to  $\tau$ . On the other hand it is closed because of the equality

$$AW(\Omega) \setminus V(\Omega, \mu) = \bigcup_{\nu \in AW(\Omega) \setminus V(\Omega, \mu)} V(\Omega, \nu) \in \tau$$

which is a consequence of (i). Now it is enough to show that  $V(\Omega, \mu)$  is connected. Let  $\mu_0, \mu_1 \in V(\Omega, \mu)$  and let  $\mu(t) := t\mu_1 + (1-t)\mu_0$  for any

$t \in ]0; 1[$ . We have  $\mu(0) = \mu_0$  and  $\mu(1) = \mu_1$ . Since the hermitian operator

$$A_{\mu_t} = tA_{\mu_1} + (1-t)A_{\mu_0}$$

the map  $]0; 1[ \ni t \mapsto A_{\mu(t)} \in L(L^2H(\Omega, \mu))$  is continuous (i.e. the map  $]0; 1[ \ni t \mapsto \langle \cdot | \cdot \rangle_{\mu(t)} \in \text{Heg}(L^2H(\Omega, \mu))$  is continuous). Then, by the definition of  $\tau_\mu$ , the map  $\mu(\cdot)$  is continuous. It means that  $V(\Omega, \mu)$  is connected.  $\square$

In general it may happen that  $(AW(\Omega), \tau)$  is not a Hausdorff space.

**Proposition 3.1.** *The topological space  $(AW(\Omega), \tau)$  is Hausdorff iff the map (3.2) is injective.*

*Proof.* ( $\Rightarrow$ ) Suppose that the map (3.2) is not injective, i.e. there exist two different weights  $\mu_1, \mu_2 \in AW(\Omega)$  such that  $L^2H(\Omega, \mu_1) = L^2H(\Omega, \mu_2)$  and  $\langle \cdot | \cdot \rangle_{\mu_1} = \langle \cdot | \cdot \rangle_{\mu_2}$ . Then the set  $\{\mu_1\}$  is not closed in  $(AW(\Omega), \tau)$  (the smallest closed set containing  $\mu_1$  contains also  $\mu_2$ ). This implies that  $(AW(\Omega), \tau)$  is not a Hausdorff space – a contradiction.

( $\Leftarrow$ ) Assume that the map (3.2) is injective. If  $\mu_1, \mu_2 \in AW(\Omega)$  and  $\mu_1 \neq \mu_2$  then: (a)  $L^2H(\Omega, \mu_1) \neq L^2H(\Omega, \mu_2)$  or (b)  $L^2H(\Omega, \mu_1) = L^2H(\Omega, \mu_2)$  and  $\langle \cdot | \cdot \rangle_{\mu_1} \neq \langle \cdot | \cdot \rangle_{\mu_2}$ . In case (a) we have  $\mu_1 \in V(\Omega, \mu_1)$ ,  $\mu_2 \in V(\Omega, \mu_2)$ ,  $V(\Omega, \mu_1) \cap V(\Omega, \mu_2) = \emptyset$  and  $V(\Omega, \mu_1), V(\Omega, \mu_2) \in \tau$ . In case (b)  $A_{\mu_1} \neq A_{\mu_2}$ . Since  $L(L^2H(\Omega, \mu_1))$  is a Hausdorff space, there exist two open sets  $\mathcal{O}_1$  and  $\mathcal{O}_2$  in this space such that  $A_{\mu_1} \in \mathcal{O}_1$ ,  $A_{\mu_2} \in \mathcal{O}_2$  and  $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ . Let

$$\mathcal{A}(\nu) := A_\nu, \quad \nu \in V(\Omega, \mu_1).$$

Since the map  $\mathcal{A} : V(\Omega, \mu_1) \rightarrow L(L^2H(\Omega, \mu_1))$  is continuous, we obtain that  $\mathcal{A}^{-1}(\mathcal{O}_1)$  and  $\mathcal{A}^{-1}(\mathcal{O}_2)$  are open sets in  $V(\Omega, \mu_1)$ ,  $\mu_1 \in \mathcal{A}^{-1}(\mathcal{O}_1)$ ,  $\mu_2 \in \mathcal{A}^{-1}(\mathcal{O}_2)$  and

$$\mathcal{A}^{-1}(\mathcal{O}_1) \cap \mathcal{A}^{-1}(\mathcal{O}_2) = \mathcal{A}^{-1}(\mathcal{O}_1 \cap \mathcal{O}_2) = \emptyset.$$

Then  $(AW(\Omega), \tau)$  is Hausdorff.  $\square$

Now we are ready to give a sufficient condition for the space  $(AW(\Omega), \tau)$  to be a Hausdorff space

**Definition 3.2.** An open set  $\Omega \subset \mathbf{C}^n$  is said to be of *bounded type* if it is biholomorphic to some bounded set  $\Omega_1 \subset \mathbf{C}^n$ .

**Theorem 3.4.** *Let  $\Omega \subset \mathbf{C}^n$  be an open set of bounded type. Then the map (3.2) is injective and in a consequence the topological space  $(AW(\Omega), \tau)$  is a Hausdorff space.*

*Proof.* It was proved in [16] (Theorem 3.1) that if  $\Omega \subset \mathbf{C}^n$  is of bounded type,  $\mu_1, \mu_2 \in AW(\Omega)$  and  $\mu_1 \neq \mu_2$  then the Bergman functions  $\mathcal{K}[\mu_1] \neq \mathcal{K}[\mu_2]$ . It was also proved in [16] (Theorem 2.1) that the function  $\mathcal{K}[\mu]$  completely determines the space  $L^2H(\Omega, \mu)$  and the scalar product  $\langle \cdot | \cdot \rangle_\mu$ . In particular if  $\mathcal{K}[\mu_1] \neq \mathcal{K}[\mu_2]$  then  $\langle \cdot | \cdot \rangle_{\mu_1} \neq \langle \cdot | \cdot \rangle_{\mu_2}$ . It means that the map (3.2) is injective.  $\square$

**Example 3.2.** Let  $\Omega$  and let  $\mu_a$  be such as in Example 3.1. Then  $\Omega$  is of bounded type and the holomorphic function

$$f(z) = \sum_{m=0}^{\infty} c_m z^m, \quad z \in \Omega,$$

is an element of  $L^2H(\Omega, \mu_a)$  iff

$$\sum_{m=0}^{\infty} \frac{|c_m|^2}{2m + a + 2} < \infty.$$

The last condition is equivalent to the inequality

$$\sum_{m=0}^{\infty} \frac{|c_m|^2}{2m + 2} < \infty$$

which does not depend on  $a$ . Hence for any  $a_1, a_2 \in (-2; +\infty)$  we have

$$L^2H(\Omega, \mu_{a_2}) = L^2H(\Omega, \mu_{a_1}) = L^2H(\Omega, \mu_1),$$

where  $\mu_1 \equiv 1$  is a weight of Lebesgue measure on  $\Omega$ . It means that for any  $a \in (-2; +\infty)$  we have  $\mu_a \in V(\Omega, \mu_1)$ .

## References

- [1] J. Burbea, P. Masani, *Banach and Hilbert spaces of vector valued functions*, Research Notes in Mathematics 90, Pitman, Boston 1984.
- [2] S. Dragomir, *On weighted Bergman kernels of bounded domains*, Studia Math. 108 (1994), 149-157.
- [3] H.P. Jakobsen, M. Vergne, *Wave and Dirac operators and representations of the conformal group*, J. Funct. Anal., 24 (1977), 52-106.

- [4] S. Janson, J. Peetre and R. Rochberg, *Hankel forms and the Fock space*, Uppsala University, Dept. of Mathematics, Report 6 (1986).
- [5] S.G. Krantz, *Function theory of several complex variables*, Intersciences-Willey, New York 1982.
- [6] E. Ligocka, *The regularity of the weighted Bergman projection*, in: Seminar of Deformation Theory 1982/84, pp. 197-203, Lecture Notes in Math. 1165, Springer, 1985.
- [7] E. Ligocka, *On the Forelli-Rudin construction and weighted Bergman projections*, Studia Math. 94 (1989), 257-272.
- [8] T. Mazur, *Canonical isometry on weighted Bergman spaces*, Pac. J. Math., 136(2) (1989), 303-310.
- [9] T. Mazur, *On the complex manifolds of Bergman type*, in: Classical Analysis, *Proceedings of the 6th Symposium, 23-29 September 1991*, pp. 132-138, T. Mazur (ed), World Scientific, Singapore-New Jersey-London-Hong Kong, 1992.
- [10] A. Odziejewicz, *On reproducing kernels and quantization of states*, Commun. Math. Phys. 114 (1988) 577-597.
- [11] A. Odziejewicz, *Coherent states and geometric quantization*, Commun. Math. Phys., 150 (1992), 385-413.
- [12] Z. Pasternak-Winiarski, *On the dependence of the reproducing kernel on the weight of integration*, J. Funct. Anal. 94 (1990) 110-134,
- [13] Z. Pasternak-Winiarski, *On weights which admit the reproducing kernels of Bergman type*, Internat. J. Math & Math. Sci., 15 (1993) 1-14.
- [14] Z. Pasternak-Winiarski, *Admissible weights and weighted Bergman functions*, in "Proceedings of the Second International Conference *Function Spaces*", Poznań 1989, Teubner-Texte zur Mathematik, B. G. Teubner Verlagsgesellschaft, Stuttgart-Leipzig 1991.
- [15] Z. Pasternak-Winiarski, *On the dependence of the Bergman function on deformations of the Hartogs domain*, Ann. Polon. Math., 55 (1991) 287-300.



- [16] Z. Pasternak-Winiarski, *Properties of the functional transformation defined by weighted Bergman kernels*, in "Singularities and Mathematical Physics" Singularity Theory Seminar, S. Janeczko (ed), Institute of Mathematics, Warsaw University of Technology, 1997.
- [17] Z. Pasternak-Winiarski, *Maps on complex manifolds into Grassmann spaces defined by reproducing kernels of Bergman type*, Demonstratio Math. 30 No 2 (1997), 465-474.

## Exotic moduli of Goursat distributions exist already in codimension three

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### Abstract

A distribution  $D$  of rank  $\geq 2$  and corank  $r \geq 2$ , on a manifold  $M$ , is Goursat when its Lie square  $[D, D]$  is a distribution of constant corank  $r - 1$ , the Lie square of  $[D, D]$  is of constant corank  $r - 2$  and so on step by step, until reaching in  $r$  steps the whole tangent bundle  $TM$ .

It has not been known whether moduli of the local classification of Goursat distributions of the type 2c from the reference work [MonZ], called ‘exotic’ in [M3], might show up already in codimension 3. (First examples of such moduli, produced in [M3], were in codimension 4.) In the note we show a concrete geometric class of codimension 3, for Goursat distributions of corank  $r = 10$ , in which there does sit an exotic module, and give a shortcut proof of this fact.

## 1 Introduction

The note deals with Goursat flags – nested sequences of  $r \geq 2$  (one says then about flags of length  $r$ ) distributions in the tangent bundle  $TM$  to a ( $C^\infty$ , or real analytic) manifold  $M$  of dimension  $n \geq r + 2$ , every bigger one being the Lie square of the preceding and having by one bigger rank. Locally, without loss of generality, one can assume that rank of  $D$  is simply 2; cf. p.462 in [MonZ].

Flags of a fixed length  $r$ , or distributions of corank  $r$ , are stratified into *geometric classes* that can be labelled, or encoded, by words of length  $r$  over the alphabet  $\{G, S, T\}$  such that a) they start with GG, and b) a letter T never goes directly after a letter G. These classes are but the geometric essence of *regions* of Jean, [J], constructed in the configuration space for the renowned kinematical system ‘car + trailers’.

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The only generic – and only one dense and open – stratum is GGG...GG, dealt with a hundred years ago (with no knowledge whatsoever of the adjacent thinner singular strata) by Engel, von Weber and E. Cartan. The codimension of a stratum is easily seen in its code – it is the number of letters *not* G in it; this property of geometric classes will often be used in the present note.

Since 1997 it has been known that Goursat distributions have numeric *moduli* of the local classification. More light on them was shed when Montgomery and Zhitomirskii predicted in [MonZ] that there should altogether exist five patterns of local prolongations of Goursat germs, denoted by them 1, 2a, 2b, 2c, and 3. Out of these five, only 2c and 3 are responsible for the appearing of moduli, whereas the patterns 1, 2a, 2b create, from germs sitting in one orbit of the local classification, only finite families of finer orbits of Goursat germs of length bigger by one.

First found moduli were of type 3; that type is simpler than 2c. They resided in the geometric classes GGGSTTGGG and GGSGSGSG. Only later, in [M3], were there produced examples of type 2c, in the classes GGSGSSGSG and GGSTTTGGGG.<sup>2</sup>

The object of the note is the geometric class  $\mathcal{C} = \text{GGGSSTGGGG}$  of Goursat distributions of corank 10. This class has been prompted by the analysis of an important family of (unimodal) Legendre<sup>3</sup> curves:  $x(t) = t^4$ ,  $p(t) = t^{11} \pm t^{13} + \lambda t^{14}$ ,  $y(t) = \int_0^t p(\tau) dx(\tau)$ ,  $\lambda \in \mathbb{R}$  – a parameter, which was recently put forward by G. Ishikawa. The computations included in the present note show that a so-called Goursat–Legendre duality, worked upon by Montgomery & Zhitomirskii, see also Thm. 2 in [Is2], holds within this precise geometric class  $\mathcal{C}$  (within its generic part, in fact).

As mentioned already, with no loss of generality one can just think about rank 2 distributions living on a 12-dimensional manifold, say  $M$ , and, as germs, sitting in  $\mathcal{C}$ . A module of type 2c will be produced, we underline, not in the entire class  $\mathcal{C}$ , but in a generic part of it, defined precisely in the next section.

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<sup>2</sup> Such examples were not yet known in the time of writing of [MonZ] and pattern 2c had been foreseen only theoretically.

<sup>3</sup> ‘Legendre’ means tangent to the contact structure  $dy - p dx = 0$  in  $\mathbb{R}^3(x, y, p)$ . In [Z] and [Is1] the reader may find a lot on *contact* classification of curves.

## 2 Kumpera-Ruiz pseudo-normal forms for the germs in $\mathcal{C}$

A Goursat distribution  $D^{10}$  around  $p \in M$ , sitting in  $\mathcal{C}$  as the germ at  $p$ , can be viewed in Kumpera–Ruiz coordinates ([KR]), say  $x^1, x^2, \dots, x^{12}$  which are centered at  $p$ . In fact, in the language of Pfaffian equations,  $D^{10} = (\omega^1, \omega^2, \dots, \omega^{10})$ , where

$$\begin{aligned} \omega^1 &= dx^2 - x^3 dx^1, & \omega^2 &= dx^3 - x^4 dx^1, & \omega^3 &= dx^4 - x^5 dx^1, & \omega^4 &= dx^1 - x^6 dx^5, \\ \omega^5 &= dx^5 - x^7 dx^6, & \omega^6 &= dx^7 - x^8 dx^6, & \omega^7 &= dx^8 - (1 + x^9) dx^6, & \omega^8 &= \\ \omega^9 &= dx^9 - (E + x^{10}) dx^6, & \omega^{10} &= dx^{10} - (a + x^{11}) dx^6, & \omega^{11} &= dx^{11} - (b + x^{12}) dx^6. \end{aligned}$$

In this description  $a, b$  and  $E$  are certain real parameters. Note (an elementary fact) that the additive constant in  $\omega^7$ , automatically nonzero in  $\mathcal{C}$ , is here normalized to 1 already. The first observation is that  $E$  can be reduced to 0. Indeed, the infinitesimal symmetries of  $D^{10}$  are parametrized – see, for inst., [M2] – by functions  $k = k(x^1, x^2, x^3)$  called contact hamiltonians (Arnold) or contactians (Lychagin). When  $E = 0$ , after due computations based on Prop. 5.2 in [M2], the symmetry parametrized by  $k$  appears to have at 0 the components

$$\begin{aligned} & -k_3 \partial_1 + k \partial_2 + k_1 \partial_3 + k_{11} \partial_4 + k_{111} \partial_5 + (4k_2 + 15k_{13}) \partial_9 \\ & + 13k_{1111} \partial_{10} + a(6k_2 + 23k_{13}) \partial_{11} + b(7k_2 + 27k_{13}) \partial_{11} | 0. \end{aligned} \quad (2.1)$$

Note the absence in this expansion of the  $\partial_6, \partial_7, \partial_8$  components, explained by the relevant letters SST in the code of  $\mathcal{C}$ . Note also that the coefficient at  $\partial_{12}$  is a combination of those at  $\partial_9$  and  $\partial_{11}$ :

$$b(7k_2 + 27k_{13}) | 0 = -\frac{b}{2}(4k_2 + 15k_{13}) + \frac{3b}{2a} a(6k_2 + 23k_{13}) | 0. \quad (2.2)$$

When the two latter vanish, so does the former. This is a key to the module hidden in  $D^{10}$ : if the additive constants at  $x^9$  (1) and  $x^{11}$  ( $a$ ) are frozen, then the constant  $b$  at  $x^{12}$  cannot be moved by symmetries embeddable in flows. Yet there are also symmetries of Goursat objects *not* embeddable in flows. . . The presence of the term  $k_{1111} \partial_{10}$  in (2.1) is, by standard techniques for Goursat objects, sufficient to conjugate the zero and *any* non-zero value of the constant  $E$ . That is, to conjugate a  $D^{10}$  with  $E = 0$  and certain values  $a$  and  $b$  to a different  $D^{10}$  with a prescribed non-zero  $E$  and certain other values  $\tilde{a}, \tilde{b}$ .

Consequently, until the end of the present note **we assume**  $E = 0$ .

The role of the parameter  $a$  is, in distinction to  $E$ , important. Using the notion of the *symmetry dimension* of a distribution  $D$  at a point  $p$ , written  $\text{SD}_p(D)$  (cf. Obs. 1 in [M1]; it is the dimension of the linear hull of all infinitesimal symmetries of  $D$  at  $p$ ), one is able to tell the value  $a = 0$  from the remaining  $a \neq 0$ . Indeed, the matrix built of coefficients: 4, 15, and 6, 23 in (2.1) is invertible (compare the equality (2.2)), and  $k$  is any smooth, or real analytic, function in the vicinity of  $0 \in \mathbb{R}^3$ . Hence it follows directly from (2.1) that

$$\text{SD}_0(D^{10}) = \begin{cases} 7, & \text{when } a = 0, \\ 8, & \text{when } a \neq 0. \end{cases} \quad (2.3)$$

Thus  $a = 0$  and any value  $a \neq 0$  are non-equivalent. Also, any two values  $a$  and  $\tilde{a}$  of different signs,  $a\tilde{a} < 0$ , give non-equivalent flag members  $D^9$ , and all the more so  $D^{10}$ . This statement is not immediate; its proof is similar to the proof of Thm. 4.2 in [M2], with (2.3) replacing the symmetry dimension data (31) and (33) in [M2]. On the other hand, any  $a \neq 0$  can be quickly normalized to  $\text{sgn}(a)$  (in the style of [M1], p. 225). Accordingly, the class  $\mathcal{C}$  splits up into the invariant parts  $\text{GGGSSTGGG}^0\text{G}$ ,  $\text{GGGSSTGGG}^-\text{G}$  and  $\text{GGGSSTGGG}^+\text{G}$ . (This is, besides, an example of the prolongation pattern 2b of [MonZ] for the germs in the class  $\text{GGGSSTGGG}$ .) In the sequel we will restrict ourselves uniquely to the  $+$  and  $-$  parts. In fact, these are open parts of  $\mathcal{C}$ , while  $\text{GGGSSTGGG}^0\text{G}$  is of codimension 1 in  $\mathcal{C}$ .

### 3 The module of type 2c in $\text{GGGSSTGGG}^\pm\text{G}$

Our object of consideration  $D^{10}$  is now given by the Pfaffian equations

$$\begin{aligned} dx^2 - x^3 dx^1 &= 0, & dx^3 - x^4 dx^1 &= 0, & dx^4 - x^5 dx^1 &= 0, & dx^1 - x^6 dx^5 &= 0, \\ dx^5 - x^7 dx^6 &= 0, & dx^7 - x^8 dx^6 &= 0, & dx^8 - (1+x^9) dx^6 &= 0, & dx^9 - x^{10} dx^6 &= 0, \\ dx^{10} - (a + x^{11}) dx^6 &= 0, & dx^{11} - (b + x^{12}) dx^6 &= 0, \end{aligned}$$

with  $a = \pm 1$  and  $b \in \mathbb{R}$ . The parameter  $a$  will be **fixed** in our arguments; they look identical in either of the two generic parts of  $\mathcal{C}$ .

**Theorem 3.1.** *In the above family of germs at  $0 \in \mathbb{R}^{12}$  of Goursat distributions with fixed value of  $a \in \{-1, 1\}$ , the value  $|b|$  is a module of the local smooth, or real analytic, classification, whereas every two values  $b$  and  $-b$  are equivalent.*

*Proof of Theorem 3.1, introductory part.* Suppose that a local diffeo  $g = (g^1, g^2, \dots, g^{12}) : (\mathbb{R}^{12}, 0) \leftrightarrow$  conjugates two distributions  $D^{10}$  as above, with parameters  $a, b$  and  $a, \tilde{b}$  ( $a = \pm 1$ ). By general considerations related to conjugating arbitrary Goursat distributions given in KR pseudo-normal forms, we know that  $g^l$  depends only on:  $x^1, x^2, x^3$  when  $l \leq 3$ , and on  $x^1, x^2, \dots, x^l$  when  $4 \leq l \leq 12$ . Moreover, because of the 4th and 5th letters in the code of  $\mathcal{C}$  being S,  $g^7(x^1, \dots, x^7) = x^7 F(x^1, \dots, x^7)$  and  $g^6(x^1, \dots, x^6) = x^6 G(x^1, \dots, x^6)$  for certain invertible at 0 functions  $F$  and  $G$ .

In the sequel we shall write simply  $g_k^l$  for  $\frac{\partial g^l}{\partial x^k}$ . For instance, the inequality  $\frac{\partial g^l}{\partial x^l} |0 \neq 0$  will henceforth be noted  $g_l^l |0 \neq 0$ .

Moreover still, we write  $D^{10} = (\partial_{12}, Y_b)$ , where the second, more involved generator is so constructed as to have no  $\partial_{12}$  component and satisfy all ten Pfaffian equations in question,

$$Y_b^T = [x^7 x^6, x^7 x^6 x^3, x^7 x^6 x^4, x^7 x^6 x^5, x^7, 1, \\ x^8, 1 + x^9, x^{10}, a + x^{11}, b + x^{12}, 0] .$$

In terms of these key vector field generators  $Y_b$  and  $Y_{\tilde{b}}$ , the conjugacy by  $g$  (which is subject to the mentioned restrictions!) means two things. Firstly,

$$Dg(x)Y_b(x) = f(x)Y_{\tilde{b}}(g(x)) + h(x)\partial_{12} \quad (3.1)$$

with certain function coefficients  $f$  and  $h$  such that  $f |0 \neq 0$ . And secondly, that  $g_{12}^{12} |0 \neq 0$ . That last information will not be used; in the occurrence, it follows easily from the previous data. When comparing coefficients at  $\partial_l$  on the both sides in (3.1), we will say: taking scalar equation "l" of (3.1).

The scalar equation "8" of (3.1) implies  $g_6^8 + g_8^8 |0 = f |0$  implies  $g_8^8 |0 = f |0$ , because  $g^8$  sits in the ideal generated by  $x^7$  and  $x^8$ . Now, by differentiating "8" of (3.1) wrt  $x^9$  at 0, one gets  $f g_9^9 |0 = g_8^8 |0$ , because  $f$  is given by "6" of (3.1) and depends only on  $x^1, \dots, x^7$ . Hence  $g_9^9 |0 = 1$  and also

$$f g_{10}^{10} |0 = 1, \quad (3.2)$$

because  $f g_{10}^{10} |0 = g_9^9 |0$  by differentiating "9" of (3.1) wrt  $x^{10}$  at 0. We keep drawing conclusions from (3.1). Its scalar equation "10" evaluated at 0 says

$$g_6^{10} + g_8^{10} + a g_{10}^{10} |0 = a f |0 .$$

The sum of first two summands on the LHS of this equation vanishes in view of (2.3) and arguments similar to those in [M2], p. 29:

$$g_6^{10} + g_8^{10} | 0 = 0. \quad (3.3)$$

This implies  $g_{10}^{10} | 0 = f | 0$  which can be coupled with (3.2). In this way

$$f | 0 = g_{10}^{10} | 0 = \pm 1. \quad (3.4)$$

Now we can differentiate "11" of (3.1) wrt  $x^{11}$  at 0, obtaining  $fg_{11}^{11} | 0 = g_{10}^{10} | 0$ , hence also, in view of (3.4),

$$g_{11}^{11} | 0 = 1. \quad (3.5)$$

Passing to the main quantities, we evaluate at 0 scalar equation "11" of (3.1),

$$g_6^{11} + g_8^{11} + a g_{10}^{11} + b g_{11}^{11} | 0 = \tilde{b} f | 0. \quad (3.6)$$

On the LHS of this equation there happens something important.

**Proposition 3.2.**  $g_6^{11} + g_8^{11} + a g_{10}^{11} | 0 = 0.$

A skeleton of proof (which altogether needs 10 pages) of this proposition is given in the next section.

With Prop. 3.2 taken for granted and upon using equalities (3.4) and (3.5), relation (3.6) boils down to precisely  $b = \pm \tilde{b}$ .

What remains to be proved in Thm. 3.1 is to construct a conjugacy of two germs  $D^{10}$  with the same value  $a$  and with the opposite values of the last parameter, i. e. with  $b$  and  $-b$ . It is an exercise (which is to be done backwards from  $\omega^{10}$  to  $\omega^1$ ) that the following *reflection*  $\Phi$  does that job, universally for all values of  $b$ :

$$\begin{aligned} \Phi(x^1, x^2, \dots, x^{12}) = \\ (x^1, -x^2, -x^3, -x^4, -x^5, -x^6, x^7, -x^8, x^9, -x^{10}, x^{11}, -x^{12}) \end{aligned}$$

Yet, needless to say, the brunt of the proof resides in justifying Proposition 3.2.

## 4 Skeleton of proof of Proposition 3.2

We have not yet used the fact that  $g$  conjugates, one to the other, two KR pseudo-normal forms with no additive constants at  $x^{10}$ . In fact, scalar equation "9" of (3.1) evaluated at 0 reads

$$g_6^9 + g_8^9 | 0 = 0. \quad (4.1)$$

In equation "8" of (3.1), the function  $g^9$  gets expressed by  $f$  and  $g^8$ . In turn, in equation "7" of (3.1),  $g^8$  gets expressed by  $f$  and  $g^7$ . In consequence, it is not long to verify that equation (4.1) boils down to

$$F_6 | 0 = f f_6 | 0 \quad (4.2)$$

(remember also that (3.4) holds). At this point the proof takes an unexpected turn and returns to the – already used – equation (3.3). That sum of partial derivatives of  $g^{10}$  at 0 vanishes due to the symmetry-dimension-related arguments. One can write this vanishing in terms of  $f$  and  $F$  as well. After a longer computation (3 handwritten pages) using also (4.2), equation (3.3) boils down to

$$(F_6)^2 + 2F_{66} | 0 = 0. \quad (4.3)$$

Now the time comes to likewise reduce the combination of partials of  $g^{11}$  showing up in Prop. 3.2, using under way the relations (4.2)–(4.3). The outcome of 6 handwritten pages of computations reads<sup>4</sup>

$$\begin{aligned} g_6^{11} + g_8^{11} + a g_{10}^{11} | 0 = \\ - 5f f_5 + 10F_5 - 4aF_6 + 15(F_6)^3 + 10F_{666} + 30F_{67} - 15f f_{67} - 5f f_{666} | 0. \end{aligned}$$

We note that, by "5" and "6" of (3.1), the function  $fF$  depends only on  $x^1, \dots, x^6$  and is affine in  $x^6$ , while  $f$  can be expressed in terms of the function  $G$  and is affine in  $x^7$ . These facts suffice to continue the process of reduction, and to obtain

$$\begin{aligned} g_6^{11} + g_8^{11} + a g_{10}^{11} | 0 \\ = \pm \left( -60G_5 - 8aG_6 + 120(G_6)^3(1 + 4G) - 60G_{666} \right) | 0 \end{aligned}$$

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<sup>4</sup> Equalities (4.2) and (4.3) are purely algebraic relations which can be used in more than one way. So the result of simplifications might be written down in different forms, too.



(the  $\pm 1$  factor on the RHS is  $f|0 = G|0 = \pm 1$ ). Having gone so far, we note that  $G = \frac{fFG}{fF}$ , hence  $G$  can be expressed in terms of  $g^1$  (the numerator) and  $g^5$  (the denominator). Remembering also that  $g^5$  is, by "4" of (3.1), affine in  $x^5$ , so that the derivative  $g_{55}^5$  vanishes, after standard computations we get

$$\begin{aligned} g_6^{11} + g_8^{11} + a g_{10}^{11} |0 & \quad (4.4) \\ & = \pm 4 \left( 2a g_1^1 g_1^5 - 30 (g_1^1 g_1^5)^3 (1 + 4g_5^5) + 90 g_1^1 (g_1^5)^3 \right) |0. \end{aligned}$$

Only at this moment we start to see something. On the RHS of (4.4) there is a common factor  $g_1^1 g_1^5 |0$ , and the first subfactor in it,  $g_1^1 |0 = fFG|0$ , clearly does not vanish (in fact it equals 1). So how about the second subfactor  $g_1^5 |0$ ? The gist of the matter is that it does vanish.

**Lemma 4.1.**  $g_1^5 |0 = 0$ .

*Proof of Lemma 4.1.* We look back at the relation (4.2) and try to express it in terms of  $g^5$ , if possible. To this end we write  $F = \frac{fF}{f}$ , knowing that the numerator (denominator) is given by equation "5" ("6") of (3.1). In the outcome, not yet decisive, (4.2) gets reduced to

$$\frac{g_1^5}{G} - 2 \frac{g_5^5}{G^2} G_6 |0 = 2G G_6 |0.$$

Recall at this moment that

$$\frac{g_1^1}{g_5^5} |0 = G|0 = f|0 = fF|0 = g_5^5 |0,$$

implying  $g_1^1 |0 = G^2 |0 = 1$ . With this information,  $G_6 |0 = -g_1^5 |0$  and the eventual simplification of (4.2), or: final simplification of (4.1), reads

$$g_1^5 g_5^5 - 2g_5^5 (-g_1^5) |0 = 2g_5^5 (-g_1^5) |0.$$

We know that  $g_5^5 |0 \neq 0$ , hence  $g_1^5 |0 = 0$ .  $\square$

In view of (4.4) and Lemma 4.1,  $g_6^{11} + g_8^{11} + a g_{10}^{11} |0 = 0$ . Prop. 3.2 is now proved.

The (skeleton of) proof of Theorem 3.1 is finished.

## References

- [Is1] G. Ishikawa; *Classifying singular Legendre curves by contactomorphisms*; J. Geom. Physics **52** (2004), 113 – 126.
- [Is2] —; *Singular Legendre curves and Goursat systems*. International Conference on Differential Equations and Dynamical Systems, Suzdal (Russia) 2004 (pp. 266–268 in Abstracts).
- [J] F. Jean; *The car with  $n$  trailers: characterisation of the singular configurations*; ESAIM: COCV **1** (1996), 241 – 266 (electronic).
- [KR] A. Kumpera, C. Ruiz; *Sur l'équivalence locale des systèmes de Pfaff en drapeau*; in: F. Gherardelli (ed.), *Monge–Ampère Equations and Related Topics*, Ist. Alta Math., Rome 1982, 201 – 248.
- [MonZ] R. Montgomery, M. Zhitomirskii; *Geometric approach to Goursat flags*; Ann. Inst. H. Poincaré – AN **18** (2001), 459 – 493.
- [M1] P. Mormul, *Contact hamiltonians distinguishing locally certain Goursat systems*; in: *Poisson Geometry*, Banach Center Publications **51**, Warsaw 2000, 219 – 230.
- [M2] —; *Goursat distributions with one singular hypersurface – constants important in their Kumpera-Ruiz pseudo-normal forms*. Preprint N° 185 Labo. de Topologie, Univ. de Bourgogne, Dijon (June 1999).
- [M3] —, *Examples of exotic moduli in local classification of Goursat flags*; Univ. Iagel. Acta Mathematica **38** (2000), 17 – 28.
- [Z] M. Zhitomirskii; *Germes of integral curves in contact 3-space, plane and space curves*. Preprint NI00043-SGT (December 2000), Newton Institute, Cambridge.

# The Euler characteristic of a link of a set defined by a Noetherian family of analytic functions

Aleksandra Nowel<sup>1</sup>

## 1 Motivation

**Sullivan** (1971):

*If  $X$  is a real algebraic set in  $\mathbb{R}^n$  and  $x \in X$ , then  $\chi(\text{lk}(x, X))$  is even.*

**Parusiński–Szafraniec** (1997), **Coste–Kurdyka** (1998):

*For any regular morphism  $\phi : X \rightarrow W$  of real algebraic sets there exist real polynomials  $g_1, \dots, g_s$  on  $W$  such that for every  $w \in W$*

$$\chi(\phi^{-1}(w)) = \text{sgn } g_1(w) + \dots + \text{sgn } g_s(w).$$

Some properties used in the proof of this result come out of the Noetherianity of the ring of real polynomials. Does it work in "non-polynomial" situations?

## 2 Problem

$\Omega$  – a semianalytic compact subset of  $\mathbb{R}^n$

$\mathcal{A}(\Omega)$  – the algebra of real analytic functions defined in an open neighbourhood of  $\Omega$

$\mathcal{F} \subset \mathcal{A}(\Omega)$  – a family of real analytic functions

$Y_\omega = \left( \bigcap_{f \in \mathcal{F}} f^{-1}(0) \right)_\omega$  – the representative of the germ at  $\omega$

$X_\omega = \{x \mid x + \omega \in Y_\omega\}_0$  – the representative of the germ at the origin

$L_\omega = \text{lk}(0, X_\omega) = (S_\epsilon^{n-1} \cap X_\omega)$  – the link of  $X_\omega$  at the origin

RESULT: there exist  $v_1, \dots, v_s \in \mathcal{A}(\Omega)$  such that

$$\frac{1}{2}\chi(L_\omega) = \sum_{i=1}^s \text{sgn } v_i(\omega).$$

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### 3 Noetherian Families

Let  $A$  be an algebra over  $\mathbb{R}$  and  $\Gamma$  – a subset of the maximal spectrum  $SM(A)$  of  $A$ .

In  $\Gamma$  we have the topology induced from  $SM(A)$ , i.e.  $F$  is closed in  $\Gamma$  if  $F = \{\gamma \in \Gamma \mid B \subset \gamma\}$  for some  $B \subset A$ .

We assume that  $A$  and  $\Gamma$  satisfy the following conditions:

- (a) for all  $\gamma \in \Gamma$  the canonical map  $\mathbb{R} \longrightarrow A/\gamma$  is an isomorphism,
- (b)  $\Gamma$  equipped with the topology of  $SM(A)$  is a Noetherian space (i.e. every decreasing sequence of closed sets in  $\Gamma$  is stationary – any closed set in  $\Gamma$  is the union of finitely many irreducible closed sets).

Assume  $a \in A, \gamma \in \Gamma$  and  $S \subset A$ . We denote

$a(\gamma) \in \mathbb{R}$  – the image of  $a$  under the map  $A \longrightarrow A/\gamma \cong \mathbb{R}$

$V(S) = \{\gamma \in \Gamma : a(\gamma) = 0 \text{ for all } a \in S\}$

closed sets in  $\Gamma \equiv$  sets  $V(S), S \subset A$

Let  $x = (x_1, \dots, x_n)$ .

$A[[x]]$  (resp.  $\mathbb{R}[[x]]$ ) – the ring of formal power series in  $x$  with coefficients in  $A$  (resp. in  $\mathbb{R}$ )

$\mathbb{R}\{x\}$  – the ring of formal power series which are convergent in some neighbourhood of the origin

$A_c[[x]]$  – the subring of  $A[[x]]$  consisting of  $\sum_{\alpha} a_{\alpha} x^{\alpha}$  such that  $\sum_{\alpha} a_{\alpha}(\gamma) x^{\alpha} \in \mathbb{R}\{x\}$  for each  $\gamma \in \Gamma$

Let  $\gamma \in \Gamma$  and  $f = \sum_{\beta} a_{\beta} x^{\beta} \in A[[x]]$ . We denote

$f_{\gamma} = \sum_{\beta} a_{\beta}(\gamma) x^{\beta} \in \mathbb{R}[[x]]$

If  $N$  is the ideal in  $A[[x]]$  generated by  $f_i$  then

$N_{\gamma}$  – the ideal in  $\mathbb{R}[[x]]$  generated by  $f_{i,\gamma}$ .

**Definition** A collection  $\mathcal{N}$  of ideals of  $\mathbb{R}[[x]]$  (resp. of  $\mathbb{R}\{x\}$ ) is called a *Noetherian family (parameterized by  $(A, \Gamma)$ )* if there exists a couple  $(A, \Gamma)$  satisfying the conditions (a) and (b) given above, and an ideal  $N \subset A[[x]]$  (resp.  $N \subset A_c[[x]]$ ) such that  $\mathcal{N} = (N_{\gamma})_{\gamma \in \Gamma}$ .

**Definition** The *Lojasiewicz exponent*  $\mathcal{L}(I)$  of an ideal  $I$  in  $\mathbb{R}\{x\}$  generated by  $f_1, \dots, f_p$  is the infimum of the set of such  $\alpha > 0$  for which there exists such  $c > 0$  that

$$\sum_{i=1}^p |f_i(x)| \geq c \varrho(x, V(I))^\alpha$$

in some neighbourhood of the origin.

**Theorem 3.1.** (El Khadiri, Tougeron 1984) *Let  $(I_\gamma)_{\gamma \in \Gamma}$  be a Noetherian family of ideals in  $\mathbb{R}\{x\}$ . Then the family of Lojasiewicz exponents  $\mathcal{L}(I_\gamma)$  of the ideals  $I_\gamma$  is bounded.*

$\Omega$  is a Noetherian space with the topology induced from  $SM(\mathcal{A}(\Omega))$  (by identifying  $\omega \in \Omega$  with the ideal  $p_\omega = \{f \in \mathcal{A}(\Omega) \mid f(\omega) = 0\}$ ,  $\{\bigcap_{f \in B} f^{-1}(0)\}_{B \subset \mathcal{A}(\Omega)}$  is the family of closed sets in  $\Omega$ ).

The pair  $(\mathcal{A}(\Omega), \Omega)$  satisfies conditions (a) and (b). Since  $\Omega$  is a Noetherian space, for every closed subset  $D$  of  $\Omega$  there exist  $f_1, \dots, f_p \in \mathcal{A}(\Omega)$  such that  $D = \bigcap_{i=1}^p f_i^{-1}(0) \cap \Omega$ , so  $D$  is an intersection of  $\Omega$  and an analytic set.

Properties of Noetherian families imply

$$\exists_{0 \leq h \in \mathcal{A}(\Omega)_c[[x]]} \forall_{\omega \in \Omega} X_\omega = V_0(h_\omega).$$

A closed (with respect to the topology induced from  $SM(\mathcal{A}(\Omega))$ ) subset of  $\Omega$  is *irreducible* if it is not a union of its two proper closed subsets. Every closed subset  $D$  of a Noetherian space  $\Omega$  has a decomposition into finitely many irreducible components, i.e.  $D = \bigcup_{i=1}^k D_i$ , where every  $D_i$  is a closed irreducible subset of  $D$  and  $D_i \not\subset \bigcup_{j \neq i} D_j$ .

We will say that a function  $g : \Omega \rightarrow \mathbb{R}$  is a *sum of signs of analytic functions* if there exist  $g_1, g_2, \dots, g_s \in \mathcal{A}(\Omega)$  s. t.  $g(\omega) = \sum_{i=1}^s \text{sgn } g_i(\omega)$ .

**Lemma 3.2.** *Assume that for each closed irreducible subset  $D \subset \Omega$  there exists a proper closed subset  $\Sigma \subset D$  such that  $g$  restricted to  $D \setminus \Sigma$  is a sum of signs of a.f. Then  $g$  is a sum of signs of a.f. on  $\Omega$ .*

*Proof.*

1. On  $D \setminus \Sigma$  we have

$$g(\omega) = \sum_{i=1}^s \text{sgn } f_i(\omega).$$

2. By induction on the number of irreducible components for a closed subset  $D' \subset \Omega$  there exists a proper closed subset  $\Sigma' \subset D'$  such that on  $D' \setminus \Sigma'$

$$g(\omega) = \sum_{i=1}^{s'} \operatorname{sgn} q_i(\omega).$$

3. From above we have

$$g(\omega) = \sum_{i=1}^{s'} \operatorname{sgn} q_i(\omega)$$

on  $\Omega \setminus \Sigma$ .

Repeat the construction on  $\Sigma'$ :

$$g(\omega) = \sum_{i=1}^{s''} \operatorname{sgn} u_i(\omega)$$

on  $\Sigma' \setminus \Sigma''$ .

We obtain:

$$\Omega \supset \Sigma' \supset \Sigma'' \supset \dots$$

Noetherianity implies this sequence being stationary, i.e. there exists  $k$  such that  $\Sigma^{(k)} = \emptyset$ .

After some gluing we obtain

$$g(\omega) = \sum_{i=1}^p \operatorname{sgn} g_i(\omega) \text{ on } \Omega.$$

## 4 Solving the problem

Arguments similar to those of Parusiński and Szafraniec imply some facts:

1.  $F_\omega = (F_\omega^1, \dots, F_\omega^n) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$  – analytic germs at the origin such that

$$\forall_{1 \leq i \leq n} \exists_{f_i \in \mathcal{A}(\Omega)_c[[x]]} \forall_{\omega \in \Omega} F_\omega^i(x) = f_i(\omega, x)$$

and  $0 \in \mathbb{R}^n$  is isolated in  $F_\omega^{-1}(0)$  for all  $\omega \in D$ .

For each closed irreducible subset  $D \subset \Omega$  there exist  $k \gg 1$ ,  $a \in \mathbb{R}$ , a proper closed  $\Sigma \subset D$ , and  $G_\omega(x) = F_\omega(x) + a(x_1^k, \dots, x_n^k)$ ,  $\omega \in D$ , such that  $G_\omega$  have the same property as  $F_\omega$ , have algebraically isolated zero at the origin, and  $\deg_0(G_\omega) = \deg_0(F_\omega)$ .

$\deg_0(F_\omega)$  is a sum of signs of a.f. on  $D \setminus \Sigma$ .

By Lemma 3.2  $\deg_0(F_\omega) = \deg_0(G_\omega)$  is a sum of signs of a.f. on  $\Omega$ .

2. If  $f \in \mathcal{A}(\Omega)$  then

$$\chi(S_\epsilon \cap \{f(x + \omega) \leq 0\}) = \chi(S_{\omega, \epsilon} \cap \{f \leq 0\}) = 1 - \deg_0 \nabla g_\omega$$

for some  $g_\omega \in \mathbb{R}\{x\}$  having an isolated critical point at the origin.

Since  $\nabla g_\omega$  satisfies the assumptions of 1.,  $\chi(S_{\omega, \epsilon} \cap \{f \leq 0\})$  is a sum of signs of a.f.

**Lemma 4.1.** *If  $f \in \mathcal{A}(\Omega)$  then*

$$\frac{1}{2}(\chi(S_{\omega, \epsilon}^{n-1} \cap \{f \geq 0\}) \pm \chi(S_{\omega, \epsilon}^{n-1} \cap \{f \leq 0\}))$$

*is a sum of signs of a.f.*

*Proof.* Let  $g(\omega, t) = tf(\omega)$ ,  $\omega$  belongs to some neighbourhood of  $\Omega$ ,  $t \in [-1; 1]$ . The set  $\Omega \times [-1, 1]$  is compact and semianalytic, so  $g \in \mathcal{A}(\Omega \times [-1, 1])$ .

One can show that for  $\epsilon$  sufficiently small

$$\chi(S_{\omega, \epsilon}^{n-1} \cap \{f \geq 0\}) = 2 - \chi(S_{(\omega, t), \epsilon}^n \cap \{g \geq 0\})$$

$$\chi(S_{\omega, \epsilon}^{n-1} \cap \{f \leq 0\}) = 2 - \chi(S_{(\omega, -t), \epsilon}^n \cap \{g \geq 0\})$$

We have

$$\chi(S_{(\omega, t), \epsilon}^n \cap \{g \geq 0\}) = \sum_{i=1}^s \operatorname{sgn} g_i(\omega, t)$$

$$\frac{1}{2}(\chi(\{f \geq 0\} \cap S_{\omega, \epsilon}^{n-1}) - \chi(\{f \leq 0\} \cap S_{\omega, \epsilon}^{n-1})) =$$

$$= \frac{1}{2} \lim_{t \rightarrow 0^+} \sum_{i=1}^s (\operatorname{sgn} g_i(\omega, -t) - \operatorname{sgn} g_i(\omega, t)).$$

Fix an irreducible component  $D_j \subset \Omega$ . We have

$$g_i(\omega, t) = t^{k_i} h_i(\omega, t),$$

where  $h_i \in \mathcal{A}(\Omega \times [-1; 1])$ ,  $h_i \not\equiv 0$  on  $\Delta_j \times \{0\}$ . Let

$$\Sigma := \{\omega \in \Delta_j \mid \forall_{i=1, \dots, s} h_i(\omega, 0) = 0\}$$

and let  $h'_i(\omega) = -h_i(\omega, 0)$ ,  $k_i$  odd,  $h_i(\omega) = 0$ ,  $k_i$  even. Then on  $\Delta_j \setminus \Sigma$

$$\frac{1}{2} \lim_{t \rightarrow 0^+} \sum_{i=1}^s (\operatorname{sgn} g_i(\omega, -t) - \operatorname{sgn} g_i(\omega, t)) = \sum_{i=1}^s \operatorname{sgn} h'_i(\omega).$$

Clearly  $h'_i \in \mathcal{A}(\Omega)$ .

By Lemma 3.2 it is also a sum of signs of a.f. on  $\Omega$ .

**Theorem 4.2.**  $\frac{1}{2} \chi(\underbrace{S_\epsilon^{n-1} \cap X_\omega}_{L_\omega})$  is a sum of signs of a.f. on  $\Omega$ .

*Proof.*

$$\begin{aligned} \frac{1}{2} \chi(L_\omega) &= \frac{1}{2} \chi(S_\epsilon^{n-1} \cap V_0(h_\omega)) = \\ &= \frac{1}{2} [\chi(S_\epsilon^{n-1} \cap \{h_\omega \leq 0\}) + \chi(S_\epsilon^{n-1} \cap \{h_\omega \geq 0\}) - \chi(S_\epsilon^{n-1})] = \\ &= \sum_{i=1}^s \operatorname{sgn} h_i(\omega) - \frac{1 + (-1)^{n-1}}{2} \end{aligned}$$

on  $\Omega$ .

## 5 Corollaries

**Corollary 5.1.**  $\chi(L_\omega)$  is even for  $\omega \in \Omega$ .

A function  $f : \Omega \rightarrow \mathbb{Z}$  is *semianalytically constructible* if it admits a representation as a finite sum

$$f = \sum_i m_i \mathbf{1}_{X_i},$$



where the  $m_i$ 's are integers, the  $X_i$ 's are semianalytic, and  $\mathbf{1}_{X_i}$  denotes the characteristic function of the set  $X_i$ .

We can define the Euler integral and the link of  $f$ :

$$\int_{\Omega} f = \sum_i m_i \chi(X_i),$$

$$\Lambda f(x) = \int_{\Omega \cap S_{x,\epsilon}^{n-1}} f,$$

where  $\epsilon$  is sufficiently small.

If  $f$  is a sum of signs of analytic functions  $f_i$ , then  $f$  is semianalytically constructible and:

$$\int_{\Omega} f = \sum_{i=1}^s (\chi(A_i) - \chi(B_i)),$$

where  $A_i = \{f_i \geq 0\}$ ,  $B_i = \{f_i \leq 0\}$ ,

$$\Lambda f(\omega) = \sum_{i=1}^s (\chi(A_i \cap S_{\omega,\epsilon}^{n-1}) - \chi(B_i \cap S_{\omega,\epsilon}^{n-1}))$$

for  $\epsilon$  sufficiently small.

**Corollary 5.2.** *The function  $\frac{1}{2}\Lambda f$  is integer-valued and it is a sum of signs of analytic functions.*

If  $a \neq 0 \neq b$ , then  $\text{sgn } a + \text{sgn } b = 1 + \text{sgn } ab \pmod{4}$ .

**Corollary 5.3.** *There exist a proper closed subset  $\Sigma \subset \Omega$ , an integer  $\mu$  and an analytic function  $v \in \mathcal{A}(\Omega)$  such that  $v$  is  $\neq 0$  on  $\Omega \setminus \Sigma$ , and for  $\omega \in \Omega \setminus \Sigma$*

$$\frac{1}{2}\chi(L_{\omega}) = \mu + \text{sgn } v(\omega) \pmod{4}.$$

## References

- [1] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry*, Berlin: Springer – Verlag (1998)
- [2] M. Coste, K. Kurdyka, *On the link of a stratum in a real algebraic set.* Topology 31, No.2 (1992), 323–336.

- [3] M. Coste, K. Kurdyka, *Le discriminant d'un morphisme de variétés algébriques réelles*, Topology 37, No.2 (1998), 393–399
- [4] D. Eisenbud, H. Levine, *An algebraic formula for the degree of a  $C^\infty$  map germ*, Ann. of Math. 106 (1977), 19–44
- [5] A. El Khadiri, J.–Cl. Tougeron, *Familles noethériennes de modules sur  $k[[x]]$  et applications*, (preprint, 1984)
- [6] A. El Khadiri, J.–Cl. Tougeron, *Familles noethériennes de modules sur  $k[[x]]$  et applications*, Bull. Sci. math. 120 (1996), 253–292
- [7] G. M. Khimshiashvili, *On the local degree of a smooth map*, Soobshch. Akad. Nauk. Gruz. SSR, 85 (1977), 309–311
- [8] S. Łojasiewicz, *Ensembles semi-analytiques*, IHES 1965
- [9] C. McCrory, A. Parusiński, *Algebraically constructible functions*, Ann. Scient. Ecole Norm. Sup. (4) 30 (1997), 527–552
- [10] C. McCrory, A. Parusiński, *Complex monodromy and the topology of real algebraic sets*, Compositio Math. 106, (1997), 211–233
- [11] R. Narasimhan, *Introduction to the theory of analytic spaces*, Lecture Notes in Mathematics, No. 25 Springer – Verlag, Berlin – New York 1966
- [12] A. Nowel, *Topological invariants of analytic sets associated with Noetherian families*, Ann. Inst. Fourier (Grenoble) 55, no. 2 (2005), 549–571
- [13] A. Parusiński, Z. Szafraniec, *Algebraically constructible functions and signs of polynomials*, Manuscripta Math. 93, no.4 (1997), 443–456
- [14] A. Parusiński, Z. Szafraniec, *On the Euler characteristic of fibres of real polynomial maps*, Singularities Symposium – Łojasiewicz 70 (Kraków, 1996; Warsaw, 1996), 175–182, Banach Center Publ., 44, Polish Acad. Sci., Warsaw, 1998
- [15] D. Sullivan, *Combinatorial invariants of analytic spaces*, Proc. Liverpool Singularities Symposium I, Lecture Notes in Math. 192, Springer – Verlag (1971), 127–138

- [16] Z. Szafraniec, *On the Euler characteristic of analytic and algebraic sets*, Topology 25, No.4 (1986), 411–414
- [17] Z. Szafraniec, *On the Euler characteristic of complex algebraic varieties*, Math. Ann. 280 (1988), 177–183
- [18] J.-Cl. Tougeron, *Idéaux de fonctions différentiables*, Springer – Verlag Berlin Heidelberg New York, 1972
- [19] H. Whitney, *Tangents to an analytic variety*, Ann. of Math. 81 (1965), 496–549

## Drapeau Theorem for differential systems

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### Abstract

Generalizing the theorem for Goursat flags, we will characterize those flags which are obtained by “Rank 1 Prolongation” from the space of 1-jets for 1 independent and  $m$  dependent variables.

## 1 Introduction

This paper is concerned with the Drapeau Theorem for differential systems. By a differential system  $(R, D)$  we mean a distribution  $D$  on a manifold  $R$ , i.e.  $D$  is a subbundle of the tangent bundle  $T(R)$ . The *derived system*  $\partial D$  of  $D$  is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where  $\mathcal{D} = \Gamma(D)$  denotes the space of sections of  $D$ . In general  $\partial D$  is obtained as a subsheaf of the tangent sheaf of  $R$  (for the precise argument see e.g. [12], [3]). Moreover higher derived systems  $\partial^i D$  are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put  $\partial^0 D = D$  by convention. In this paper a differential system  $(R, D)$  is called regular if  $\partial^i D$  are subbundles of  $T(M)$  for every  $i \geq 1$ .

We say that  $(R, D)$  is an  $m$ -flag of length  $k$  if  $(R, D)$  is regular and has a derived length  $k$ , i.e.  $\partial^k D = T(R)$ ;

$$D \subset \partial D \subset \dots \subset \partial^{k-2} D \subset \partial^{k-1} D \subset \partial^k D = T(R),$$

such that  $\text{rank } D = m + 1$  and  $\text{rank } \partial^i D = \text{rank } \partial^{i-1} D + m$  for  $i = 1, \dots, k$ . In particular  $\dim R = (k + 1)m + 1$ .

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In particular  $(R, D)$  is called a *Goursat flag* (un drapeau de Goursat) of length  $k$  when  $m = 1$ . Historically, by Engel, Goursat and Cartan, it is known that a Goursat flag  $(R, D)$  of length  $k$  is locally isomorphic, at a generic point, to the canonical system  $(J^k(M, 1), C^k)$  on the  $k$ -jet spaces of 1 independent and 1 dependent variable (for the definition of the canonical system  $(J^k(M, 1), C^k)$ , see §2). The characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [2] for the first order systems and in [12] and [13] for higher order systems for  $n$  independent and  $m$  dependent variables. However, it was first explicitly exhibited by A. Giaro, A. Kumpera and C. Ruiz in [6] that a Goursat flag of length 3 has singularities and the research of singularities of Goursat flags of length  $k$  ( $k \geq 3$ ) began as in [9]. To this situation, R. Montgomery and M. Zhitomirskii constructed the “Monster Goursat manifold” by successive application of the “Cartan prolongation of rank 2 distributions” ([4]) to a surface and showed that every germ of a Goursat flag  $(R, D)$  of length  $k$  appears in this “Monster Goursat manifold” in [8], by first exhibiting the following Sandwich Lemma for  $(R, D)$ :

$$\begin{array}{ccccccc} D & \subset & \partial D & \subset \dots \subset & \partial^{k-2} D & \subset \partial^{k-1} D \subset & \partial^k D = T(R) \\ \cup & & \cup & & \cup & & \\ \text{Ch}(D) & \subset & \text{Ch}(\partial D) & \subset \dots \subset & \text{Ch}(\partial^2 D) & \subset \dots \subset & \text{Ch}(\partial^{k-1} D) \end{array}$$

where  $\text{Ch}(\partial^i D)$  is the Cauchy characteristic system of  $\partial^i D$  and  $\text{Ch}(\partial^i D)$  is a subbundle of  $\partial^{i-1} D$  of corank 1 for  $i = 1, \dots, k-1$ . Here the *Cauchy Characteristic System*  $\text{Ch}(C)$  of a differential system  $(R, C)$  is defined by

$$\text{Ch}(C)(x) = \{X \in C(x) \mid X]d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \text{ for } i = 1, \dots, s\},$$

where  $C = \{\omega_1 = \dots = \omega_s = 0\}$  is defined locally by defining 1-forms  $\{\omega_1, \dots, \omega_s\}$ . Moreover, after [8], P. Mormul defined the notion of a *special  $m$ -flag* of length  $k$  for  $m \geq 2$  to characterize those  $m$ -flags which are obtained by successive application of the “generalized Cartan prolongation” to the space of 1-jets of 1 independent and  $m$  dependent variables.

The main purpose of this paper is first to clarify the procedure of “Rank 1 Prolongation” of an arbitrary differential system  $(R, D)$  of rank  $m+1$ , and to give good criteria for an  $m$ -flag of length  $k$  to be special, i.e. to be locally isomorphic to the  $k$ -th Rank 1 Prolongation  $(P^k(M), C^k)$  of a manifold  $M$

of dimension  $m + 1$ . More precisely we will show for an  $m$ -flag of length  $k$  and for  $m \geq 3$ :

**Corollary 5.8.** *An  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 3$  is locally isomorphic to  $(P^k(M), C^k)$  if and only if  $\partial^{k-1}D$  is of Cartan rank 1, and moreover for  $m \geq 4$ , if and only if  $\partial^{k-1}D$  is of Engel rank 1.*

Here, the *Cartan rank* of  $(R, C)$  is the smallest integer  $\rho$  such that there exist 1-forms  $\{\pi^1, \dots, \pi^\rho\}$ , which are independent modulo  $\{\omega_1, \dots, \omega_s\}$  and satisfy

$$d\alpha \wedge \pi^1 \wedge \dots \wedge \pi^\rho \equiv 0 \pmod{\omega_1, \dots, \omega_s} \quad \text{for } \forall \alpha \in \mathcal{C}^\perp = \Gamma(C^\perp),$$

where  $C = \{\omega_1 = \dots = \omega_s = 0\}$ . Furthermore the *Engel (half) rank* of  $(R, C)$  is the smallest integer  $\rho$  such that

$$(d\alpha)^{\rho+1} \equiv 0 \pmod{\omega_1, \dots, \omega_s} \quad \text{for } \forall \alpha \in \mathcal{C}^\perp,$$

Moreover we will show for an  $m$ -flag of length  $k$  and for  $m \geq 2$ :

**Corollary 6.3.** *An  $m$ -flag  $(R, D)$  of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$  if and only if there exists a completely integrable subbundle  $F$  of  $\partial^{k-1}D$  of corank 1.*

For this purpose, we will first review the geometric construction of jet spaces in §2 and clarify the procedure of Rank 1 Prolongation in §3. In §4, we will analyze the notion of a special  $m$ -flag of length  $k$  and reestablish the local characterization of  $(P^k(M), C^k)$  by utilizing the Realization Lemma [12]. In §5 and §6, we will show the above criteria (the Drapeau Theorem) for an  $m$ -flag of length  $k$ .

## 2 Geometric construction of jet spaces

In this section, we will briefly recall the geometric construction of jet bundles in general, following [12] and [13], which is our basis for the later considerations.

Let  $M$  be a manifold of dimension  $m + n$ . Fixing the number  $n$ , we form the space of  $n$ -dimensional *contact elements* to  $M$ , i.e. the Grassmann bundle  $J(M, n)$  over  $M$  consisting of  $n$ -dimensional subspaces of tangent spaces to  $M$ . Namely,  $J(M, n)$  is defined by

$$J(M, n) = \bigcup_{x \in M} J_x, \quad J_x = \text{Gr}(T_x(M), n),$$

where  $\text{Gr}(T_x(M), n)$  denotes the Grassmann manifold of  $n$ -dimensional subspaces in  $T_x(M)$ . Let  $\pi : J(M, n) \rightarrow M$  be the bundle projection. The *canonical system*  $C$  on  $J(M, n)$  is, by definition, the differential system of codimension  $m$  on  $J(M, n)$  defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M, n)) \mid \pi_*(v) \in u\} \subset T_u(J(M, n)) \xrightarrow{\pi_*} T_x(M),$$

where  $\pi(u) = x$  for  $u \in J(M, n)$ .

Let us describe  $C$  in terms of a canonical coordinate system in  $J(M, n)$ . Let  $u_o \in J(M, n)$ . Let  $(x_1, \dots, x_n, z^1, \dots, z^m)$  be a coordinate system on a neighborhood  $U'$  of  $x_o = \pi(u_o)$  such that  $dx_1, \dots, dx_n$  are linearly independent when restricted to  $u_o \subset T_{x_o}(M)$ . We put

$$U = \{u \in \pi^{-1}(U') \mid dx_1|_u, \dots, dx_n|_u \text{ are linearly independent}\}.$$

Then  $U$  is a neighborhood of  $u_o$  in  $J(M, n)$ . Here  $dz^\alpha|_u$  is a linear combination of  $dx_i|_u$ 's, i.e.  $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$ . Thus, there exist unique functions  $p_i^\alpha$  on  $U$  such that  $C$  is defined on  $U$  by the following 1-forms:

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha dx_i \quad (\alpha = 1, \dots, m),$$

where we identify  $z^\alpha$  and  $x_i$  on  $U'$  with their lifts on  $U$ . The system of functions  $(x_i, z^\alpha, p_i^\alpha)$  ( $\alpha = 1, \dots, m, i = 1, \dots, n$ ) on  $U$  is called a *canonical coordinate system* of  $J(M, n)$  subordinate to  $(x_i, z^\alpha)$ .

$(J(M, n), C)$  is the (geometric) 1-jet space and especially, in case  $m = 1$ , is the so-called contact manifold. Let  $M, \widehat{M}$  be manifolds of dimension  $m + n$  and  $\varphi : M \rightarrow \widehat{M}$  be a diffeomorphism. Then  $\varphi$  induces the isomorphism  $\varphi_* : (J(M, n), C) \rightarrow (J(\widehat{M}, n), \widehat{C})$ , i.e. the differential map  $\varphi_* : J(M, n) \rightarrow J(\widehat{M}, n)$  is a diffeomorphism sending  $C$  onto  $\widehat{C}$ . The reason why the case  $m = 1$  is special is explained by the following theorem of Bäcklund.

**Theorem (Bäcklund)** *Let  $M$  and  $\widehat{M}$  be manifolds of dimension  $m + n$ . Assume  $m \geq 2$ . Then, for an isomorphism  $\Phi : (J(M, n), C) \rightarrow (J(\widehat{M}, n), \widehat{C})$ , there exists a diffeomorphism  $\varphi : M \rightarrow \widehat{M}$  such that  $\Phi = \varphi_*$ .*

The essential part of this theorem is to show that  $F = \text{Ker } \pi_*$  is the covariant system of  $(J(M, n), C)$  for  $m \geq 2$ . Namely an isomorphism  $\Phi$  sends  $F$  onto  $\widehat{F} = \text{Ker } \widehat{\pi}_*$  for  $m \geq 2$ . For the proof, we refer the reader to Theorem 1.4 in [13].

In case  $m = 1$ , it is a well known fact that the group of isomorphisms of  $(J(M, n), C)$ , i.e. the group of contact transformations, is larger than the group of diffeomorphisms of  $M$ . Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number  $m$  of dependent variables is 1 or greater.

(1) Case  $m = 1$ . We should start from a contact manifold  $(J, C)$  of dimension  $2n + 1$ , which is locally a space of 1-jets for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space  $(L(J), E)$  as follows: We consider the Lagrange-Grassmann bundle  $L(J)$  over  $J$  consisting of all  $n$ -dimensional integral elements of  $(J, C)$ :

$$L(J) = \bigcup_{u \in J} L_u \subset J(J, n),$$

where  $L_u$  is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space  $(C(u), d\varpi)$ . Here  $\varpi$  is a local contact form on  $J$ . Namely,  $v \in J(J, n)$  is an integral element if and only if  $v \subset C(u)$  and  $d\varpi|_v = 0$ , where  $u = \pi(v)$ . Let  $\pi : L(J) \rightarrow J$  be the projection. Then the canonical system  $E$  on  $L(J)$  is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where  $\pi(v) = u$  for  $v \in L(J)$ . We have  $\partial E = \pi_*^{-1}(C)$  and  $\text{Ch}(C) = \{0\}$  (cf. [12]). Hence we get  $\text{Ch}(\partial E) = \text{Ker } \pi_*$ , which implies the Bäcklund theorem for  $(L(J), E)$  (cf. [12]).

Now we put

$$(J^2(M, n), C^2) = (L(J(M, n)), E),$$

where  $M$  is a manifold of dimension  $n + 1$ .

(2) Case  $m \geq 2$ . Since  $F = \text{Ker } \pi_*$  is a covariant system of  $(J(M, n), C)$ , we define  $J^2(M, n) \subset J(J(M, n), n)$  by

$$J^2(M, n) = \{n\text{-dim. integral elements of } (J(M, n), C) \text{ transversal to } F\},$$

$C^2$  is defined as the restriction to  $J^2(M, n)$  of the canonical system on  $J(J(M, n), n)$ .

Now the higher order (geometric) jet spaces  $(J^{k+1}(M, n), C^{k+1})$  for  $k \geq 2$  are defined (simultaneously for all  $m$ ) by induction on  $k$ . Namely, for  $k \geq 2$ , we define  $J^{k+1}(M, n) \subset J(J^k(M, n), n)$  and  $C^{k+1}$  inductively as follows:

$$J^{k+1}(M, n) = \{n\text{-dim. integral el. of } (J^k(M, n), C^k) \text{ transv. to } \text{Ker}(\pi_{k-1}^k)_*\},$$



where  $\pi_{k-1}^k : J^k(M, n) \rightarrow J^{k-1}(M, n)$  is the projection. Here we have

$$\text{Ker } (\pi_{k-1}^k)_* = \text{Ch } (\partial C^k),$$

and  $C^{k+1}$  is defined as the restriction to  $J^{k+1}(M, n)$  of the canonical system on  $J(J^k(M, n), n)$ . Then we have ([12],[13])

$$\begin{array}{ccccccc} C^k & \subset \dots \subset & \partial^{k-2} C^k & \subset \partial^{k-1} C^k & \subset \partial^k C^k & = T(J^k(M, n)) \\ & & \cup & & \cup & & \cup \\ \{0\} & = \text{Ch } (C^k) & \subset \text{Ch } (\partial C^k) & \subset \dots \subset \text{Ch } (\partial^{k-1} C^k) & \subset & F \end{array}$$

where  $\text{Ch } (\partial^{i+1} C^k)$  is a subbundle of  $\partial^i C^k$  of corank  $n$  for  $i = 0, \dots, k-2$ , and when  $m \geq 2$ ,  $F$  is a subbundle of  $\partial^{k-1} C^k$  of corank  $n$ . The transversality conditions are expressed as

$$C^k \cap F = \text{Ch } (\partial C^k) \quad \text{for } m \geq 2, \quad C^k \cap \text{Ch } (\partial^{k-1} C^k) = \text{Ch } (\partial C^k) \quad \text{for } m = 1$$

By the above diagram and the rank condition, the jet spaces  $(J^k(M, n), C^k)$  can be characterized as higher order contact manifolds as in [12] and [13].

Here we observe that, if we drop the transversality condition in our definition of  $J^k(M, n)$  and collect all  $n$ -dimensional integral elements, we may have some singularities in  $J^k(M, n)$  in general. However, since every 2-form vanishes on 1-dimensional subspaces, in case  $n = 1$  the integrability condition for  $v \in J(J^{k-1}(M, 1), 1)$  reduces to  $v \subset C^{k-1}(u)$  for  $u = \pi_{k-1}^k(v)$ . Hence in this case we can safely drop the transversality condition in the above construction, as in the next section, which constitutes the key construction for the Drapeau theorem in later considerations.

### 3 Rank 1 Prolongation

Let  $(R, D)$  be a differential system, i.e.  $R$  is a manifold of dimension  $s+m+1$  and  $D$  is a subbundle of  $T(R)$  of rank  $m+1$ . Starting from  $(R, D)$ , we define  $(P(R), \widehat{D})$  as follows (cf. [4]):

$$P(R) = \bigcup_{x \in R} P_x \subset J(R, 1),$$

where

$$P_x = \{1\text{-dim. integral el. of } (R, D)\} = \{u \subset D(x) \mid 1\text{-dim. subspaces}\} \cong \mathbb{P}^m.$$

Let  $p : P(R) \rightarrow R$  be the projection. We define the canonical system  $\widehat{D}$  on  $P(R)$  by

$$\widehat{D}(u) = p_*^{-1}(u) = \{v \in T_u(P(R)) \mid p_*(v) \in u\} \subset T_u(P(R)) \xrightarrow{p_*} T_x(R),$$

where  $p(u) = x$  for  $u \in P(R)$ .

We call  $(P(R), \widehat{D})$  the *prolongation of rank 1* (or *Rank 1 Prolongation* for short) of  $(R, D)$ . Then  $P(R)$  is a manifold of dimension  $2m + s + 1$  and  $\widehat{D}$  is a differential system of rank  $m + 1$ . In case  $(R, D) = (M, T(M))$ , we have  $(P(M), \widehat{D}) = (J(M, 1), C)$ . Moreover

$$J^2(M, 1) \subset P(J(M, 1)) \subset J(J(M, 1), 1)$$

As for the prolongation of rank 1, we have

**Proposition 3.1.** *Let  $(R, D)$  be a differential system of rank  $m + 1$  and let  $(P(R), \widehat{D})$  be the prolongation of rank 1 of  $(R, D)$ . Then  $\widehat{D}$  is of rank  $m + 1$ ,  $\partial\widehat{D} = p_*^{-1}(D)$  and  $\text{Ch}(\widehat{D})$  is trivial. Moreover, if  $\text{Ch}(D)$  is trivial, then  $\text{Ch}(\partial\widehat{D})$  is a subbundle of  $\widehat{D}$  of corank 1.*

*Proof.* Let  $s + m + 1$  be the dimension of  $R$ . For  $x \in R$ , let  $\{\varpi^\beta, \theta^\alpha\}$  ( $\alpha = 1, \dots, m + 1$ ,  $\beta = 1, \dots, s$ ) be a coframe on a neighborhood  $U$  of  $x$  such that

$$D = \{\varpi^1 = \dots = \varpi^s = 0\}.$$

$p^{-1}(U)$  is covered by  $m + 1$  open sets  $\widehat{U}_i = \{v \in p^{-1}(U) \mid \theta^i|_v \neq 0\}$  in  $P(R)$  :

$$p^{-1}(U) = \widehat{U}_1 \cup \dots \cup \widehat{U}_{m+1}.$$

For  $v \in \widehat{U}_i$ ,  $v$  is a 1-dimensional subspace of  $T_x(R)$ ,  $x = p(v)$ . Hence, restricting  $\theta^\alpha$  to  $v$ , we have

$$\theta^\alpha|_v = p_i^\alpha(v)\theta^i|_v \quad \text{for } \alpha = 1, \dots, \check{i}, \dots, m + 1$$

where  $\check{\phantom{i}}$  over a symbol means that the symbol is deleted. These  $p_i^\alpha$  ( $\alpha = 1, \dots, \check{i}, \dots, m + 1$ ) constitute a fiber coordinate system on  $\widehat{U}_i$ .

Now we put

$$\pi_i^\alpha = \theta^\alpha - p_i^\alpha \theta^i \quad \text{for } \alpha = 1, \dots, \check{i}, \dots, m+1.$$

Then we have

$$\widehat{D} = \{p^* \varpi^1 = \dots = p^* \varpi^s = \pi_i^\alpha = 0 \quad (\alpha = 1, \dots, \check{i}, \dots, m+1)\}.$$

Since  $d\varpi^\beta, d\theta^\alpha$  are 2-forms on  $M$ ,  $d\varpi^\beta|_u = 0, d\theta^\alpha|_u = 0$  for  $u \in P(R)$ . This implies that

$$d\varpi^\beta \equiv d\theta^\alpha \equiv 0 \pmod{\varpi^1, \dots, \varpi^s, \pi_i^\alpha} \quad (\alpha = 1, \dots, \check{i}, \dots, m+1),$$

where we write  $\varpi^\beta, \theta^\alpha$  instead of  $p^* \varpi^\beta, p^* \theta^\alpha$ , respectively.

Thus the structure equation for  $\widehat{D}$  reads

$$\begin{cases} d\varpi^\beta \equiv 0 & \pmod{\varpi^1, \dots, \varpi^s, \pi_i^\alpha} \quad (\alpha = 1, \dots, \check{i}, \dots, m+1) \\ d\pi_i^\alpha \equiv \theta^i \wedge dp_i^\alpha & \pmod{\varpi^1, \dots, \varpi^s, \pi_i^\alpha} \quad (\alpha = 1, \dots, \check{i}, \dots, m+1) \end{cases}$$

Therefore

$$\partial \widehat{D} = \{\varpi^1 = \dots = \varpi^s = 0\},$$

$$\text{Ch}(\widehat{D}) = \{\varpi^1 = \dots = \varpi^s = \pi_i^\alpha = \theta^i = dp_i^\alpha = 0 \quad (\alpha = 1, \dots, \check{i}, \dots, m+1)\}$$

These equations imply that  $\partial \widehat{D} = p_*^{-1}(D)$  and  $\text{Ch}(\widehat{D})$  is trivial.

Moreover, if  $\text{Ch}(D)$  is trivial, it follows that

$$\text{Ch}(\partial \widehat{D}) = \text{Ch}(p_*^{-1}(D)) = p_*^{-1}(\text{Ch}(D)) = \text{Ker } p_*$$

Then, by the very definition of canonical system  $\widehat{D}$ , it follows that  $\text{Ch}(\partial \widehat{D})$  is a subbundle of  $\widehat{D}$  of corank 1.  $\square$

This proposition implies that, starting from any differential system  $(R, D)$ , we can repeat the procedure of Rank 1 Prolongation. Let  $(P^1(R), D^1)$  be the prolongation of rank 1 of  $(R, D)$ . Then  $(P^k(R), D^k)$  is defined inductively as the prolongation of rank 1 of  $(P^{k-1}(R), D^{k-1})$ , which is called *k-th prolongation of rank 1* of  $(R, D)$ . Moreover, starting from a manifold  $M$  of dimension  $m+1$ , we put

$$(P^k(M), C^k) = (P(P^{k-1}(M)), \widehat{C}^{k-1})$$

where  $(P^1(M), C^1) = (J(M, 1), C)$ . When  $m = 1$ ,  $(P^k(M), C^k)$  are called “monster Goursat manifolds” in [8].

Here we observe that the above proposition also implies

**Proposition 3.2.** *Let  $(R, D)$  be an  $m$ -flag of length 1, i.e.  $\dim R = 2m + 1$ ,  $\text{rank } D = m + 1$  and  $\partial D = T(R)$ . Then the  $k$ -th prolongation  $(P^k(R), D^k)$  of rank 1 of  $(R, D)$  is an  $m$ -flag of length  $k + 1$ . Namely,  $D^k$  satisfies  $\text{rank } D^k = m + 1$ ,  $\text{rank } \partial^{i+1} D^k = \text{rank } \partial^i D^k + m$  for  $i = 0, \dots, k$  and  $\partial^{k+1} D^k = T(P^k(R))$ .*

Schematically we have the following diagram;

$$\begin{array}{ccccccc}
D^k & \subset & \partial D^k & \subset & \dots & \subset & \partial^{k-1} D^k & \subset & \partial^k D^k & \subset & \partial^{k+1} D^k = T(P^k(R)) \\
& & \downarrow p_*^k & & & & \downarrow p_*^k & & \downarrow p_*^k & & \downarrow p_*^k \\
& & D^{k-1} & \subset & \dots & \subset & \partial^{k-2} D^{k-1} & \subset & \partial^{k-1} D^{k-1} & \subset & \partial^k D^{k-1} = T(P^{k-1}(R)) \\
& & & & \downarrow p_*^{k-1} & & \downarrow p_*^{k-1} & & \downarrow p_*^{k-1} & & \downarrow p_*^{k-1} \\
& & & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & & & & \vdots & & \vdots & & \vdots \\
& & & & & & \downarrow p_*^2 & & \downarrow p_*^2 & & \downarrow p_*^2 \\
& & & & & & D^1 & \subset & \partial D^1 & \subset & \partial^2 D^1 = T(P^1(R)) \\
& & & & & & & & \downarrow p_*^1 & & \downarrow p_*^1 \\
& & & & & & & & D & \subset & \partial D = T(R)
\end{array}$$

where  $p^i : P^i(R) \rightarrow P^{i-1}(R)$  is the projection. Here we note

$$\partial^k D^k = (p_0^k)_*^{-1}(D),$$

where  $p_0^k : P^k(R) \rightarrow R$  is the projection.

## 4 Special $m$ -flags of length $k$

An  $m$ -flag  $(R, D)$  ( $m \geq 2$ ) of length  $k$  is called a *special  $m$ -flag* if: there exists a completely integrable subbundle  $F$  of  $\partial^{k-1} D$  of corank 1 that contains  $\text{Ch}(\partial^{k-1} D)$ ,  $\text{Ch}(\partial^i D)$  is a subbundle of  $\partial^{i-1} D$  of corank 1 for  $i = 1, \dots, k-1$ ,

and  $\text{Ch}(D)$  is trivial. In other words one should have the following diagram for  $(R, D)$ :

$$\begin{array}{ccccccc} D & \subset & \partial D & \subset \dots \subset & \partial^{k-2} D & \subset \partial^{k-1} D \subset & \partial^k D = T(R) \\ & & \cup & & \cup & & \cup \\ \{0\} = \text{Ch}(D) & \subset & \text{Ch}(\partial D) & \subset & \text{Ch}(\partial^2 D) & \subset \dots \subset & \text{Ch}(\partial^{k-1} D) \subset & F \end{array}$$

where  $\text{rank } \partial^i D = \text{rank } \partial^{i-1} D + m$  for  $i = 1, \dots, k$  and  $\text{rank } D = m + 1$ .

First, by repeated use of Rank 1 prolongations starting from a manifold  $M$  of dimension  $m + 1$ , we obtain by Proposition 3.1,

**Proposition 4.1.**  *$(P^k(M), C^k)$  is a special  $m$ -flag of length  $k$ .*

Conversely, by utilizing the following Realization Lemma, we will show that every special  $m$ -flag of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$ .

**Realization Lemma ([12], p.122)** *Let  $R$  and  $M$  be manifolds. Assume that the quadruple  $(R, D, p, M)$  satisfies the following conditions:*

- (i)  *$p$  is a map of  $R$  into  $M$  of constant rank.*
- (ii)  *$D$  is a differential system on  $R$  such that  $F = \text{Ker } p_*$  is a subbundle of  $D$  of corank  $n$ .*

*Then there exists a unique map  $\psi$  of  $R$  into  $J(M, n)$  satisfying  $p = \pi \cdot \psi$  and  $D = \psi_*^{-1}(C)$ . Furthermore, let  $u$  be any point of  $R$ . Then  $\psi$  is in fact defined by*

$$\psi(u) = p_*(D(u)) \quad \text{as a point of } Gr(T_x(M), n), \quad x = \pi(u),$$

*and satisfies*  $\text{Ker}(\psi_*)_u = F(u) \cap \text{Ch}(D)(u)$ .

**Theorem 4.2.** *A special  $m$ -flag  $(R, D)$  of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$ , where  $M$  is a manifold of dimension  $m + 1$ . Moreover  $F$  is unique for  $(R, D)$ .*

*Proof.* Let  $(R, D)$  be a special  $m$ -flag of length  $k$ . Matters being of local nature, we may assume that the leaf space  $M = R/F$  of the foliation  $F$  defined on  $R$  is a manifold of dimension  $2m + 1$  so that  $p : R \rightarrow M$  is a submersion and  $\text{Ker } p_* = F$ . Putting  $p = \psi^0$ , we will define maps  $\psi^i : R \rightarrow P^i(M)$  such that  $\text{Ker } \psi_*^i = \text{Ch}(\partial^{k-i} D)$  for  $i = 1, \dots, k$  as follows:

First, Realization Lemma for the quadruple  $(R, \partial^{k-1}D, p, M)$  gives us the map  $\psi^1$  of  $R$  into  $P^1(M) = J(M, 1)$  such that  $(\psi_*^1)^{-1}(C^1) = \partial^{k-1}D$  and  $\text{Ker}(\psi^1)_* = \text{Ch}(\partial^{k-1}D)$ . By dimension count, we see that  $\psi^1$  is locally a submersion of  $R$  onto  $P^1(M)$ . If the maps  $\psi^j : R \rightarrow P^j(M)$  such that  $\text{Ker} \psi_*^j = \text{Ch}(\partial^{k-j}D)$  are defined for  $j = 1, \dots, i-1$ , Realization Lemma for  $(R, \partial^{k-i}D, \psi^{i-1}, P^{i-1}(M))$  gives us the map  $\psi^i$  of  $R$  into  $P^i(M)$  such that  $(\psi_*^i)^{-1}(C^i) = \partial^{k-i}D$  and  $\text{Ker}(\psi^i)_* = \text{Ch}(\partial^{k-i}D)$ . Thus, for  $i = k$ , we obtain the map  $\psi^k$  of  $R$  into  $P^k(M)$  such that  $(\psi_*^k)^{-1}(C^k) = D$  and  $\text{Ker}(\psi^k)_* = \text{Ch}(D) = \{0\}$ . Then, by dimension count,  $\psi^k$  is a local isomorphism of  $(R, D)$  onto  $(P^k(M), C^k)$ .

For the uniqueness of  $F$ , we first observe that for a special  $m$ -flag  $(R, D)$  of length 1  $\psi^1$  is an isomorphism of  $(R, D)$  onto  $(J(M, 1), C)$ . In this case the uniqueness of  $F$  follows from Proposition 1.3 in [13], which gives the characterization of the covariant system  $F$ . For a special  $m$ -flag  $(R, D)$  of length  $k$  ( $k \geq 2$ ) we consider, locally, the leaf space  $\bar{J} = R/\text{Ch}(\partial^{k-1}D)$  by  $\text{Ch}(\partial^{k-1}D)$ . Let  $\bar{p} : R \rightarrow \bar{J}$  be the projection. On  $\bar{J}$  we have differential systems  $\bar{D} = \partial^{k-1}D/\text{Ch}(\partial^{k-1}D)$  and  $\bar{F} = F/\text{Ch}(\partial^{k-1}D)$  such that  $\bar{F}$  is a completely integrable subbundle of  $\bar{D}$  of corank 1 and  $\text{Ch}(\bar{D})$  is trivial, i.e.  $(\bar{J}, \bar{D})$  is a special  $m$ -flag of length 1. Then the uniqueness of  $F = \bar{p}_*^{-1}(\bar{F})$  follows from that of  $\bar{F}$ . This completes the proof of Theorem.  $\square$

**Remark 4.3.** After [8], P. Mormul first defined the notion of special  $m$ -flags of length  $k$  for  $m \geq 2$  in a slightly different form in [10] (cf. Theorem 6.2), generalizing the works of [7] or [11]. The above theorem was first observed by him in Remark 3 [10].

In view of Theorem 4.2, our task is to characterize the special  $m$ -flags among  $m$ -flags of length  $k$ , which will be accomplished in the following sections.

## 5 Main Theorem ( $m \geq 3$ )

Let  $(R, D)$  be an  $m$ -flag of length 1, i.e.  $R$  is a manifold of dimension  $2m + 1$  such that  $\text{rank } D = m + 1$  and  $\partial D = T(R)$ . By definition,  $(R, D)$  is a special  $m$ -flag ( $m \geq 2$ ) if there exists a completely integrable subbundle  $F$  of  $D$  of

corank 1 and  $\text{Ch}(D)$  is trivial. Then, by Realization Lemma,  $(R, D)$  is locally isomorphic to  $(P^1(M), C^1) = (J(M, 1), C)$ , where  $M = R/F$  is (locally) the leaf space of the foliation  $F$  on  $R$ . In case  $m = 1$  it is easy to see that a 1-flag of length 1 is a contact manifold of dimension 3. 2-flags of length 1 have peculiar aspects and were extensively studied in [5] (cf. §6). Now we start with the following characterization of special  $m$ -flags of length 1 for  $m \geq 3$ .

**Proposition 5.1.** *An  $m$ -flag  $(R, D)$  of length 1 for  $m \geq 3$  is a special  $m$ -flag if and only if  $D$  is of Cartan rank 1.*

Here, the *Cartan rank* of  $(R, D)$  is the smallest integer  $\rho$  such that there exist 1-forms  $\{\pi^1, \dots, \pi^\rho\}$ , which are independent modulo  $\{\eta^1, \dots, \eta^m\}$  and satisfy

$$d\alpha \wedge \pi^1 \wedge \dots \wedge \pi^\rho \equiv 0 \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \forall \alpha \in \mathcal{D}^\perp = \Gamma(D^\perp),$$

where  $D = \{\eta^1 = \dots = \eta^m = 0\}$ .

*Proof of Proposition 5.1.* First, assume that  $(R, D)$  is special. Then we can take local defining 1-forms  $\{\eta^1, \dots, \eta^m, \omega\}$ , which are independent at each point, such that

$$D = \{\eta^1 = \dots = \eta^m = 0\}, \quad F = \{\eta^1 = \dots = \eta^m = \omega = 0\}.$$

Since  $F$  is completely integrable,  $d\eta^\beta \equiv 0 \pmod{\eta^1, \dots, \eta^m, \omega}$  for  $\beta = 1, \dots, m$ . Hence there exist 1-forms  $\{\varpi^1, \dots, \varpi^m\}$  such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \beta = 1, \dots, m.$$

This implies that  $D$  is of Cartan rank 1.

Conversely, assume that the Cartan rank of  $(R, D)$  is 1. Let us take local defining 1-forms  $\{\eta^1, \dots, \eta^m\}$  of  $D$  as above:

$$D = \{\eta^1 = \dots = \eta^m = 0\}.$$

Since the Cartan rank of  $D$  is 1, there exists a 1-form  $\omega$ , which is independent modulo  $\{\eta^1, \dots, \eta^m\}$  such that

$$\omega \wedge d\eta^\beta \equiv 0 \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \beta = 1, \dots, m.$$

Hence there exist 1-forms  $\{\varpi^1, \dots, \varpi^m\}$  such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \beta = 1, \dots, m.$$

Then, from  $\text{rank } \partial D = \text{rank } D + m$ , it follows that  $\{\eta^1, \dots, \eta^m, \omega, \varpi^1, \dots, \varpi^m\}$  are linearly independent. Taking exterior derivative of both sides of the above mod equality, we get

$$0 \equiv d\omega \wedge \varpi^\beta \pmod{\eta^1, \dots, \eta^m, \omega} \quad \text{for } \beta = 1, \dots, m.$$

Hence, from  $m \geq 3$ , we obtain  $d\omega \equiv 0 \pmod{\eta^1, \dots, \eta^m, \omega}$ . Putting

$$F = \{\eta^1 = \dots = \eta^m = \omega = 0\},$$

we have

$$d\eta^\beta \equiv d\omega \equiv 0 \pmod{\eta^1, \dots, \eta^m, \omega} \quad \text{for } \beta = 1, \dots, m.$$

Thus  $F$  is completely integrable. Moreover

$$\text{Ch}(D) = \{\eta^1 = \dots = \eta^m = \omega = \varpi^1 = \dots = \varpi^m = 0\}$$

implies  $\text{Ch}(D)$  is a subbundle of  $F$  of corank  $m$ . Hence  $\text{Ch}(D)$  is trivial. This completes the proof of Proposition.  $\square$

**Remark 5.2.** *As a characterization of 1-jet spaces, Bryant's normal form theorem is well known ([2], [3]). This theorem in 1 independent variable case says that an  $m$ -flag  $(R, D)$  of length 1 for  $m \geq 3$  is a special  $m$ -flag if and only if  $D$  is of Engel (half-)rank 1 and  $\text{Ch}(D)$  is trivial. Here the Engel rank of  $(R, D)$  is the smallest integer  $\rho$  such that*

$$(d\alpha)^{\rho+1} \equiv 0 \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \forall \alpha \in \mathcal{D}^\perp,$$

where  $D = \{\eta^1 = \dots = \eta^m = 0\}$ . Here we observe that we cannot replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m = 3$ , as the following example shows: Let  $(y^1, y^2, y^3, x^0, x^1, x^2, x^3)$  be a coordinate system of  $R$ . Let us take a coframe  $\{\eta^1, \eta^2, \eta^3, \theta^i (i = 0, \dots, 3)\}$  as follows:

$$\eta^1 = dy^1 + x^2 dx^3, \quad \eta^2 = dy^2 + x^3 dx^1, \quad \eta^3 = dy^3 + x^1 dx^2, \quad \theta^i = dx^i.$$



Then, for  $D = \{\eta^1 = \eta^2 = \eta^3 = 0\}$ , we have

$$\begin{cases} d\eta^1 & \equiv \theta^2 \wedge \theta^3 \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^2 & \equiv \theta^3 \wedge \theta^1 \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^3 & \equiv \theta^1 \wedge \theta^2 \pmod{\eta^1, \eta^2, \eta^3}. \end{cases}$$

Thus  $(R, D)$  is a 3-flag of length 1 such that  $(R, D)$  is of Engel rank 1 and has non-trivial  $\text{Ch}(D)$ .

However, we can replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m \geq 4$ , as the following Lemma implies.

**Lemma 5.3.** *Let  $V$  be a vector space over  $\mathbb{R}$ . Let  $\omega_1, \dots, \omega_r \in \wedge^2 V$  be 2-forms such that  $\{\omega_1, \dots, \omega_r\}$  are linearly independent and  $\omega_i \wedge \omega_j = 0$  for  $1 \leq i < j \leq r$ . Then*

(1) *In case  $r = 2$  there exist vectors  $v_0, v_1, v_2 \in V$ , which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2.$$

(2) *In case  $r = 3$  one of the following holds:*

(i) *There exist vectors  $v_1, v_2, v_3 \in V$ , which are linearly independent, such that*

$$\omega_1 = v_2 \wedge v_3, \quad \omega_2 = v_3 \wedge v_1, \quad \omega_3 = \pm v_1 \wedge v_2.$$

(ii) *There exist vectors  $v_0, v_1, v_2, v_3 \in V$ , which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2, \quad \omega_3 = v_0 \wedge v_3.$$

(3) *In case  $r \geq 4$  there exist vectors  $v_0, \dots, v_r \in V$ , which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2, \quad \dots, \quad \omega_r = v_0 \wedge v_r.$$

In case  $m = 1$  every Goursat flag of length  $k$  ( $k \geq 2$ ) is a special 1-flag, i.e. the Sandwich Lemma holds automatically ([8]). By contrast, we need some condition for an  $m$ -flag of length 2 ( $m \geq 2$ ) to be special, as the following example shows.

**Example 5.4.** Let  $R$  be a manifold of dimension  $3m + 1$  ( $m \geq 2$ ), and let  $(x^\alpha, y^\beta, z^\beta)$  ( $\alpha = 0, 1, \dots, m$ ,  $\beta = 1, \dots, m$ ) be a coordinate system on  $R$ . For a fixed  $a \in \{0, 1, \dots, m-2\}$ , let us take a coframe  $\{\eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m, \theta^0, \dots, \theta^m\}$  as follows:

$$\begin{aligned} \theta^\alpha &= dx^\alpha & , \quad \eta^\gamma dz^\gamma + y^\gamma dx^0 - \frac{1}{2}(x^0)^2 dx^\gamma & \quad (\gamma = 1, \dots, m-a-1) \\ \zeta^\beta &= dy^\beta + x^0 dx^\beta & , \quad \eta^\delta = dz^\delta + y^{\delta-1} dx^{\delta-1} & \quad (\delta = m-a, \dots, m) \end{aligned}$$

We consider  $D = \{\eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^m = 0\}$ . Then we have

$$\begin{cases} d\eta^\beta \equiv 0 & \pmod{\eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m} & \text{for } \beta = 1, \dots, m, \\ d\zeta^\beta \equiv \theta^0 \wedge \theta^\beta & \pmod{\eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m} & \text{for } \beta = 1, \dots, m. \end{cases}$$

$$\begin{cases} d\eta^\gamma \equiv \zeta^\gamma \wedge \theta^0 & \pmod{\eta^1, \dots, \eta^m} & \text{for } \gamma = 1, \dots, m-a-1, \\ d\eta^\delta \equiv \zeta^{\delta-1} \wedge \theta^{\delta-1} & \pmod{\eta^1, \dots, \eta^m} & \text{for } \delta = m-a, \dots, m. \end{cases}$$

Hence we get

$$\begin{aligned} \partial D &= \{\eta^1 = \dots = \eta^m = 0\} & , \quad \partial^2 D = T(R) & , \quad \text{Ch}(\partial D) = \\ &= \{\eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^{m-1} = \theta^0 = \theta^{m-a-1} = \dots = \theta^{m-1} = 0\} \end{aligned}$$

Thus,  $(R, D)$  is an  $m$ -flag of length 2, but  $\text{Ch}(\partial D)$  is not a subbundle of  $D$ . Moreover  $\text{rank Ch}(\partial D)$  is  $m-a$ .

In order to get good control over  $\text{Ch}(\partial D)$ , we prepare the following proposition, which gives us the Sandwich Lemma for  $m \geq 3$ .

**Proposition 5.5.** Let  $(R, D)$  be a regular differential system such that  $\text{rank } \partial^2 D = \text{rank } \partial D + m$  and  $\text{rank } \partial D = \text{rank } D + m$ . Assume  $m \geq 3$  and the Cartan rank of  $\partial D$  is 1, then  $\text{Ch}(\partial D)$  is a subbundle of  $D$  of corank 1. Moreover the Cartan rank of  $D$  is 1

In view of Lemma 5.3, we can replace the Cartan rank 1 condition by the Engel rank 1 condition when  $m \geq 4$  (cf. Remark 5.6).

*Proof.* Let  $x$  be any point of  $R$ . By the rank condition, there exist linearly independent 1-forms  $\{\pi^i, \eta^\beta, \zeta^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  defined on a neighborhood  $U$  of  $x$ , where  $s = \text{corank } \partial^2 D$ , such that

$$\begin{aligned} \partial^2 D &= \{\pi^1 = \dots = \pi^s = 0\}, \\ \partial D &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = 0\}, \\ D &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^m = 0, \}. \end{aligned}$$

$$\begin{cases} d\pi^i \equiv 0, & d\eta^\beta \not\equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m} \\ d\eta^\beta \equiv 0, & d\zeta^\beta \not\equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m} \end{cases}$$

Since the Cartan rank of  $\partial D$  is 1, there exist 1-forms  $\{\omega, \varpi^1, \dots, \varpi^m\}$  on a neighborhood  $V \subset U$  of  $x$  such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m}$$

From rank  $\partial^2 D = \text{rank } \partial D + m$  it follows that  $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent at each  $y \in V$ . Then we have

$$\text{Ch}(\partial D) = \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \omega = \varpi^1 = \dots = \varpi^m = 0\},$$

Thus  $\text{Ch}(\partial D)$  is a subbundle of  $\partial D$  of corank  $m + 1$ .

Now the structure equation for  $D$  implies

$$\omega \wedge \varpi^\beta \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}.$$

First of all, we claim: *There exists no open neighborhood  $V' \subset V$  of  $x$  such that  $\omega$  vanishes identically on  $V'$  modulo  $\mathcal{D}^\perp$ .* Assume the contrary, i.e. there exists  $V'$  such that  $\omega_{V'} \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}$ . Then we may assume  $\omega = \zeta^1$ , so that

$$d\eta^\beta \equiv \zeta^1 \wedge \varpi^\beta \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m}$$

Taking the exterior derivative of both sides of this mod equation, we obtain

$$0 \equiv d\zeta^1 \wedge \varpi^\beta \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1}.$$

Since  $\{\pi^i, \eta^\beta, \zeta^1, \varpi^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent and  $m \geq 3$ , we get

$$d\zeta^1 \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1},$$

which contradicts the structure equation for  $D$ .

Now we divide the proof according to the dependence of  $\omega_x$  modulo  $D^\perp(x)$ .

(1)  $\omega_x \not\equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}$ .

From  $\omega \wedge \varpi^\beta \equiv 0 \pmod{\mathcal{D}^\perp}$ , we have

$$\varpi^\beta \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m, \omega}.$$

Hence we have

$$\varpi^\beta \equiv \sum_{\gamma=1}^m a_\gamma^\beta \zeta^\gamma \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \omega}.$$

Since  $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent, it follows that  $\det(a_\gamma^\beta(x)) \neq 0$ . Therefore

$$\begin{aligned} \text{Ch}(\partial D) &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \omega = \varpi^1 = \dots = \varpi^m = 0\} = \\ &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^m = \omega = 0\} \subset D. \end{aligned}$$

Thus  $\text{Ch}(\partial D)$  is a completely integrable subbundle of  $D$  of corank 1, so  $d\zeta^\beta \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m, \omega}$ . Hence we have

$$d\zeta^\beta \equiv \omega \wedge \theta^\beta \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m},$$

Since  $\text{rank } \partial D = \text{rank } D + m$ ,  $\{\pi^i, \eta^\beta, \zeta^\beta, \omega, \theta^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent and the Cartan rank of  $D$  is 1.

(2)  $\omega_x \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}$ .

Since  $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent, there exists  $\beta_0 \in \{1, \dots, m\}$  such that  $\varpi_x^{\beta_0} \not\equiv 0 \pmod{D^\perp(x)}$ . We may shrink our neighborhood  $V$  of  $x$  so that  $\varpi_y^{\beta_0} \not\equiv 0 \pmod{D^\perp(y)}$  for each  $y \in V$ . Then, from  $\omega \wedge \varpi^{\beta_0} \equiv 0 \pmod{\mathcal{D}^\perp}$ , we have

$$\omega \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m, \varpi^{\beta_0}}.$$

Moreover we claim:

$$\varpi^\beta \wedge \varpi^{\beta_0} \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m},$$

holds on  $V$  for each  $\beta \in \{1, \dots, m\}$ .

In fact, for each  $y \in V$ , we consider the following two cases.

(a)  $\omega_y \not\equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}$ .

From  $\omega \wedge \varpi^\beta \equiv 0 \pmod{\mathcal{D}^\perp}$ , we have  $\varpi_y^\beta \equiv \lambda^\beta \omega_y \pmod{D^\perp(y)}$ . Since  $\lambda^{\beta_0} \neq 0$ , we get  $\omega_y \equiv \lambda \varpi_y^{\beta_0}$  for  $\lambda \neq 0$ . Hence  $\varpi_y^\beta \wedge \varpi_y^{\beta_0} \equiv 0 \pmod{D^\perp(y)}$ , as desired.

(b)  $\omega_y \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}$ .

Assume the contrary, i.e. that there exists  $\gamma \in \{1, \dots, m\}$  such that  $\varpi_y^\gamma \wedge \varpi_y^{\beta_0} \not\equiv 0 \pmod{D^\perp(y)}$ . Then we may take a neighborhood  $V_0 \subset V$  of  $y$  so that

$$\varpi_z^\gamma \wedge \varpi_z^{\beta_0} \not\equiv 0 \pmod{D^\perp(z)},$$

for each  $z \in V_0$ . However  $\omega$  cannot vanish identically on  $V_0$ , as shown above. Hence there exists a point  $z_0 \in V_0$  such that  $\omega_{z_0} \not\equiv 0 \pmod{D^\perp(z_0)}$ . Then, as in (a), we get  $\varpi_{z_0}^\gamma \wedge \varpi_{z_0}^{\beta_0} \equiv 0 \pmod{D^\perp(z_0)}$ , which is a contradiction.

Since  $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent, we obtain

$$\begin{aligned} \text{Ch}(\partial D) &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \omega = \varpi^1 = \dots = \varpi^m = 0\} = \\ &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^m = \varpi^{\beta_0} = 0\} \subset D. \end{aligned}$$

Thus  $\text{Ch}(\partial D)$  is a completely integrable subbundle of  $D$  of corank 1. Moreover, as in (1), the Cartan rank of  $D$  is 1. This completes the proof of Proposition.  $\square$

**Remark 5.6.** *We cannot replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m = 3$ , as the following example shows: Let  $(z^1, z^2, z^3, y^1, y^2, y^3, x^0, x^1, x^2, x^3)$  be a coordinate system of  $R$ . Let us take a coframe  $\{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3, \theta^0, \theta^1, \theta^2, \theta^3\}$  as follows:*

$$\begin{aligned} \eta^1 &= dz^1 + y^1 dx^0, & \eta^2 &= dz^2 + y^2 dy^1, & \eta^3 &= dz^3 + x^0 dy^2, \\ \zeta^1 &= dy^1 - x^1 dx^0, & \zeta^2 &= dy^2 - x^2 dx^0, & \zeta^3 &= dy^3 - x^3 dx^0, \\ \theta^0 &= dx^0, & \theta^1 &= dx^1, & \theta^2 &= dx^2, & \theta^3 &= dx^3. \end{aligned}$$

We consider  $D = \{\eta^1 = \eta^2 = \eta^3 = \zeta^1 = \zeta^2 = \zeta^3 = 0\}$ . Then we have

$$\begin{cases} d\eta^\beta \equiv 0 & \pmod{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3} & \text{for } \beta = 1, 2, 3, \\ d\zeta^\beta \equiv \theta^0 \wedge \theta^\beta & \pmod{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3} & \text{for } \beta = 1, 2, 3. \end{cases}$$

$$\begin{cases} d\eta^1 \equiv \zeta^1 \wedge \theta^0 & \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^2 \equiv (\zeta^2 + x^2 \theta^0) \wedge (\zeta^1 + x^1 \theta^0) & \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^3 \equiv \theta^0 \wedge \zeta^2 & \pmod{\eta^1, \eta^2, \eta^3}. \end{cases}$$

Hence we get

$$\begin{aligned}\partial D &= \{\eta^1 = \eta^2 = \eta^3 = 0\} \quad , \quad \partial^2 D = T(R), \\ \text{Ch}(\partial D) &= \{\eta^1 = \eta^2 = \eta^3 = \zeta^1 = \zeta^2 = \theta^0 = 0\}.\end{aligned}$$

Thus,  $(R, D)$  is a 3-flag of length 2 such that the Engel rank of  $\partial D$  is 1, but  $\text{Ch}(\partial D)$  is not a subbundle of  $D$ .

However, by Lemma 5.3, we can replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m \geq 4$ .

By utilizing the above proposition repeatedly, we obtain

**Theorem 5.7.** *An  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 3$  is a special  $m$ -flag if and only if  $\partial^{k-1}D$  is of Cartan rank 1. Moreover, an  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 4$  is a special  $m$ -flag if and only if  $\partial^{k-1}D$  is of Engel rank 1.*

*Proof.* The ‘only if’ part follows from the existence of the completely integrable subbundle  $F$  of  $\partial^{k-1}D$  of corank 1 for the special  $m$ -flag as in the proof of Proposition 5.1.

For the ‘if’ part, first, the proof of Proposition 5.1 shows the existence of a completely integrable subbundle  $F$  of  $\partial^{k-1}D$ , which contains  $\text{Ch}(\partial^{k-1}D)$ . By repeated application of the previous proposition, we obtain that  $\text{Ch}(\partial^{i+1}D)$  is a subbundle of  $\partial^i D$  of corank 1 for  $i = 0, \dots, k-2$ . Thus we are left to show that  $\text{rank } D = \text{rank } \text{Ch}(D) + m + 1$ .

Let us take defining 1-forms of  $D$ ,  $\partial D$  and  $\text{Ch}(\partial D)$  such that

$$\begin{aligned}\partial D &= \{\pi^1 = \dots = \pi^s = 0\} \quad , \quad D = \{\pi^1 = \dots = \pi^s = \zeta^1 = \dots = \zeta^m = 0\}, \\ \text{Ch}(\partial D) &= \{\pi^1 = \dots = \pi^s = \zeta^1 = \dots = \zeta^m = \omega = 0\}.\end{aligned}$$

where  $s$  is the corank of  $\partial D$ . Since  $\text{Ch}(\partial D)$  is completely integrable, we have

$$d\zeta^\alpha \equiv 0 \pmod{\pi^1, \dots, \pi^s, \zeta^1, \dots, \zeta^m, \omega}, \quad \text{for } \alpha = 1, \dots, m.$$

Therefore, there exist 1-forms  $\{\theta^1, \dots, \theta^m\}$  such that

$$\begin{cases} d\pi^i \equiv 0, & \pmod{\pi^1, \dots, \pi^s, \zeta^1, \dots, \zeta^m} \quad \text{for } i = 1, \dots, s, \\ d\zeta^\alpha \equiv \omega \wedge \theta^\alpha, & \pmod{\pi^1, \dots, \pi^s, \zeta^1, \dots, \zeta^m} \quad \text{for } \alpha = 1, \dots, m. \end{cases}$$

Then, from  $\text{rank } \partial D = \text{rank } D + m$  it follows that  $\{\pi^i, \zeta^\alpha, \omega, \theta^\alpha (i = 1, \dots, s, \alpha = 1, \dots, m)\}$  are linearly independent. Hence

$$\text{Ch}(D) = \{\pi^1 = \dots = \pi^s = \zeta^1 = \dots = \zeta^m = \omega = \theta^1 = \dots = \theta^m = 0\}.$$

Thus  $\text{rank } D = \text{rank } \text{Ch}(D) + m + 1$ . This completes the proof of the Theorem.  $\square$

Hence, by Theorem 4.2, we obtain the Drapeau Theorem for  $m \geq 3$

**Corollary 5.8.** *Let  $M$  be a manifold of dimension  $m + 1$ . An  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 3$  is locally isomorphic to  $(P^k(M), C^k)$  if and only if  $\partial^{k-1}D$  is of Cartan rank 1, and moreover for  $m \geq 4$ , if and only if  $\partial^{k-1}D$  is of Engel rank 1.*

## 6 Integrable subbundle of corank 1

Let  $(R, D)$  be a 2-flag of length 1. Then it can be shown ([5]) that there exists a local coframe  $\{\eta^1, \eta^2, \theta^0, \theta^1, \theta^2\}$  such that  $D = \{\eta^1 = \eta^2 = 0\}$ ,

$$\begin{cases} d\eta^1 & \equiv \theta^0 \wedge \theta^1 & (\text{mod } \eta^1, \eta^2), \\ d\eta^2 & \equiv \theta^0 \wedge \theta^2 & (\text{mod } \eta^1, \eta^2). \end{cases}$$

Thus the Cartan rank of  $(R, D)$  is always 1 and we have the covariant system  $F = \{\eta^1 = \eta^2 = \theta^0 = 0\}$  of  $D$  of corank 1 (cf. [14]). As is well known,  $F$  is not necessarily completely integrable.

As for a 2-flag of length 2, we observe that, in Example 5.4, putting  $m = 2$ , we obtain the following structure equation for  $D = \{\eta^1 = \eta^2 = \zeta^1 = \zeta^2 = 0\}$ , where

$$\begin{aligned} \eta^1 dz^1 + y^1 dx^0 - \frac{1}{2}(x^0)^2 dx^1, \quad \eta^2 = dz^2 + y^1 dx^1 \\ \zeta^1 = dy^1 + x^0 dx^1, \quad \zeta^2 = dy^2 + x^0 dx^2, \quad \theta^0 = dx^0, \quad \theta^1 = dx^1, \quad \theta^2 = dx^2, \end{aligned}$$

$$\begin{cases} d\eta^\beta & \equiv 0 & (\text{mod } \eta^1, \eta^2, \zeta^1, \zeta^2) & \text{for } \beta = 1, 2, \\ d\zeta^\beta & \equiv \theta^0 \wedge \theta^\beta & (\text{mod } \eta^1, \eta^2, \zeta^1, \zeta^2) & \text{for } \beta = 1, 2. \end{cases}$$

$$\begin{cases} d\eta^1 & \equiv \zeta^1 \wedge \theta^0 & (\text{mod } \eta^1, \eta^2), \\ d\eta^2 & \equiv \zeta^1 \wedge \theta^1 & (\text{mod } \eta^1, \eta^2). \end{cases}$$

Thus  $\partial D = \{\eta^1 = \eta^2 = 0\}$  and the Cartan rank of  $\partial D$  is 1, whereas  $Ch(\partial D)$  is not a subbundle of  $D$ . This shows that the statement of Proposition 5.5 is false for  $m = 2$ .

To cover the case  $m = 2$ , we strengthen the hypothesis of Proposition 5.5 as in the following.

**Proposition 6.1.** *Let  $(R, D)$  be a regular differential system such that  $\text{rank } \partial^2 D = \text{rank } \partial D + m$  and  $\text{rank } \partial D = \text{rank } D + m$ . Assume that there exists a completely integrable subbundle  $F$  of  $\partial D$  of corank 1, then  $\text{Ch}(\partial D)$  is a subbundle of  $D$  of corank 1.*

*Proof.* Let  $x$  be any point of  $R$ . By the rank condition, there exist linearly independent 1-forms  $\{\pi^i, \eta^\beta, \zeta^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  defined on a neighborhood  $U$  of  $x$ , where  $s = \text{corank } \partial^2 D$ , such that

$$\begin{aligned} \partial^2 D &= \{\pi^1 = \dots = \pi^s = 0\}, \\ \partial D &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = 0\}, \\ D &= \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^m = 0\}. \end{aligned}$$

$$\begin{cases} d\pi^i \equiv 0, & d\eta^\beta \not\equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m} \\ d\eta^\beta \equiv 0, & d\zeta^\beta \not\equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m} \end{cases}$$

Moreover, since  $F$  is a subbundle of  $\partial D$  of corank 1, there exists a 1-form  $\omega$  such that  $\{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \omega\}$  are linearly independent and

$$F = \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \omega = 0\}$$

Since  $F$  is completely integrable, we have  $d\eta^\beta \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \omega}$ . Hence there exist 1-forms  $\{\varpi^1, \dots, \varpi^m\}$  on a neighborhood  $V \subset U$  of  $x$  such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m}$$

From  $\text{rank } \partial^2 D = \text{rank } \partial D + m$  it follows that  $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \dots, s, \beta = 1, \dots, m)\}$  are linearly independent at each  $y \in V$ . Then we have

$$\text{Ch}(\partial D) = \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \omega = \varpi^1 = \dots = \varpi^m = 0\} \subset F,$$

Thus  $\text{Ch}(\partial D)$  is a subbundle of  $F$  of corank  $m$ .

Now the structure equation for  $D$  implies

$$\omega \wedge \varpi^\beta \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}.$$

First of all, we claim: *There exists no open neighborhood  $V' \subset V$  of  $x$  such that  $\omega$  vanishes identically on  $V'$  modulo  $\mathcal{D}^\perp$ .* Assume the contrary, i.e. there exists  $V'$  such that  $\omega_{V'} \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m}$ .



Then we may assume  $\omega = \zeta^1$  on  $V'$ . Since  $F$  is completely integrable and  $F = \{\pi^1 = \dots = \pi^s = \eta^1 = \dots = \eta^m = \zeta^1 = 0\}$ , we get

$$d\zeta^1 \equiv 0 \pmod{\pi^1, \dots, \pi^s, \eta^1, \dots, \eta^m, \zeta^1},$$

which contradicts the structure equation for  $D$ .

The rest of the proof is quite similar to that of Proposition 5.5, hence it is omitted.  $\square$

The above proposition is also obtained independently by Adachi [1].

By utilizing the above proposition repeatedly, we obtain

**Theorem 6.2.** *An  $m$ -flag  $(R, D)$  of length  $k$  is a special  $m$ -flag if and only if there exists a completely integrable subbundle  $F$  of  $\partial^{k-1}D$  of corank 1. Moreover,  $F$  is unique for  $(R, D)$ .*

*Proof.* The ‘only if’ part is trivial. For the ‘if’ part, by repeated application of the above Proposition, we obtain that  $F \supset \text{Ch}(\partial^{k-1}D)$  and  $\text{Ch}(\partial^{i+1}D)$  is a subbundle of  $\partial^i D$  of corank 1 for  $i = 0, \dots, k-2$ . Thus we are left to show that  $\text{rank } D = \text{rank } \text{Ch}(D) + m + 1$ , but the proof is the same as in Theorem 5.7. The uniqueness of  $F$  follows from Theorem 4.2.  $\square$

Hence, by Theorem 4.2, we obtain the following Drapeau Theorem for  $m \geq 2$ .

**Corollary 6.3.** *Let  $M$  be a manifold of dimension  $m + 1$ . An  $m$ -flag  $(R, D)$  of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$  if and only if there exists a completely integrable subbundle  $F$  of  $\partial^{k-1}D$  of corank 1.*

## References

- [1] Adachi, J.: Global stability of special multi-flags, in preparation
- [2] Bryant, R. : Some aspect of the local and global theory of Pfaffian systems, Thesis, University of North Carolina, Chapel Hill, 1979

- [3] Bryant, R., Chern, S., Gardner, R., Goldschmidt, H. and Griffiths, P.: Exterior Differential Systems, MSRI Publ. vol. **18**, Springer Verlag, Berlin 1991
- [4] Bryant, R. and Hsu, L.: Rigidity of integral curves of rank 2 distributions, *Invent. Math.* **144** (1993), 435–461
- [5] Cartan, E. : Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, *Ann. École Normale*, **27** (1910), 109–192
- [6] Giaro, A., Kumpera, A. et Ruiz, C.: Sur la lecture correcte d’un résultat d’Élie Cartan, *C.R.Acad.Sc.Paris*, **287**, Sér.A (1978), 241–244.
- [7] Kumpera, A. and Rubin, J.L.: Multi-flag systems and ordinary differential equations, *Nagoya Math.J.* **166** (2002), 1–27.
- [8] Montgomery, R. and Zhitomirskii, M.: Geometric approach to Goursat flags, *Ann.Inst. H.Poincaré-AN* **18** (2001), 459–493.
- [9] Mormul,P.: Goursat Flags:Classification of codimension-one singularities, *J.of Dynamical and Control Systems*, **6**, No. 3(2000), 311–330
- [10] Mormul, P.: Multi-dimensional Cartan prolongation and special k-flags, *Banach center Publ.* vol.**65** (2004), 157–178.
- [11] Pasillas-Lépine, W. and Respondek, W. : Contact systems and corank one involutive subdistributions, *Acta Appl. Math.* **69** (2001), 105–128
- [12] Yamaguchi, K.: Contact geometry of higher order, *Japan. J. Math.* **8** (1982), 109–176.
- [13] Yamaguchi, K.: Geometrization of jet bundles, *Hokkaido Math. J.* **12** (1983), 27–40.
- [14] Yamaguchi, K.:  $G_2$ -geometry of overdetermined systems of second order, *Trends in Mathematics (Analysis and Geometry in Several Complex Variables)* (1999), Birkhäuser, Boston, 289–314.

## The formal orbital normal forms for the nilpotent singularity. The case of generalized saddle

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### 1 Introduction

Takens [Ta] began the study of germs of vector fields of the form

$$\dot{x} = y + \dots, \quad \dot{y} = \dots \tag{1.1}$$

where dots denote non-linear terms. Such singularities are called nilpotent or Bogdanov-Takens singularities. In this paper the situation with complex equations, i.e.  $(x, y) \in (\mathbf{C}^2, 0)$ , and complex time is considered.

Takens had shown that there exists a formal change of coordinates  $x, y$ , which reduces (1.1) to

$$\dot{x} = y + a(x), \quad \dot{y} = b(x) \tag{1.2}$$

where  $a(x) = a_r x^r + a_{r+1} x^{r+1} + \dots$ ,  $r \geq 2$  and  $b(x) = b_{s-1} x^{s-1} + b_s x^s + \dots$ ,  $s \geq 3$  are formal power series. The form (1.2), which is called Takens prenormal form, is not the final normal form with respect to the orbital equivalences.

The systematic study of complex nilpotent singularities from the orbital classification point of view has begun with the works of Cerveau and Moussu [CeMo], Elizarov et al. [EISV], Loray and Meziani [Lo,LoMe,Me] and Stróżyńska and Żołądek [StZo1,St,StZo2]. In particular, in [Lo,StZo1] a complete formal orbital normal form was obtained for the case  $s < 2r$  (with  $a_r b_s \neq 0$ ), called generalized cusp. In [St] a complete formal orbital normal form was obtained for the case  $2r < s$ , called generalized saddle-node. This short article is devoted to present the main results of the paper [StZo2] considering the orbital classification of nilpotent singularities in the remaining case

$$s = 2r$$

called generalized saddle.

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## 2 The results

Let  $V_H = (y + ax^r)\partial_x + bx^{2r-1}\partial_y$  be a two-parameter family of vector fields, i.e.

$$\dot{x} = y + ax^r, \quad \dot{y} = bx^{2r-1} \quad (2.1)$$

in the complex space. We assume that  $(a, b) \in \mathbf{C}^2 \setminus (0, 0)$ .

Applying the orbital change  $x \rightarrow \alpha x$ ,  $y \rightarrow \beta y$ ,  $dt \rightarrow \gamma dt$ , we find a new system

$$\dot{x} = \frac{\beta\gamma}{\alpha}y + a\alpha^{r-1}\gamma x^r, \quad \dot{y} = \frac{b\alpha^{2r-1}\gamma}{\beta}x^{2r-1}$$

and assuming  $\beta\gamma/\alpha = 1$ , we get a change

$$(a, b) \rightarrow (\nu a, \nu^2 b), \quad \nu \in \mathbf{C}^*.$$

Therefore we shall treat the system (2.1) as parametrized by the elements  $[a : b]$  of the one-dimensional weighted projective line  $\mathbf{C}P_w^1 := (\mathbf{C}^2 \setminus (0, 0))/\mathbf{C}^*$  (with the  $\mathbf{C}^*$ -action as above). The representatives of the orbits  $\mathbf{C}^* \cdot [\xi : \eta]$  can be chosen, for example, as follows:

$$(a, b) = \begin{cases} (1, \eta/\xi^2) & \text{if } [\xi : \eta] \neq [0 : 1], \\ (0, 1) & \text{otherwise} \end{cases} \quad (2.2)$$

(this section is not continuous at the point  $[0 : 1]$ ).

Putting  $z = x^r$  and dividing by  $x^{r-1}$ , we get from (2.1) the linear system

$$\dot{z} = raz + ry, \quad \dot{y} = bz \quad (2.3)$$

We call (2.3) principal linear system.

Its eigenvalues are equal

$$\lambda_{1,2} = \frac{r}{2} \left( a \pm \sqrt{a^2 + 4b/r} \right)$$

Their ratio

$$\lambda = \frac{a - \sqrt{a^2 + 4b/r}}{a + \sqrt{a^2 + 4b/r}}$$

is an algebraic function on  $\mathbf{C}P_w^1$ .

**Proposition 2.1**

The critical point  $z = y = 0$  of the system (2.3) is:

1. a  $(k : l)$ -resonant node ( $\lambda = \frac{k}{l}$ ,  $k, l \in \mathbf{N}$ ) iff

$$[a : b] = [(k + l) : -klr];$$

2. a  $(k : -l)$ -resonant saddle ( $\lambda = -\frac{k}{l}$ ) iff

$$[a : b] = [(k - l) : klr];$$

3. a focus ( $\lambda \notin \mathbf{R}$ ) iff

$$\frac{a^2}{b} \in \mathbf{C} \setminus (-\infty, -\frac{4}{r}] \setminus [0, \infty);$$

4. a non-resonant node iff

$$\frac{a^2}{b} < -\frac{4}{r}, \quad [a : b] \neq [(\mu + 1) : -\mu r], \quad \mu \in \mathbf{Q};$$

5. a non-resonant saddle iff

$$0 \leq \frac{a^2}{b}, \quad [a : b] \neq [(\mu + 1) : -\mu r], \quad \mu \in \mathbf{Q};$$

6. a saddle-node ( $\lambda = 0$ ) iff

$$[a : b] = [1 : 0].$$

Note that the case  $\lambda = -1$  corresponds to  $a = 0$ , i.e. the generalized cusp case; we can treat it as a particular case of a saddle.

**Definition**

Two germs  $V, V'$  of analytic vector fields in  $(\mathbf{C}^2, 0)$  are formally (analytically) orbitally equivalent iff there is a formal (analytic) diffeomorphism  $\mathcal{G}$  of  $(\mathbf{C}^2, 0)$  transforming the phase curves of  $V$  to the phase curves of  $V'$ . This means that there is a formal (analytic) function  $\psi$ ,  $\psi(0) \neq 0$  such that  $\psi \cdot V' = \mathcal{G}_*^{-1} V \circ \mathcal{G}$ .

In the following theorems there is a classification of vector fields of the form

$$V = V_H + W \tag{2.4}$$

where  $V_H$  is given in (2.1) (with the parameters  $a, b$  defined by the agreement (2.2) and  $W$  is of higher order with respect to the quasi-homogeneous gradation  $\widetilde{deg}$  defined by

$$\widetilde{deg}x = -\widetilde{deg}\partial_x = 1, \quad \widetilde{deg}y = -\widetilde{deg}\partial_y = r$$

Thus  $\widetilde{deg}V_H = r - 1$ .

Denote also

$$E_H = x\partial_x + ry\partial_y \tag{2.5}$$

the quasi-homogeneous Euler vector field.

**Theorem 2.2**

Assume that  $V$  in (2.4) is such that the principal linear system (2.3) corresponding to  $V_H$  is either non-resonant (cases 3, 4 or 5 of Proposition 2.1.) or a  $(k : l)$ -resonant node with  $k, l > 1$ ,  $\gcd(k, l) = 1$ .

Then it is formally orbitally equivalent to one of the following orbitally non-equivalent fields:

- (i)  $V_H$  or
- (ii) the field

$$V_H + x^t(1 + \phi(x))\partial_x \tag{2.6}$$

where  $t \not\equiv 0 \pmod{r}$ ,  $t > r$  and

$$\phi(x) = \sum_{j \not\equiv -t \pmod{r}} a_j x^j \tag{2.7}$$

is a formal power series,  $\phi(0) = 0$ .

Two vector fields  $V$  and  $V'$  of the form (2.6) with exponents  $t, t'$  and series  $\phi, \phi'$  are orbitally equivalent iff  $t = t'$  and  $\phi'(x) \equiv \phi(\alpha x)$  for some constant  $\alpha$  satisfying  $\alpha^{t-r} = 1$ .

**Theorem 2.3**

Assume that the principal linear system (2.3) is a  $(k : 1)$ -resonant node.

Then  $V$  is formally orbitally equivalent to one of the following orbitally non-equivalent fields:

- (i)  $V_H$  or
- (ii) the field

$$V = V_H + x^{kr}\partial_x \tag{2.8}$$

for  $k > 1$  or  
 (iii) the field

$$V = V_H + cx^{kr}\partial_x + x^t(1 + \phi(x))\partial_x \quad (2.9)$$

where  $t \not\equiv 0 \pmod{r}$ ,  $t > r$ ,  $c = 0$  for  $k = 1$  and

$$\phi(x) = \sum_{\substack{j \not\equiv -t \pmod{r}, \\ j \neq kr}} a_j x^j$$

Two vector fields of the form (2.9) with parameters  $c, c'$ , exponents  $t, t'$  and series  $\phi(x), \phi'(x)$  are orbitally equivalent iff  $t = t'$ ,  $c' = \alpha^{(k-1)r}c$  and  $\phi'(x) \equiv \phi(\alpha x)$  for some  $\alpha$  satisfying  $\alpha^{t-r} = 1$ .

The field  $V$  with formal orbital forms:  $V_H$  for  $k = 1$ , (2.8) and (2.9) with  $c \neq 0$  have only one analytic separatrix.

#### Theorem 2.4

Assume that the principal linear system (2.3) is a  $(k_0 : -l_0)$ -resonant saddle with  $\gcd(k_0, l_0) = 1$ .

Then  $V$  is formally orbitally equivalent to one of the following orbitally non-equivalent fields:

- (i)  $V_H$  or
- (ii) the field (2.6) with  $t \not\equiv 0 \pmod{r}$ ,  $t > r$  and

$$\phi(x) = \sum_{\substack{j+t \pmod{r} \neq 0, \\ j+t \pmod{r_0} = r, \\ j \pmod{r_0} \neq 0}} a_j x^j$$

where  $r_0 = (k_0 + l_0)r$  or

- (iii) the field (2.6) with  $t = r + n_0 r_0$  for some integer  $n_0$  and

$$\phi(x) = a_{n_0 r_0} x^{n_0 r_0}$$

or

- (iv) the field (2.6) with  $t = r + n_0 r_0$  and

$$\phi(x) = a_{n_0 r_0} x^{n_0 r_0} + a_{j_0} x^{j_0} + \sum_{\substack{j > j_0 \\ j \pmod{r} \neq 0 \\ j \neq j_0 + n_0 r_0}} a_j x^j$$

where  $a_{j_0} \neq 0$  for some  $j_0 > 0$ ,  $j_0 \neq 0 \pmod{r}$ .

Two vector fields  $V = V_H + x^t(1 + \phi(x))\partial_x$  and  $V' = V_H + x^{t'}(1 + \phi'(x))\partial_x$  are orbitally equivalent iff  $t = t'$  and  $\phi'(x) = \phi(\alpha x)$  for some number  $\alpha$  satisfying  $\alpha^{t-r} = 1$ .

### 3 The resolution of the singularity and the method of the proof

The resolution of the singularity  $x = y = 0$  of the (formal) vector fields described in Theorems 2.2-2.4 uses the quasi-homogeneous blowing-up, which means rewriting the vector field  $V = V_H + W$  (see (2.4)) in the variables  $x, u = y/x^r$  and division by  $x^{r-1}$ . We get the system

$$\dot{x} = x(u + a) + O(x^2), \quad \dot{u} = b - aru - ru^2 + O(x) \quad (3.1)$$

We have the singular points  $p_{1,2} : x = 0, u = u_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 + 4b/r})$ . Since  $b \neq 0$ , we have  $u_{1,2} \neq 0, \infty$ .

The case  $p_1 = p_2$  occurs when  $a^2 + 4b/r = 0$ , which corresponds to the  $(1 : 1)$ -resonant node in the principal linear system (2.3). The singular point  $p_1 = p_2$  is a saddle-node for the system (3.1). Its center separatrix is  $x = 0$  and its strong separatrix takes the form  $u = -\frac{a}{2} + O(x)$  and is analytic. It is known that a saddle-node can have at most two analytic separatrices.

In the case  $a^2 + 4b/r \neq 0$  each point  $p_{1,2}$  has the following eigenvalues:  $\lambda_1(p_{1,2}) = \frac{1}{2}(a \pm \sqrt{a^2 + 4b/r})$  in the  $x$ -direction and  $\lambda_2(p_{1,2}) = \mp r \sqrt{a^2 + 4b/r}$  in the  $u$ -direction. When the principal linear system is non-resonant, the corresponding ratios  $\lambda(p_{1,2}) = (\lambda_2/\lambda_1)(p_{1,2})$  are also not rational.

In the resonant cases  $[a : b] = [(k+l) : -klr]$ ,  $k > |l|$  and  $l$  positive or negative, we have  $\lambda(p_1) = \frac{(k-l)r}{-k}$  and  $\lambda(p_2) = \frac{(k-l)r}{l}$ .

In the case of resonant saddle of the principal linear system ( $l < 0$ ) the points  $p_{1,2}$  are also saddles, each with two analytic separatrices.

In the case of resonant node of the principal linear system ( $0 < l < k$ ) the point  $p_1$  is a saddle, while  $p_2$  is a  $((k-l)r : l)$ -resonant node. When  $l = 1$  the ratio  $\lambda(p_2) = (k-1)r$  is natural. It turns out that the term  $cx^{kr}\partial_x$  in the vector field (2.9) gives the resonant term  $-rcu_2x^{(k-1)r}\partial_u$  in the local expansion of the vector field (3.1) near  $p_2$ . Its presence causes local non-linearizability of (3.1) near  $p_2$  and absence of separatrices (different from the



divisor  $x = 0$ ) of the point  $p_2$ . More precisely, let us write (3.1) in the form

$$\frac{du}{dx} = \frac{-r(u - u_1)(u - u_2) - rcx^{(k-1)r} + \dots}{x(u + a + cx^{(k-1)r} + \dots)}$$

and look for a separatrix in the form  $u = u_2 + dx^{(k-1)r} + \dots$ . Substituting it into the above equation gives the following contradiction:  $d(k-1)r = (\lambda_2 d - rcu_2)/\lambda_1$  in the coefficients before  $x^{(k-1)r-1}$  (this explains the statement about the absence of two separatrices in Theorem 2.3.).

In the other cases of resonant node of the principal linear system there are two possibilities: either  $\lambda(p_2) \notin \mathbf{N}$  and there exists a separatrix or  $\lambda(p_2) \in \mathbf{N}$  (e.g. when  $l|r$ ) and it is not obvious whether the point  $p_2$  has non-trivial analytic separatrix.

In the case of generalized cusp the formal normal form was proved in [Lo,StZo1] in two different ways. In [St] another, direct, method was used in the case of generalized saddle-node (i.e.  $b = 0$  in (2.1)). It turns out that only the method from [StZo1] is general enough to give a unified proof of all Theorems 2.2-2.4.

The orbital changes rely on an application of conjugations

$$V \rightarrow \mathcal{P}_V(Z) := (Ad_{\exp Z})_* V$$

and multiplications

$$V \rightarrow (1 + \chi)V$$

Here  $Z = Z_1 \partial_x + Z_2 \partial_y$  is a vector field (formal or analytic),  $\exp Z$  is the phase flow diffeomorphism (after time 1) and  $\chi$  is a function (formal or analytic).

Note that if  $Z$  is parallel to  $V$ ,  $Z = \kappa(x, y)V$ , then the map  $\exp Z$  preserves the phase portrait of  $V$  and the field  $\mathcal{P}_V(\kappa V)$  is also parallel to  $V$ . In order to avoid this ambiguity, one uses the notion of a bivector field introduced by Bogdanov [Bo].

If  $V = V_1 \partial_x + V_2 \partial_y$ , then define the bivector field

$$Z \wedge V = \Omega \partial_x \wedge \partial_y$$

where  $\Omega = V_2 Z_1 - V_1 Z_2$ . One can say that  $\Omega$  measures the component of  $Z$  transversal to  $V$ . If  $\Omega = 0$  and  $V$  has isolated singularity, then  $Z = \kappa V$  for some function  $\kappa$ .

This suggests that one should consider the map

$$Z \rightarrow \frac{\mathcal{P}_V(Z) \wedge V}{\partial_x \wedge \partial_y} \tag{3.2}$$

from the space  $(\mathbf{C}[[x, y]])^2$  of formal vector fields to the space  $\mathbf{C}[[x, y]]$  of formal functions. Since the map  $Z \rightarrow \mathcal{P}_V(Z)$  is non-linear, the map (3.2) cannot be factorized to a map from  $\mathbf{C}[[x, y]]$  to  $\mathbf{C}[[x, y]]$ . But the linear part of  $\mathcal{P}_V(\cdot)$  is equal to  $-ad_V(\cdot)$  and we have well defined linear map

$$\mathcal{L}_V \Omega = -\frac{ad_V Z \wedge V}{\partial_x \wedge \partial_y}, \quad \Omega = Z \wedge V$$

In [StZo1] it is shown that

$$\mathcal{L}_V \Omega = \dot{\Omega} - \text{div} V \cdot \Omega$$

where  $\dot{\Omega} = V(\Omega) = \partial \Omega / \partial V$ .

Theorems 2.2-2.4 can be proved using only the linear operator  $\mathcal{L}_V$ . Of course, one shall use also the quasi-homogeneous gradation *deg* and eliminate recursively the terms of growing quasi-homogeneous degree.

If  $U = U_1 \partial_x + U_2 \partial_y$  is the part of  $V$  ("transversal" to  $V$ ), which should be reduced, then we have bivector homological equation

$$\mathcal{L}_V \Omega + \Theta = 0, \quad \text{where } \Theta = \frac{U \wedge V}{\partial_x \wedge \partial_y} \quad (3.3)$$

Having solved the bivector homological equation, one gets some function  $\Omega$ . Having the function  $\Omega$ , one finds the vector field from the equation  $Z_1 V_2 - Z_2 V_1 = \Omega$ ; assuming  $V_1 = y + ax^r + \dots$ ,  $V_2 = bx^{2r-1} + \dots$  it can be noticed that the solution exists (provided that the expansion of  $\Omega$  begins with the terms of degree  $\widetilde{\text{deg}} \geq r + 2$ ). The application of  $(Ad_{\exp Z})_* V$  eliminates  $U$ , leaving only the terms of the form  $\sigma(x, y)V$ . The latter can be reduced using the multiplication  $(1 - \sigma)V$ . The terms non-linear in  $Z$  are of higher quasi-homogeneous degree and are eliminated in the further steps of recursive process.

The above short explanation shows that the key of the orbital normal form reduction is the solution of bivector homological equation (3.3). The whole algorithm of reduction is divided into three general steps. The first is to approximate the homological operator  $\mathcal{L}_V$  by  $\mathcal{L}_{V_H}$ , determine its kernel in  $\mathbf{C}[[x, y]]$  and a subspace complementary to its image. Next, using the operator  $\mathcal{L}_{V_H + x^t \partial_x}$  to  $\Omega$ 's from  $\ker \mathcal{L}_{V_H}$  one reduces some additional terms (here  $x^t \partial_x$  is the first term not reduced in the previous step). In the third and last step one uses the operator  $\mathcal{L}_{V_H + x^t \partial_x + \text{const} \cdot x^u \partial_u}$  in a similar way. This finishes the short sketch of the proof; the complete proof can be found in [StZo2].

## References

- [Bo] R.I. Bogdanov, *Local orbital normal forms of vector fields on a plain*, Trans. Petrovski Sem. 5 (1979), 50-85
- [CeMo] D. Cerveau, R. Moussu, *Groupes d'automorphismes de  $(\mathbf{C}, 0)$  et équations différentielles  $ydy + \dots = 0$* , Bull. Soc. Math. France 116 (1988), 459-488
- [EISV] P.M. Elizarov, Yu.S. Il'yashenko, A.A. Shcherbakov, S.M. Voronin, *Finitely generated groups of germs of one-dimensional conformal mappings and invariants for complex singular points of analytic foliations of the complex plane*, Yu.S. Il'yashenko (Ed.) "Nonlinear Stokes Phenomena", Advances in Soviet Mathematics, vol. 14, AMS, Providence, RI (1993), 57-105
- [Lo] F. Loray, *Réduction formelle des singularités cuspidales de champs de vecteurs analytiques*, J. Differential Equations 158 (1999), 152-173
- [LoMe] F. Loray, R. Meziani, *Classification de certains feuilletages associés à un cusp*, Bol. Soc. Brasil. Mat. 21 (1994), 93-106
- [Me] R. Meziani, *Classification analytique d'équations différentielles  $ydy + \dots = 0$  et espace de modules*, Bol. Soc. Brasil. Mat 25 (1996), 23-53
- [St] E. Stróżyńska, *The analytic and formal normal forms for the nilpotent singularity. The generalized saddle-node case*, Bull. Sci. Math. 126 (2002), 555-579
- [StZo1] E. Stróżyńska, H. Żołądek, *The analytic and formal normal forms for the nilpotent singularity*, J. Differential Equations 179 (2002), 479-537
- [StZo2] E. Stróżyńska, H. Żołądek, *Orbital formal normal forms for general Bogdanov-Takens singularity*, J. Differential Equations 193 (2003), 239-259
- [Ta] F. Takens, *Singularities of vector fields*, Publ. Math. IHES 43 (1974), 47-100

## Surfaces which contain many circles

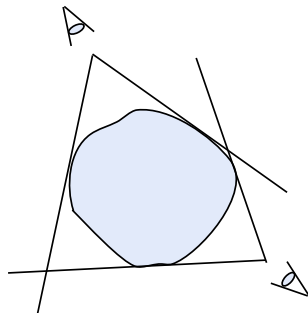
*Nobuko Takeuchi*<sup>1</sup>

### 1 Introduction

"A sphere" is a familiar shape. Most people would describe a sphere as a round object. A sphere always looks round, as we know. But there is a question whether a shape which appears round from any angle is always a sphere?

### 2 Two analyses of shapes which always look round

Part I: If one standing in any position is able to see the object in question as a circular form, then one may decide that the object is a sphere.



Circular cones may be used to distinguish whether the object in question is in fact a circular form. Then the vertex of the cone may be applied at every point. One may also use circular cylinders, if the object in question is examined from a distance.

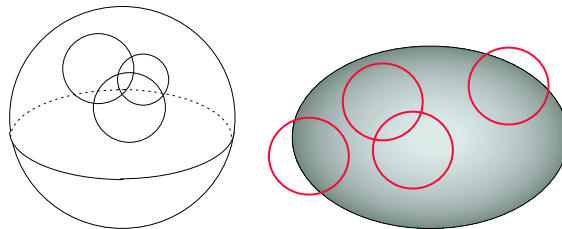
We already know that an object which only looks round from limited positions may not be called a sphere, such as an ellipsoid of revolution.

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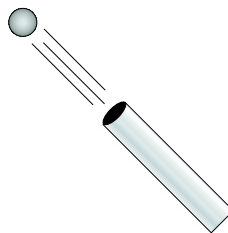
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Professor Shigetake Matsuura dealt with this problem in his 1980-1981 articles published in Japanese in the Tokyo-based Japanese mathematical journal called Suugaku seminar. He concluded that only shapes which appear circular from every angle are spheres.

Part II: If a circle in  $E^3$  of any given radius can be pasted perfectly in any position on the object in question, then one may decide that the object is a sphere.

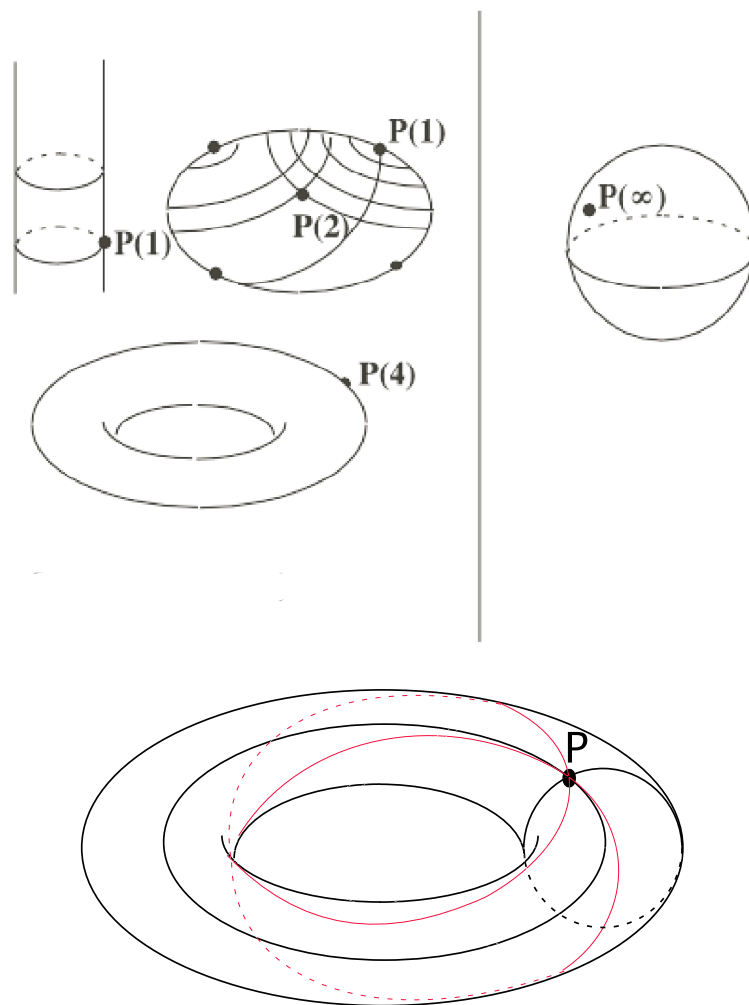


During the war, the Japanese military used cannon balls. The balls needed to be perfect spheres, that is objects which may be viewed as circular from any angle in order to function properly. If they were not perfect spheres, they would fail to hit their target. Only perfect spheres proved to be useful weapons. Subsequently, the military developed a methodology for examining the shapes of the newly made balls. They would paste multiple circles onto the surface of the ball in question. If a circle could be placed on any part of the ball's surface and fit properly, they would conclude that the ball was a perfect sphere and thus include it in their weaponry supplies.



We know that a sphere in  $E^3$  is characterized as a closed surface which contains an infinite number of circles in  $E^3$  through each point. But we do not know a surface other than a sphere or a plane which contains many circles through each point of it.

In the following figures,  $(n)$  is the number of circles through a point  $P$  on the surface.



### 3 Conjectures and theorems

In 1980, Richard Blum found a closed  $C^\infty$  surface of genus one which contains six circles through each point, and he gave a conjecture:

**Conjecture 1 (R. Blum)** *A closed  $C^\infty$  surface in  $E^3$  which contains seven circles through each point is a sphere.*

However he did not produce original affirmative theorems for this conjecture.

In 1984, Koichi Ogiue and Ryoichi Takagi showed that

**Theorem 3.1 (K. Ogiue and R. Takagi).** *A  $C^\infty$  surface in  $E^3$  is (a part of) a plane or a sphere if it contains two circles through each point which are tangent to each other.*

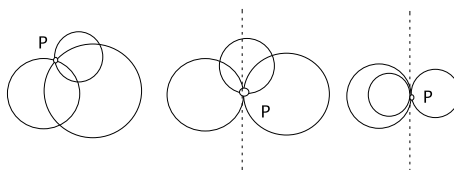
Additionally, drawing from the fact that an ellipsoid contains two circles through each point except only at four points, they postulated that

**Conjecture 2 (K. Ogiue and R. Takagi)** *A simply connected complete  $C^\infty$  surface in  $E^3$  is a plane or a sphere if it contains two circles through each point.*

We have the following partial affirmative results toward conjectures 1 and 2:

**Theorem 3.2.** *A simply connected complete  $C^\infty$  surface in  $E^3$  is a plane or a sphere, if it contains three circles through each point.*

**Theorem 3.3.** *A  $C^\infty$  surface in  $E^3$  is (a part of) a plane or a sphere if it contains three circles through each point, any two of which are tangent to each other or have two points in common.*



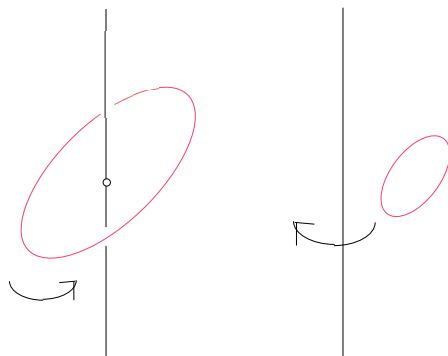
**Theorem 3.4.** *A closed  $C^\infty$  surface of genus one in  $E^3$  cannot contain seven circles through each point.*

Our next theorem usefully demonstrates the theories of our forefathers in their testing of the functionality of cannon balls used during the war.

**Theorem 3.5 (K. Ogiue and N. Takeuchi).** *A smooth ovaloid in  $E^3$  is a sphere if the surface contains a circle of an arbitrary but fixed radius through each point.*

## 4 Examples of surfaces which contain many circles

**Example 1 (Hula hoop surfaces)** *A hula hoop surface is a smooth surface obtained by revolving a circle around a suitable axis.*



Let  $\gamma(a, b, r)$ ,  $r > 0$  be a circle on the  $xy$ -plane defined by

$$(x - a)^2 + (y - b)^2 = r^2$$

and let  $\gamma(a, b, r, \alpha)$  be the circle obtained by tilting  $\gamma(a, b, r)$  around the diameter parallel to the  $x$ -axis by the angle  $\alpha$ ,  $-\frac{\pi}{2} < \alpha \leq \frac{\pi}{2}$ . It is easily seen that  $\gamma(a, b, r, \alpha)$  is given by

$$\begin{aligned} x &= a + r \cos \theta \\ y &= b + r \cos \alpha \sin \theta \\ z &= r \sin \alpha \sin \theta \end{aligned}$$

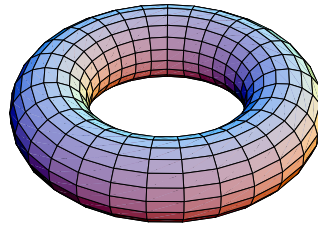
Let  $H(a, b, r, \alpha)$  be the surface obtained by rotating  $\gamma(a, b, r, \alpha)$  around the  $z$ -axis. Then it is easily seen that  $H(a, b, r, \alpha)$  is a smooth surface if and only if  $a = b = 0$  and  $\alpha = \frac{\pi}{2}$  or  $a \neq 0$  and  $(a^2 - r^2) \cos^2 \alpha + b^2 \neq 0$ . We see that  $H(0, 0, r, \frac{\pi}{2})$  is a sphere and otherwise  $H(a, b, r, \alpha)$  is topologically a torus. Note that  $H(a, b, r, \alpha)$  contains at least two circles through each point, one is latitudinal circle and the other is a rotated  $\gamma(a, b, r, \alpha)$ . We denote  $\gamma(a, b, r, \alpha) \sim \gamma(\bar{a}, \bar{b}, \bar{r}, \bar{\alpha})$  when two circles are congruent under the rotation around the  $z$ -axis. For example,  $\gamma(a, b, r, \alpha) \sim \gamma(-a, -b, r, -\alpha)$ . It is clear that if  $\gamma(a, b, r, \alpha) \sim \gamma(\bar{a}, \bar{b}, \bar{r}, \bar{\alpha})$ , then  $H(a, b, r, \alpha) = H(\bar{a}, \bar{b}, \bar{r}, \bar{\alpha})$ . We see that  $H(a, b, r, \alpha)$  is obtained by revolving the curve on the  $xz$ -plane defined by

$$\begin{aligned} x &= \sqrt{(a + r \cos \theta)^2 + (b + r \cos \alpha \sin \theta)^2} \\ y &= 0 \\ z &= r \sin \alpha \sin \theta \end{aligned}$$



and hence that  $H(a, b, r, \alpha)$  is defined by the equation

$$\begin{aligned} (x^2 + y^2 + z^2)^2 - \frac{4b \cos \alpha}{\sin \alpha} (x^2 + y^2 + z^2)z - 2(a^2 + b^2 + r^2)(x^2 + y^2) \\ - 2(a^2 + b^2 + r^2 - \frac{2a^2 + 2b^2 \cos^2 \alpha}{\sin^2 \alpha})z^2 + \frac{4b \cos \alpha}{\sin \alpha} (a^2 + b^2 + r^2)z \\ + (a^2 + b^2 + r^2)^2 - 4a^2r^2 = 0 \end{aligned}$$



Then, we can see a hula hoop surface which is not a sphere contains exactly four or five circles through each point.

**Theorem 4.1.** *A compact smooth surface of revolution which contains at least two circles through each point is a hula hoop surface.*

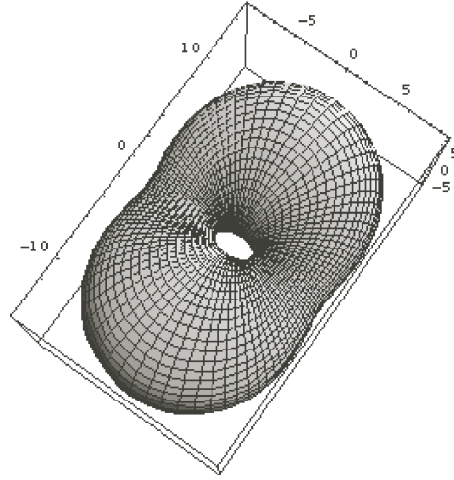
**Corollary 4.2.** *There exists no compact smooth surface of revolution which contains exactly  $k$  circles through each point for  $k = 2, 3, 6, 7$ .*

**Corollary 4.3.** *A compact smooth surface of revolution which contains exactly four circles through each point is an ordinary torus.*

**Example 2 (Blum's surface)** *R. Blum's surfaces are defined by a quartic equation of the form:*

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 + a = 0, (a_1 \geq a_2 > 0, a_3 < -\sqrt{a})$$

*They contain exactly four (if  $a_1 = a_2, a_3 = -\sqrt{a}$ ), five (if  $a_1 = a_2, a_3 \neq -\sqrt{a}$ , or if  $a_1 \neq a_2, a_3 = -\sqrt{a}$ ) or six (if  $a_1 \neq a_2, a_3 \neq -\sqrt{a}$ ) circles through each point.*



Furthermore, we must recognize the fact that cyclides contain many circles.

**Example 3 (Cyclides)** A cyclide is a surface in  $E^3$  defined by a quartic equation of the form:

$$(x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^2 + x_2^2 + x_3^2) \sum_{i=1}^3 b_i x_i + \sum_{i,j=1}^3 a_{ij} x_i x_j + 2 \sum_{i=1}^3 a_i x_i + a = 0$$

An ordinary torus gives a typical example and quadratic surfaces are considered as singular examples. A closed  $C^\infty$  surface of genus one which contains six circles through each point, found by R. Blum, is also one of cyclides.

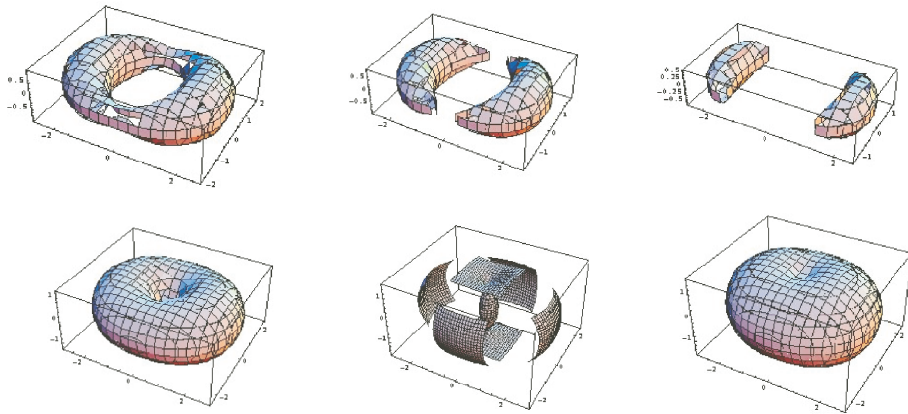
Subsequently we developed the following theorems.

**Theorem 4.4.** A non-singular cyclide is conformally equivalent to a cyclide of the form:

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1 x_1^2 - 2a_2 x_2^2 - 2a_3 x_3^2 + a = 0 \quad (a \neq 0)$$

which is topologically a torus, a sphere or two spheres. A cyclide with singularities is conformally equivalent to a quadratic surface.

**Theorem 4.5.** A cyclide contains  $n$  circles through each non-umbilic point and  $n - 1$  circles through each isolated umbilic point unless it is a sphere or a pair of two spheres, where  $n = 1, 2, 3, 4, 5$  or  $6$ .



## References

- [1] R. Blum, Circles on surfaces in the Euclidean 3-space, Lecture Note in Math. **792**, Springer, 1980 , 213–221.
- [2] K. Ogiue and R. Takagi, A submanifold which contains many extrinsic circles, Tsukuba J. Math. **8** , 1984 , 171–182.
- [3] Z.A. Melzak, Concrete Mathematics, John Wiley, 1973, p. 155.
- [4] T. Cecil, Lie sphere geometry, Springer, 1992.
- [5] G. Darboux, Principes de Géométrie Analytique, Garthier-Villars, 1917.
- [6] J. Milnor, Morse theory, Ann. of Math. Stud. vol.51, Princeton Univ.Press, 1963.
- [7] N. Takeuchi, A sphere as a surface which contains many circles, Journal of Geometry **24** , 1985 , 123–130.
- [8] N. Takeuchi, A closed surface of genus one in  $E^3$  cannot contain seven circles through each point, Proc.AMS. **100** , 1987 , 145–147.
- [9] N. Takeuchi, A sphere as a surface which contains many circles II, Journal of Geometry **34** , 1989 , 195–200.
- [10] K. Ogiue and N. Takeuchi, Hulahoop Surfaces , Journal of Geometry **46** (1993) 127–132.

- [11] K. Ogiue and N. Takeuchi, A sphere can be characterized as a smooth ovaloid which contains one circle through each point, *Journal of Geometry* **49**, 1994, 163–165.
- [12] N. Takeuchi, Cyclides, *Hokkaido Mathematical Journal* , **29**(1), 2000, 119–148.

# The Euler number of the normalization of a certain hypersurface with quasi-ordinary singularities

Shoji Tsuboi<sup>1 2 3</sup>

## Abstract

In [T2] and [T3] we have proved a numerical formula which gives the Euler number of the (non-singular) normalization  $X$  of an algebraic threefold with ordinary singularities  $\bar{X}$  in  $P^4(\mathbf{C})$ . In the proof of this formula, we have used a Lefschetz pencil of hyperplane sections on  $\bar{X}$ , and calculated the *Segre classes* of the singular subscheme of  $\bar{X}$  in order to compute the *class* (number) of  $\bar{X}$ , i.e. the degree of the top *Mather class* of  $\bar{X}$  in  $P^4(\mathbf{C})$ . In this article we will show that this method also works for a wider class of hypersurfaces in  $P^4(\mathbf{C})$  to compute the Euler number of their normalizations.

## 1 An example of a hypersurface with quasi-ordinary singularities in $P^4(\mathbf{C})$

Let  $H_i$  ( $1 \leq i \leq 3$ ) be non-singular hypersurfaces of degrees  $r_i$  ( $1 \leq i \leq 3$ ), respectively, in the complex projective 4-space  $P^4(\mathbf{C})$  such that they are in general position at every point where they intersect. Let  $f_i$  ( $1 \leq i \leq 3$ ) be the homogeneous polynomial of degree  $r_i$  which defines the hypersurface  $H_i$ . We may assume  $r_1 \geq r_2 \geq r_3$  because of symmetry. We choose and fix a positive integer  $n$  with  $n \geq 2r_1 + 2r_2$ . Let  $\bar{X}$  be a hypersurface in  $P^4(\mathbf{C})$  defined by the equation

$$F := Af_1f_2f_3 + B(f_1f_2)^2 + C(f_2f_3)^2 + D(f_3f_1)^2 = 0, \quad (1.1)$$

where  $A, B, C$  and  $D$  are homogeneous polynomials of five variables of respective degrees  $n - r_1 - r_2 - r_3$ ,  $n - 2r_1 - 2r_2$ ,  $n - 2r_2 - 2r_3$  and  $n - 2r_3 - 2r_1$ .

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We put  $D_{\overline{X}}^{(ij)} := H_i \cap H_j$  ( $1 \leq i < j \leq 3$ ) and  $D_{\overline{X}} := \bigcup_{1 \leq i < j \leq 3} D_{\overline{X}}^{(ij)}$ . Then, by Bertini's theorem,  $\overline{X}$  is non-singular outside  $D_{\overline{X}}$  if we choose sufficiently generic  $A, B, C$  and  $D$ .

**Proposition 1.1.** *If the homogeneous polynomials  $A, B, C$  and  $D$  are chosen sufficiently generic, then  $\overline{X}$  is locally isomorphic to one of the following germs of three-dimensional hypersurface singularities at the origin of  $\mathbf{C}^4$  at every point of  $\overline{X}$ :*

- (i)  $w = 0$  (simple point),
- (ii)  $zw = 0$  (ordinary double point),
- (iii)  $yzw = 0$  (ordinary triple point),
- (iv)  $xy^2 - z^2 = 0$  (cuspidal point),
- (v)  $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$  (degenerate ordinary triple point),

where  $(x, y, z, w)$  are the coordinates on  $\mathbf{C}^4$ .

For the proof we refer to [T1].

## 2 The singularity $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$

We consider the following affine threefold:

$$\mathbf{C}^4 \supset T : f := (xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \quad (2.1)$$

where  $(x, y, z, w)$  are the coordinates on  $\mathbf{C}^4$ . As shown in [T1],  $T$  has an ordinary triple point at  $(0, 0, 0, w)$  if  $w \neq 0$ . Hence, we may think of the singularity  $(T, 0)$  of  $T$  at the origin of  $\mathbf{C}^4$  as a *degenerate ordinary triple point*.

Normalization: Let

$$P^3(\mathbf{C}) \supset S : f := (xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \quad (2.2)$$

be the hypersurface in  $P^3(\mathbf{C})$ , defined by the same polynomial  $f$  that defines  $T$ . This surface  $S$  is classically known as the *Steiner surface*. The surface  $S$  is obtained by projecting  $P^2(\mathbf{C})$  embedded in  $P^5(\mathbf{C})$  by the 2-fold Veronese map to  $P^3(\mathbf{C})$ . Indeed, if we denote by  $V$  the image of  $P^2(\mathbf{C})$  in  $P^5(\mathbf{C})$  by the map  $v : P^2(\mathbf{C}) \rightarrow P^5(\mathbf{C})$  defined by

$$\begin{aligned} (\xi_0 : \xi_1 : \xi_2) \in P^2(\mathbf{C}) &\mapsto (\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_1\xi_2) \\ &= (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in P^5(\mathbf{C}), \end{aligned}$$

then the surface  $S$  coincides with the image of  $V$  by the linear projection  $\bar{p} : \mathbf{P}^5(\mathbf{C}) \rightarrow \mathbf{P}^3(\mathbf{C})$  defined by

$$\begin{aligned} (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathbf{P}^5(\mathbf{C}) &\mapsto (y_0 : y_1 : y_2 : -(x_0 + x_1 + x_2)) \\ &= (x : y : z : w) \in \mathbf{P}^3(\mathbf{C}). \end{aligned} \quad (2.3)$$

Applying the calculation similar to that in [T1], we can see that  $S$  is an algebraic surface with *ordinary singularities*, whose singular locus  $D_S$  is  $\{x = y = 0\} \cup \{y = z = 0\} \cup \{z = x = 0\}$ , and that  $S$  has one ordinary triple point at  $[0 : 0 : 0 : 1]$ , six cuspidal points at  $[0 : 0 : \pm 2 : 1]$ ,  $[0 : \pm 2 : 0 : 1]$ ,  $[\pm 2 : 0 : 0 : 1]$ , and ordinary double points at other points of  $D_S$ . We denote by  $C_S$  the *cone* over  $S$ , which is nothing but  $T$ . We denote by  $C_V$  the cone over  $V$ . Since  $V$  is a non-singular, *projectively normal* subvariety in  $\mathbf{P}^5(\mathbf{C})$ ,  $(C_V, 0)$  is a *normal* singular point (cf. [H], Exercise 3.4 (e), p.394). Hence, if we denote by  $p : \mathbf{C}^6 \rightarrow \mathbf{C}^4$  the linear projection induced by  $\bar{p} : \mathbf{P}^5(\mathbf{C}) \rightarrow \mathbf{P}^3(\mathbf{C})$  in (2.3), and by  $n : C_V \rightarrow C_S (= T)$  the restriction of  $p$  to  $C_V$ , then  $n : C_V \rightarrow C_S$  gives the normalization of  $(T, 0)$ .  $(C_V, 0)$  becomes non-singular after a single blowing-up. Indeed, if we denote by  $\hat{\tau}$  the blowing-up  $\widehat{\mathbf{C}}^6 \rightarrow \mathbf{C}^6$  at the origin of  $\mathbf{C}^6$ ,  $\widehat{\mathbf{C}}^6$  can be identified with  $[H_{\mathbf{P}^5(\mathbf{C})}]^{-1}$ , where  $[H_{\mathbf{P}^5(\mathbf{C})}]$  denotes the line bundle on  $\mathbf{P}^5(\mathbf{C})$  determined by a hyperplane  $H_{\mathbf{P}^5(\mathbf{C})}$  in  $\mathbf{P}^5(\mathbf{C})$ . Furthermore, the proper inverse image  $\widehat{C}_V$  of  $C_V$  by  $\tau$  (resp. the exceptional divisor  $E := \tau^{-1}(0)$ ) can be identified with  $[H_{\mathbf{P}^5(\mathbf{C})}]_{|V}^{-1} \simeq [H_{\mathbf{P}^2(\mathbf{C})}]^{-2}$  (resp. the zero cross-section of the line bundle  $L := [H_{\mathbf{P}^2(\mathbf{C})}]^{-2}$  on  $\mathbf{P}^2(\mathbf{C})$ ), where  $[H_{\mathbf{P}^5(\mathbf{C})}]_{|V}^{-1}$  denotes the restriction of  $[H_{\mathbf{P}^5(\mathbf{C})}]^{-1}$  to  $V$ . From this fact, it follows that  $E^2 = -2H_{\mathbf{P}^2(\mathbf{C})}$ , where  $H_{\mathbf{P}^2(\mathbf{C})}$  denotes a hyperplane in  $\mathbf{P}^2(\mathbf{C})$ .

**Theorem 2.1.**  $(C_V, 0)$  is

- (i) *rational, and so Cohen-Macaulay,*
- (ii) *“rigid” under small deformations,*
- (iii) *Gorenstein of index two,*
- (iv) *terminal, and so canonical,*
- (v) *quasi-ordinary, that is there is a finite morphism  $(C_V, 0) \rightarrow \mathbf{C}^3$  whose branching locus is contained in the hypersurface of  $\mathbf{C}^3$  defined by  $x_1 x_2 x_3 = 0$ , where  $(x_1, x_2, x_3)$  denote the coordinates on  $\mathbf{C}^3$*

**Proof:** Here we give only the proof of the assertion (v). For the proofs of the rest of the assertions, we refer to [T1] and [T4]. Let  $X(n, k) := \mathbf{C}^n / \mu_k$ , where  $\mu_k$  is the cyclic group of  $k$ -th root of 1, acting by  $\epsilon : (x_1, \dots, x_n) \rightarrow (\epsilon x_1, \dots, \epsilon x_n)$ . Then the affine cone over the  $k$ -fold Veronese embedding of  $\mathbf{P}^{n-1}(\mathbf{C})$  is isomorphic to  $X(n, k)$ . The map

$$\mathbf{C}^n \ni (x_1, \dots, x_n) \rightarrow (x_1^k, \dots, x_n^k) \in \mathbf{C}^n$$

factors through  $X(n, k)$  and induces a quasi-ordinary projection  $X(n, k) \rightarrow \mathbf{C}^n$ . Since  $(C_V, 0)$  is isomorphic to  $X(3, 2)$ , it is quasi-ordinary.  $\square$

Hypersurface section: Let

$$\begin{aligned} \mathbf{C}^4 &\supset H : w = f(x, y, z), \\ \mathbf{C}^6 &\supset p^*H : x_0 + x_1 + x_2 + f(y_0, y_1, y_2) = 0, \end{aligned}$$

where  $f$  is a sufficiently *generic* holomorphic function defined in a small open neighborhood of the origin with  $f(0, 0, 0) = 0$ , and  $p$  is the linear projection  $\mathbf{C}^6 \rightarrow \mathbf{C}^4$  induced by  $\bar{p} : \mathbf{P}^5(\mathbf{C}) \rightarrow \mathbf{P}^3(\mathbf{C})$  as before. Let

$$\begin{aligned} T \cap H &: (xy)^2 + (yz)^2 + (zx)^2 + xyzf(x, y, z) = 0, \\ C_V \cap p^*H &: \text{the intersection of } C_V \text{ with } p^*H. \end{aligned}$$

**Proposition 2.2.**  $(C_V \cap p^*H, 0)$  is normal, and so  $p|_{C_V \cap p^*H} : (C_V \cap p^*H, 0) \rightarrow (T \cap H, 0)$  gives the normalization of  $(T \cap H, 0)$ .

**Proof:** Since  $x_0 + x_1 + x_2 + f(y_0, y_1, y_2)$  is a non-zero divisor in  $\mathcal{O}_{C_V, 0}$ ,  $\text{Prof } \mathcal{O}_{C_V \cap p^*H, 0} = \text{Prof } \mathcal{O}_{C_V, 0} - 1 = 2$ . Hence  $(C_V \cap p^*H, 0)$  is normal.  $\square$

**Proposition 2.3.**  $(T \cap H, 0)$  becomes a surface with only ordinary double points by the blowing-up at the origin, and a generic hypersurface section of  $(T \cap H, 0)$  is an ordinary quadruple point of a curve.

**Proof:** The tangent cone  $C_0(T \cap H)$  to  $T \cap H$  at the origin 0 of  $\mathbf{C}^4$  is given by

$$(xy)^2 + (yz)^2 + (zx)^2 + xyz(ax + by + cz) = 0, \quad (2.4)$$

where  $a, b, c$  are sufficiently generic complex numbers. We denote by  $\bar{C}$  the curve in  $\mathbf{P}^2(\mathbf{C})$  defined by the equation (2.4). By Bertini's theorem, the curve



$\overline{C}$  is singular only at the three points  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$  and  $[0 : 0 : 1]$ , if we take sufficiently *generic* complex numbers  $a, b, c$ . Furthermore, we may assume that these are ordinary double points. Therefore, since  $C_0(T \cap H)$  is the cone over  $\overline{C}$ , the assertion follows.  $\square$

**Proposition 2.4.**  $C_V \cap p^*C_0(H)$  is isomorphic to the cone over the twisted rational curve of degree 4 in  $P^4(\mathbf{C})$ .

**Proof:** Since the defining equation of  $p^*C_0(H)$  in  $\mathbf{C}^6$  is  $x_0 + x_1 + x_2 + ay_0 + by_1 + cy_2 = 0$ , the tangent cone  $C_0(C_V \cap p^*H)$  to  $C_V \cap p^*H$  at the origin 0 of  $\mathbf{C}^6$  is given by

$$C_V \cap p^*C_0(H) = C_{V \cap \overline{p^*C_0(H)}}, \quad (2.5)$$

where  $\overline{C_0(H)}$  denotes the hyperplane in  $P^3(\mathbf{C})$  defined by  $C_0(H)$ . Note that  $\overline{p^*C_0(H)}$  is nothing but the hyperplane in  $P^5(\mathbf{C})$  defined by  $p^*C_0(H)$ . The pull-back of  $V \cap \overline{C_0(H)}$  by the 2-fold Veronese embedding  $v : P^3(\mathbf{C}) \rightarrow P^5(\mathbf{C})$  is

$$C_Q : Q(\xi_0, \xi_1, \xi_2) = \xi_0 + \xi_1 + \xi_2 + a\xi_0\xi_1 + b\xi_0\xi_2 + c\xi_1\xi_2 = 0.$$

We may assume that the quadric  $C_Q$  is non-singular, since the complex numbers  $a, b, c$  are sufficiently *generic*. Then there exist quadratic forms  $p_0(s, t), p_1(s, t), p_2(s, t)$  of two variables  $s, t$  so that if we define the 2-fold Veronese map  $u$  from  $P^1(\mathbf{C})$  to  $P^2(\mathbf{C})$  by

$$(s : t) \in P^1(\mathbf{C}) \mapsto (p_0(s, t) : p_1(s, t) : p_2(s, t)) = (\xi_0 : \xi_1 : \xi_2) \in P^2(\mathbf{C}), \quad (2.6)$$

then the image of  $P^1(\mathbf{C})$  by the map  $u$  coincides with the quadric  $C_Q$ . The quadratic forms  $p_i$  ( $0 \leq i \leq 2$ ) satisfy

$$p_0^2 + p_1^2 + p_2^2 + ap_0p_1 + bp_0p_2 + cp_1p_2 = 0, \quad (2.7)$$

and the composite map  $v \circ u : P^1(\mathbf{C}) \rightarrow P^5(\mathbf{C})$  gives rise to an isomorphism between  $P^1(\mathbf{C})$  and  $V \cap \overline{p^*C_0(H)}$  in  $P^5(\mathbf{C})$ . The relation among  $p_i p_j$ 's ( $0 \leq i \leq j \leq 2$ ) in (2.7) is the only one linear relation among  $p_i p_j$ 's, because the existence of another linear relation among  $p_i p_j$ 's linearly independent of (2.7) would contradict the fact that the map  $u$  in (2.6) is an embedding. Therefore  $V \cap \overline{p^*C_0(H)}$  is isomorphic to the image of  $P^1(\mathbf{C})$  into  $P^4(\mathbf{C})$  by the 4-fold Veronese map. Then we are done because of (2.5).  $\square$

### 3 Lefschetz pencil on $\overline{X}$ and Euler number of the normalization of $\overline{X}$

Lefschetz pencil on  $\overline{X}$ : Throughout this section we denote by  $\overline{X}$  an irreducible hypersurface in  $P^4(\mathbf{C})$ , which is defined locally at every point of  $\overline{X}$  by one of the equations from (i) through (vi) in Proposition 1.1 with respect to a suitable local holomorphic coordinate system  $(x, y, z, w)$ . We denote by  $\overline{D}$  the double point locus (surface) of  $\overline{X}$ , i.e. the singular locus of  $\overline{X}$ , by  $\overline{T}$  the triple point locus (curve) of  $\overline{X}$ , by  $\overline{C}$  the cuspidal point locus (curve) of  $\overline{X}$ , and by  $\Sigma\overline{q}$  the quadruple point locus of  $\overline{X}$ . Let  $P_\infty$  be a 2-dimensional linear subspace of  $P^4(\mathbf{C})$  such that  $\overline{C}_\infty := P_\infty \cap \overline{X}$  is an irreducible curve with ordinary double points in  $P_\infty \simeq P^2(\mathbf{C})$ . Let  $P$  be a 1-dimensional linear subspace of  $P^4(\mathbf{C})$  situated in twisted position with respect to  $P_\infty$ , i.e. the linear subspace  $L(P_\infty, P)$  generated by  $P_\infty$  and  $P$  coincides with  $P^4(\mathbf{C})$ . Let  $\pi : \overline{X} \setminus \overline{C}_\infty \rightarrow P$  be the linear projection with center  $\overline{C}_\infty$ , i.e.,  $\pi(x) := H_x \cap P$  for  $x \in \overline{X} \setminus \overline{C}_\infty$ , where  $H_x = L(x, P_\infty)$  is the hyperplane generated by  $x$  and  $P_\infty$ . We put  $\overline{X}_\lambda := H_\lambda \cap \overline{X}$  for  $\lambda \in P$  and put  $\overline{\mathcal{L}} := \bigcup_{\lambda \in P} \overline{X}_\lambda$ , which is a linear pencil on  $\overline{X}$  with the base point locus  $Bs(\overline{\mathcal{L}}) = \overline{C}_\infty$ . Let  $n_{\overline{X}} : X \rightarrow \overline{X}$  be the normalization map, and  $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$  the pull-back of  $\overline{\mathcal{L}}$  to  $X$  by  $n_{\overline{X}}$ . By use of the argument similar to that in [T2], we obtained the following:

**Proposition 3.1.** *If we take  $P_\infty$  sufficiently general, then there exists a finite set of points  $Q := \{\lambda_1, \dots, \lambda_q\}$  of  $P$  such that:*

- (i)  $\overline{X}_{\lambda_i}$  contains only one quadruple point of  $\overline{X}$ , which we denote by  $\overline{q}_{\lambda_i}$ , and  $X_{\lambda_i} := n_{\overline{X}}^{-1}(\overline{X}_{\lambda_i})$  is non-singular outside  $n_{\overline{X}}^{-1}(\overline{q}_{\lambda_i})$  for any  $i$  with  $1 \leq i \leq q$ .
- (ii)  $\overline{X}_\lambda$  contains no quadruple point of  $\overline{X}$  for any point  $\lambda \in P \setminus Q$ .
- (iii) There exists a finite set of points  $\{\mu_1, \dots, \mu_c\}$  of  $P \setminus Q$  such that:
  - (a)  $X_\lambda := n_{\overline{X}}^{-1}(\overline{X}_\lambda)$  is non-singular for  $\lambda \in P \setminus Q$  with  $\lambda \neq \mu_i$  ( $1 \leq i \leq c$ ), and
  - (b)  $X_{\mu_i}$  is a surface with only one isolated ordinary double point which is contained in  $X \setminus n_{\overline{X}}^{-1}(\overline{C}_\infty)$  for any  $i$  with  $1 \leq i \leq c$ , where  $c$  is the class (number) of  $\overline{X}$  in  $P^4(\mathbf{C})$ , i.e. the degree of the top polar class  $[M_3]$  of  $\overline{X}$  in  $P^4(\mathbf{C})$ .

In the sequel we assume that the linear pencil  $\overline{\mathcal{L}} := \bigcup_{\lambda \in P} \overline{X}_\lambda$  is such as in Proposition 3.1. Then the linear pencil  $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$ , the pull-back of  $\overline{\mathcal{L}} := \bigcup_{\lambda \in P} \overline{X}_\lambda$  to  $X$  by the normalization map  $n_{\overline{X}} : X \rightarrow \overline{X}$ , has  $C_\infty := n^{-1}(\overline{C}_\infty)$  as its base point locus. Let  $\sigma : \widehat{X} \rightarrow X$  be the blowing-up along  $C_\infty$ , and  $\widehat{\mathcal{L}} := \bigcup_{\lambda \in P} \widehat{X}_\lambda$ , the proper inverse of  $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$ . Then  $\widehat{\mathcal{L}}$  gives a fibering of  $\widehat{X}$  over  $P \simeq P^1(\mathbf{C})$ . Therefore the Euler number  $\chi(\widehat{X})$  of  $\widehat{X}$  is given by

$$\begin{aligned} \chi(\widehat{X}) &= \chi(P^1(\mathbf{C}))\chi(\widehat{X}_\lambda) + \sum_{i=1}^q (\chi(\widehat{X}_{\lambda_i}) - \chi(\widehat{X}_\lambda)) + \sum_{j=1}^c (\chi(\widehat{X}_{\mu_j}) - \chi(\widehat{X}_\lambda)) \\ &= 2\chi(\widehat{X}_\lambda) - c + \sum_{i=1}^q (\chi(\widehat{X}_{\lambda_i}) - \chi(\widehat{X}_\lambda)) \\ &= 2\chi(X_\lambda) - c + \sum_{i=1}^q (\chi(X_{\lambda_i}) - \chi(X_\lambda)), \end{aligned}$$

where  $\widehat{X}_\lambda$  and  $X_\lambda$  denote generic members of  $\widehat{\mathcal{L}}$  and  $\mathcal{L}$  respectively. Here the second equality above follows from the fact that a topological 2-cycle vanishes when  $\lambda \rightarrow \mu_j$  for  $j = 1, \dots, c$ . We put  $\widehat{E} := \sigma^{-1}(C_\infty)$ . Then, since  $\widehat{X} \setminus \widehat{E} \simeq X \setminus C_\infty$ ,

$$\begin{aligned} \chi(\widehat{X}) - \chi(X) &= \chi(\widehat{E}) - \chi(C_\infty) \\ &= \chi(P^1(\mathbf{C}))\chi(C_\infty) - \chi(C_\infty) \\ &= \chi(C_\infty) \end{aligned}$$

Hence,

$$\chi(X) = 2\chi(X_\lambda) - \chi(C_\infty) - c + \sum_{i=1}^q (\chi(X_{\lambda_i}) - \chi(X_\lambda)). \quad (3.1)$$

Since  $X_\lambda$  is the normalization of a surface with ordinary singularities  $\overline{X}_\lambda$  in  $P^3(\mathbf{C})$ , by the classical formula,

$$\chi(X_\lambda) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma,$$

and, since  $C_\infty$  is the normalization of the plane curve  $\overline{C}_\infty$  whose degree is equal to  $n$  and has  $m$  ordinary double points,

$$\chi(C_\infty) = 2 - 2g(C_\infty) = 2 - (n - 1)(n - 2) + 2m,$$

where  $n := \deg \bar{X}$ ,  $m := \deg \bar{D}$ ,  $t := \deg \bar{T}$  and  $\gamma := \deg \bar{C}$ .

**Lefschetz pencil on  $\bar{S}$ :** In the sequel we denote by  $\bar{S}$  one of  $\bar{X}_{\lambda_j}$ , ( $j = 1, \dots, q$ ), i.e. an irreducible hypersurface in  $P^3(\mathbf{C})$  which is locally isomorphic to one of the following germs of two dimensional hypersurface singularities at the origin of  $\mathbf{C}^3$  at every point of  $\bar{S}$ :

- (i)  $z = 0$  (simple point),
- (ii)  $yz = 0$  (ordinary double point),
- (iii)  $xyz = 0$  (ordinary triple point),
- (iv)  $xy^2 - z^2 = 0$  (cuspidal point),
- (v)  $(xy)^2 + (yz)^2 + (zx)^2 + xyz\phi(x, y, z) = 0$  (confluence of three ordinary double points),

where  $(x, y, z)$  are the coordinates on  $\mathbf{C}^3$ , and  $\phi$  is a sufficiently *generic* holomorphic function defined in a small open neighborhood of the origin with  $\phi(0, 0, 0) = 0$ . Furthermore,  $\bar{S}$  has the singularity (v) at just one point. We denote by  $n_{\bar{S}} : S \rightarrow \bar{S}$  the normalization of  $\bar{S}$ . Similarly as in the case of  $\bar{X}$ , we have the following:

**Proposition 3.2.** *There exists a linear pencil of hyperplane sections  $\bar{\mathcal{L}}_S := \bigcup_{\lambda \in P} \bar{S}_\lambda$  ( $P \simeq P^1(\mathbf{C})$ ) on  $\bar{S}$ , satisfying the following conditions:*

- (i) *The base point locus  $Bs(\bar{\mathcal{L}}_S)$  is  $n$  distinct points ( $n := \deg \bar{X}$ ).*
- (ii) *There exists just one point  $\lambda_0 \in P$  such that:*
  - (a)  *$\bar{S}_{\lambda_0}$  is a plane curve of degree  $n$ , having only one ordinary quadruple point, which we denote by  $\bar{q}_{\lambda_0}$ , and  $m-4$  ordinary double points ( $m := \deg \bar{D}$ ) as singularities, and*
  - (b)  *$S_{\lambda_0} := n_{\bar{S}}^{-1}(\bar{S}_{\lambda_0})$  is non-singular outside  $n_{\bar{S}}^{-1}(\bar{q}_{\lambda_0})$ .*
- (iii) *There exists a finite set of points  $\{\mu_1, \dots, \mu_{c_{\bar{S}}}\}$  of  $P - \{\lambda_0\}$  such that:*
  - (c)  *$S_\lambda := n_{\bar{S}}^{-1}(\bar{S}_\lambda)$  is non-singular for any  $\lambda \in P - \{\lambda_0\}$  with  $\lambda \neq \mu_i$  ( $1 \leq i \leq c_{\bar{S}}$ ), and*
  - (d)  *$S_{\mu_i}$  ( $1 \leq i \leq c_{\bar{S}}$ ) is a curve with only one ordinary double point which is not contained in  $S \setminus n_{\bar{S}}^{-1}(Bs(\bar{\mathcal{L}}_S))$ , where  $c_{\bar{S}}$  is the class (number) of  $\bar{S}$  in  $P^3(\mathbf{C})$ , i.e. the degree of the top polar class  $[M_2]$  of  $\bar{S}$  in  $P^3(\mathbf{C})$ .*

By the same argument as in the case of  $\overline{X}$ , we have

$$\chi(S) = 2\chi(S_\lambda) + c_{\overline{S}} - n + (\chi(S_{\lambda_0}) - \chi(S_\lambda)), \quad (3.2)$$

where  $S_\lambda$  denotes a *generic* member of  $\mathcal{L}_S := \bigcup_{\lambda \in P} S_\lambda$ , the pull-back of  $\overline{\mathcal{L}}_S := \bigcup_{\lambda \in P} \overline{S}_\lambda$  to  $S$  by the normalization map  $n_{\overline{S}} : S \rightarrow \overline{S}$ .

**Lemma 3.3.**

$$\chi(S_{\lambda_0}) - \chi(S_\lambda) = 1.$$

**Proof:** We denote by  $n_0 : S_{\lambda_0}^* \rightarrow \overline{S}_{\lambda_0}$  the normalization of  $\overline{S}_{\lambda_0}$ , and by  $\overline{q}_{\lambda_0}$  the quadruple point of  $\overline{S}_{\lambda_0}$ . Since  $\overline{S}_{\lambda_0}$  is a plane curve of degree  $n$  with one ordinary quadruple point and  $m - 4$  ordinary double points, the genus  $g(S_{\lambda_0}^*)$  of the normalization  $S_{\lambda_0}^*$  of  $\overline{S}_{\lambda_0}$  is given by

$$g(S_{\lambda_0}^*) = \frac{1}{2}(n-1)(n-2) - m - 2. \quad (3.3)$$

Hence,

$$\chi(S_{\lambda_0}^*) = 2 - 2g(S_{\lambda_0}^*) = 2 - (n-1)(n-2) + 2(m+2). \quad (3.4)$$

Since  $S_{\lambda_0}$  is obtained by pushing forward the four distinct points  $n_0^{-1}(\overline{q}_{\lambda_0})$  on  $S_{\lambda_0}^*$  to the one point  $n_{\overline{S}_{\lambda_0}}^{-1}(\overline{q}_{\lambda_0})$ ,

$$\chi(S_{\lambda_0}^*) - \chi(S_{\lambda_0}) = 3. \quad (3.5)$$

Hence, by (3.3), (3.4) and (3.5),

$$\chi(S_{\lambda_0}) = 2 - (n-1)(n-2) + 2m + 1. \quad (3.6)$$

On the other hand, since  $\overline{S}_\lambda$  is a plane curve of degree  $n$  with  $m$  ordinary double points,

$$\chi(S_\lambda) = 2 - (n-1)(n-2) + 2m. \quad (3.7)$$

Therefore, by (3.6) and (3.7), we have  $\chi(S_{\lambda_0}) - \chi(S_\lambda) = 1$ .  $\square$

## 4 Calculation of Segre classes of singular subschemes

Throughout this section  $\overline{X}$  and  $\overline{S}$  are the same as in the previous section. In the sequel we will calculate the Segre classes of the singular subscheme of  $\overline{X}$  in  $P^4(\mathbf{C})$  (resp.  $\overline{S}$  in  $P^3(\mathbf{C})$ ) to know the *class* (number)  $c$  of  $\overline{X}$  in  $P^4(\mathbf{C})$  (resp. the *class* (number)  $c_{\overline{S}}$  of  $\overline{S}$  in  $P^3(\mathbf{C})$ ). By Piene's formula ([P]), the polar classes of  $\overline{X}$  (resp.  $\overline{S}$ ) are described by use of its Segre classes. For the definition of Segre classes and their basic properties we refer to [T2].

Segre classes of the singular subscheme of  $\overline{X}$  in  $P^4(\mathbf{C})$ : Throughout this section we fix the notation as follows:

$Y := P^4(\mathbf{C})$  : the complex projective 4-space,

$\overline{J}$  : the singular subscheme of  $\overline{X}$  defined by the Jacobian ideal of  $\overline{X}$ ,

$\overline{D}$  : the double point locus (surface) of  $\overline{X}$ , i.e. the singular locus of  $\overline{X}$ ,

$\overline{T}$  : the triple point locus (curve) of  $\overline{X}$ , which is equal to the singular locus of  $\overline{D}$ ,

$\overline{C}$  : the cuspidal point locus (curve) of  $\overline{X}$ ,

$\Sigma\overline{q}$  : the quadruple point locus of  $\overline{X}$ ,

$n_{\overline{X}} : X \rightarrow \overline{X}$  : the normalization of  $\overline{X}$ ,

$f : X \rightarrow Y$  : the composition of the normalization map  $n_{\overline{X}}$  and the inclusion

$\overline{i} : \overline{X} \hookrightarrow Y$ ,

$J$  : the scheme-theoretic inverse of  $\overline{J}$  by  $f$ ,

$D, T$  and  $C$  : the inverse images of  $\overline{D}, \overline{T}$  and  $C$  by  $f$ , respectively,

$\Sigma q$  : the inverse image of  $\Sigma\overline{q}$  by  $f$ .

We consider the following diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \tau \downarrow & & \downarrow \sigma \\ X & \xrightarrow{f} & Y, \end{array} \quad (4.1)$$

where:

$\sigma : Y' \rightarrow Y$  : the blowing-up of  $Y$  along the quadruple point locus  $\Sigma\overline{q}$  of  $\overline{X}$ ,

$\tau : X' \rightarrow X$  : the blowing-up of  $X$  along  $\Sigma q$ ,

$f' : X' \rightarrow Y'$  : the map which makes the diagram above commute.

We put

$\overline{X}'$  : the proper inverse image of  $\overline{X}$  by  $\sigma$ , which is nothing but  $f'(X')$ ,

$\bar{J}'$ : the singular subscheme of  $\bar{X}'$  defined by the Jacobian ideal of  $\bar{X}'$ ,  
 $\bar{D}'$ ,  $\bar{T}'$  and  $\bar{C}'$ : the proper inverse images of  $\bar{D}$ ,  $\bar{T}$  and  $\bar{C}$  by  $\sigma$ , respectively,  
 $\bar{E} := \sigma^{-1}(\Sigma\bar{q}) = \Sigma_{\bar{q}}E_{\bar{q}}$ : the exceptional divisor of the blowing-up  $\sigma$ , where  
 $\bar{E}_q$  denotes the exceptional divisor corresponding to a quadruple point  $\bar{q}$ ,  
 $D'$ ,  $T'$  and  $C'$ : the inverse images of  $D$ ,  $T$  and  $C$  by  $\tau$ , respectively, which  
are nothing but the inverse images of  $\bar{D}'$ ,  $\bar{T}'$  and  $\bar{C}'$  by  $f'$ , respectively.

Note that  $\bar{X}'$  is a threefold with ordinary singularities. We denote by  $\bar{J}''$  the scheme-theoretic inverse of  $\bar{J}$  by the map  $\sigma_{|\bar{X}'} : \bar{X}' \rightarrow \bar{X}$ , the restriction of  $\sigma$  to  $\bar{X}'$ . Calculating directly by use of local coordinates, we have

$$\bar{J}'' = \bar{J}' + 3\bar{X}' \cdot \bar{E}.$$

We denote by  $J''$  the scheme-theoretic inverse of  $\bar{J}''$  by the map  $f''$ . Then we have

$$J'' = D' + 3E + C',$$

which also comes from a direct calculation, using the concrete description of the map  $f''$  in terms of local coordinates. If we put  $D'' := D' + 3E$ , then by the formula (3.1) in [T2] (p. 284), we have the following equalities concerning the Segre classes of  $J''$  in  $X'$ :

$$\begin{cases} s(J'', X')_2 = [D''] \\ s(J'', X')_1 = -[D'']^2 + [C'] \\ s(J'', X')_0 = [D'']^3 - c_1(N_{C'/X'}) \cap [C'] - 3D'' \cdot C' \end{cases} \quad (4.2)$$

Since

$$\begin{aligned} [D'']^2 &= [D']^2 + 6D' \cdot E + 9[E]^2, \\ [D'']^3 &= [D']^3 + 9[D']^2 \cdot E + 27D' \cdot [E]^2 + 27[E]^3, \\ D'' \cdot C' &= D' \cdot C' + 3E \cdot C', \end{aligned}$$

and since

$$\begin{cases} f'_*[D']^2 = \bar{X}' \cdot \bar{D}' + 3\bar{T}' - \bar{C}', \\ f'_*[D']^3 = [\bar{X}']^2 \cdot [\bar{D}'] - 2[\bar{D}']^2 + 5\bar{X}' \cdot \bar{T}' - \bar{X}' \cdot \bar{C}', \\ f'_*(c_1(N_{C'/X'}) \cap [C']) = -K_{Y'} \cdot \bar{C}' - \bar{X}' \cdot \bar{C}' + k_{\bar{C}'}, \\ f'_*(D' \cdot C') = 0, \end{cases}$$

which are the results in [T2], pushing forward the Segre classes  $s(J'', X')_i$  ( $0 \leq i \leq 2$ ) in (4.2), we have the following:

**Proposition 4.1.** *The Segre classes of the subscheme  $\bar{J}''$  in  $\bar{X}'$  are given as follows:*

- (i)  $s(\bar{J}'', \bar{X}')_2 = 2\bar{D}' + 3\bar{X}' \cdot \bar{E}$ ,
- (ii)  $s(\bar{J}'', \bar{X}')_1 = -\bar{X}' \cdot \bar{D}' - 3\bar{T}' + 2\bar{C}' - 12\bar{D}' \cdot \bar{E} - 9\bar{X}' \cdot [\bar{E}]^2$ ,
- (iii)  $s(\bar{J}'', \bar{X}')_0 = [\bar{X}']^2 \cdot \bar{D}' - 2[\bar{D}']^2 + 5\bar{X}' \cdot \bar{T}' + K_{Y'} \cdot \bar{C}' - [k_{\bar{C}'}] + 9\bar{X}' \cdot \bar{D}' \cdot \bar{E} + 27\bar{T}' \cdot \bar{E} - 18\bar{C}' \cdot \bar{E} + 54\bar{D}' \cdot [\bar{E}]^2 + 27\bar{X}' \cdot [\bar{E}]^3$ ,

where  $K_{Y'}$  is the canonical divisor of  $Y'$ , and  $k_{\bar{C}'}$  is that of  $\bar{C}'$ .

**Lemma 4.2.**

- (i)  $\bar{X}' \cdot \bar{E} = 4j_*(\Sigma_{\bar{q}}[H_{\bar{q}}])$ ,      (ii)  $\bar{D}' \cdot \bar{E} = 3j_*(\Sigma_{\bar{q}}[H_{\bar{q}}]^2)$ ,
- (iii)  $\bar{T}' \cdot \bar{E} = j_*(\Sigma_{\bar{q}}[H_{\bar{q}}]^3)$ ,      (iv)  $\bar{C}' \cdot \bar{E} = 6j_*(\Sigma_{\bar{q}}[H_{\bar{q}}]^3)$ ,

where  $H_{\bar{q}}$  denotes a hyperplane in  $E_{\bar{q}}$ , the exceptional divisor corresponding to a quadruple point  $\bar{q}$  of  $\bar{X}$ , and  $j : \bar{E} \hookrightarrow Y'$  the inclusion map.

**Proof:**  $\bar{X}$  is locally isomorphic to the cone over the Steiner surface  $\bar{S}$  at a quadruple point  $\bar{q}$ , and so the assertions follows.  $\square$

Since the multiplicity of  $\bar{X}$  at each quadruple point  $\bar{q}$  is four, we have

$$\sigma^*[\bar{X}] = \bar{X}' + 4\bar{E},$$

and since the multiplicity of  $\bar{D}$  at each quadruple point  $\bar{q}$  is 3, by the *blow-up formula* ([F], Theorem 6.7, p.116 and Corollary 6.7, p.117), we have

$$\sigma^*[\bar{D}] = \bar{D}' + 3j_*[\Sigma_{\bar{q}}H_{\bar{q}}].$$

Calculating push-forward of the Segre classes  $s(\bar{J}'', \bar{X}')_i$  ( $0 \leq i \leq 2$ ) in Proposition 4.1 by  $\sigma$ , using the facts above and Lemma 4.2, we have the following:

**Proposition 4.3.** *The Segre classes of the singular subscheme  $\bar{J}$  of  $\bar{X}$  are given as follows:*

- (i)  $s(\bar{J}, \bar{X})_2 = 2\bar{D}$ ,
- (ii)  $s(\bar{J}, \bar{X})_1 = -\bar{X} \cdot \bar{D} - 3\bar{T} + 2\bar{C}$ ,



$$(iii) \quad s(\overline{J}, \overline{X})_0 = [\overline{X}]^2 \cdot \overline{D} - 2[\overline{D}]^2 + 5\overline{X} \cdot \overline{T} + K_Y \cdot \overline{C} - \sigma_*[k_{\overline{C}'}] - 59[\Sigma\overline{q}],$$

where  $K_Y$  is the canonical divisor of  $Y$ , and  $\sigma_*[k_{\overline{C}'}]$  is the direct image of the canonical divisor  $k_{\overline{C}'}$  of  $\overline{C}'$  by the map  $\sigma|_{\overline{C}'} : \overline{C}' \rightarrow \overline{C}$ .

**Corollary 4.4.** *Let  $\overline{X}_0$  be a hypersurface in  $P^4(\mathbf{C})$  whose degrees of the various singular loci are the same as those of  $\overline{X}$  we are considering in this article, but without quadruple points. Then:*

$$c - c_0 = \deg[k_{\overline{C}'}] - \deg[k_{\overline{C}_0}] + 59\#[\Sigma\overline{q}],$$

where  $c$  (resp.  $c_0$ ) denotes the class (number) of  $\overline{X}$  (resp.  $\overline{X}_0$ ) in  $P^4(\mathbf{C})$ ,  $\overline{C}_0$  the cuspidal point locus (curve) of  $\overline{X}_0$ , and  $k_{\overline{C}_0}$  the canonical divisor of  $\overline{C}_0$ .

**Proof:** By Piene's formula,

$$c = (n-1)^3 \deg \overline{X} - 3(n-1)^2 \deg s_2 - 3(n-1) \deg s_1 - \deg s_0$$

Hence, by Proposition 4.3 above and Proposition 3.6 in [T2], the assertion follows.  $\square$

Segre classes of the singular subscheme of  $\overline{S}$  in  $P^3(\mathbf{C})$ : To calculate the Segre classes of the singular subscheme of  $\overline{S}$ , we consider the following diagram instead of the diagram (4.1):

$$\begin{array}{ccc} S' & \xrightarrow{g'} & Z' \\ \tau_S \downarrow & & \downarrow \sigma_S \\ \overline{S} & \xrightarrow{g} & Z, \end{array} \quad (4.3)$$

where:

$Z := P^3(\mathbf{C})$ ,

$g : S \rightarrow Z$  : the composite of the normalization map  $n_S : S \rightarrow \overline{S}$  and the inclusion  $\bar{\iota} : \overline{S} \hookrightarrow Z$ ,

$\sigma_S : Z' \rightarrow Z$  : the blowing-up of  $Z$  at the quadruple point  $\overline{q}$  of  $\overline{S}$ ,

$\tau_S : S' \rightarrow S$  : the blowing-up of  $S$  at  $q := g^{-1}(\overline{q})$ ,

$g' : S' \rightarrow Z'$  : the map which makes the diagram (4.3) commute.

We put

$\overline{D}_S$  : the double point locus (curve) of  $\overline{S}$ , i.e. the singular locus of  $\overline{S}$ ,

$\Sigma\bar{\iota}$  : the triple point locus of  $\overline{S}$ , which is equal to the singular locus of  $\overline{D}_S$ ,

$\Sigma\bar{c}$  : the cuspidal point locus of  $\bar{S}$ ,  
 $\bar{S}'$  : the proper inverse image of  $\bar{S}$  by  $\sigma_S$ , which is nothing but  $g'(S')$ ,  
 $\bar{D}'_S, \Sigma\bar{t}'$  and  $\Sigma\bar{c}'$ : the proper inverse images of  $\bar{D}_S, \Sigma\bar{t}$  and  $\Sigma\bar{c}$  by  $\sigma_S$ , respectively,  
 $D_S, \Sigma t$  and  $\Sigma c$ : the proper inverse images of  $\bar{D}_S, \Sigma\bar{t}$  and  $\Sigma\bar{c}$  by  $g$ , respectively,  
 $D'_S, \Sigma t'$  and  $\Sigma c'$ : the proper inverse images of  $D_S, \Sigma t$  and  $\Sigma c$  by  $\tau_S$ , respectively, which is nothing but the proper inverse images of  $\bar{D}'_S, \Sigma\bar{t}'$  and  $\Sigma\bar{c}'$  by  $g'$ , respectively.  
 $\bar{J}, \bar{J}', J, J', \bar{E}$  and  $E$  for  $\bar{S}$  are similarly defined as in the case of  $\bar{X}$ .

Note that  $\bar{S}'$  is an algebraic surface with ordinary singularities. In the sequel we assume that  $\bar{S}$  is defined by the equation in (2.4) at each quadruple point of  $\bar{S}$ . We are allowed to do this, because the Segre classes of a singular subscheme depend only on its tangent cone. We denote by  $\bar{J}''$  the scheme-theoretic inverse of  $\bar{J}$  by the map  $\sigma_{S|\bar{S}'} : \bar{S}' \rightarrow \bar{S}$ , the restriction of  $\sigma_S$  to  $\bar{S}'$ . Calculating directly by use of local coordinates, we have

$$\bar{J}'' = \bar{J}' + 3\bar{S}' \cdot \bar{E}.$$

We denote by  $J''$  the scheme-theoretic inverse of  $\bar{J}''$  by the map  $g'$ . Then we have

$$J'' = D'_S + 3E + \Sigma c',$$

which also comes from a direct calculation, using the concrete description of the map  $g'$  in terms of local coordinates. If we put  $D''_S := D'_S + 3E$ , then by Proposition 3.2 in [T2] (p. 284) ([F], Proposition 9.2, p.161), we have the following equalities concerning the Segre classes of  $J''$  in  $S'$ :

$$\begin{cases} s(J'', S')_1 = [D''_S] \\ s(J'', S')_0 = -[D''_S]^2 + [\Sigma c'] \end{cases} \quad (4.4)$$

**Proposition 4.5.** *The Segre classes of the subscheme  $\bar{J}''$  in  $\bar{S}'$  are given as follows:*

- (i)  $s(\bar{J}'', \bar{S}')_1 = 2[\bar{D}'_S] + 3\bar{S}' \cdot \bar{E}$ ,
- (ii)  $s(\bar{J}'', \bar{S}')_0 = -\bar{S}' \cdot \bar{D}'_S - 3[\Sigma\bar{t}'] + 2[\Sigma\bar{c}'] - 12\bar{D}'_S \cdot \bar{E} - 9\bar{S}' \cdot [\bar{E}]^2$ .

**Proof:**

$$\begin{aligned}
s(\overline{J}, \overline{S}')_1 &= g'_* s(\overline{J}'', \overline{S}')_1 \\
&= g'_*[D''_s] = g'_*[D'_s] + 3g'_*[E] \\
&= 2[\overline{D}'_S] + 3\overline{S}' \cdot \overline{E}.
\end{aligned}$$

This proves the assertion (i). Since  $\overline{S}'$  is an algebraic surface with ordinary singularities, we have

$$g'_*[D'_s]^2 = \overline{S}' \cdot \overline{D}'_S - [\Sigma \overline{c}'_S] + 3[\Sigma \overline{t}'_S]$$

(cf. [F], Example 9.3.7, p.168). Hence

$$\begin{aligned}
g'_*[D''_s]^2 &= g'_*[D'_s + 3E]^2 \\
&= g'_*[D'_s]^2 + 6g'_*[D'_s] + 9g'_*[E]^2 \\
&= \overline{S}' \cdot \overline{D}'_S - [\Sigma \overline{c}'_S] + 3[\Sigma \overline{t}'_S] + 12\overline{D}'_S \cdot \overline{E} + 9\overline{S}' \cdot [\overline{E}]^2.
\end{aligned}$$

Therefore, calculating push-forward of  $s(\overline{J}'', \overline{S}')_0$  in (4.4) by  $g'$ , we have the assertion (ii).  $\square$

**Lemma 4.6.**

$$(i) \quad \overline{S}' \cdot \overline{E} = 4k_*[\overline{H}], \quad (ii) \quad \overline{D}'_S \cdot \overline{E} = 3k_*[\overline{H}]^2,$$

where  $\overline{H}$  denotes a hyperplane in  $\overline{E}$ , the exceptional divisor corresponding to the quadruple point  $\overline{q}$  of  $\overline{S}$ , and  $k : \overline{E} \hookrightarrow Z'$  the inclusion map.

**Proof:**  $\overline{S}$  can be considered to be locally isomorphic to the cone over a plane curve  $\overline{C}$  of degree four which has three ordinary double points at the quadruple point  $\overline{q}$  (cf. Proposition 2.3), and so the assertions follow.  $\square$

Since the multiplicity of  $\overline{S}$  at the quadruple point  $\overline{q}$  is four, we have

$$\sigma^*[\overline{S}] = \overline{S}' + 4\overline{E}.$$

Calculating push-forward of the Segre classes  $s(\overline{J}', \overline{S}')_i$  ( $0 \leq i \leq 1$ ) in Proposition 4.5 by  $\sigma$ , using the fact above and Lemma 4.6, we have the following:

**Proposition 4.7.** *The Segre classes of the singular subscheme  $\overline{J}$  of  $\overline{S}$  are given as follows:*

- (i)  $s(\overline{J}, \overline{S})_1 = 2\overline{D}_S,$
- (ii)  $s(\overline{J}, \overline{S})_0 = -\overline{S} \cdot \overline{D}_S - 3[\Sigma\overline{t}] + 2[\Sigma\overline{c}] + 12[\overline{q}].$

**Corollary 4.8.** *The effect of the existence of the quadruple point  $\overline{q}$  to the class (number)  $c_{\overline{S}}$  of  $\overline{S}$  in  $P^3(\mathbf{C})$  is  $-12$ .*

**Proof:** By Piene's formula,

$$c_{\overline{S}} = (n - 1)^2 \deg \overline{S} - 2(n - 1) \deg s_1 - \deg s_0$$

Therefore, by Proposition 4.7 the assertion follows.  $\square$

**Corollary 4.9.** *Let  $X_{\lambda_i}$  be a member of the linear system  $\mathcal{L} := \bigcup_{\lambda \in P} X_{\lambda}$  for which  $\overline{X}_{\lambda_i}$  containing a quadruple point of  $\overline{X}$ , and  $X_{\lambda}$  a generic member of  $\mathcal{L}$ . Then*

$$\chi(X_{\lambda_i}) - \chi(X_{\lambda}) = -11$$

**Proof:** We set  $S := X_{\lambda_i}$ ,  $\overline{S} := \overline{X}_{\lambda_i}$ ,  $S_0 := X_{\lambda}$  and  $\overline{S}_0 := \overline{X}_{\lambda}$ . Then by (3.2),

$$\chi(S) - \chi(S_0) = c_{\overline{S}} - c_{\overline{S}_0} + \chi(S_{\lambda_0}) - \chi(S_{\lambda}),$$

where  $c_{\overline{S}}$ ,  $c_{\overline{S}_0}$  are the class numbers of  $\overline{S}$  and  $\overline{S}_0$  in  $P^3(\mathbf{C})$ , respectively,  $S_{\lambda_0}$  is the member of the linear pencil  $\mathcal{L}_S := \bigcup_{\lambda \in P} S_{\lambda}$  for which  $\overline{S}_{\lambda_0}$  containing the quadruple point of  $\overline{S}$ , and  $S_{\lambda}$  is a generic member of  $\mathcal{L}_S$ . Therefore, by Corollary 4.8 and Lemma 3.3, we obtain the assertion.  $\square$

## 5 A conclusion

**Theorem 5.1.** *Let  $\overline{X}_0$  be a hypersurface in  $P^4(\mathbf{C})$  whose degrees of the various singular loci are the same as those of  $\overline{X}$  we are considering in this article, but without quadruple points, and let  $X_0$  be the normal model of  $\overline{X}_0$ . Then:*

$$\chi(X) - \chi(X_0) = \deg k_{\overline{C}_0} - \deg k_{\overline{C}'} - 70\#[\Sigma\overline{q}],$$

where  $k_{\overline{C}_0}$  is the canonical divisor of the cuspidal point locus (curve)  $\overline{C}_0$  of  $\overline{X}_0$ , and  $k_{\overline{C}'}$  is that of the normal model  $\overline{C}'$  of the cuspidal point locus (curve)  $\overline{C}$  of  $\overline{X}$ .

**Proof:** By (3.1), Corollary 4.4 and Corollary 4.9,

$$\begin{aligned}\chi(X) - \chi(X_0) &= -(c - c_0) + \sum_{i=1}^q (\chi(X_{\lambda_i}) - \chi(X_{\lambda})) \\ &= -(\deg k_{\overline{C}'} - \deg k_{\overline{C}_0} + 59\#[\Sigma\overline{q}]) - 11\#[\Sigma\overline{q}]\end{aligned}$$

where  $c$  and  $c_0$  are the *class numbers* of  $\overline{X}$  and  $\overline{X}_0$  in  $P^4(\mathbf{C})$  respectively.  $\square$

## References

- [A-S] Aroca F. and Snoussi J., *Normal quasi-ordinary singularities*, Séminaires et Congrès **10**, Soc. Math. France, 1-9 (2005)
- [F] Fulton W., *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 2, A series of Modern Surveys in Mathematics, Springer (1984)
- [P] Piene R., *Polar classes of singular varieties*, Ann. scient. Éc. Norm. Sup., 4<sup>e</sup> série, t. 11, 247-276 (1978)
- [S] Schlessinger M., *Rigid singularities*, Proceedings of the Conference on Complex Analysis, Rice University Studies, 59, No. 1, 147-162 (1973)
- [H] Hartshorne R., *Algebraic Geometry*, Springer, New York, (1977)
- [T1] Tsuboi S., *On certain hypersurfaces with non-isolated singularities in  $P^4(\mathbf{C})$* , Proc. Japan Acad., 79A, No. 1, 1-4 (2003)
- [T2] Tsuboi S., *The Euler number of the normalization of an algebraic threefold with ordinary singularities*, “Geometric Singularity Theory”, Banach Center Publications **65**, Polish Academy of Sciences, Warszawa, 273-289 (2004)
- [T3] Tsuboi S., *The Chern numbers of the normalization of an algebraic threefold with ordinary singularities*, Séminaires et Congrès **10**, Soc. Math. France, 351-372 (2005)
- [T4] Tsuboi S., *Normalization of a certain degenerate ordinary triple point of dimension three*, to appear in the Proceedings of the 12th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications, Kyushu Univ. Press (2005)

# A new projective invariant associated to the special parabolic points of surfaces and to swallowtails

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**Abstract.** We show some generic (robust) properties of smooth surfaces immersed in the real 3-space (Euclidean, affine or projective), in the neighbourhood of a *godron*: an isolated parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. With the help of these properties and a projective invariant that we associate to each godron we present all possible local configurations of the *flecnodal curve* at a generic swallowtail in  $\mathbb{R}^3$ . We present some global results, for instance: *In a hyperbolic disc of a generic smooth surface, the flecnodal curve has an odd number of transverse self-intersections.*

## 1 Introduction

A generic smooth surface in  $\mathbb{R}^3$  has three (possibly empty) parts: an open *hyperbolic domain* at which the Gaussian curvature  $K$  is negative, an open *elliptic domain* at which  $K$  is positive and a *parabolic curve* at which  $K$  vanishes. A *godron* is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. We present various robust geometric properties of generic surfaces, associated to the godrons. For example (Theorem 2):

*Any smooth curve of a surface of  $\mathbb{R}^3$  tangent to the parabolic curve at a godron  $g$  has at least 4-point contact with the tangent plane of the surface at  $g$ .*

The line formed by the inflection points of the asymptotic curves in the hyperbolic domain is called *flecnodal curve*. The next theorem is well known.

**Theorem 1.** ([14, 10, 13, 7, 9, 5]) *At a godron of a generic smooth surface the flecnodal curve is (simply) tangent to the parabolic curve.*

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For a generic smooth surface we have the following global result (Proposition 5 and Theorem 6):

*A closed parabolic curve bounding a hyperbolic disc has a positive even number of godrons, and the flecnodal curve lying in that disc has an odd number of transverse self-intersections.*

The *conodal curve* of a surface  $S$  is the closure of the locus of points of contact of  $S$  with its *bitangent planes* (planes which are tangent to  $S$  at least at two distinct points). It is well known ([14, 10]) that:

*At a godron of a generic smooth surface the conodal curve is (simply) tangent to the parabolic curve.*

So the parabolic, flecnodal and conodal curves of a surface are mutually tangent at the godrons. At each godron, these three tangent curves determine a projective invariant  $\rho$ , as a cross-ratio (see the cr-invariant below). We show all possible configurations of these curves at a godron, according to the value of  $\rho$  (Theorem 4). There are six generic configurations, see Fig. 2.

The invariant  $\rho$  and the geometric properties of the godrons presented here are useful for the study of the local affine (projective) differential properties of swallowtails. So, for example, we present all generic configurations of the flecnodal curve in the neighbourhood of a swallowtail point of a surface of  $\mathbb{R}^3$  in general position (see Theorem 8 and Fig. 7).

The paper is organised as follows. In Section 2, we recall the classification of points of a generic smooth surface in terms of the order of contact of the surface with its tangent lines. In Section 3, we give some definitions and present our results. Finally, in Section 4, we give the proofs of the theorems.

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## 2 Projective properties of smooth surfaces

The points of a generic smooth surface in the real 3-space (projective, affine or Euclidean) are classified in terms of the contact of the surface with its tangent lines. In this section, we recall this classification and some terminology.

A generic smooth surface  $S$  is divided in three (possibly empty) parts:

- (E) An open *domain of elliptic points*: there is no real tangent line exceeding 2-point contact with  $S$ ;
- (H) An open *domain of hyperbolic points*: there are two such lines, called

*asymptotic lines* (their directions at the point of tangency are called *asymptotic directions*); and

(P) A smooth *curve of parabolic points*: a unique, but double, asymptotic line.

The *parabolic curve*, divides  $S$  into the *elliptic* and *hyperbolic domains*.

In the closure of the hyperbolic domain there is:

(F) A smooth immersed *flecnodal curve*: it is formed by the points at which an asymptotic tangent line exceeds 3-point contact with  $S$ .

One may also encounter isolated points of the following four types: (g) A *godron* is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve; (sh) A *special hyperbolic point* is a point of the simplest self-intersection of the flecnodal curve; (b) A *biflection point* is a point of the flecnodal curve at which one asymptotic tangent exceeds 4-point contact with  $S$ ; (se) A *special elliptic point* is a real point in the elliptic domain of the simplest self-intersection of the complex conjugate flecnodal curves associated to the complex conjugate asymptotic lines. In Fig. 1 the hyperbolic domain is represented in gray colour and the elliptic one in white. The flecnodal curve has a left branch  $F_l$  (white) and a right branch  $F_r$  (black). These branches will be defined in the next section.

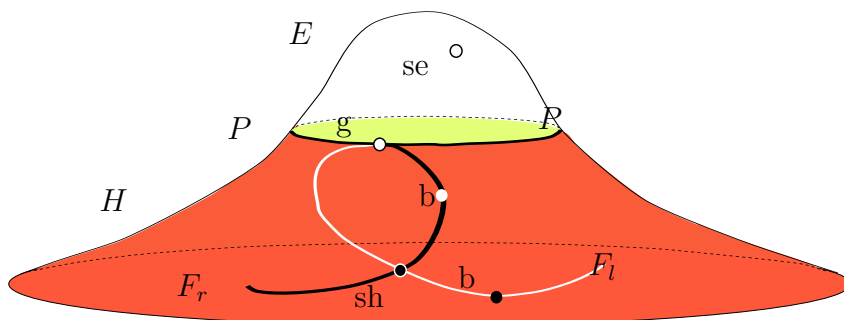


Figure 1: The 8 tangential singularities of a generic smooth surface.

The term “godron” is due to R. Thom [9]. In other papers one can find the terms “special parabolic point” or “cusp of the Gauss map”. We keep Thom’s terminology since it is shorter. Here we will study the local projective differential properties of the godrons.

The above 8 classes of tangential singularities, Theorem 1 and all the theorems presented in this paper are projectively invariant and are robust features of a smooth surface, that is, they are stable in the sense that under a sufficiently small perturbation (taking derivatives into account) they do not vanish but only deform slightly. Seven of these classes were known at the end of the 19th century in the context of the enumerative geometry of



complex algebraic surfaces, with prominent works of Cayley, Zeuthen and Salmon, see [14]. For these seven classes, the normal forms of surfaces at such points up to the 5-jet, under the group of projective transformations, were independently found by E.E. Landis ([7]) and O.A. Platonova ([13]). The special elliptic points were found by D. Panov ([8]).

For surfaces in  $\mathbb{R}^3$ , these tangential singularities depend only on the affine structure of  $\mathbb{R}^3$  (because they depend only on the contact with lines), that is, they are independent of any Euclidean structure defined on  $\mathbb{R}^3$  and of the Gaussian curvature of the surface which could be induced by such a Euclidean structure.

Besides the smooth surfaces, we also consider surfaces admitting wave front singularities (Section 3.6) and we study the behaviour of the flecnodal curve near the swallowtail points.

### 3 Statement of results

Consider the pair of fields of asymptotic directions in the hyperbolic domain. An *asymptotic curve* is an integral curve of a field of asymptotic directions.

**Left and right asymptotic and flecnodal curves.** Fix an orientation in the 3-space  $\mathbb{R}P^3$  (or in  $\mathbb{R}^3$ ). The two asymptotic curves passing through a point of the hyperbolic domain of a generic smooth surface can be distinguished in a natural geometric way: One twists like a left screw and the other like a right screw. More precisely, a regularly parametrised smooth curve is said to be a *left (right) curve* if its first three derivatives at each point form a negative (resp. a positive) frame.

**Proposition 1.** *At a hyperbolic point of a surface one asymptotic curve is left and the other one is right.*

A proof is given (for generic surfaces) in Euclidean Remark below.

The hyperbolic domain is therefore foliated by a family of left asymptotic curves and by a family of right asymptotic curves. The corresponding asymptotic tangent lines are called respectively *left* and *right asymptotic lines*.

**Definition 1.** The *left (right) flecnodal curve*  $F_l$  (resp.  $F_r$ ) of a surface  $S$  consists of the points of the flecnodal curve of  $S$  whose asymptotic line, having higher order of contact with  $S$ , is a left (resp. right) asymptotic line.

The following statement (complement to Theorem 1) is used and implicitly proved (almost explicitly) in [16, 18]. A proof is given in Section 4, see Fig. 1:

**Proposition 2.** *A godron separates locally the flecnodal curve into its right and left branches.*

**Definition 2.** A *flattening* of a generic curve is a point at which the first three derivatives are linearly dependent. Equivalently, a flattening is a point at which the curve has at least 4-point contact with its osculating plane.

The flattenings of a generic curve are isolated points separating the right and left intervals of that curve.

**Euclidean Remark.** *If we fix an arbitrary Euclidean structure in the affine oriented space  $\mathbb{R}^3$ , then the lengths of the vectors and the angles between vectors are defined. Therefore, for such Euclidean structure, the torsion  $\tau$  of a curve and the Gaussian curvature  $K$  of a surface are defined. In this case a point of a curve is right, left or flattening if the torsion at that point satisfies  $\tau > 0$ ,  $\tau < 0$  or  $\tau = 0$ , respectively. The Gaussian curvature  $K$  on the hyperbolic domain of a smooth surface is negative. The Beltrami-Enepper Theorem states that the values of the torsion of the two asymptotic curves passing through a hyperbolic point with Gaussian curvature  $K$  are given by  $\tau = \pm\sqrt{-K}$ . This proves Proposition 1.*

**Definition 3.** An *inflection* of a (regularly parametrised) smooth curve is a point at which the first two derivatives are linearly dependent. Equivalently, an inflection is a point at which the curve has at least 3-point contact with its tangent line.

A generic curve in the affine space  $\mathbb{R}^3$  has no inflection. However, a generic 1-parameter family of curves can have isolated parameter values for which the corresponding curve has one isolated inflection.

**Theorem 2.** *Let  $S$  be a generic smooth surface. All smooth curves of  $S$  which are tangent to the parabolic curve at a godron  $g$  have either a flattening or an inflection at  $g$ , and their osculating plane is the tangent plane of  $S$  at  $g$ .*

The proof of Theorem 2 is given in Section 4.

**Corollary 1.** *The godrons of a generic smooth surface are flattenings of its parabolic and flecnodal curves.*

**Remark 1.** *The converse is not true: A flattening of the parabolic curve is not necessarily a godron.*

### 3.1 The cr-invariant and classification of godrons

**The conodal curve.** Let  $S$  be a smooth generic surface. A *bitangent plane* of  $S$  is a plane which is tangent to  $S$  at least at two distinct points. The *conodal curve*  $D$  of a surface  $S$  is the closure of the locus of points of contact of  $S$  with its bitangent planes.

At a godron of  $S$ , the curve  $D$  is simply tangent to the curves  $P$  (parabolic) and  $F$  (flecnodal). This fact will be clear from our calculation of  $D$  for Platonova's normal forms of godrons.

**The projective invariant.** At any godron  $g$ , there are three tangent smooth curves  $F$ ,  $P$  and  $D$ , to which we will associate a projective invariant:

Consider the Legendrian curves  $L_F$ ,  $L_P$ ,  $L_D$  and  $L_g$  (of the 3-manifold of contact elements of  $S$ ,  $PT^*S$ ) consisting of the contact elements of  $S$  tangent to  $F$ ,  $P$ ,  $D$  and to the point  $g$ , respectively (the contact elements of  $S$  tangent to a point are just the contact elements of  $S$  at that point, that is,  $L_g$  is the fibre over  $g$  of the natural projection  $PT^*S \rightarrow S$ ). These four Legendrian curves are tangent to the same contact plane  $\Pi$  of  $PT^*S$ . The tangent directions of these curves determine four lines  $l_F$ ,  $l_P$ ,  $l_D$  and  $l_g$ , through the origin of  $\Pi$ .

**Definition 4.** The *cr-invariant*  $\rho(g)$  of a godron  $g$  is defined as the cross-ratio of the lines  $l_F$ ,  $l_P$ ,  $l_D$  and  $l_g$  of  $\Pi$ :

$$\rho(g) = (l_F, l_P, l_D, l_g).$$

**Platonova's normal form.** According to Platonova's Theorem [13], in the neighbourhood of a godron, a surface can be sent by projective transformations to the normal form

$$z = \frac{y^2}{2} - x^2y + \lambda x^4 + \varphi(x, y) \quad (\text{for some } \lambda \neq 0, \frac{1}{2}) \quad (G1)$$

where  $\varphi$  is the sum of homogeneous polynomials in  $x$  and  $y$  of degree greater than 4 and (possibly) of flat functions.

**Theorem 3.** *Let  $g$  be a godron, with cr-invariant value  $\rho$ , of a generic smooth surface  $S$ . Put  $S$  (after projective transformations) in Platonova's normal form (G1). Then the coefficient  $\lambda$  equals  $\rho/2$ .*

It turns out that among the 2-jets of the curves in  $S$ , tangent to  $P$  at a godron, there is a special 2-jet at which "something happens". We introduce it in the following lemma.

**Tangential Map and Separating 2-jet.** Let  $g$  be a godron of a generic smooth surface  $S$ . The *tangential map* of  $S$ ,  $\tau_S : S \rightarrow (\mathbb{R}P^3)^\vee$ , associates to each point of  $S$  its tangent plane at that point. The image  $S^\vee$  of  $\tau_S$  is called the *dual surface* of  $S$ .

Write  $J^2(g)$  for the set of all 2-jets of curves of  $S$  tangent to  $P$  at  $g$ . By the *image of a 2-jet  $\gamma$  in  $J^2(g)$  under the tangential map  $\tau_S$*  we mean the image, under  $\tau_S$ , of any curve of  $S$  whose 2-jet is  $\gamma$ . By Theorem 2, all the

2-jets of  $J^2(g)$  (and also the 3-jets of curves on  $S$  tangent to  $P$  at  $g$ ) are curves lying in the tangent plane of  $S$  at  $g$ . In suitable affine coordinates, the elements of  $J^2(g)$  can be identified with the curves  $t \mapsto (t, ct^2, 0)$ ,  $c \in \mathbb{R}$ .

**Separating 2-jet Lemma.** *There exists a unique 2-jet  $\sigma$  in  $J^2(g)$  (which we call the separating 2-jet at  $g$ ) satisfying the following properties:*

- (a) *The images, under  $\tau_S$ , of all elements of  $J^2(g)$  different from  $\sigma$  are cusps of  $S^\vee$  sharing the same tangent line  $l_g^\vee$ , at  $\tau_S(g)$ .*
- (b) *The image of  $\sigma$  under  $\tau_S$  is a singular curve of  $S^\vee$  whose tangent line at  $\tau_S(g)$  is different from  $l_g^\vee$ .*
- (c) (separating property): *The images under  $\tau_S$  of any two elements of  $J^2(g)$ , separated by  $\sigma$ , are cusps pointing in opposite directions.*

**Remark 2.** *Once a godron with cr-invariant  $\rho$  of a smooth surface is sent (by projective transformations) to the normal form  $z = y^2/2 - x^2y + \rho x^4/2 + \varphi(x, y)$ , the separating 2-jet is independent of  $\rho$ : It is given by the equation  $y = x^2$ , in the  $(x, y)$ -plane.*

For generic values of  $\rho$  the curves  $F$ ,  $P$  and  $D$  are simply tangent to each other. However, for isolated values of  $\rho$  two of these curves may have higher order of tangency and then some bifurcation occurs. We will look for the values of  $\rho$  at which ‘something happens’.

**Theorem 4.** *Let  $g$  be a godron of a generic smooth surface  $S$ . There are six possible generic configurations of the curves  $F$ ,  $P$  and  $D$  with respect to the separating 2-jet and to the asymptotic line at  $g$ . They are represented in Fig. 2. The actual configuration at  $g$  depends on which of the following six open intervals the cr-invariant  $\rho(g)$  belongs to, respectively:  $(1, \infty)$ ,  $(\frac{2}{3}, 1)$ ,  $(\frac{1}{2}, \frac{2}{3})$ ,  $(0, \frac{1}{2})$ ,  $(-\frac{1}{2}, 0)$  or  $(-\infty, -\frac{1}{2})$ .*

## 3.2 The index of a godron

**Definition 5.** A godron is said to be *positive* or *of index +1* (resp. *negative* or *of index -1*) if at the neighbouring parabolic points the half-asymptotic lines, directed to the hyperbolic domain, point towards (resp. away from) the godron. See Fig. 3.

**The asymptotic double of the hyperbolic domain.** A godron  $g$  can be positive or negative, depending on the index of the direction field, which is naturally associated to  $g$ , on the *asymptotic double*  $\mathcal{A}$  of  $S$ : The *asymptotic double* of  $S$  is the surface  $\mathcal{A}$  in the manifold of contact elements of  $S$ ,  $PT^*S$ , consisting of the field of asymptotic directions. It doubly covers

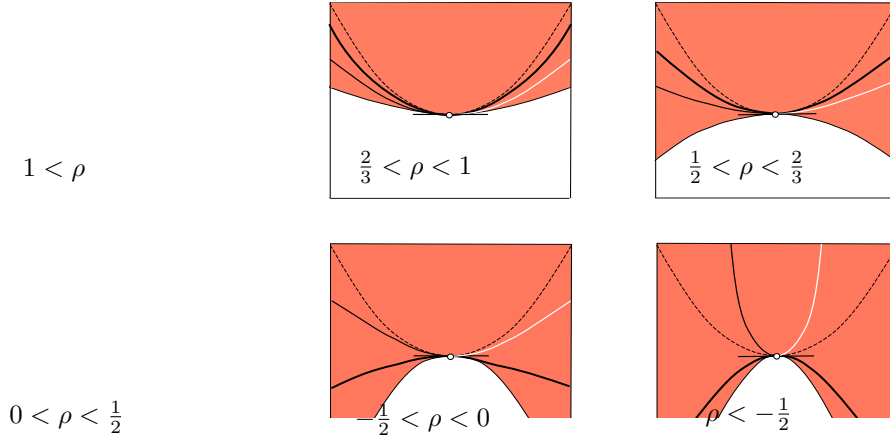


Figure 2: The configurations of the curves  $F$  (half-white half-black curves),  $P$  (boundary between white and gray domains),  $D$  (thick curves), the separating 2-jet (broken curves) and the asymptotic line (horizontal segments) at generic godrons.

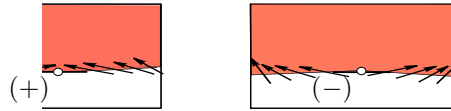


Figure 3: A positive godron and a negative godron.

the hyperbolic domain, and its projection to  $S$  has a fold singularity over the parabolic curve. There is an *asymptotic lifted field of directions* on the surface  $\mathcal{A}$ , constructed in the following way. At each point of the contact manifold  $PT^*S$  a contact plane is applied, in particular at each point of  $\mathcal{A}$ . Consider a point of the smooth surface  $\mathcal{A}$  and assume that the tangent plane of  $\mathcal{A}$  at this point does not coincide with the contact plane. Then these two planes intersect along a straight line tangent to  $\mathcal{A}$ . The same holds at all nearby points in  $\mathcal{A}$ . This defines a smooth direction field on  $\mathcal{A}$  which vanishes only at the points where those planes coincide: over the godrons.

If  $g$  is a positive godron, then the index of this direction field at its singular point equals  $+1$ , the point being a node or a focus; if  $g$  is negative, the index equals  $-1$  and the point is a saddle. See Fig. 4.

**Proposition 3.** *A godron  $g$  is positive (negative) if and only if the value of its cr-invariant  $\rho$  satisfies:  $\rho(g) > 1$  (resp.  $\rho(g) < 1$ ).*

**Corollary of Proposition 3 and Theorem 4.**

- (a) *In the neighbourhood of a positive godron the hyperbolic domain is locally convex.*
- (b) *There exist negative godrons for which the neighbouring hyperbolic domain is locally convex.*

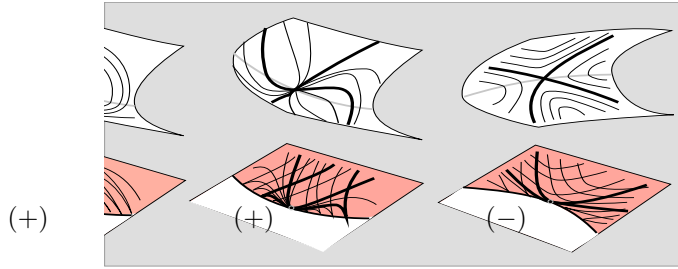


Figure 4: The asymptotic double of the hyperbolic domain near a godron.

(c) In case of item (b), the flecnodal curve lies locally between  $P$  and  $D$  (see Fig. 2). Moreover, we have:  $\frac{2}{3} < \rho < 1$ .

Items (a) and (b) of this corollary are due to F. Aicardi [1].

**Corollary 2.** *All godrons of a cubic surface in  $\mathbb{R}P^3$  are negative.*

*Proof.* By the definitions of asymptotic curve and of flecnodal curve, any straight line contained in a smooth surface is both an asymptotic curve and a connected component of the flecnodal curve of that surface.

Let  $S$  be a generic algebraic surface of degree 3. At a point of the flecnodal curve, an asymptotic line has at least 4-point contact with  $S$ . Since  $S$  is a cubic surface, this line must lie completely in  $S$ . So the flecnodal curve of  $S$  consists of straight lines.

At a godron  $g$  of  $S$ , the tangent line to the parabolic curve (that is, the flecnodal curve) lies in the hyperbolic domain. Thus the neighbouring elliptic domain is locally convex. Therefore, by the above corollary,  $g$  is negative.  $\square$

### 3.3 Locating the left and right branches of $F$

**Remark on the co-orientation of the elliptic domain.** *Each connected component of the elliptic domain is ‘naturally’ co-oriented: At each elliptic point the surface lies locally in one of the two half-spaces determined by its tangent plane at that point. This half-space, which we name the positive half-space, determines a natural co-orientation on each connected component of the elliptic domain. By continuity, the natural co-orientation extends to the parabolic points. At the parabolic points a positive half-space is therefore also defined.*

This simple observation has strong topological consequences. For example:

**Proposition 4.** *The elliptic domain of any smooth surface in the 3-space (Euclidean, affine or projective) can not contain a Möbius strip.*

In the neighbourhood of a godron  $g$  of a smooth surface  $S$ , we can distinguish explicitly which branch of the flecnodal curve is the right branch and which is the left one. For this, we need only to know the index of  $g$  and the co-orientation of  $S$  given by the positive half-space at  $g$ :

Let  $g$  be a godron of a generic smooth surface  $S$ . Take an affine coordinate system  $x, y, z$  such that the  $(x, y)$ -plane is tangent to  $S$  at  $g$ , and the  $x$ -axis is tangent to the parabolic curve at  $g$  (thus also tangent to  $F$  at  $g$ ). Direct the positive  $z$ -axis to the positive half-space at  $g$ . Direct the positive  $y$ -axis towards the neighbouring hyperbolic domain. Finally, direct the positive  $x$ -axis in such way that any basis  $(e_x, e_y, e_z)$  of  $x, y, z$  form a positive frame for the fixed orientation of  $\mathbb{R}^3$  (or of  $\mathbb{R}P^3$ ).

So one can locally parametrise the flecnodal curve at  $g$  by projecting it to the  $x$ -axis.

**Theorem 5.** *Under the above parametrisation, the left and right branches of the flecnodal curve at  $g$  correspond locally to the negative and positive semi-axes of the  $x$ -axis, respectively, if and only if  $g$  is a positive godron. The opposite correspondence holds for a negative godron.*

In other words, if you stand on the tangent plane of  $S$  at  $g$  in the positive half-space and you are looking from the elliptic domain to the hyperbolic one, then you see the right (left) branch of the flecnodal curve on your right hand side if and only if  $g$  is a positive (resp. negative) godron. So the index of  $g$  determines and is determined by the side on which the right branch of  $F$  is located.

**Remark 3.** *Proposition 2 and Theorem 5 (which are local theorems) together with the natural co-orientation of the elliptic domain, are the key elements to prove the global theorem (Theorem 6) of Section 3.5. They imply that some (global) configurations of the flecnodal curve are forbidden. So, for example, there is no surface having a hyperbolic disc in which the left and right branches of the flecnodal curve do not intersect.*

### 3.4 Degenerated godrons: $\rho = 0$ and $\rho = 1$

After the preceding sections, a natural question arises: What happens if the cr-invariant equals 0 or 1?

The godrons for which the cr-invariant equals 0 or 1 are degenerated godrons. We will explain the meaning of these degeneracies and describe the behaviour of such degenerated godrons under a small perturbation of the surface inside a generic one-parameter family of smooth surfaces.

**The case  $\rho = 0$ .** If  $\rho = 0$  then the normal form that we have used above defines just a cubic surface, which is absolutely not generic: the asymptotic line at that godron has ‘infinite point-contact’ with the surface and coincides with the flecnodal curve. In order to understand the behaviour of the flecnodal curve at the godrons with  $\rho = 0$ , we need to add some terms of degree 5, which become important and break the symmetry. In fact, one can prove that *at a generic godron with  $\rho = 0$  the asymptotic line has 5-point contact with the surface*. We name such a point a *flec-godron*.

The godron of the surface  $z = \frac{y^2}{2} - x^2y + x^5$  is a generic flec-godron. To understand better the geometry of a flec-godron, we will perturb this surface inside a generic one-parameter family of smooth surfaces in which the parameter is the cr-invariant  $\rho$ :

$$z = \frac{y^2}{2} - x^2y + \frac{\rho}{2}x^4 + x^5.$$

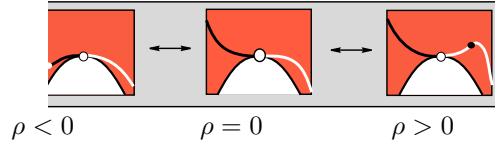


Figure 5: The transition at a flex-godron:  $\rho = 0$ .

The flecnodal and parabolic curves of this surface are depicted in Figure 5 for  $\rho < 0$ ,  $\rho = 0$  and  $\rho > 0$ .

Let  $S_t$ ,  $t \in \mathbb{R}$ , a generic one-parameter family of smooth surfaces such that for  $t = 0$  the surface  $S_0$  has a godron with cr-invariant  $\rho = 0$  (for instance the above family). Then, for  $t < 0$ , there is a biinflection point  $b_t$  which lies, say, in the right branch of the flecnodal curve of  $S_t$  and, as  $t$  goes to 0, the point  $b_t$  is ‘approaching’ the godron of  $S_t$ . At  $t = 0$  the biinflection point  $b_0$  coincides with the godron of  $S_0$ . For  $t > 0$ , the biinflection point  $b_t$  reappears but in the left branch and, as  $t$  is increasing, the point  $b_t$  is ‘going away’ from the godron of  $S_t$ . Moreover, at  $t = 0$  the flecnodal curve has an inflection. See Fig.5. Note that the index of the godron of  $S_t$ , for  $|t|$  sufficiently small, is negative.

**The case  $\rho = 1$ .** If  $\rho = 1$ , then we also have a degenerate godron, which we name *bigodron*: it is the collapse (or the birth) of two godrons of opposite indices. When  $\rho = 1$  the normal form that we used above is not convenient since it is degenerate:  $z = \frac{1}{2}(y - x^2)^2$ . For this reason the parabolic and flecnodal curves coincide with the curve  $y = x^2$  in the  $(x, y)$ -plane (this curve is sent to a point under the tangential map of  $S$ ). In order to have a generic polynomial of degree four, one must add another term of degree four:  $z = \frac{1}{2}(y - x^2)^2 \pm x^3y$ . Now, the bigodron obtained is generic (among the



bigodrons:  $\rho = 1$ ): the parabolic and flecnodal curves have 4-point contact and the whole flecnodal curve is either left or right, according to the sign  $+$  or  $-$  of the term  $\pm x^3y$ , respectively (see the central part of Fig. 6). To understand better the geometry of a bigodron, we will perturb this surface inside a generic one-parameter family of smooth surfaces:

$$z = \frac{1}{2}(y - x^2)^2 \pm x^3y + \varepsilon x^3.$$

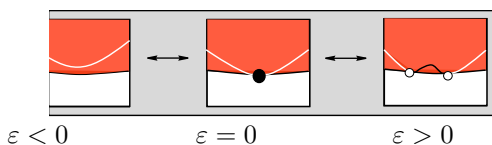


Figure 6: The transition at bigodron:  $\rho = 1$ .

The flecnodal and parabolic curves of this surface are depicted in Figure 6 for  $\varepsilon < 0$ ,  $\varepsilon = 0$  and  $\varepsilon > 0$ . When the parameter  $\varepsilon$  is negative the flecnodal curve is left and does not touch the parabolic curve, while when  $\varepsilon$  is positive the flecnodal curve touches the parabolic curve at two neighbouring godrons of opposite index, and a small segment of the right flecnodal has appeared between these godrons.

So there are two types of bigodrons: a bigodron is said to be *left (right)* if it corresponds to a bifurcation in which a small segment of the left branch of the flecnodal curve is born or vanishes.

### 3.5 Elliptic discs and hyperbolic discs of surfaces

The following global theorem holds for any generic smooth surface:

**Theorem 6.** *In any hyperbolic disc (bounded by a Jordan parabolic curve), there is an odd number of special hyperbolic points (transverse crossings of the left and right branches of the flecnodal curve).*

The cubic surfaces in  $\mathbb{R}P^3$  provide examples of surfaces having elliptic discs whose bounding parabolic curves have 0, 1, 2 or 3 negative godrons: According to Segre [15], a generic cubic surface diffeomorphic to the projective plane contains four parabolic curves (each one bounding an elliptic disc) and six godrons. According to [4], Shustin had proved that the distribution of the godrons among the four parabolic curves is  $6 = 0 + 1 + 2 + 3$ . By Corollary 3.5, all these godrons are negative.

There exist smooth surfaces having an elliptic disc whose bounding parabolic curve has 4 negative godrons:

**Example 1.** The algebraic surface given by the equation

$$z = (x^2 - 1)(y^2 - 1)$$

has an elliptic disc whose bounding parabolic curve contains 4 godrons, all negative.

For a parabolic curve bounding a hyperbolic disc the situation is quite different:

**Proposition 5.** *The sum of the indices of the godrons on the parabolic curve bounding a hyperbolic disc (of a generic surface) equals two. In particular, such parabolic curve contains a positive even number of godrons.*

*Proof.* Write  $H$  for the closure of the hyperbolic disc. The asymptotic double  $\mathcal{A}$  is a sphere. Its Euler characteristic equals 2. By Poincaré Theorem, the sum of indices of all singular points of the direction field on  $\mathcal{A}$  equals 2.  $\square$

In fact, Proposition 2 implies the following propositions:

**Proposition 6.** *At each connected component of the hyperbolic domain the flecnodal curve consists of closed curves, each of them having an even number (possibly zero) of godrons (that is, of contact points with the boundary of that domain). The godrons decompose these closed curves into left and right segments.*

**Corollary 3.** *The boundary of each connected component of the hyperbolic domain of a generic surface has an even number of godrons.*

Corollary 3 is the main result of [6]. Unfortunately the proof given in [6] is not correct since it is based in the following statement: *the Euler characteristic of a connected component  $H$  of the hyperbolic domain equals the number of godrons in  $\partial H$  at which the hyperbolic domain is locally convex (the asymptotic line has contact with  $\partial H$  exterior to  $H$ ) minus the number of godrons in  $\partial H$  at which the elliptic domain is locally convex (the asymptotic line has contact with  $\partial H$  interior to  $H$ ).* This statement is wrong: 1) In the bigodron bifurcation (see Section 3.4) two godrons are born (or killed); 2) At both godrons the contact of the asymptotic line with  $\partial H$  is exterior to  $H$  and 3) This bifurcation does not change the Euler characteristic of  $H$ .

### 3.6 Godrons and Swallowtails

**Tangential Map and Swallowtails.** It is well known (c.f. [14]) that under the tangential map of  $S$  the parabolic curve of  $S$  corresponds to the cuspidal edge of  $S^\vee$ , the conodal curve of  $S$  corresponds to the self-intersection line of  $S^\vee$  (this follows from the definitions of dual surface and conodal curve) and a godron corresponds to a swallowtail point.

**Legendrian Remark.** *The most natural approach to the singularities of the tangential map is via Arnold's theory of Legendrian singularities [3]. The image of a Legendrian map is called the front of that map. The tangential map of a surface is a Legendre map, and so it can be expected to have only Legendre singularities. Thus for a surface in general position, the only singularities of its dual surface (i.e. of its front) can be: self-intersection lines, cuspidal edges and swallowtails. So the godrons are the most complicated singularities of the tangential map of a generic surface.*

**Definition of Front.** In this paper, a *front in general position* is a surface whose singularities, and the singularities of its dual surface, are at most: self-intersection lines, cuspidal edges and swallowtails. Moreover, we require that the parabolic curve never passes through a swallowtail point (the same requirement for the dual front).

**The invariant of a swallowtail.** We can associate a projective invariant (a number) to a swallowtail point  $s$  of a front  $S$ : We apply the tangential map of  $S$  (in a neighbourhood of  $s$ ) to obtain a locally smooth surface  $S^\vee$  having a godron with cr-invariant  $\rho$ . The number  $\rho(s) := \rho$  is associated to the swallowtail  $s$ .

*The tangential map of  $S$  sends the elliptic (hyperbolic) domain of  $S$  to the elliptic (resp. hyperbolic) domain of  $S^\vee$ .* Thus the hyperbolic and elliptic domains of a front in general position are separated by the cuspidal edge (and by the parabolic curve). This implies that there are two types of swallowtails:

**Definition 6.** A swallowtail point of a generic front is said to be *hyperbolic* (*elliptic*) if, locally, the self-intersection line of that front is contained in the hyperbolic (resp. elliptic) domain.

The proofs of the following theorems show that the configurations of the curves  $F$ ,  $P$  and  $D$  at a godron have a relevant meaning for the local (projective, affine or Euclidean) differential properties of the swallowtails.

**Theorem 7.** *The dual of a surface at a positive godron is an elliptic swallowtail. The dual of a surface at a negative godron is a hyperbolic swallowtail.*

*Proof.* By Proposition 3, a godron  $g$  is positive (negative) if and only if its cr-invariant satisfies  $\rho(g) > 1$  (resp.  $\rho(g) < 1$ ).

By Theorem 4,  $\rho(g) > 1$  (resp.  $\rho(g) < 1$ ) if and only if the conodal curve at  $g$  lies locally in the elliptic (hyperbolic) domain.

Finally, since the tangential map sends the elliptic (hyperbolic) domain to the elliptic (resp. hyperbolic) domain of the dual surface, it is evident that the conodal curve at  $g$  lies locally in the elliptic (hyperbolic) domain if and only if the dual surface is an elliptic (resp. hyperbolic) swallowtail.  $\square$

**Theorem 8.** *In the neighbourhood of a swallowtail point  $s$  of a front in general position, the flecnodal curve  $F$  has a cusp whose tangent direction coincides with that of the cuspidal edge. The point  $s$  separates  $F$  locally into its left and right branches. There are four possible generic configurations of  $F$  in the neighbourhood of  $s$  (see Fig. 7):*

(e) *For an elliptic swallowtail the flecnodal curve is a cusp lying in the small domain bounded by the cuspidal edge ( $\rho(s) \in (0, 1)$ ).*

*There are 3 different generic types of hyperbolic swallowtails.*

( $h_1$ ) *Each branch of the cuspidal edge is separated from the self-intersection line by one branch of the flecnodal curve ( $\rho(s) \in (-\frac{1}{2}, 0)$ ).*

( $h_2$ ) *The self-intersection line lies between the two branches of the flecnodal curve and separates them from branches of the cuspidal edge. The cusp of the flecnodal curve points in the same direction as the cusp of the cuspidal edge ( $\rho(s) \in (-\frac{1}{2}, 0)$ ).*

( $h_3$ ) *The cusp of the flecnodal curve and the cusp of the cuspidal edge are pointing in opposite directions ( $\rho(s) \in (-\infty, -\frac{1}{2})$ ).*

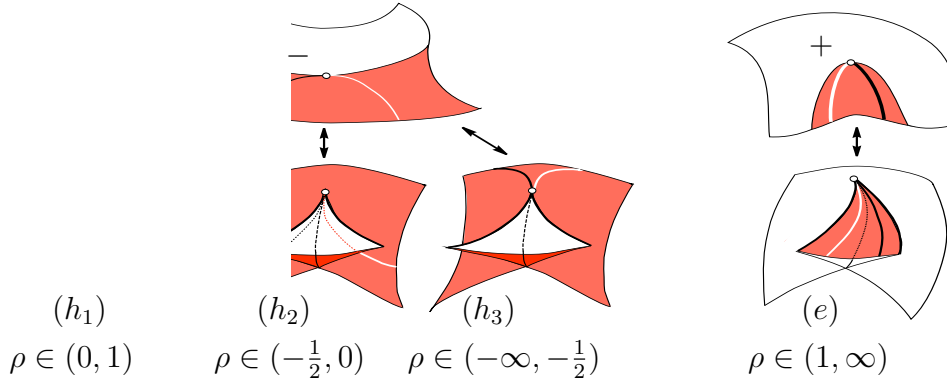


Figure 7: A godron of a smooth surface and its dual surface: a swallowtail.

## 4 The proofs of the theorems

**Preparatory conventions and results.** In the sequel, we will consider the surface  $S$  as the graph of a smooth function  $z = f(x, y)$ , where  $x, y, z$  form an affine coordinate system. The asymptotic directions satisfy the equation:

$$f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2 = 0.$$

For  $dy = pdx$ , this equation takes the form

$$A^f(x, y, p) = f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0. \quad (1)$$

Equation (1) is called the *asymptote-equation* of  $f$ .

In what follows, we will assume without loss of generality that the point under consideration in the  $(x, y, p)$ -space is the origin: by a translation and a rotation in the  $(x, y)$ -plane, we can take  $(x, y) = (0, 0)$  and  $p = 0$ , respectively.

Moreover, we will take an affine coordinate system  $x, y, z$  such that the  $(x, y)$ -plane is tangent to  $S$  at the point under consideration. Thus we will have the conditions

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0. \quad (2)$$

The parabolic curve of the surface  $z = f(x, y)$  is the restriction of the graph of  $f$  to the discriminant curve (in the  $(x, y)$ -plane) of equation (1). That is, the parabolic curve is determined by the equations

$$A^f(x, y, p) = 0 \quad \text{and} \quad A_p^f(x, y, p) = 0. \quad (*)$$

The fact that a godron is a folded singularity of (1) implies that

$$A_x^f(0, 0, 0) = 0. \quad (**)$$

The conditions (\*) and (\*\*), at the origin in the  $(x, y, p)$ -space, imply that

$$f_{xx} = f_{xy} = f_{xxx} = 0 \quad (3)$$

at the origin in the  $(x, y)$ -plane.

The choice of a coordinate system such that the  $x$ -axis is an asymptotic direction of  $S$  at the origin is equivalent to our assumption that the point under consideration in the  $(x, y, p)$ -space is the origin.

So the  $x$ -axis is tangent to the parabolic curve at the godron.

## 4.1 Proof of Theorem 2

Let  $\gamma(t) = (x(t), y(t), z(t))$  be a curve on  $S$ , where  $z(t) = f(x(t), y(t))$ , which is tangent to the parabolic curve at the origin, that is,

$$\dot{y}(0) = 0. \quad (4)$$

Since all our calculations and considerations take place at the origin  $(x, y) = (0, 0)$  and at  $t = 0$ , we will omit to write this explicitly.

Evidently conditions (2) imply  $\dot{z} = f_x \dot{x} + f_y \dot{y} = 0$ . The equality

$$\ddot{z} = f_x \ddot{x} + f_y \ddot{y} + (f_{xx} \dot{x}^2 + 2f_{xy} \dot{x} \dot{y} + f_{yy} \dot{y}^2)$$

together with conditions (2), (3) and (4) imply that  $\ddot{z} = 0$ . This proves that the plane  $z = 0$  is osculating.

Finally, the equality

$$\begin{aligned} \ddot{z} &= f_x \ddot{x} + f_y \ddot{y} + 3(f_{xx} \dot{x} \ddot{x} + f_{xy} (\dot{x} \dot{y} + \ddot{x} \dot{y}) + f_{yy} \dot{y} \ddot{y}) \\ &\quad + f_{xxx} \dot{x}^3 + 3f_{xxy} \dot{x}^2 \dot{y} + 3f_{xyy} \dot{x} \dot{y}^2 + f_{yyy} \dot{y}^3 \end{aligned}$$

together with conditions (2), (3) and (4), imply that  $\ddot{z} = 0$ , proving that the first three derivatives of  $\gamma$  at  $t = 0$  are linearly dependent (all of them lie in the  $(x, y)$ -plane). So  $\gamma$  has a flattening or an inflection at the origin, according to the linear independence or dependence, respectively, of its first two derivatives at  $t = 0$ .  $\square$

## 4.2 Preliminary remarks and computations

We recall that Platonova's Theorem [13] implies that at a godron of a generic smooth surface  $S$ , there is an affine coordinate system such that  $S$  is locally given by

$$z = \frac{y^2}{2} - x^2 y + \lambda x^4 + \varphi(x, y) \quad (\text{for some } \lambda \neq \frac{1}{2}, 0) \quad (G1)$$

where  $\varphi$  is the sum of homogeneous polynomials in  $x$  and  $y$  of degree greater than 4 and (possibly) of flat functions.

The information we need about  $S$  (for the proofs of our theorems) is contained in its 4-jet. The term  $\varphi$  in (G1) only breaks slightly the symmetry, but it does not contain additional information. Thus, in the proofs of our theorems, we will systematically use Platonova's normal form of the the 4-jet of  $S$ . The reader can easily verify that the term  $\varphi$  has no influence on our arguments.

First we need to calculate the curves  $F$ ,  $P$  and  $D$ . For that we need the second partial derivatives of the functions  $f(x, y; \lambda) = \frac{y^2}{2} - x^2 y + \lambda x^4$ :

$$f_{xx} = -2y + 12\lambda x^2, \quad f_{xy} = -2x, \quad f_{yy} = 1. \quad (H)$$

The asymptote-equations of the surfaces  $z = \frac{y^2}{2} - x^2 y + \lambda x^4$  are therefore given by

$$A^f(x, y, p; \lambda) = (12\lambda x^2 - 2y) - 4xp + p^2 = 0. \quad (5)$$

We are interested in the configurations of the curves  $F$ ,  $P$  and  $D$  at the godron  $g$ . According to Theorem 2, these curves have at least 4-point contact with the  $(x, y)$ -plane. We will thus consider the curves  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$  on the  $(x, y)$ -plane, whose images by  $f$  are  $F$ ,  $P$  and  $D$ , respectively. These plane curves have the same 2-jets as  $F$ ,  $P$  and  $D$ , respectively.

**The parabolic curve.** The equations (\*) of 4.1 imply that  $\bar{P}$  is given by the Hessian of  $f$ ,  $f_{xy}^2 - f_{xx}f_{yy} = 0$ . From (H) one obtains that  $\bar{P}$  is the parabola

$$y = 2(3\lambda - 1)x^2.$$

**The flecnodal curve.** According to [16, 18], the curve  $\bar{F}$  associated to the surface  $z = f(x, y)$  is obtained from the intersection of the surfaces

$$A^f(x, y, p) = 0 \quad \text{and} \quad I^{A^f}(x, y, p) := (A_x^f + pA_y^f)(x, y, p) = 0,$$

in the  $(x, y, p)$ -space, by the projection of this intersection to the  $(x, y)$ -plane, along the  $p$ -direction. From (5) one obtains

$$I^{A^f}(x, y, p) = 6(4\lambda x - p).$$

Combining the equation  $p = 4\lambda x$  with (5) one obtains that  $\bar{F}$  is the parabola

$$y = 2\lambda(4\lambda - 1)x^2.$$

**The conodal curve.** Since Platonova's normal form is symmetric with respect to the  $x$ -direction, the bitangent planes in the neighbourhood of  $g$  are invariant under the reflection  $(x, y, z) \mapsto (-x, y, z)$ . Thus the points of the conodal curve satisfy  $f_x(x, y; \lambda) = 0$ . That is,  $-2x(y - 2\lambda x^2) = 0$ . Thus the curve  $\bar{D}$  is the parabola

$$y = 2\lambda x^2.$$

### 4.3 Proof of Theorem 3

We consider the parabolas  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$  as graphs of functions  $y = y(x)$ . The Legendrian curves  $L_F$ ,  $L_P$  and  $L_D$  in the  $(x, y, p)$ -space  $J^1(\mathbb{R}, \mathbb{R})$  (which is the space of 1-jets of the real functions  $y(x)$  of one real variable) are tangent to the contact plane  $\Pi$  at the origin (parallel to the plane  $y = 0$ ). The slope of the tangent line at the origin, of each of these Legendrian curves, equals twice the second derivative at zero of the function  $y = y(x)$  associated to the corresponding parabola, that is, equals twice the coefficient of that parabola (note that the term  $\varphi$  in (G1) will contribute with higher order terms which will have no influence on these coefficients).

The Legendrian curve consisting of the contact elements tangent to the origin is vertical. Write  $l_g$  for its tangent line. The cross-ratio of the tangent lines  $l_F$ ,  $l_P$ ,  $l_D$  and  $l_g$  is given in terms of the coefficients  $c$  of the parabolas  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$  by

$$\rho(g) = (l_F, l_P, l_D, l_g) = \frac{c(F) - c(D)}{c(P) - c(D)} = \frac{2\lambda(4\lambda - 1) - 2\lambda}{2(3\lambda - 1) - 2\lambda} = 2\lambda.$$

This proves Theorem 3. □

**Rewriting the equations in terms of  $\rho$ .** After Theorem 3, we rewrite Platonova's normal forms of the 4-jet of  $S$  at a godron and the equations of the curves  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$  in terms of the cr-invariant  $\rho$ :

$$z = \frac{y^2}{2} - x^2y + \rho \frac{x^4}{2} \quad (\rho \neq 1, 0). \quad (R)$$

$$y = (3\rho - 2)x^2; \quad (P)$$

$$y = \rho(2\rho - 1)x^2; \quad (F)$$

$$y = \rho x^2. \quad (D)$$

#### 4.4 Proof of Separating 2-jet Lemma

An easy way to compute (and to see) the dual surface of  $S \subset \mathbb{R}P^3$ , viewed as a surface lying in the same space  $\mathbb{R}P^3$ , is by the 'polar duality map' with respect to a quadric. With this map, the calculations are simpler if the considered quadric is a paraboloid of revolution (see [17]). Moreover, if the surface  $S$  is the graph of a function  $z = f(x, y)$ , then the polar duality map with respect to the paraboloid  $z = \frac{1}{2}(x^2 + y^2)$  coincides with the classical Legendre transform of  $f$ . So the dual surface has the following parametrisation:

$$\tau_f : (x, y) \mapsto (f_x(x, y), f_y(x, y), xf_x(x, y) + yf_y(x, y) - f(x, y)).$$

In the case of the surfaces  $S_\rho$  given in (R), one obtains

$$\tau_\rho : (x, y) \mapsto \left( -2xy + 2\rho x^3, y - x^2, \frac{y^2}{2} - 2x^2y + 3\rho \frac{x^4}{2} \right).$$

The images of our plane curves  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$ , under  $\tau_\rho$ , are exactly the flecnodal curve, the cuspidal edge and the self-intersection line of the dual surface  $S_\rho^\vee$ , respectively. Since  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$  are parabolas, we state the proposition:

**Lemma 1.** *The image of the parametrised parabola  $t \mapsto (t, ct^2)$ , under  $\tau_\rho$ , is the parametrised space curve (lying on  $S^\vee$ ):*

$$\alpha_\rho^c : t \mapsto \left( 2(\rho - c)t^3, (c - 1)t^2, \left( \frac{c^2}{2} - 2c + \frac{3}{2}\rho \right) t^4 \right).$$

*Proof.* This is a direct application of the above Legendre duality map  $\tau_\rho$ . □



The above parametrisation implies that the curves  $\alpha_\rho^c$  have at least 4-point contact with the  $(x, y)$ -plane at  $t = 0$ . In order to study the behaviour of the curves  $\alpha_\rho^c$  for different values of  $c$  (for a fixed value of the cr-invariant  $\rho$ ), we will consider their projection to the  $(x, y)$ -plane along the  $z$ -direction:

$$\gamma_\rho^c : t \mapsto (2(\rho - c)t^3, (c - 1)t^2). \quad (6)$$

Clearly,  $\gamma_\rho^c(t) - \alpha_\rho^c(t) = O(t^4)$ .

**Lemma 2.** *Fix a value of the godron invariant  $\rho$ . The images of all parabolas  $y = cx^2$ ,  $c \neq 1$ , under the composition of  $\tau_\rho$  with the projection  $(x, y, z) \mapsto (x, y)$ , are cusps pointing down if  $c > 1$  and pointing up if  $c < 1$ . These cusps are semi-cubic if  $c \neq \rho$  and (very) degenerate if  $c = \rho$ .*

*The image of the parabola  $y = x^2$  ( $c = 1$ ) under the above composition is the  $x$ -axis if  $\rho \neq 1$  and it is the origin if  $\rho = 1$ .*

*Proof.* Lemma 2 and Separating Lemma follow from parametrisation (6).  $\square$

**Remark 4.** *It is clear from Lemma 2 that the behaviour of the curve  $\tau_\rho(\bar{F})$ ,  $\tau_\rho(\bar{P})$  or  $\tau_\rho(\bar{D})$  in  $S_\rho^\vee$ , changes drastically when the coefficient  $c_F(\rho)$ ,  $c_P(\rho)$  or  $c_D(\rho)$ , respectively, passes through the value 1.*

## 4.5 Proof of Theorem 4

The projection of  $S_\rho$  to the  $(x, y)$ -plane, along the  $z$ -axis, is a local diffeomorphism. So the configuration of the curves  $F$ ,  $P$  and  $D$  with respect to the asymptotic line and the separating 2-jet at  $g$ , on the surface  $S$ , is equivalent to the configuration of the parabolas  $\bar{F}$ ,  $\bar{P}$  and  $\bar{D}$  with respect to the parabolas  $y = 0 \cdot x^2 = 0$  and  $y = 1 \cdot x^2$  (see Remark 2), on the  $(x, y)$ -plane.

Given a value of  $\rho$ , this configuration is determined by the order, on the real line, of the coefficients of these five parabolas:

$$c_F = \rho(2\rho - 1), \quad c_P = 3\rho - 2, \quad c_D = \rho, \quad c_{al} = 0, \quad c_\sigma = 1.$$

The graphs of these coefficients, as functions of  $\rho$ , are depicted in Fig. 8.

Using the formulas of the coefficients  $c_F$ ,  $c_P$  and  $c_D$  (or from Fig. 8) one obtains by straightforward and elementary calculations that:

$$\begin{aligned} \rho \in (1, \infty) & \iff 0 < 1 < c_D < c_P < c_F; \\ \rho \in (\frac{2}{3}, 1) & \iff 0 < c_P < c_F < c_D < 1; \\ \rho \in (\frac{1}{2}, \frac{2}{3}) & \iff c_P < 0 < c_F < c_D < 1; \\ \rho \in (0, \frac{1}{2}) & \iff c_P < c_F < 0 < c_D < 1; \\ \rho \in (-\frac{1}{2}, 0) & \iff c_P < c_D < 0 < c_F < 1; \\ \rho \in (-\infty, -\frac{1}{2}) & \iff c_P < c_D < 0 < 1 < c_F. \end{aligned}$$

This proves Theorem 4.  $\square$

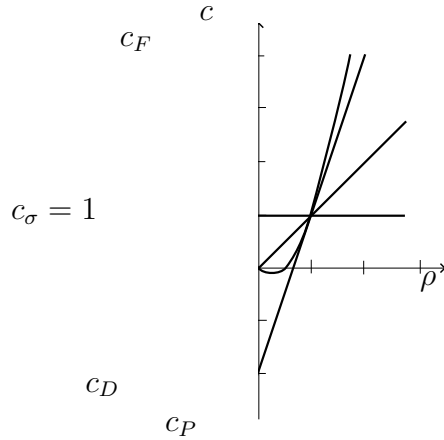


Figure 8: The coefficients  $c_F$ ,  $c_P$  and  $c_D$  as functions of the invariant  $\rho$ .

## 4.6 Proof of Proposition 3

Consider the family of surfaces  $S_\rho$  given by (R). By (P), the slope  $m$  of the tangent lines of the curve  $\bar{P}$  is given by:

$$m(x) = 2(3\rho - 2)x.$$

The slope  $p$  of the (double) asymptotic lines on the parabolic curve, projected to the  $(x, y)$ -plane, is given by the equation  $A_p^f(x, y, p; \frac{\rho}{2}) = 0$ , that is,

$$p(x) = 2x.$$

The points of the positive  $y$ -axis, near the origin, are hyperbolic points of the surface  $S_\rho$  of (R). So the hyperbolic domain of  $S_\rho$  lies locally in the upper side of the parabolic curve. Therefore  $g$  is a positive (negative) godron if and only if the difference of slopes  $(p - m)$  is a decreasing (resp. increasing) function of  $x$ , at  $x = 0$ .

Consequently, the equation  $(p - m)'(0) = -6(\rho - 1)$  implies that the godron  $g$  is positive for  $\rho > 1$  and negative for  $\rho < 1$ , proving Proposition 3.  $\square$

## 4.7 Proof of Proposition 2 and of Theorem 5

**Preliminary remarks on the asymptotic-double (see Section 3.2).**

The asymptotic double  $\mathcal{A}$  of the surface  $S$  is foliated by the integral curves of the asymptotic lifted field of directions. By definition of the lifted field, the asymptotic curves of  $S$  are the images of these integral curves under the natural projection  $PT^*S \rightarrow S$  (sending each contact element to its point of contact) and, under this projection, the asymptotic double  $\mathcal{A}$  (of  $S$ ) doubly covers the hyperbolic domain with a fold singularity over the parabolic curve.

Write  $\tilde{P}$  for the curve of  $\mathcal{A}$  which projects over  $P$  (that is, the curve formed by the fold points in  $\mathcal{A}$  of the above projection). The surface  $\mathcal{A} \setminus \tilde{P}$ , has two (not necessarily connected) components, noted by  $\mathcal{A}_l$  and  $\mathcal{A}_r$ , separated by  $\tilde{P}$ . The integral curves on the component  $\mathcal{A}_l$ , are projected over the left asymptotic curves and the integral curves on the component  $\mathcal{A}_r$  are projected over the right ones. We call these components the *left component* and the *right component*, respectively, of  $\mathcal{A} \setminus \tilde{P}$ .

Now, consider the surface  $S$  as the graph of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $z = f(x, y)$ , and take the projection  $\pi : (x, y, z) \rightarrow (x, y)$ , along the  $z$ -axis. The derivative of  $\pi$  sends the contact elements of  $S$  onto the contact elements of  $\pi(S) \subset \mathbb{R}^2$  and it induces a contactomorphism  $PT^*S \rightarrow PT^*\mathbb{R}^2$  sending  $\mathcal{A}$  to a surface  $\tilde{\mathcal{A}}$  in  $PT^*\mathbb{R}^2$ , which doubly covers (under the natural projection  $PT^*\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ) the image in  $\mathbb{R}^2$  of the hyperbolic domain. We still call the surface  $\tilde{\mathcal{A}} \subset PT^*\mathbb{R}^2$  the asymptotic-double of  $S$ . This surface consists of the contact elements of the  $(x, y)$ -plane satisfying the following equation:

$$f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = 0. \quad (*)$$

In order to handle the the asymptotic double  $\tilde{\mathcal{A}}$ , we take an ‘affine’ chart of  $PT^*\mathbb{R}^2$ . The space of 1-jets of the real functions of one real variable  $J^1(\mathbb{R}, \mathbb{R})$  (with coordinates  $x, y, p$ ) has a natural contact structure (defined by the 1-form  $\alpha = dy - pdx$ ) and it parametrises almost all contact elements of  $\mathbb{R}^2$ : The contact element with slope  $p_0 \neq \infty$  at the point  $(x_0, y_0)$  of the plane of the variables  $(x, y)$  is represented by the point  $(x_0, y_0, p_0)$  in  $J^1(\mathbb{R}, \mathbb{R})$ . The asymptotic-double  $\tilde{\mathcal{A}}$  is the surface in  $J^1(\mathbb{R}, \mathbb{R})$  given by the equation

$$A^f(x, y, p) := f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0, \quad (7)$$

(obtained from  $(*)$  by taking  $p = dy/dx$ ). Moreover, the solutions of the implicit differential equation (7) are the images (by  $\pi$ ) of the asymptotic curves of  $S$ . Equation (7) is called the *asymptote-equation* of  $f$ .

The curve  $\tilde{P}$  is the discriminant curve (see c.f. [2]) of the implicit differential equation  $A^f(x, y, p) = 0$  and it is determined by the pair of equations  $A^f(x, y, p) = 0$  and  $A_p^f(x, y, p) = 0$ .

Below, the images on the plane, under the map  $\pi : (x, y, z) \mapsto (x, y)$ , of the parabolic and flecnodal curves, of the godrons and of the special hyperbolic and special elliptic points of  $S$ , will be called the same, that is, parabolic curves, etc. One obtains the original objects by applying the function  $f$  and taking the graph.

*Proof of Proposition 2.* Write  $\tilde{F}$  for the intersection of  $\tilde{\mathcal{A}}$  ( $A^f(x, y, p) = 0$ ) with the surface given by the equation  $I^{A^f}(x, y, p) = 0$ . As we mentioned in **4.2** the flecnodal curve in the  $(x, y)$ -plane is the image of the curve  $\tilde{F}$  under the projection  $(x, y, p) \mapsto (x, y)$ . The points of (transverse) intersection of

the curves  $\tilde{F}$  and  $\tilde{P}$  project over the godrons of  $S$ . So, over a godron the curve  $\tilde{P}$  locally separates  $\tilde{F}$  (see Fig. 9).

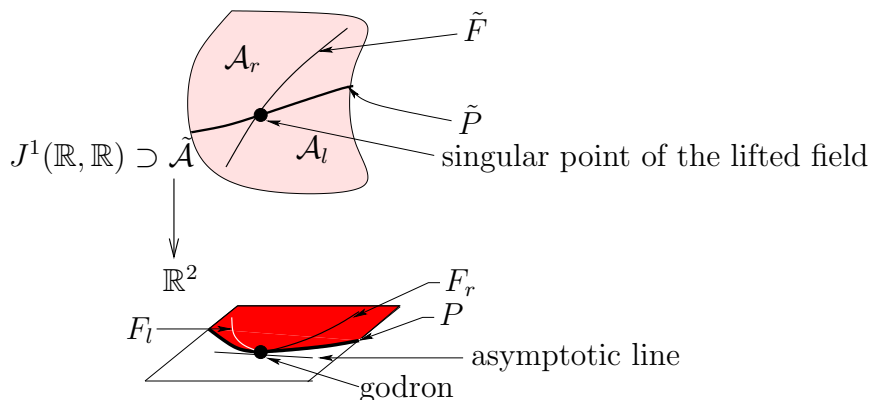


Figure 9: The projection  $\pi : \mathcal{A} \rightarrow S$ , the curves  $\tilde{P}$ ,  $\tilde{F}$ ,  $P$  and  $F$ .

That is,  $\tilde{F}$  has one branch on the left component of  $\tilde{\mathcal{A}}$  and other branch on the right component. This implies that a godron separates locally the left and right branches of the flecnodal curve, proving Proposition 2.  $\square$

*Proof of Theorem 5.* Consider a godron  $g$  with cr-invariant  $\rho$  of a smooth surface. To prove Theorem 5 we need to know the values of  $\rho$  for which the curves  $\tilde{P}$  and  $\tilde{F}$  are tangent. Of course, such non-generic values correspond to godrons of non-generic surfaces. To found these values we only need to know the tangent directions of these curves over  $g$ . The tangent lines of these curves belong to the tangent plane of  $\tilde{\mathcal{A}}$  at  $\tilde{g}$  (the point over  $g$  in the  $(x, y, p)$ -space), which is also the contact plane at  $\tilde{g}$ . So it suffices to take the 4-jet of  $S$  at  $g$ .

Take the normal form considered above:

$$z = \frac{y^2}{2} - x^2y + \rho \frac{x^4}{2}. \quad (R)$$

The coordinates  $(x, y, z)$  of this normal form satisfy the conditions considered in Theorem 5.

Since the point  $\tilde{g}$  is the origin, the tangent lines to the curves  $\tilde{P}$  and  $\tilde{F}$  belong to the  $(x, p)$ -plane. The surface  $A_p^f(x, y, p) = 0$  is the plane given by the equation  $p = 2x$ , which is independent of  $\rho$ . The surface  $I^{A^f}(x, y, p) = 0$  is the plane given by the equation  $p = 2\rho x$ . So the curves  $\tilde{P}$  and  $\tilde{F}$  are tangent only for  $\rho = 1$  (in this case we have the collapse of two godrons).

By Proposition 3, this implies that the side on which the right branch of the flecnodal curve will lie depends only on the index of the godron.

To see explicitly on which side of the  $x$ -axis the right branch of the flecnodal curve lies for a negative godron, it is enough to look at an example. We will take a godron of a cubic surface (whose index is  $-1$ , after Corollary 3.5).

The osculating plane of an asymptotic curve at a point of a surface is the tangent plane to the surface at that point. Using this fact, one defines the “osculating plane” of a straight line lying in a surface.

In this way, a segment of a straight line lying in a surface is said to be a left (right) curve, if the tangent plane to the surface along that segment twists like a left (resp. right) screw.

The  $x$ -axis is an asymptotic (and flecnodal) curve of the cubic surface  $z = y^2/2 - x^2y$ . One easily verifies that the positive half-axis is a left asymptotic curve. This proves Theorem 5.  $\square$

## 4.8 Proof of Theorem 6

First, we will prove Theorem 6 for the case in which the parabolic curve bounding the hyperbolic disc has only two godrons.

**Lemma 3.** *If the parabolic curve bounding a hyperbolic disc  $H$  (of a generic smooth surface) has exactly two godrons, then the disc  $H$  contains an odd number of special hyperbolic points.*

Write  $g_1$  and  $g_2$  for the godrons lying on  $\partial H$ . By Proposition 5, both  $g_1$  and  $g_2$  are positive godrons.

**Claim 1.** *If two vectors  $v_1$  and  $v_2$  are tangent to  $F$  at  $g_1$  and  $g_2$ , respectively, and both are pointing from  $F_l$  to  $F_r$ , then  $v_1$  and  $v_2$  orient the parabolic curve  $\partial H$  in the same way.*

*Proof.* Since all neighbouring elliptic points of the parabolic curve  $\partial H$  belong to the same connected component of the elliptic domain, they have the same “natural” co-orientation (given by the tangent plane). Since both godrons are positive, Claim 1 follows from Theorem 5.  $\square$

*Proof of Lemma 3.* Write  $f_r$  for the connected component of  $F_r$  which starts at  $g_1$ . Since there are only two godrons on  $\partial H$ ,  $f_r$  is a segment ending in  $g_2$ . This segment separates  $H$  into two parts, which we name  $A$  and  $B$ . The connected component of  $F_l$  starting in  $g_1$ ,  $f_l$ , is also a segment ending in  $g_2$ . Claim 1 implies that if in the neighbourhood of  $g_1$  the segment  $f_l$  lies in  $A$ , then, in the neighbourhood of  $g_2$ , it lies in  $B$ . Thus  $f_l$  crosses  $f_r$  an odd number of times.

If  $H$  contains other connected components of  $F_l$  and  $F_r$ , then there are (possibly) additional special hyperbolic points in  $H$ . Apart from  $f_l$  and  $f_r$ , the only connected components of  $F_l$  and  $F_r$  in  $H$  are closed curves

(possibly empty). But the number of intersection points of a closed curve of  $F_r$  (lying  $H$ ) with  $f_l$ , or with a closed curve of  $F_l$ , is even. Thus the number of intersection points of  $F_l$  with  $F_r$  is odd.  $\square$

*Proof of Theorem 6.* To prove the general case of Theorem 6, we will consider the closure of the hyperbolic disc, the parabolic curve  $\partial H$  and the connected components of  $F_l$  and  $F_r$  lying in  $H$  as a diagram  $\Delta$ . We will prove in a purely combinatorial manner that the number of intersection points of  $F_l$  with  $F_r$  is odd. For this, we will transform the diagram  $\Delta$  using two “moves”, which are elementary changes (of two types) of local diagrams, that preserve the number of intersection points of  $F_l$  with  $F_r$ :

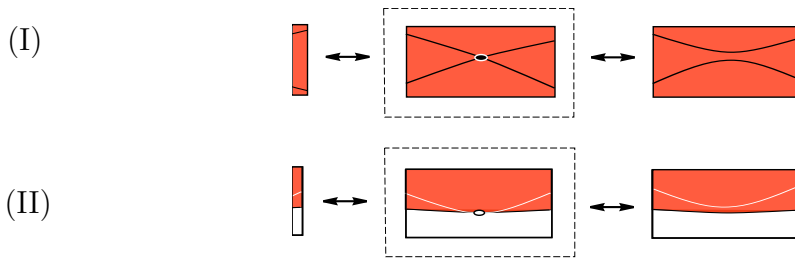


Figure 10: The two elementary moves of diagrams. The moves with opposite choice of colours of the flecnodal curve are also possible.

These moves are depicted in Fig. 10, where an intermediate singular diagram is marked by a dotted box.

Write  $G^+$  and  $G^-$  for the number of positive and negative godrons on  $\partial H$ , respectively. Since the asymptotic covering of  $\bar{H}$  is a sphere,  $G^+ - G^- = 2$ .

If  $G^- = 0$ , the theorem is proved in Lemma 3. So suppose  $G^- > 0$ .

Consider a pair of godrons  $g_+$  and  $g_-$  of opposite index, which are consecutive on  $\partial H$ . Two vectors tangent to  $\partial H$  and pointing from  $F_l$  to  $F_r$ , one at  $g_+$  and the other at  $g_-$ , provide different orientations of  $\partial H$  (see Claim 1).

Consider the segment of parabolic curve joining  $g_+$  to  $g_-$ , and which does not contain other godrons. The local diagram in the tubular neighbourhood of this segment of the parabolic curve is depicted in the left side of Fig. 11.

**Step 1.** In this tubular neighbourhood we deform the black curves starting in  $g_+$  and  $g_-$ , in order to approach one to the other (the central diagram of Fig. 11). Now we apply a move of type I to this diagram in order to obtain a new diagram in which the connected component of  $F_r$  starting in  $g_+$  will be a segment ending in  $g_-$  and lying in the tubular neighbourhood of the considered segment of the parabolic curve.

**Step 2.** Applying a move of type II to the local diagram obtained in Step 1, one obtains a new diagram without the pair of godrons  $g_+$  and  $g_-$ .

Applying  $G^-$  times the above process, one obtains a final diagram having only two positive godrons. Theorem 6 is proved by applying Lemma 3 to

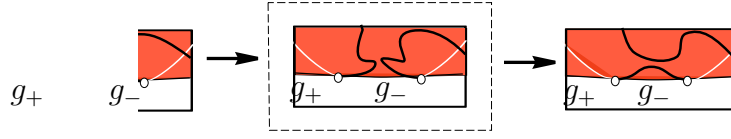


Figure 11: A deformation of  $F_r$  and a move if type I.

this final diagram (note that also the proof of Lemma 3 depends only on the combinatorial properties of the initial diagram).  $\square$

## 4.9 Proof of Theorem 8

To prove Theorem 8 we will use the fact that *the tangential map of  $S$  sends the flecnodal curve of  $S$  onto the flecnodal curve of  $S^\vee$ .*

The dual of a front  $\tilde{S}$  in general position at a swallowtail point  $s$  is a godron of a generic (locally) smooth surface. So Theorem 1 and Separating Lemma imply that the flecnodal curve of  $\tilde{S}$  has a cusp at  $s$  having the same tangent line that the cuspidal edge of  $\tilde{S}$ . Now, by Proposition 2, the swallowtail point separates the flecnodal curve into its left and right branches.

The configuration formed by the flecnodal curve, the cuspidal edge and the self-intersection line of  $\tilde{S}$  at the swallowtail point  $s$ , is determined by the configuration formed by the curves  $F$ ,  $P$ ,  $D$  and the separating 2-jet on the (locally smooth) dual surface  $\tilde{S}^\vee$ , at its godron  $g = s^\vee$ .

Since the asymptotic line is not considered in the concerned configurations, we can eliminate the number 0 (corresponding to the asymptotic line) from the six inequalities of the proof of Theorem 4. One obtains four distinct inequalities, corresponding to four open intervals for the values of  $\rho$  :

$$\begin{aligned}
 \rho \in (1, \infty) & \iff 1 < c_D < c_P < c_F; \\
 \rho \in (0, 1) & \iff c_P < c_F < c_D < 1; \\
 \rho \in (-\frac{1}{2}, 0) & \iff c_P < c_D < c_F < 1; \\
 \rho \in (-\infty, -\frac{1}{2}) & \iff c_P < c_D < 1 < c_F.
 \end{aligned}$$

Using Separating Lemma and the configurations of Theorem 4 (not considering the asymptotic line) one obtains that these four configurations correspond to the four configurations (of Theorem 8) for the flecnodal curve, the cuspidal edge and the self-intersection line in the neighbourhood of a swallowtail point of a front in general position.  $\square$

**Remark 5.** *When this paper was almost finished, I visited l'École Normale Supérieure de Lyon to give a talk about the results of [18] and of this paper. Few days before my talk, E. Ghys and D. Serre have found the book [12] on the history of thermodynamics in Netherlands. It describes a part of Korteweg's work ([10, 11]) about the godrons (called plaits in [12]), the parabolic curve*

and the conodal curve. According to [12], Korteweg had also described the bifurcations of the parabolic and conodal curves when two godrons are born or disappear, for an evolving surface. His mathematical work on the theory of surfaces was motivated by thermodynamical problems.

## References

- [1] **Aicardi F.**, *Geometrical Properties of Generic Real surfaces at the Special Parabolic Points*, Preprint.
- [2] **Arnol'd V.I.**, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York Heidelberg Berlin, 1983. (Russian version: Nauka, 1978.)
- [3] **Arnol'd V.I., Varchenko A.N., Gussein-Zade S.M.**, *Singularities of Differentiable Maps*, Vol. 1, Birkhäuser (1986). (Russian version: Nauka, 1982)
- [4] **Arnol'd V.I.**, *Topological problems of the theory of wave propagation*, Russian Math. Surveys **51:1** (1996) pp. 1–47.
- [5] **Banchoff T., Gaffney T., McCrory**, *Cusps and Gauss mappings*, Research Notes in Math., No. 55, Pitman, Boston (1982).
- [6] **Banchoff T., Thom R.**, *Sur les points paraboliques des surfaces*, C. R. Acad. Sci. Paris, **291** Sér. A (1980) 503–505.
- [7] **Landis E.E.**, *Tangential singularities*, Funct. Anal. Appl. **15:2** (1981), 103–114.
- [8] **Panov D.A.**, *Special Points of Surfaces in the Three-Dimensional Projective Space*, Funct. Anal. Appl. **34:4** (2000) 276–287.
- [9] **Kergosien Y.L., Thom R.**, *Sur les points paraboliques des surfaces*, C. R. Acad. Sci. Paris, **290** Sér. A (1980) 705–710.
- [10] **Korteweg D.J.**, *Sur les points de plissement*, Arch. Néerl. **24** (1891) 57–98.
- [11] **Korteweg D.J.**, *La théorie générale des plis*, Arch. Néerl. **24** (1891) 295–368.
- [12] **Levelt Sengers J.**, *How fluids unmix: Discoveries by the School of Van der Waals and Kamerlingh Onnes*, KNAW, Amsterdam 2002.
- [13] **Platonova O.A.**, *Singularities of the mutual position of a surface and a line*, Russ. Math. Surv. **36:1** (1981) 248–249. Zbl.458.14014.
- [14] **Salmon G.**, *A Treatise in Analytic Geometry of Three Dimensions*, Chelsea Publ. (1927).
- [15] **Segre B.**, *The Nonsingular Cubic Surfaces*, Clarendon Press, Oxford (1942).
- [16] **Uribe-Vargas R.**, *Singularités symplectiques et de contact en géométrie différentielle des courbes et des surfaces*, PhD. Thesis, Université Paris 7, 2001, (In English).
- [17] **Uribe-Vargas R.**, *On Polar Duality, Lagrange and Legendre Singularities and Stereographic Projection to Quadrics*, Proc. London Math. Soc.(3) **87** (2003) 701–724.
- [18] **Uribe-Vargas R.**, *Surface Evolution, Implicit Differential Equations and Pairs of Legendrian Fibrations*, Preprint.



## Centre symmetry sets and other invariants of algebraic sets

*Mariusz Zajac*<sup>1</sup>

The aim of the present note is to reformulate some classical concepts of the theory of planar curves in the language of algebraic geometry. In Sec. 1 we shall present the definitions and several basic properties of the centre symmetry sets of a closed planar curve. Sec. 2 recalls other classical notions, namely that of the centre of curvature, and the evolute, i.e. the locus of the centres of curvature, and shows how they can be expressed in purely algebro-geometric terms. Finally, in Sec. 3 we shall indicate the way of combining the methods, namely the algebraic definition of the centre symmetry set and the anti-centre symmetry set. In particular we shall study the ACSS of a cubic curve. We shall suggest some possibilities for future research and mention some difficulties that are likely to appear.

The author wishes to thank the CIRM at Luminy for the hospitality and Peter Giblin for discussions and clarifying some points.

### 1 The centre symmetry sets

Let us consider a smooth curve  $C \in \mathbf{R}^2$ . If  $C$  is an *oval*, i.e. a closed curve without inflection points, we can say that a point  $O$  is the *centre of symmetry* of  $C$  if it is the midpoint of all chords passing through  $O$ . This is, obviously, a very restrictive property and a generic oval has no centre of symmetry. One can, however, generalize this notion and replace the classical centre of symmetry by a certain set that can be defined for any oval  $C$  and reduces to a single point if  $C$  is centrally symmetric. The main idea is to draw the segments joining those pairs of points on  $C$  at which the tangents are parallel. This one-parameter family of chords is depicted in Fig. 1 for the bounded connected component of the cubic curve  $x^2 = y^3 - y$  (this is the first interesting case, as obviously all closed conics, i.e. ellipses, are centrally symmetric).

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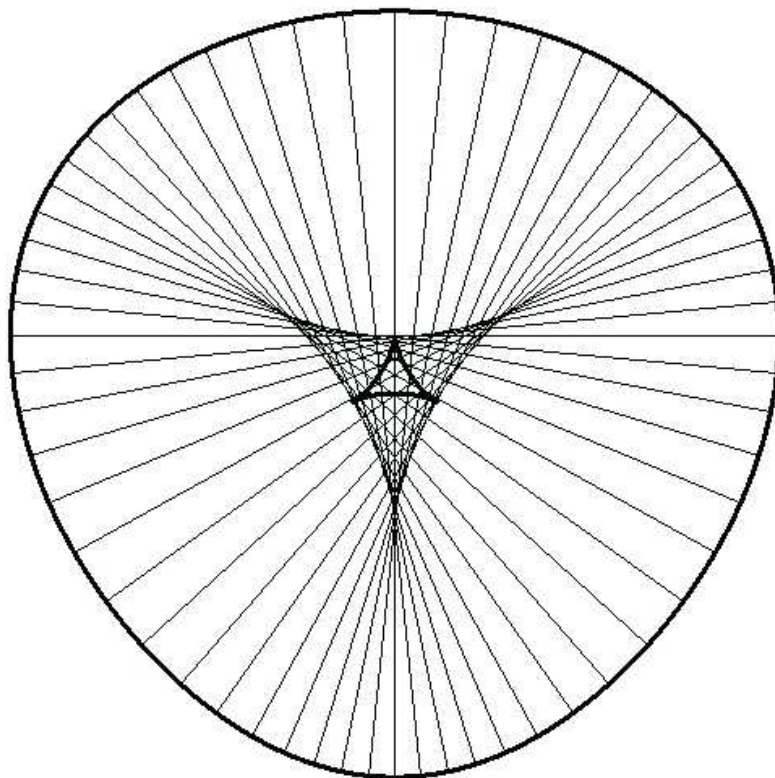


Figure 1: Symmetry sets of an oval

We see that the following two definitions seem natural:

- the envelope of the above-mentioned family of chords (the larger curve with three cusps in Fig. 1) will be called the *centre symmetry set* (CSS);
- the set of midpoints of the chords considered (the smaller curve with three cusps in Fig. 1) will be called the *anti-centre symmetry set* (ACSS).

Of course if  $C$  has a centre of symmetry  $O$  then both the CSS and the ACSS reduce to the point  $O$ . These sets are also invariant under the affine transformations of  $\mathbf{R}^2$  because we only use the notions of tangency, parallel lines and the midpoint of a segment.

The differential properties of these sets were studied in [4], [1] and [2], and in the case of a generic oval they can be summarized as follows:

- apart from a finite number of cusps both sets are smooth curves with no inflection points;
- the CSS and the ACSS have the same odd number of cusps;
- the cusps of the ACSS are the midpoints of the chords joining those pairs of points at which the curvatures of  $C$  are equal;
- the cusps of the CSS lie on the chords joining a point of large curvature with a point of small curvature (more precisely the ratio of curvatures must have a critical point).

## 2 Focal loci

In this section we shall recall the concept of the *focal locus* of a curve (traditionally referred to as the *evolute* of the curve), which is the locus of the centres of curvature of this curve. We shall generally follow the exposition of [5], restricting ourselves, however, to the case of algebraic curves in the (affine or projective) plane.

There are several equivalent ways of defining the centre of curvature of a curve in elementary differential geometry. We choose one of them because it can easily be rewritten in purely algebraic terms.

Let  $\gamma : (a, b) \rightarrow \mathbf{R}^2$  be a smooth curve parametrized by the arclength, and let  $T(t)$  and  $N(t)$  be the unit vectors tangent and normal to  $\gamma$  at  $\gamma(t)$ , respectively. If we now define the following function:

$$e : (a, b) \times \mathbf{R} \rightarrow \mathbf{R}^2; : (t, r) \mapsto \gamma(t) + rN(t),$$

i.e. the *endpoint map* associating to every normal vector of length  $r$  beginning at  $\gamma(t)$  its end, then using the well-known Frenet formulae:

$$\gamma'(t) = T(t), : T'(t) = k(t)N(t), : N'(t) = -k(t)T(t),$$

where  $k(t)$  is the curvature, we can easily prove the following

**Proposition 1.** *The evolute of a planar curve is equal to the set of critical values of the endpoint mapping.*

It should be mentioned here that the notions of critical and regular values are invariant under any diffeomorphic change of coordinates in the domain. Therefore the above proposition will remain equally valid if we take any smooth parametrization  $t$  instead of the arclength or any smooth normal vector field  $N(t)$  instead of the vectors of length 1. However, the evolute is invariant under isometries only, and not all affine transformations, because we always require that  $N(t)$  should be orthogonal to the tangent.

## Focal loci of algebraic curves

As usual, the most natural setting for dealing with algebraic curves is the (complex) projective plane. If we consider a polynomial  $f(x, y)$  of degree  $d$  and its homogeneous counterpart  $F(X, Y, Z) = Z^d f(X/Z, Y/Z)$  then instead of the affine normal line to  $C = \{F(x, y) = 0\}$  at a regular point  $P = (x_0, y_0) \in C$  with the equation

$$F_x(x_0, y_0) \cdot (y - y_0) - F_y(x_0, y_0) \cdot (x - x_0) = 0$$

we can talk of its projectivization: the line in  $\mathbf{P}^2$  joining  $P = (x_0 : y_0 : 1)$  with the point at infinity  $P_\infty = (F_X(P) : F_Y(P) : 0)$ .

The projectivization of the endpoint mapping  $e$  is now

$$E : C \times \mathbf{P}^1 \rightarrow \mathbf{P}^2;$$

$$((X : Y : Z), (\lambda : \mu)) \mapsto (\lambda X + \mu F_X(X, Y, Z) : \lambda Y + \mu F_Y(X, Y, Z) : \lambda Z).$$

In this setting the focal locus is the critical value set of an explicitly defined rational mapping defined on a smooth algebraic variety. Therefore the evolute of an algebraic curve is also an algebraic set. More detailed analysis of this set, which can be found in [5], leads for instance to the following

**Theorem 1.** *Let  $X \subset \mathbf{P}_{\mathbb{C}}^2$  be a general algebraic curve of degree  $d > 1$  and  $Z$  its focal locus. Then  $\deg Z = 3d(d - 1)$ .*

Note that the degree of the real affine view of the focal locus can be equal to  $3d(d - 1)$  or lower, e.g. for an ellipse the algebraic degree of the evolute equals  $3d(d - 1) = 6$  but for a parabola it is 3.

## 3 Algebraic approach to centre symmetry sets

Let  $C \subset \mathbf{R}^2$  be a smooth curve. As we see from the very definition of CSS and ACSS, the most important objects are pairs of points on  $C$  with parallel tangents. Let us define

$$S = \{(P, Q) \in C \times C : \text{the tangents to } C \text{ at } P \text{ and } Q \text{ are parallel}\}.$$

If the projectivized equation of  $C$  is  $F = 0$ , this amounts to saying that these tangents meet at a point with  $Z = 0$ , or equivalently

$$S = \{(P, Q) \in C \times C : F_X(P)F_Y(Q) = F_X(Q)F_Y(P)\}.$$

The set  $S$  can be rather complicated in the general (nonconvex) case, but if  $C$  is an oval then  $S$  is the sum of the diagonal  $\{(P, P)\}$  and the set of

pairs of opposite points  $\{(P, P')\}$ . Thus topologically  $S$  has two components isomorphic to  $C$  itself, but they cannot be distinguished algebraically, which is a serious drawback (see below).

Once we have defined  $S$  we can define both CSS and ACSS in algebraic terms.

**Proposition 2.** *The ACSS of a convex algebraic curve is (a connected component of) the image of  $S$  under the 'midpoint mapping'*

$$m((X_1 : Y_1 : 1), (X_2 : Y_2 : 1)) = (X_1 + X_2 : Y_1 + Y_2 : 2),$$

whereas the CSS is (a connected component of) the set of critical values of the restriction  $j|_{S \times \mathbf{P}^1}$ , where  $j$  is the 'chord mapping'

$$j((X_1 : Y_1 : 1), (X_2 : Y_2 : 1), (\lambda : \mu)) = (\lambda X_1 + \mu X_2 : \lambda Y_1 + \mu Y_2 : \lambda + \mu).$$

Prop. 2 can be proved analogously to Prop. 1.

### 3.1 ACSS of an algebraic curve – cubic case

In the affine setting we are tempted to say that the point  $(x, y)$  belongs to the ACSS of the curve  $C = \{f(x, y) = 0\}$  if the following equations hold for some  $x_1, y_1, x_2, y_2$  :

$$\begin{cases} f(x_1, y_1) = 0 \\ f(x_2, y_2) = 0 \\ f_x(x_1, y_1)f_y(x_2, y_2) = f_x(x_2, y_2)f_y(x_1, y_1) \\ x = \frac{x_1 + x_2}{2} \\ y = \frac{y_1 + y_2}{2} \end{cases} \quad (3.1)$$

If we eliminate  $x_1, y_1, x_2, y_2$  from the above system, we should in principle obtain the equation defining the ACSS. However, as mentioned before, if  $(x, y) = (x_1, y_1) = (x_2, y_2) \in C$  then the parallel tangents condition is tautologically fulfilled, so in fact the system (3.1) defines the union of the ACSS and the curve  $C$  itself. There are also other interesting phenomena, which will be visible in the example.

Let  $f(x, y) = x^3 - 3x - y^2$ . The elimination of  $x_1, y_1, x_2$  and  $y_2$  from (3.1), performed with the Singular package, gives the following 12-th degree equation for  $x$  and  $y$  :

$$(x^3 - 3x - y^2)(8x^9 - 12x^6y^2 - 30x^7 + 6x^3y^4 + 3x^4y^2 - y^6 + 42x^5 + 6xy^4 + 6x^2y^2 - 26x^3 - y^2 + 6x) = 0$$

The divisibility of this equation by  $f(x, y)$  agrees with our previous analysis, and the zero set of the second factor is shown in Fig. 2 as a thin line together

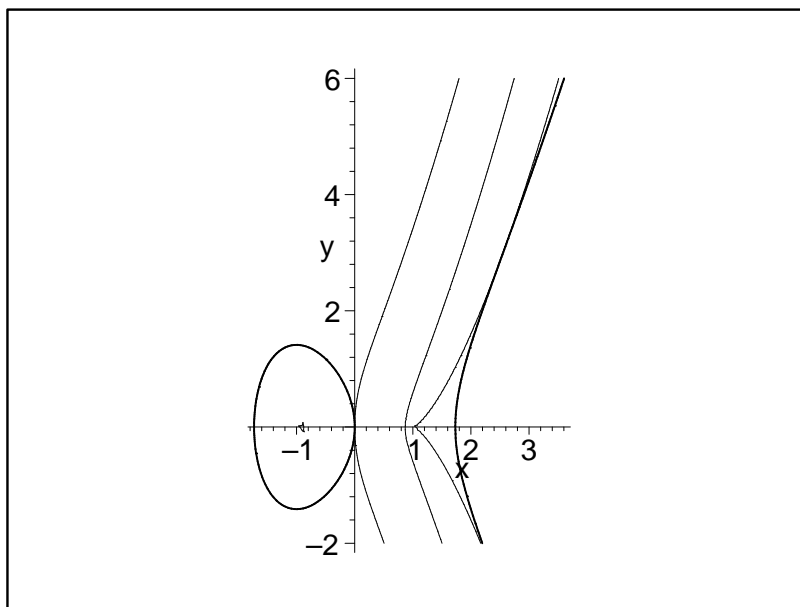


Figure 2: ACSS of  $x^3 - 3x - y^2 = 0$

with the original (thick) curve consisting of an oval and an unbounded branch with two inflection points.

Let us make some observations:

- First and foremost, what we see is the real picture of the complex ACSS, i.e.  $x$  and  $y$  are real, but  $x_1, y_1, x_2$  and  $y_2$  need not be.
- Beside the ACSS of the oval we can see three unbounded branches. Their appearance is easy to understand: for a fixed point  $(x, y) \in C$  with large coordinates  $x, y$  and almost vertical tangent line there are three other points where the tangents are parallel, and they are approximately  $(-\sqrt{3}, 0), (0, 0), (\sqrt{3}, 0)$ . Therefore as  $(x, y)$  goes to infinity, the three respective midpoints tend asymptotically to three 'parallel' cubic curves.
- From the equation of the ACSS one can directly compute its singular points. It turns out that in the complex domain there are ten of them, four of which have real coordinates, namely the three cusps of the ACSS of the oval:  $(-1, 0)$  and  $(-0.8475, \pm 0.153036)$  (hardly visible in Fig. 2, but cf. Fig. 1) plus the point  $(1, 0)$ , whose appearance is confusing at first sight, as it does not seem to be the midpoint of any chord of  $C$ . However, in precisely the same way as  $(-1, 0)$  is the midpoint of the chord joining the points  $(-1, \pm\sqrt{2})$ , the cusp  $(1, 0)$  is the midpoint of the chord between  $(1, \sqrt{2}i)$  and  $(1, -\sqrt{2}i)$ . Indeed,

apart from the obvious symmetry  $(x, y) \mapsto (x, -y)$  there is also another one:  $(x, y) \mapsto (-x, iy)$ .

- In fact, the segments of the rightmost branch between its cusp  $(1, 0)$  and the inflection points of  $C$  consist of real points that are midpoints of pairs of conjugate complex points. Let us discuss it in detail.

In order to find a point on  $C$  with tangent parallel to a given line  $y = ax + b$  we have to solve the system of equations

$$\begin{cases} x^3 - 3x - y^2 = 0 \\ \frac{3(x^2-1)}{2y} = a \end{cases} \quad (3.2)$$

Substituting  $y$  from the second equation to the first one we get an equation of order 4, which can have 2,3 or 4 real solutions, depending on  $a$ . Geometrically speaking, the two limiting values of  $a$  are just the slopes of the tangents at the inflection points of  $C$  : there are four parallel tangents going in 'more vertical' directions but only two in 'more horizontal' ones. Nevertheless, even if there are only two real solutions to the system (3.2), there are also two such points  $(x, y)$  with complex conjugate coordinates, and their midpoint is real.

For a general cubic curve, however, the degree of the ACSS can be higher.

**Theorem 2.** *Let  $X \subset \mathbf{R}^2$  be a general algebraic curve of degree 3 and  $Z$  its ACSS. Then  $\deg Z = 12$ .*

This means that performing the procedure of eliminating  $x_1, y_1, x_2, y_2$  from the system (3.1) for a general cubic polynomial  $f(x, y)$  gives  $G(x, y) = 0$ , where  $G$  is a divisible by  $f$  polynomial of degree 15, so the degree of  $G/f$  is 12.

Before sketching a proof of this result, let us show why the curve  $x^3 - 3x - y^2 = 0$  is not general for this problem. One can easily see that the system (3.2) for a general cubic leads to an equation of degree 6, not 4, and indeed, a cubic can have as much as six parallel tangents. This is not the case for  $x^3 - 3x - y^2 = 0$  because its projectivization  $X^3 - 3XZ^2 - Y^2Z = 0$  has at infinity an inflection point  $(0 : 1 : 0)$  with the tangent  $Z = 0$ . Then  $(0 : 1 : 0)$  is a double solution of the projectivization of (3.2) for any  $a$ . One could also say informally that the line at infinity  $Z = 0$  is parallel to any line  $Y = aX + bZ$  because two lines are called parallel whenever their intersection lies at infinity.

In Fig. 3 and 4 we see more examples of the ACSS of cubic curves. Please pay attention to the neighbourhoods of the inflection points. As a rule only one branch consists of midpoints of visible (real) chords.

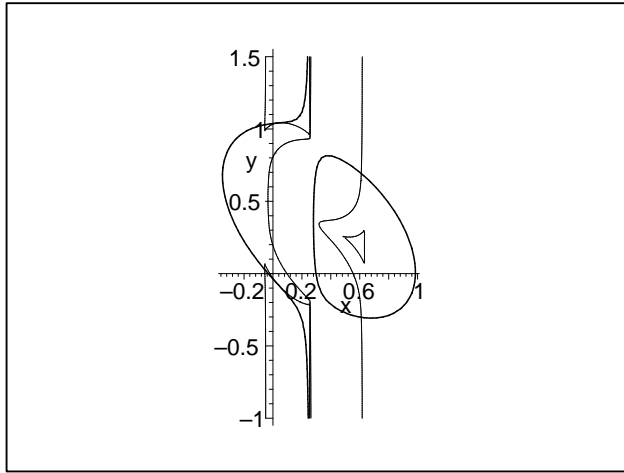


Figure 3: ACSS of  $(x^2 + xy + y^2)(x - \frac{1}{4}) + \frac{1}{100} = 0$

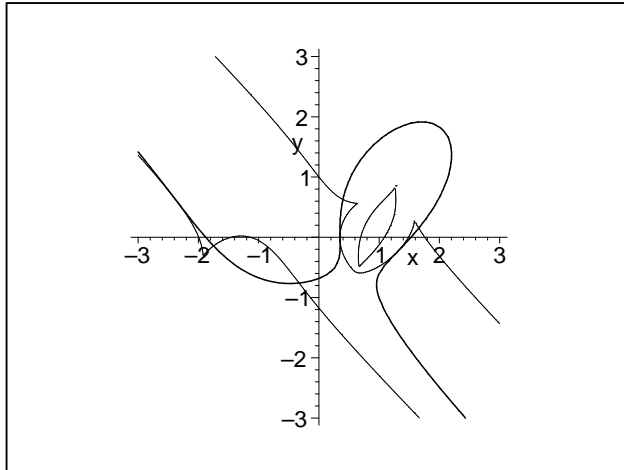


Figure 4: ACSS of  $x^3 + y^3 - 3xy - 3x + y + 1 = 0$

### 3.2 ACSS of an algebraic curve – another approach

We intend to derive an algebraic condition for an arbitrary point  $P = (x_0, y_0)$  to belong to the ACSS of the curve  $C = \{f(x, y) = 0\}$ . By translation invariance this happens whenever  $(0, 0)$  belongs to the ACSS of the shifted curve  $f_P(x, y) = f(x + x_0, y + y_0) = 0$ . The coefficients of  $f_P$  are polynomials in  $x_0$  and  $y_0$ , therefore it suffices to find a condition for  $(0, 0)$  to belong to the ACSS of the curve  $C = \{f(x, y) = 0\}$  algebraic in the coefficients of  $f$ .

However, the origin  $(0, 0)$  belongs to the ACSS of  $C$  if and only if:

for some  $x$  and  $y$  both points  $(x, y)$  and  $(-x, -y)$  belong to  $C$  and the respective tangents are parallel, or equivalently:

for some  $x$  and  $y$  the point  $(x, y)$  belongs both to  $C$  and to the symmetric



curve  $-C = \{f(-x, -y) = 0\}$  and the respective tangents coincide, which means exactly:

the curves  $C$  and  $-C$  have at least one nontransversal intersection point.

We can also observe that the system

$$\begin{cases} f(x, y) = 0 \\ f(-x, -y) = 0 \end{cases} \quad (3.3)$$

is equivalent to

$$\begin{cases} f_e(x, y) = \frac{f(x, y) + f(-x, -y)}{2} = 0 \\ f_o(x, y) = \frac{f(x, y) - f(-x, -y)}{2} = 0 \end{cases} \quad (3.4)$$

where  $f_e$  and  $f_o$  denote the sums of monomials of even (respectively odd) degrees included in  $f$ .

In particular for a cubic equation  $f = \sum_{i+j \leq 3} a_{ij}x^i y^j$  we obtain

$$\begin{cases} a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{00} = 0 \\ a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{10}x + a_{01}y = 0 \end{cases}$$

We solve this system in the following steps:

We complete the square in the first equation, i.e. perform a linear change of coordinates in order to have  $a_{11} = 0$ . By abuse of notation the new  $x$  and  $y$  will have unchanged names.

We solve the first equation for  $y^2$  obtaining  $y^2 = Ax^2 + B$  and substitute  $Ax^2 + B$  for  $y^2$  and  $(Ax^2 + B)y$  for  $y^3$  in the second equation, obtaining

$$\begin{cases} y^2 = Ax^2 + B \\ Cx^3 + Dx^2y + Ex + Fy = 0 \end{cases}$$

(here  $A, \dots, F$  are some rational functions of the  $a_{ij}$ 's).

The second equation gives now

$$y = -\frac{Cx^2 + E}{Dx^2 + F}x$$

and from the first equation we have

$$\left(\frac{Cx^2 + E}{Dx^2 + F}\right)^2 x^2 = Ax^2 + B \quad (3.5)$$

which, when multiplied by the denominator, becomes a cubic equation in  $x^2$ . This equation has a multiple root if and only if its discriminant is 0, and that imposes a polynomial condition on  $A, \dots, F$ , hence also on the  $a_{ij}$ 's.

The preceding sketch omits some details, but it shows in principle that we can derive not only the algebraic equation of the ACSS of any fixed cubic curve, but also the general formula transforming any set of 10 coefficients  $a_{ij}$  into the equation of the ACSS. Computations using Maple showed that this equation has in fact degree 12.

### 3.3 Critical points of the ACSS

From the above considerations it is clear that the ACSS has a critical point if and only if the system (3.3), or equivalently (3.4), has a solution of multiplicity at least 3, which gives an extra algebraic condition that the cubic equation in  $x^2$ , obtained from (3.5) should have a triple root rather than only a double one.

It is worth pointing out that the ACSS appeared in [3] under the name MPTL (mid-parallel-tangents locus) as an auxiliary set for studying one of generalizations of the axis of symmetry, namely the AESS (affine envelope symmetry set), which is the set of the centres of conics having two points of (at least) triple contact with the given curve (so called 3+3 conics). According to Propositions 2.4.7 and 2.4.9 of [3] the critical points of the ACSS are simultaneously critical points of the AESS and they are the centres of 3+3 conics with parallel tangents.

### 3.4 CSS of an algebraic curve – preliminary remarks

In this part we shall skip the details of computation and show only the general idea that one can treat the CSS in the same way as the ACSS in 3.2 and try to derive a condition for  $(0, 0)$  to belong to the CSS of the curve  $C = \{f(x, y) = 0\}$ .

According to [4] and [1], any point on the CSS of  $C$  divides its respective chord in the ratio equal to the ratio of the curvatures of  $C$  at the ends of this chord. In other words the origin  $(0, 0)$  belongs to the CSS of  $C$  if and only if there exists such a negative real  $k$  that the curves  $C$  and  $kC = \{f(kx, ky) = 0\}$  have an intersection point with common tangents and curvatures, i.e. a triple intersection point (side remark: in the oval case there can be no such positive real  $k \neq 1$ ).

This amounts to considering the system of equations:

$$\begin{cases} f(x, y) = 0 \\ f(kx, ky) = 0, \end{cases} \quad (3.6)$$

which unfortunately cannot be transformed to such a special form as (3.4) because of the existence of  $k$ . However if  $\deg f = n$ , then the following equivalent system

$$\begin{cases} k^n f(x, y) - f(kx, ky) = 0 \\ f(x, y) - f(kx, ky) = 0 \end{cases} \quad (3.7)$$

is simpler than (3.6) because the first equation has now degree  $n - 1$  and the second one has no constant term.

Generally speaking, there are several ways of checking whether two given plane curves have a triple intersection point. One of them requires equating the derivatives  $y'(x)$  and  $y''(x)$  for both curves computed by the Implicit Function Theorem; another method can be applied when one of the curves has a rational parametrization (this is the case for (3.7) when  $n = 3$ , since then the first equation has degree 2) – it involves substituting this parametrization to the equation to the other curve and writing the discriminant conditions in order to check if the resulting polynomial has a triple root.

Quite obviously, although we could in principle write the corresponding algebraic equations explicitly, they are much more complicated than in the ACSS case. Last but not least, we still have to eliminate  $k$  from these equations, and remember that there may exist nonreal numbers  $k$  for which the system (3.6) has a triple real solution  $(x, y)$ .

It should be hoped, however, that for some special classes of curves the computations could simplify considerably – e.g. for the curves of constant width the ACSS coincides with the focal locus. These aspects will be dealt with in the further research.

### 3.5 Final remarks

- A single point on the evolute corresponds to (a neighbourhood of) a single point on the original curve  $C$ , while one point on the CSS or ACSS reflects properties of two points on  $C$ .
- One algebraic equation usually defines a sum of several disjoint ovals, some of which may be convex and some nonconvex. The maximal number of ovals grows proportionally to the square of the degree of  $C$ , and therefore the maximal number of components of CSS and ACSS is asymptotically proportional to the fourth power of this degree.
- Finally, though the classical methods of algebraic geometry were used successfully in [5], one must be aware that some extra structure on  $\mathbf{P}^2$  is necessary. For instance, in order to define parallel tangents and the midpoint of a segment we must fix a projective line  $Z = 0$ , and the evolute requires some notion of distance or orthogonality (in fact there exist also so called affine normals and affine evolutes, but we do not address them here).

The above remarks suggest that, although various types of symmetry sets arise quite naturally and find various applications e.g. in image recognition and processing, their detailed algebraic study encounters serious obstacles.

## References

- [1] P. Giblin, P. Holtom, *The centre symmetry set*, in: Janeczko, Stanislaw (ed.) et al., *Geometry and topology of caustics – CAUSTICS '98*, Banach Cent. Publ. 50 (1999), 91–105
- [2] P. Giblin, V. Zakalyukin, *Singularities of centre symmetry sets*, Proc. London Math. Soc. (3) 90 (2005), 132–166
- [3] P. Holtom, *Affine-invariant symmetry sets*, PhD thesis, available at <http://www.liv.ac.uk/~pjgiblin>
- [4] S. Janeczko, *Bifurcations of the center of symmetry*, Geom. Dedicata 60, No.1 (1996), 9–16
- [5] F. Trifogli, *Focal loci of algebraic hypersurfaces: A general theory*, Geom. Dedicata 70, No.1 (1998), 1–26