On Fermat curves and maximal nodal curves

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Introduction

A nodal curve C is an irreducible plane curve of degree n which contains only nodes ($= A_1$ singularities). A nodal curve is called *a maximal nodal curve* if it is rational and nodal. By Plücker formula, it must contain $\frac{(n-1)(n-2)}{2}$ nodes to be maximal. In the space of polynomials of two variables, a maximal nodal curve can be understood as a generalization of a Chebycheff polynomial. In our paper [**?**], we constructed a maximal nodal curve of join type $f(x) + g(y) = 0$ using a Chebycheff polynomial $f(x)$ and a similar polynomial $g(y)$ that has one maximal value and two minimal values.

Construction from Fermat curveand the Dual Geometry

In this paper, we present another extremely simple way, for a given integer $n > 2$, to construct a maximal nodal curve D_{n-1} with a beautiful symmetry, as a bi-product of the geometry of the Fermat curve $x^n + y^n + 1 = 0$. A smooth point P of a plane curve C is called *a flex point of flex-order* $k - 2$, $k \geq 3$ if the tangent line T_P at P and C intersect with intersection multiplicity k. The maximal nodal curve D_n, which we construct in this paper, contains 3 flexes of flex-order $n - 2$ and it is symmetric with respect to the permutation of three variables U, V, W.

Zariski Conjecture

By a special case of Zariski and Fulton ([**?**, **?**]), $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/n\mathbb{Z}$ if C is a maximal nodal curve of degree n. The examples D_n provide an alternate proof. Zariski observed ([**?**]) that the fundamental group of the complement of an irreducible curve C of degree n is abelian if C has a flex of flex-order either n or $n-1$. Since the moduli of maximal nodal curves of degree n is irreducible by Harris ([**?**]), the claim follows.

Construction

For the construction, we start from the Fermat curve $\mathcal{F}_n : x^n + y^n + 1 = 0$ and study singularities of the dual curve $\check{\mathcal{F}}_n$. The Fermat curve and the dual curve $\check{\mathcal{F}}_n$ have canonical $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ actions, thus the defining polynomial of $\check{\mathcal{F}}_n$ is written as $h(u^n, v^n) = 0$ for a polynomial $h(u, v)$ of degree $n - 1$. The curve $h(u, v) = 0$ defines our maximal nodal curve D_{n-1}. Geometrically this is the quotient of the dual curve $\check{\mathcal{F}}_n$ by the above action. Moreover the curve D_{n-1} is explicitly parametrized as

$$D_{n-1} : \quad u(t) = t^{n-1}, \qquad v(t) = (-1 - t)^{n-1}.$$

The Gauss map and the dual curves

We consider an irreducible plane curve C of degree n, $C:\ f(x,y)=0\subset\mathbb{C}^2$. Its homogenization $F(X,Y,Z)=0$ defines the projective curve C of degree n in $\mathbb{P}^2$ where $F(X,Y,Z)=f(X/Z,Y/Z)Z^n$. For a smooth point $P=(a,b,c)\in C$, the tangent line is defined by $F_X(P)X+F_Y(P)Y+F_Z(P)Z=0$ where F_X,F_Y,F_Z are derivatives in the corresponding variables. The dual projective plane $\check{\mathbb{P}}^2$ has the dual coordinates U,V,W. In the dual projective plane $\check{\mathbb{P}}^2$, we usually work in the affine space $\{W\neq 0\}$ with the coordinates (u,v) where $u=U/W$, $v=V/W$. The Gauss map associated with C is defined by

$$G_F : C\to\check{\mathbb{P}}^2,\quad G_F(P)=(F_X(P)\colon F_Y(P):F_Z(P)).$$

Thus in the affine coordinates (x,y), $P=(x,y)\in C$ is mapped into $G_f(P)=(f_x(x,y):f_y(x,y):-xf_x(x,y)\,-\,yf_y(x,y))$.

The class formula

$$\textit{Class} - \textit{formula}0\check{n} = n(n-1) - \sum_{P\in\Sigma(C)} (\mu(C,P) + m(C,P) - 1)$$

If C is non-singular, we have $\check{n} = n(n-1)$.

Cyclic action We assume that there exists a polynomial $g(x,y)$ such that $f(x,y) = g(x^m, y^s)$ $m \geq s$ for some positive integers $m, s \geq 2$. Under this assumption, we consider the action on $\mathbb{P}^2$ of the product of cyclic groups $\mathbb{Z}_{m,s} := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z}$, which is defined as follows. Let $\omega_\ell := \exp(2\pi i/\ell)$ and we identify the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$ with the multiplicative subgroup of $\mathbb{C}^*$ generated by ω_ℓ. The action is defined by

$$\psi : \mathbb{Z}_{m,s} \times \mathbb{P}^2 \to \mathbb{P}^2, \quad (\gamma, (x,y)) \mapsto (x\,\omega_m^j, y\,\omega_s^k), \text{ where } \gamma = (\omega_m^j, \omega_s^k)$$

In the homogeneous coordinates, this action is written as

$$(\gamma, (X:Y:Z)) \mapsto (X\omega_n^j : Y\omega_s^k : Z)$$

Action on Dual space

Define the action of $\mathbb{Z}_{m,s}$ on the dual projective plane similarly:

$$\check{\psi}(\gamma,(u,v)) = (\omega_m^j u, \omega_s^k v) \text{ or},$$
$$\check{\psi}(\gamma,(U:V:W)) = (\omega_m^j U : \omega_s^k V : W).$$

Then by an easy computation,

$$\begin{aligned} G_f(P) = \quad & (mx^{m-1}g_x(x^m,y^s) : sy^{s-1}g_y(x^m,y^s) : \\ & -mx^m g_x(x^m,y^s) - sy^s g_y(x^m,y^s)) \\ G_f(P^\gamma) = \quad & (m(\omega_m^j x)^{m-1}g_x(x^m,y^s) : s(\omega_s^k y)^{s-1}g_y(x^m,y^s) : \\ & -mx^m g_x(x^m,y^s) - sy^s g_y(x^m,y^s)) \\ & = G_f(P)^{1/\gamma} \end{aligned}$$

Uop-Down Isomorphism

Proposition: The dual curve is invariant by the $\mathbb{Z}_{m,s}$-action. This implies that $\check{f}(u,v)$ can be written as $h(u^m, v^s)$ using some polynomial $h(u,v)$. Note that $h(u,v)$ is not the defining polynomial of the dual curve of $g(x,y)=0$ in general. However we have the following fundamental result.

Theorem Let $C(g) := \{(x,y); g(x,y)=0\} \subset \mathbb{P}^2$ and $D := \{(u,v); h(u,v)=0\} \subset \check{\mathbb{P}}^2$.
Then there exists a canonical birational mapping $\Phi_{m,s} : C(g) \to C(h)$.

Proof

Let $\pi_{m,s} : \mathbb{P}^2 \to \mathbb{P}^2$, $\check{\pi}_{m,s} : \check{\mathbb{P}}^2 \to \check{\mathbb{P}}^2$ be the branched covering map defined by

$$\pi_{m,s}(X : Y : Z) = (X^m : Y^s Z^{m-s} : Z^m),\ \pi_{m,s}(x, y) = (x^m,\ y^s)$$
$$\check{\pi}_{m,s}(U : V : W) = (U^m : V^s W^{m-s} : W^m),\ \check{\pi}_{m,s}(u,\ v) = (u^m,\ v^s)$$

$\pi_{m,s} : C \to C(g)$ and $\check{\pi}_{m,s} : \check{C} \to C(h)$.

$$\begin{array}{ccc} C : f(x,y) = 0 & \xrightarrow{G_f} & \check{C} : \check{f}(u,v) = 0 \\ \downarrow \pi_{m,s} & & \downarrow \check{\pi}_{m,s} \\ C(g) : g(x,y) = 0 & \xrightarrow{\Phi_{m,s}} & C(h) : h(u,v) = 0 \end{array}$$

Proof-bis

Let us consider the multi-valued section of $\pi_{m,s}$:

$$\lambda:\ C(g)\to C,\ \lambda(x,y)=(x^{1/m},\ y^{1/s}).$$

The composition $\Phi_{m,s}:=\check{\pi}_{m,s}\circ G_f\circ\lambda:\ C(g)\to C(h)$ is a well-defined single valued rational mapping and it does not depend on the choice of λ. In fact, it is given by:

$$\Phi_{m,s}(x,y)=$$
$$\left(\frac{m^m\,x^{m-1}g_x(x,y)^m}{(-m\,x\,g_x(x,y)-\ s\,y\,g_y(x,y))^m},\frac{s^s\,y^{s-1}g_y(x,y)^s}{(-m\,x\,g_x(x,y)-\ s\,y\,g_y(x,y))^s}\right)$$

Similarly we consider a (multi-valued) section $\check{\lambda}:\mathbb{C}^2\to\mathbb{C}^2$, $\check{\lambda}(u,v)=(u^{1/m},v^{1/s})$, of $\check{\pi}_{m,s}$ and the composition $\Psi_{m,s}=\pi_{m,s}\circ G_{\check{f}}\circ\check{\lambda}:C(h)\to C(g)$,

$$\Psi_{m,s}(u,v)=$$
$$\left(\frac{m^m\,u^{m-1}(h_u(u,v))^m}{(-m\,u\,h_u(u,v)-\ s\,v\,h_v(u,v))^m},\frac{s^s\,v^{s-1}(h_v(u,v))^s}{(-m\,u\,h_u(u,v)-\ s\,v\,h_v(u,v))^s}\right).$$

Proof-bis

It is easy to observe that $\Phi_{m,s}$, $\Psi_{m,s}$ satisfies $\Psi_{m,s} \circ \Phi_{m,s} = \mathrm{id}_{C(g)}$ and $\Phi_{m,s} \circ \Psi_{m,s} = \mathrm{id}_{C(h)}$ as the $G_{\check{f}}$ and G_f are mutually inverse. For example, the equality $\Psi_{m,s} \circ \Phi_{m,s} = \mathrm{id}_{C(h)}$ is shown as follows. Put $(x', y') := \lambda(x, y)$ and $(u, v) := G_f(x', y')$. Then

$$
\begin{aligned}
\Psi_{m,s} \circ \Phi_{m,s}(x, y) \quad &= \pi_{m,s} \circ (G_{\check{f}} \circ \check{\lambda} \circ \check{\pi}_{m,s})(u, v) \\
&= \pi_{m,s} \circ G_{\check{f}}((u, v)^{\gamma}) \quad \exists \gamma \in \mathbb{Z}_{m,s} \\
&= \pi_{m,s} \circ (G_{\check{f}}(u, v))^{1/\gamma} \\
&= \pi_{m,s}((x', y')^{1/\gamma}) \\
&= \pi_{m,s}(x', y') = (x, y)
\end{aligned}
$$

Singularities of the dual curves

We recall basic properties for the dual curve which we use later.
First case:Singularity from the singular points of C. Suppose that P is a singular point of C. Then $G_f(P)$ is a singular point of $\check{C}$. The exceptional case is when the topological equivalence class of (C, P) is $B_{k,k-1}$, $k \geq 3$. The Gauss image of P is a flex point of flex-order $k-2$.

Flex points

The locus of the flex points are described by $\mathrm{Hess}(F)(X, Y, Z) = F(X, Y, Z) = 0$ where $\mathrm{Hess}(F)(X, Y, Z)$ is the hessian of F:

$$\mathrm{Hess}(F)(X, Y, Z) = \begin{vmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{vmatrix}$$

Thus by Bézout theorem, we have

$$\sharp(\text{flex points}) = 3n(n-2)$$

where the number is counted with multiplicity.

Second case

There is another singularity which is produced from a special point of C. There are two such special points: flex points (already described above) and points with multi-tangent lines.
A smooth point $P \in C$ gives a *multi-tangent line* if the tangent line T_P is also tangent to C at some other point $Q \in C$, so $T_Q = T_P$. The most common one is a *bi-tangent line*. If P is a bi-tangent point (so there is another point $Q \in C$ so that $I(C, T_P; Q) = 2$ and any other intersections $C \cap T_P$ are transverse), its image by the Gauss map is a node (i.e., A_1). If it has q-tangent points, the image is topologically equivalent to a Brieskorn singularity $B_{q,q}$. This singularity has q smooth local branches intersecting transversely.

Dual Curve

The degree $\check{n}$ of $\check{C}$ is

$$\check{d} = n(n-1) - \sum_P (\mu(P) + m(C,P) - 1)$$

Assuming that C is smooth and flex points of C are generic i.e., their flex-order are 3, and that C has only bi-tangent lines, the classical formula tells that
class-formula1

$$\sharp(\text{bi-tangents}) = \frac{(\check{n}-1)(\check{n}-2)}{2} - 3n(n-2) - \frac{(n-1)(n-2)}{2},$$
$$\check{n} = n(n-1).$$

For more general situation

$$\frac{(\check{n}-1)(\check{n}-2)}{2} - \sum_{Q \in \Sigma(\check{C})} \delta(\check{C}, Q) = \frac{(n-1)(n-2)}{2} - \sum_{P \in \Sigma(C)} \delta(C, P)$$

Local (or global) parametrization

We assume that C is locally irreducible at P and C is parametrized as

$$x = x(t),\ y = y(t), \quad |t| \leq 1$$

where x, y are the affine coordinate $x = X/Z$, $y = Y/Z$. Then the local branch that is the image of the local irreducible germ (C, P) has the parametrization at $G_f(P)$ (see [**?**], for example)

$$U(t) = y'(t),\ V(t) = -x'(t),\ W(t) = x'(t)y(t) - x(t)y'(t) \quad (1)$$

If $\check{C}$ is locally irreducible at $G_f(P)$, the above parametrization describes the local germ $(\check{C}, G_f(P))$. Equivalently in the affine coordinates $(u, v) = (U/W, V/W)$, the parametrization is given as

$$u(t) = \frac{y'(t)}{x'(t)y(t) - x(t)y'(t)}, \quad v(t) = \frac{-x'(t)}{x'(t)y(t) - x(t)y'(t)} \quad (2)$$

Geometry of Fermat curves

In this section, we study the Fermat curve of degree n:

$$\mathcal{F}_n : F(X,Y,Z) = X^n + Y^n + Z^n = 0.$$

We denote the degree of the dual curve $\check{\mathcal{F}}_n$ by $\check{n}$. Note that $\check{n} = n(n-1)$. There is an obvious $\mathbb{Z}_{n,n}$ acts on $\mathcal{F}_n$ and $\check{\mathcal{F}}_n$. Note that $\mathcal{F}_n$ has 3n flexes of flex-order $n-2$ at

$$P_{1,j} := (0 : \xi_j : 1),\ P_{2,j} := (\xi_j : 0 : 1),\ P_{3,j} := (1 : \xi_j : 0),$$
$$j = 0, \dots, n-1$$
$$\text{where} \quad \xi_j = \exp((2j+1)\sqrt{-1}/n)$$

The tangent line at $P_{1,j}$ is defined by $y = \xi_j$ and it produces a $B_{n,n-1}$ singularity on $\check{\mathcal{F}}_n$ at $(U : V : W) = (0 : 1 : -\xi_j)$. The situation is exactly the same for other flexes through a permutation of coordinates.

Bi-tangents.

Now we consider bi-tangent (or multi-tangent) lines on $\mathcal{F}_n$. The dual curve $\check{\mathcal{F}}_n$ has genus $\frac{(n-1)(n-2)}{2}$ and $3n$ $B_{n,n-1}$ singularities coming from flex points. Then by the formula (**??**), the number of the bi-tangent lines τ should be

$$\begin{aligned}\tau &= \frac{(\check{n}-1)(\check{n}-2)}{2} - 3n \times \frac{(n-1)(n-2)}{2} - \frac{(n-1)(n-2)}{2} \\ &= \frac{n^2(n-2)(n-3)}{2}\end{aligned}$$

Calculation of bi-tangents

The Fermat curve $\mathcal{F}_n$ has $\frac{n^2(n-2)(n-3)}{2}$ bi-tangent lines.

Proof.

Let $\omega := \exp(2\pi\sqrt{-1}/(n-1))$.
Suppose that $P = (a, b),\ Q = (a', b') \in \mathcal{F}_n$ are bi-tangent points.
The tangent line at P is given by $a^{n-1}x + b^{n-1}y + 1 = 0$. Thus
$G_f(P) = (a^{n-1} : b^{n-1} : 1)$ and the assumption implies

$$a^n + b^n + 1 = (a')^n + (b')^n + 1 = 0, \quad a^{n-1} = (a')^{n-1}, \quad b^{n-1} = (b')^{n-1}$$

Thus we can write $a' = a\omega^k, \quad b' = b\omega^j$ for some integers
$0 < j, k < n-1$ and $a^n(\omega^k - \omega^j) = (\omega^j - 1)$. As we assume that
$P \neq Q$ and $P, Q \in \mathbb{C}^2$, we may assume that $j \neq k$ and $k, j \neq 0$.
Thus putting $\beta_{j,k} := \frac{\omega^j - 1}{\omega^k - \omega^j}$, we get:

$$a^n = \beta_{j,k},\ b^n = -1 - \beta_{j,k}, \quad a' = a\omega^k,\ b' = b\omega^j$$

for some $1 \leq j,\ k \leq n-1,\ k \neq j$.

Proof-bis

For any $0 < j < n-1$, put $j_c = n-1-j$. Observe that $\omega^{-j} = \omega^{j_c}$.
Put $\alpha_{j,k} := \frac{\omega^k - 1}{1-\omega^j}$. Then we have

$$\beta_{j,k} = \frac{\omega^j - 1}{\omega^k - \omega^j} = \alpha_{(j-k)_c, j_c}$$

The complex number $\beta_{j,k}$, or equivalently $\alpha_{j,k}$, $1 \leq j, k \leq n-1$ and $j \neq k$ are all distinct.

Geometry of the dual Fermat curve $\check{\mathcal{F}}_n$

Let $\check{f}(u,v)=0$ (and put $\check{F}(U,V,W)$ be its homogenization) be the defining affine (resp. homogeneous) polynomial of the dual curve where u,v are affine coordinates defined by $u=U/W,\ v=V/W$. As $\mathcal{F}_n$ is a symmetric polynomial with $\mathbb{Z}_{n,n}$ action, $\check{F}(U,V,W)$ is a symmetric polynomial of degree $n(n-1)$ with $\mathbb{Z}_{n,n}$ action. (Recall that $\mathbb{Z}_{n,n}=\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$.)
Namely we can write that $\check{f}(u,v)=h(u^n,v^n)$ for some symmetric polynomial $h(u,v)$ of degree $n-1$. We have already observed that $\check{n}=n(n-1)$ and the singularities are $3n$ $B_{n,n-1}$ singularities, and $n^2(n-2)(n-3)/2$ nodes i.e., A_1-singularities. On each coordinate axis $U=0,\ V=0$ and $W=0$, there are exactly n $B_{n,n-1}$ singularities. The tangent line at $P_{1,j}$ is defined by $y=\xi_j$ and its Gauss image is a $B_{n,n-1}$ singularity at $(U:V:W)=(0:1:-\xi_j)$.

Construction of maximal nodal curves

We will give an explicit construction of such a maximal nodal curve as an application of the Fermat curve. Let $\check{F}(U, V, W)$ be the defining polynomial of $\check{\mathcal{F}}_n$ and write it as $\check{F}(U, V, W) = H(U^n, V^n, W^n)$ with $H(U, V, W)$ is a polynomial of degree $n-1$. Then we consider the curve of degree $n-1$ defined by $H(U, V, W) = 0$. We denote it as D_{n-1}. We claim that D_{n-1} is a maximal nodal curve of degree $n-1$. In fact, the rationality follows from the rationality of the line $L: x + y + 1 = 0$ and by Theorem **??**. As $\check{\mathcal{F}}_n$ has $n^2(n-1)(n-2)/2$ nodes outside of the union of coordinate axis $UVW = 0$ and they are invariant by the $(\mathbb{Z}/n\mathbb{Z})^2$ action. We consider n^2-fold branched covering $\check{\pi}_{n,n} : \mathbb{P}^2 \to \mathbb{P}^2$ as before. The image of $n^2(n-2)(n-3)/2$ nodes is now $(n-2)(n-3)/2$ nodes on D_{n-1}. Thus D_{n-1} is maximal nodal.

Theorem

The curve D_{n-1} is a maximal nodal curve and is parametrized as follows.

$$D_{n-1} \ : \ u(t) \, = \, t^{n-1}, \quad v(t) \, = \, (-1-t)^{n-1}$$

It has 3 flexes of flex-order $n-1$ on each coordinate axis whose tangent lines are the coordinate axes. The defining polynomial $h(u,v)$ of D_{n-1} is given by:

$$h(u,v) = \mathrm{Resultant}\,(u-u(t), v-v(t), t).$$

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