

# SINGULARITIES OF AFFINE EQUIDISTANTS: EXTRINSIC GEOMETRY OF SURFACES IN 4-SPACE.

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ABSTRACT. For a generic embedding of a smooth closed surface  $M$  into  $\mathbb{R}^4$ , the subset of  $\mathbb{R}^4$  which is the affine  $\lambda$ -equidistant of  $M$  appears as the discriminant set of a stable mapping  $M \times M \rightarrow \mathbb{R}^4$ , hence their stable singularities are  $A_k$ ,  $k = 2, 3, 4$ , and  $C_{2,2}^\pm$ . In this paper, we characterize these stable singularities of  $\lambda$ -equidistants in terms of the bi-local extrinsic geometry of the surface, leading to a geometrical study of the set of weakly parallel points on  $M$ .

## 1. INTRODUCTION

When  $M$  is a smooth closed curve on the affine plane  $\mathbb{R}^2$ , the set of all midpoints of chords connecting pairs of points on  $M$  with parallel tangent vectors is called the *Wigner caustic* of  $M$ , or the *area evolute* of  $M$ , or still, the *affine 1/2-equidistant* of  $M$ . The 1/2-equidistant is generalized to any  $\lambda$ -equidistant, denoted  $E_\lambda(M)$ ,  $\lambda \in \mathbb{R}$ , by considering all chords connecting pairs of points of  $M$  with parallel tangent vectors and the set of all points of these chords which stand in the  $\lambda$ -proportion to their corresponding pair of points on  $M$ .

The definition of the affine  $\lambda$ -equidistant of  $M$  is generalized to the cases when  $M$  is an  $n$ -dimensional closed submanifold of  $\mathbb{R}^q$ , with  $q \leq 2n$ , by considering the set of all  $\lambda$ -points of chords connecting pairs of points on  $M$  whose direct sum of tangent spaces do not coincide with  $\mathbb{R}^q$ , the so-called *weakly parallel pairs* on  $M$ . In the particular case of  $M^2 \subset \mathbb{R}^4$ , a weakly parallel pair on the surface  $M$  can be either 1-parallel (when the tangent spaces span a 3-space) or 2-parallel, which is the case of true parallelism, also called strong parallelism.

Affine equidistants of smooth submanifolds appear naturally in the definition of affine-invariant global centre symmetry sets of these submanifolds and, particularly the Wigner caustic, also appear naturally in

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some problems in mathematical physics. In every case, precise knowledge of their singularities is an important issue [11, 8, 9, 4, 3, 2]. Thus, stable singularities of affine equidistants of  $M^n \subset \mathbb{R}^q$  have been extensively studied [1, 7, 8, 9, 10, 4, 3, 2], culminating in its complete classification for all pairs  $(2n, q)$  of nice dimensions [5].

On the other hand, not so much is known with respect to the interpretation for the realization of these stable singularities in terms of the extrinsic geometry of  $M^n \subset \mathbb{R}^q$ . The case of curves on the plane has long been well understood [1, 7], just as for hypersurfaces [8]. Another instance that has been completely worked out refers to a Lagrangian surface  $M^2$  in symplectic  $\mathbb{R}^4$ , for its Wigner caustic on shell, that is, the part of its  $1/2$ -equidistant that is close to and contains  $M$  [3]. A geometric study of the Wigner caustic on shell for general surfaces in  $\mathbb{R}^4$  has also been partly worked out in [10].

In this paper, we extend the extrinsic geometric study of the realization of affine equidistants to the case of general (off-shell)  $\lambda$ -equidistants of any smooth surface  $M^2 \subset \mathbb{R}^4$ . Our paper is organized as follows:

First, Section 2 reviews basic definitions and characterizations of affine equidistants. The presentation is based on [5]. Then, basic facts on extrinsic geometry of surfaces in 4-space are recalled in Section 3.

Our geometric study is presented in Sections 4 and 5. First, in Section 4 we describe the realization of singularities of affine equidistants in terms of the bi-local extrinsic geometry of the surface. The main result for the case of 1-parallel pairs is presented in Theorem 4.3, while Theorems 4.4 and 4.5 present the results for the 2-parallel case.

This is followed in Section 5 by a complementary study of the set of weakly parallel points on  $M$ . We start by using the Grassmannian of 2-planes in 4-space, cf. Propositions 5.1 and 5.2 and Theorem 5.3, leading to the final detailed description of the set of weakly parallel points presented in Corollary 5.4, Theorem 5.8 and Proposition 5.9.

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## 2. SINGULARITIES OF AFFINE EQUIDISTANTS: OVERVIEW

In this section, we summarize the material that is presented in [5] in greater detail, in order to describe, characterize and classify the singularities of affine  $\lambda$ -equidistants of smooth submanifolds.

**2.1. Definition of affine equidistants.** Let  $M$  be a smooth closed  $n$ -dimensional submanifold of the affine space  $\mathbb{R}^q$ , with  $q \leq 2n$ . Let  $\alpha, \beta$  be points of  $M$  and denote by  $\tau_{\alpha-\beta} : \mathbb{R}^q \ni x \mapsto x + (\alpha - \beta) \in \mathbb{R}^q$  the translation by the vector  $(\alpha - \beta)$ .

**Definition 2.1.** A pair of points  $\alpha, \beta \in M$  ( $\alpha \neq \beta$ ) is called a **weakly parallel** pair if

$$T_\alpha M + \tau_{\alpha-\beta}(T_\beta M) \neq \mathbb{R}^q.$$

A weakly parallel pair  $\alpha, \beta \in M$  is called  **$k$ -parallel** if

$$\dim(T_\alpha M \cap \tau_{\alpha-\beta}(T_\beta M)) = k.$$

If  $k = n$  the pair  $\alpha, \beta \in M$  is called **strongly parallel**, or just **parallel**. We also refer to  $k$  as the **degree of parallelism** of the pair  $(\alpha, \beta)$ .

**Definition 2.2.** A **chord** passing through a pair  $\alpha, \beta$ , is the line

$$l(\alpha, \beta) = \{x \in \mathbb{R}^q | x = \lambda\alpha + (1 - \lambda)\beta, \lambda \in \mathbb{R}\},$$

but we sometimes also refer to  $l(\alpha, \beta)$  as a chord *joining*  $\alpha$  and  $\beta$ .

**Definition 2.3.** For a given  $\lambda$ , an **affine  $\lambda$ -equidistant** of  $M$ ,  $E_\lambda(M)$ , is the set of all  $x \in \mathbb{R}^q$  such that  $x = \lambda\alpha + (1 - \lambda)\beta$ , for all weakly parallel pairs  $(\alpha, \beta)$  in  $M$ .  $E_\lambda(M)$  is also called a **momentary equidistant** of  $M$ . Whenever  $M$  is understood, we write  $E_\lambda$  for  $E_\lambda(M)$ .

Note that, for any  $\lambda$ ,  $E_\lambda(M) = E_{1-\lambda}(M)$  and in particular  $E_0(M) = E_1(M) = M$ . Thus, the case  $\lambda = 1/2$  is special:

**Definition 2.4.**  $E_{1/2}(M)$  is called the **Wigner caustic** of  $M$  [1, 15].

**2.2. Characterization of affine equidistants by projection.** Consider the product affine space:  $\mathbb{R}^q \times \mathbb{R}^q$  with coordinates  $(x_+, x_-)$  and the tangent bundle to  $\mathbb{R}^q$ :  $T\mathbb{R}^q = \mathbb{R}^q \times \mathbb{R}^q$  with coordinate system  $(x, \dot{x})$  and standard projection  $\pi : T\mathbb{R}^q \ni (x, \dot{x}) \rightarrow x \in \mathbb{R}^q$ .

**Definition 2.5.**  $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$ , a  **$\lambda$ -chord transformation**

$$\Gamma_\lambda : \mathbb{R}^q \times \mathbb{R}^q \rightarrow T\mathbb{R}^q, (x^+, x^-) \mapsto (x, \dot{x})$$

is a linear diffeomorphism defined by:

$$(2.1) \quad x = \lambda x^+ + (1 - \lambda)x^-, \quad \dot{x} = x^+ - x^-.$$

**Remark 2.6.** The choice of linear equation for  $\dot{x}$  in (2.1) is not unique, but this is the simplest one. Among other possibilities, the choice  $\dot{x} = \lambda x^+ - (1 - \lambda)x^-$  is particularly well suited for the study of affine equidistants of *Lagrangian* submanifolds in symplectic space [4].

Now, let  $M$  be a smooth closed  $n$ -dimensional submanifold of the affine space  $\mathbb{R}^q$  ( $2n \geq q$ ) and consider the product  $M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$ . Let  $\mathcal{M}_\lambda$  denote the image of  $M \times M$  by a  $\lambda$ -chord transformation,

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M) ,$$

which is a  $2n$ -dimensional smooth submanifold of  $T\mathbb{R}^q$ .

Then we have the following general characterization:

**Theorem 2.7** ([4]). *The set of critical values of the standard projection  $\pi : T\mathbb{R}^q \rightarrow \mathbb{R}^q$  restricted to  $\mathcal{M}_\lambda$  is  $E_\lambda(M)$ .*

**Definition 2.8.**  $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$ , the  $\lambda$ -point map is the projection

$$\Psi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathbb{R}^q , (x^+, x^-) \rightarrow x = \lambda x^+ + (1 - \lambda) x^- .$$

**Remark 2.9.** Because  $\Psi_\lambda = \pi \circ \Gamma_\lambda$  we can rephrase Theorem 2.7: *the set of critical values of the projection  $\Psi_\lambda$  restricted to  $M \times M$  is  $E_\lambda(M)$ .*

**2.3. Characterization of affine equidistants by contact.** In the literature, if  $M \subset \mathbb{R}^2$  is a smooth curve, the Wigner caustic  $E_{1/2}(M)$  has been described in various ways, one of which says that, if  $\mathcal{R}_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denotes reflection through  $a \in \mathbb{R}^2$ , then  $a \in E_{1/2}(M)$  when  $M$  and  $\mathcal{R}_a(M)$  are not transversal [1, 15]. We generalize this description for every  $\lambda$ -equidistant of submanifolds of more arbitrary dimensions.

**Definition 2.10.**  $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$ , a  $\lambda$ -reflection through  $a \in \mathbb{R}^q$  is the map

$$(2.2) \quad \mathcal{R}_a^\lambda : \mathbb{R}^q \rightarrow \mathbb{R}^q , x \mapsto \mathcal{R}_a^\lambda(x) = \frac{1}{\lambda}a - \frac{1 - \lambda}{\lambda}x$$

**Remark 2.11.** A  $\lambda$ -reflection through  $a$  is not a reflection in the strict sense because  $\mathcal{R}_a^\lambda \circ \mathcal{R}_a^\lambda \neq id : \mathbb{R}^q \rightarrow \mathbb{R}^q$ , instead,

$$\mathcal{R}_a^{1-\lambda} \circ \mathcal{R}_a^\lambda = id : \mathbb{R}^q \rightarrow \mathbb{R}^q ,$$

so that, if  $a = a_\lambda = \lambda a^+ + (1 - \lambda)a^-$  is the  $\lambda$ -point of  $(a^+, a^-) \in \mathbb{R}^{2q}$ ,

$$\mathcal{R}_{a_\lambda}^\lambda(a^-) = a^+ , \mathcal{R}_{a_\lambda}^{1-\lambda}(a^+) = a^- .$$

Of course, for  $\lambda = 1/2$ ,  $\mathcal{R}_a^{1/2} \equiv \mathcal{R}_a$  is a reflection in the strict sense.

Now, let  $M$  be a smooth  $n$ -dimensional submanifold of  $\mathbb{R}^q$ , with  $2n \geq q$ . Also, let  $M^+$  be a germ of submanifold  $M$  around  $a^+$ , let  $M^-$  be a germ of submanifold  $M$  around  $a^-$  and let  $a = a_\lambda = \lambda a^+ + (1 - \lambda)a^-$  be the  $\lambda$ -point of  $(a^+, a^-) \in M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$ .

Then, the following characterization is immediate:

**Proposition 2.12.** *The following conditions are equivalent:*

- (i)  $a \in E_\lambda(M)$

- (ii)  $M^+$  and  $\mathcal{R}_a^\lambda(M^-)$  are not transversal at  $a^+$
- (iii)  $M^-$  and  $\mathcal{R}_a^{1-\lambda}(M^+)$  are not transversal at  $a^-$ .

Therefore, the study of the singularities of  $E_\lambda(M) \ni 0$  can be proceeded via the study of the contact between  $M^+$  and  $\mathcal{R}_0^\lambda(M^-)$  or, equivalently, the contact between  $\mathcal{R}_0^{1-\lambda}(M^+)$  and  $M^-$ .

**2.4. Singularities of contact.** Let  $N_1, N_2$  be germs at  $x$  of smooth  $n$ -dimensional submanifolds of the space  $\mathbb{R}^q$ , with  $2n \geq q$ . We describe  $N_1, N_2$  in the following way:

- $N_1 = f^{-1}(0)$ , where  $f : (\mathbb{R}^q, x) \rightarrow (\mathbb{R}^{q-n}, 0)$  is a submersion-germ,
- $N_2 = g(\mathbb{R}^n)$ , where  $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^q, x)$  is an embedding-germ.

**Definition 2.13.** A **contact map** between submanifold-germs  $N_1, N_2$  is the following map-germ  $\kappa_{N_1, N_2} : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{q-n}, 0)$ , where  $\kappa_{N_1, N_2} = f \circ g$ .

Let  $\tilde{N}_1, \tilde{N}_2$  be another pair of germs at  $\tilde{x}$  of smooth  $n$ -dimensional submanifolds of the space  $\mathbb{R}^q$ , described in the same way as  $N_1, N_2$ .

**Definition 2.14.** The contact of  $N_1$  and  $N_2$  at  $x$  is of the same **contact-type** as the contact of  $\tilde{N}_1$  and  $\tilde{N}_2$  at  $\tilde{x}$  if  $\exists$  a diffeomorphism-germ  $\Phi : (\mathbb{R}^q, x) \rightarrow (\mathbb{R}^q, \tilde{x})$  s.t.  $\Phi(N_1) = \tilde{N}_1$  and  $\Phi(N_2) = \tilde{N}_2$ . We denote the contact-type of  $N_1$  and  $N_2$  at  $x$  by  $\mathcal{K}(N_1, N_2, x)$ .

**Theorem 2.15** ([14]).  $\mathcal{K}(N_1, N_2, x) = \mathcal{K}(\tilde{N}_1, \tilde{N}_2, \tilde{x})$  if and only if the contact maps  $f \circ g$  and  $\tilde{f} \circ \tilde{g}$  are  $\mathcal{K}$ -equivalent.

**Definition 2.16.** We say that  $N_1$  and  $N_2$  are  **$k$ -tangent** at  $x = 0$  if

$$\dim(T_0 N_1 \cap T_0 N_2) = k.$$

If  $k$  is maximal, that is,  $k = \dim(T_0 N_1) = \dim(T_0 N_2)$ , we say that  $N_1$  and  $N_2$  are **tangent** at 0.

**Remark 2.17.** In the context of affine equidistants,  $E_\lambda(M)$ , note that  $N_1 = M^+$  and  $N_2 = \mathcal{R}_0^\lambda(M^-)$  are  **$k$ -tangent** at 0 if and only if  $T_{a^+} M^+$  and  $T_{a^-} M^-$  are  **$k$ -parallel**, where  $\lambda a^+ + (1 - \lambda) a^- = 0 \in E_\lambda(M)$ .

**Proposition 2.18** ([5]). If  $N_1$  and  $N_2$  are  $k$ -tangent at 0 then the corank of the contact map  $\kappa_{N_1, N_2}$  is  $k$ .

### 3. EXTRINSIC GEOMETRY OF SURFACES IN 4-SPACE: OVERVIEW

In this section, we recall basic definitions and results on the extrinsic geometry of smooth surfaces in 4-space. See [12, 13] for details.

Let  $f : U \rightarrow \mathbb{R}^4$  be a local parametrisation of  $M$ , where  $U$  is an open subset of  $\mathbb{R}^2$ . Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$  be a positively oriented orthonormal frame in  $\mathbb{R}^4$  such that at any  $y = (y_1, y_2) \in U$ ,  $\{\mathbf{e}_1(y), \mathbf{e}_2(y)\}$  is a basis for the tangent plane  $T_p M$  and  $\{\mathbf{e}_3(y), \mathbf{e}_4(y)\}$  is a basis for the normal plane  $N_p M$  at  $p = f(y)$ .

**Definition 3.1.** The **second fundamental form** of  $M$  at  $p$  is the vector valued quadratic form  $\Pi_p : T_p M \rightarrow N_p M$  associated to the normal component of the second derivative  $d^2 f$  of  $f$  at  $p$ , that is,

$$\Pi_p = \langle d^2 f, \mathbf{e}_3 \rangle \mathbf{e}_3 + \langle d^2 f, \mathbf{e}_4 \rangle \mathbf{e}_4.$$

Let  $a = \langle \mathbf{e}_3, f_{y_1 y_1} \rangle$ ,  $b = \langle \mathbf{e}_3, f_{y_1 y_2} \rangle$ ,  $c = \langle \mathbf{e}_3, f_{y_2 y_2} \rangle$ ,  $e = \langle \mathbf{e}_4, f_{y_1 y_1} \rangle$ ,  $f = \langle \mathbf{e}_4, f_{y_1 y_2} \rangle$ ,  $g = \langle \mathbf{e}_4, f_{y_2 y_2} \rangle$ .

Then, with this notation, we can write

$$\Pi_p(\mathbf{u}) = (au_1^2 + 2bu_1u_2 + cu_2^2)\mathbf{e}_3 + (eu_1^2 + 2fu_1u_2 + gu_2^2)\mathbf{e}_4,$$

where  $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \in T_p M$ .

The matrix  $\alpha = \begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$  is called the matrix of the second fundamental form with respect to the orthonormal frame  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ .

**Definition 3.2.** The second fundamental form of  $M$  at  $p$ , along a normal vector field  $\nu$  is the quadratic form  $\Pi_p^\nu : T_p M \rightarrow \mathbb{R}$  defined by

$$\Pi_p^\nu(\mathbf{u}) = \langle \Pi_p(\mathbf{u}), \mathbf{v} \rangle, \quad \mathbf{u} \in T_p M, \mathbf{v} = \nu(p) \in N_p M,$$

where  $\Pi_p(\mathbf{u}) : T_p M \rightarrow N_p M$  is the second fundamental form at  $p$ .

Let  $S^1$  be the unit circle in  $T_p M$  parametrized by  $\theta \in [0, 2\pi]$ . Denote by  $\gamma_\theta$  the curve obtained by intersecting  $M$  with the hyperplane at  $p$  composed by the direct sum of the normal plane  $N_p M$  and the straight line in the tangent direction represented by  $\theta$ . Such curve is called *normal section of  $M$  in the direction  $\theta$* .

**Definition 3.3.** The **curvature ellipse** is the image of the mapping

$$\begin{aligned} \eta : S^1 &\longrightarrow N_p M \\ \theta &\longmapsto \eta(\theta) \end{aligned}$$

where  $\eta(\theta)$  is the curvature vector of  $\gamma_\theta$ .

Scalar invariants of the extrinsic geometry of surfaces in  $\mathbb{R}^4$  can be defined using the coefficients of the second fundamental form. For instance the Gaussian curvature

$$(3.1) \quad \mathcal{G}_M = ac - b^2 + eg - f^2$$

and the  $\Delta$  function

$$(3.2) \quad \Delta_M = \frac{1}{4} \det \begin{bmatrix} a & 2b & c & 0 \\ e & 2f & g & 0 \\ 0 & a & 2b & c \\ 0 & e & 2f & g \end{bmatrix}.$$

Although neither  $\Delta_M$  nor  $\mathcal{G}_M$  is an affine invariant (a chosen metric was used to define them), the following proposition allows for an affine-invariant classification of a point  $p \in M \subset \mathbb{R}^4$ .

**Proposition 3.4** ([3], Proposition 4.18). *The sign of  $\Delta_M$  is an affine invariant. When  $\text{rank}\{II_{(p)}\} = 1$ , the sign of  $\mathcal{G}_M$  is also an affine invariant.*

**Definition 3.5.** A point  $p \in M$  is called

- (i) **parabolic** if  $\Delta_M(p) = 0$ ,
- (ii) **elliptic** if  $\Delta_M(p) > 0$ ,
- (iii) **hyperbolic** if  $\Delta_M(p) < 0$ .

**Definition 3.6.** A parabolic point  $p \in M$  is called

- (i-i) **point of nondegenerate ellipse**, if  $\text{rank}\{II_{(p)}\} = 2$ .

When  $\text{rank}\{II_{(p)}\} = 1$ ,  $p$  is an inflection point. In this case, it is

- (i-ii) **inflection point of real type**, if  $\mathcal{G}_M(p) < 0$ ,
- (i-iii) **inflection point of flat type**, if  $\mathcal{G}_M(p) = 0$ .
- (i-iv) **inflection point of imaginary type**, if  $\mathcal{G}_M(p) > 0$ ,

**Definition 3.7.** A direction  $\mathbf{v} \in N_p M$  is a **binormal direction** at  $p$  if the second fundamental form  $II_p^\mathbf{v}$  along the  $\mathbf{v}$  direction is a degenerate quadratic form. In this case, a direction  $\mathbf{u} \in T_p M$  in the kernel of  $II_p^\mathbf{v}(\mathbf{u})$  is called an **asymptotic direction**.

**Definition 3.8.** For a surface  $M \subset \mathbb{R}^4$ ,  $p \in M$  and  $\mathbf{u} \in T_p M$ ,  $\mathbf{v} \in N_p M$ , we say that  $(\mathbf{u}, \mathbf{v})$  is a **contact pair** of  $M$  at  $p$  if  $\mathbf{v}$  is a binormal direction at  $p$  and  $\mathbf{u}$  is an asymptotic direction associated to  $\mathbf{v}$ .

**Proposition 3.9** ([13], Lemma 3.2). *Let  $M$  be a surface in  $\mathbb{R}^4$ ,*

- 1) *For a hyperbolic point  $p \in M$ , there are exactly 2 contact pairs at  $p$ .*
- 2) *For an elliptic point  $p \in M$ , there are no contact pairs at  $p$ .*
- 3) *For a parabolic point  $p \in M$ ,*
  - i) *if  $p$  is a point of nondegenerate ellipse, then there exists only one contact pair at  $p$ .*
  - ii) *if  $p$  is an inflection point, then there exists only one  $\mathbf{v} \in N_p M$  such that, for all  $\mathbf{u} \in T_p M$ ,  $(\mathbf{u}, \mathbf{v})$  is a contact pair at  $p$ .*

#### 4. EXTRINSIC GEOMETRY OF SURFACES IN 4-SPACE AND SINGULARITIES OF THEIR AFFINE EQUIDISTANTS

We now present the geometric interpretation for the realizations of stable singularities of affine equidistants of surfaces in  $\mathbb{R}^4$ .

We first recall the following result from [5]:

**Theorem 4.1** ([5], Theorem 5.2). *There exists a residual set  $\mathcal{S}$  of embeddings  $i : M^2 \rightarrow \mathbb{R}^4$ , such that the map  $\Psi_\lambda : M \times M \setminus \Delta \rightarrow \mathbb{R}^4$  is locally stable, where  $\Psi_\lambda(x, y) = \lambda i(x) + (1 - \lambda)i(y)$  and  $\Delta$  is the diagonal in  $M \times M$ .*

**Definition 4.2.** We say that  $i : M^2 \rightarrow \mathbb{R}^4$  is a **generic embedding** if  $i \in \mathcal{S}$ .

Because the codimension of each singularity of  $\Psi_\lambda$  is at most 4, the possible stable singularities of affine equidistants of surfaces in  $\mathbb{R}^4$  are:

$$A_1, A_2, A_3, A_4 \text{ for 1-parallelism, } C_{2,2}^\pm \text{ for 2-parallelism.}$$

For the reader's convenience, we recall the normal forms of these stable singularities  $(\mathbb{R}^4, 0) \rightarrow (\mathbb{R}^4, 0)$  in the table below:

Notation	Normal form
$A_2$	$(u_1, u_2, u_3, y^2)$
$A_\mu, 2 \leq \mu \leq 4$	$(u_1, u_2, u_3, y^{\mu+1} + \sum_{i=1}^{\mu-1} u_i y^i)$
$C_{2,2}^+$	$(u_1, u_2, x^2 + u_1 y, y^2 + u_2 x)$
$C_{2,2}^-$	$(u_1, u_2, x^2 - y^2, xy + u_1 x + u_2 y)$

We refer to [5], where all possible stable singularities of affine equidistants are classified for submanifolds  $M^n \subset \mathbb{R}^q$ , with  $(2n, q)$  an arbitrary pair of nice dimensions, for all possible degrees of parallelism.

In this paper, we focus on investigating the conditions for realizing these equidistant singularities  $A_\mu$ ,  $1 \leq \mu \leq 4$  and  $C_{2,2}^\pm$  from the extrinsic geometry of a generic embedding of smooth surface  $M \subset \mathbb{R}^4$ .

In this specific case we substitute submanifold-germs  $N_1$  and  $N_2$  of Section 2 by  $N_1 = M^+$  and  $N_2 = \mathcal{R}_0^\lambda(M^-)$ , or equivalently by  $N_1 = M^-$  and  $N_2 = \mathcal{R}_0^{1-\lambda}(M^+)$ , where  $M_+$  is the surface-germ of  $M$  around  $a^+ \in M \subset \mathbb{R}^4$  and  $M_-$  is the surface-germ of  $M$  around  $a^- \in M \subset \mathbb{R}^4$ , with  $\lambda a^+ + (1 - \lambda)a^- = 0$ .

**4.1. Bi-local geometry of weakly parallel pairs and singularities of affine equidistants.** We start by looking at the bi-local geometry of 1-parallel pairs.



Suppose  $(a^+, a^-)$  is a pair of 1-parallel points. Then, we can choose coordinates in a neighbourhood of  $a^+$  and  $a^-$  as follows:

$$(4.1) \quad \begin{aligned} \Phi^+ : (\mathbb{R}^2, 0) &\rightarrow (\mathbb{R}^4, a^+) \\ (y, z) &\mapsto a^+ + (y, z, \phi(y, z), \psi(y, z)), \end{aligned}$$

$$j^1\phi(0, 0) = j^1\psi(0, 0) = 0.$$

$$(4.2) \quad \begin{aligned} \Phi^- : (\mathbb{R}^2, 0) &\rightarrow (\mathbb{R}^4, a^-) \\ (u, v) &\mapsto a^- + (u, \xi(u, v), \zeta(u, v), v), \end{aligned}$$

$j^1\xi(0, 0) = j^1\zeta(0, 0) = 0$ . In these coordinates, the local expression of the map  $\Psi_\lambda|_{M \times M}$  is given by

$$\begin{aligned} \Psi_\lambda|_{M \times M} : (\mathbb{R}^2, 0) \times (\mathbb{R}^2, 0) &\rightarrow (\mathbb{R}^4, 0) \\ ((y, z), (u, v)) &\mapsto (\lambda y + (1 - \lambda)u, \lambda z + (1 - \lambda)\xi(u, v), \\ &\quad \lambda\phi(y, z) + (1 - \lambda)\zeta(u, v), \lambda\psi(y, z) + (1 - \lambda)v) \end{aligned}$$

where, to simplify, we have assumed  $\lambda a^+ + (1 - \lambda)a^- = 0$ , for fixed  $\lambda$ .

In order to construct the contact map, we first reflect  $(M^-, a^-)$  through the point 0 to get  $\mathcal{R}_0^\lambda(M^-)$ , parametrized as

$$\mathcal{R}_0^\lambda(\Phi^-)(u, v) = a^+ - \left( \frac{(1 - \lambda)}{\lambda}u, \frac{(1 - \lambda)}{\lambda}\xi(u, v), \frac{(1 - \lambda)}{\lambda}\zeta(u, v), \frac{(1 - \lambda)}{\lambda}v \right).$$

The contact map  $\mathcal{K}^\lambda : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  is then given by

$$(4.3) \quad \begin{aligned} \mathcal{K}^\lambda(y, z) = & \left( z + \frac{1 - \lambda}{\lambda}\xi\left(\frac{-\lambda}{1 - \lambda}y, \frac{-\lambda}{1 - \lambda}\psi(y, z)\right), \right. \\ & \left. \phi(y, z) + \frac{1 - \lambda}{\lambda}\zeta\left(\frac{-\lambda}{1 - \lambda}y, \frac{-\lambda}{1 - \lambda}\psi(y, z)\right) \right). \end{aligned}$$

The following theorem distinguishes the  $A_\mu$ ,  $1 \leq \mu \leq 4$  singularities of equidistants, in terms of the bi-local geometry of  $M$ .

**Theorem 4.3.** *Let  $a^+ \in M^+$ ,  $a^- \in M^-$ , so that  $\lambda a^+ + (1 - \lambda)a^- = 0$  is a singular point of  $\Psi_\lambda|_{M \times M}$ . For a pair of vectors  $(\mathbf{u}, \mathbf{v})$  in  $\mathbb{R}^4$ , such that  $\mathbf{u}$  is in the direction of 1-parallelism of  $(a^+, a^-)$  and  $\mathbf{v} \in N_{a^+}M^+ \cap N_{a^-}M^-$  is in the common normal direction, let  $\eta_+$  and  $\eta_-$  be the normal curvature of  $M^+$  and  $\mathcal{R}_0^\lambda(M^-)$  along  $\mathbf{v}$  in the common direction  $\mathbf{u}$ . Then 0 is a singular point of  $\Psi_\lambda|_{M \times M}$  of type  $A_k$  if and only if*

$$(4.4) \quad \eta_+^{(j)}(0) = (-1)^{j+1} \left( \frac{\lambda}{1 - \lambda} \right)^{j+1} \eta_-^{(j)}(0), \quad j = 0, \dots, k - 1,$$

$$(4.5) \quad \eta_+^{(k)}(0) \neq (-1)^{k+1} \left( \frac{\lambda}{1 - \lambda} \right)^{k+1} \eta_-^{(k)}(0),$$

where  $\eta_+^{(j)}$  and  $\eta_-^{(j)}$  denote the  $j$ -order derivatives of  $\eta_+$  and  $\eta_-$  respectively.

*Proof.* We can solve the first equation  $\mathcal{K}_1^\lambda = 0$  in (4.3), as  $z = z(y)$ , so that the contact map  $\mathcal{K}^\lambda$  is  $\mathcal{K}$ -equivalent to the suspension of

$$(4.6) \quad \theta_\lambda : \mathbb{R} \rightarrow \mathbb{R} \\ y \mapsto \phi(y, z(y)) + \frac{1-\lambda}{\lambda} \zeta\left(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}\psi(y, z(y))\right).$$

The point 0 is a singularity of type  $A_k$  of  $\theta_\lambda$  if and only if

$$(4.7) \quad \frac{\partial^j \phi}{\partial y^j}(0) = (-1)^{j-1} \left( \frac{\lambda}{1-\lambda} \right)^{j-1} \frac{\partial^j \zeta}{\partial y^j}(0), \quad j = 1, \dots, k,$$

$$(4.8) \quad \frac{\partial^j \phi}{\partial y^j}(0) \neq (-1)^{j-1} \left( \frac{\lambda}{1-\lambda} \right)^{j-1} \frac{\partial^j \zeta}{\partial y^j}(0), \quad j = k+1,$$

noting that condition (4.7) for  $j = 1$  is the condition of 1-parallelism.

Letting  $\alpha_+$  and  $\alpha_-$  be curves in  $M^+$  and  $\mathcal{R}_0^\lambda(M^-)$  given by

$$\begin{aligned} \alpha_+(y) &= (y, z(y), \phi(y, z(y), \psi(y, z(y))) \\ \alpha_-(y) &= (y, \frac{\lambda-1}{\lambda} \xi\left(\frac{\lambda}{\lambda-1}y, \frac{\lambda}{\lambda-1}\psi(y, z(y))\right), \\ &\quad \frac{\lambda-1}{\lambda} \zeta\left(\frac{\lambda}{\lambda-1}y, \frac{\lambda}{\lambda-1}\psi(y, z(y))\right), \psi(y, z(y))) \end{aligned}$$

and letting  $\eta_+(y)$  and  $\eta_-(y)$  be the projections of the normal curvatures of  $\alpha_+$  and  $\alpha_-$  in the common normal direction  $\mathbf{v}$ , then

$$\eta_+(y) = \frac{\partial^2 \phi}{\partial y^2}(y, z(y)) \text{ and } \eta_-(y) = \frac{\partial^2 \zeta}{\partial y^2}(y, z(y)).$$

So, equations (4.7)-(4.8) reduce to equations (4.4)-(4.5).  $\square$

We now look at the bi-local description of 2-parallel pairs.

Suppose  $(a^+, a^-)$  is a pair of 2-parallel points. Then, we can choose coordinates in a neighbourhood of  $a^+$  and  $a^-$  as follows:

$$(4.9) \quad \begin{aligned} \Phi^+ : (\mathbb{R}^2, 0) &\rightarrow (\mathbb{R}^4, a^+) \\ (y, z) &\mapsto a^+ + (y, z, \phi(y, z), \psi(y, z)), \end{aligned}$$

$$j^1 \phi(0, 0) = j^1 \psi(0, 0) = 0.$$

$$(4.10) \quad \begin{aligned} \Phi^- : (\mathbb{R}^2, 0) &\rightarrow (\mathbb{R}^4, a^-) \\ (u, v) &\mapsto a^- + (u, v, \xi(u, v), \zeta(u, v)), \end{aligned}$$

$$j^1 \xi(0, 0) = j^1 \zeta(0, 0) = 0.$$

Again, for simplicity we assume that for  $\lambda$  fixed,  $\lambda a^+ + (1-\lambda)a^- = 0$ . Now the contact map  $\mathcal{K}^\lambda : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  is

$$(4.11) \quad \begin{aligned} \mathcal{K}^\lambda(y, z) = & (\phi(y, z) + \frac{1-\lambda}{\lambda} \xi(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}z), \\ & \psi(y, z) + \frac{1-\lambda}{\lambda} \zeta(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}z)) \end{aligned}$$

Let the contact surface  $\mathcal{C}^\lambda \subset \mathbb{R}^4$  be the graph of the contact map  $\mathcal{K}^\lambda$ .

If  $0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$  is a singular point of type  $C_{2,2}^+$  of the contact map  $\mathcal{K}^\lambda$ , then  $\Delta_{\mathcal{C}^\lambda}(0) < 0$  [13]. It follows that  $\mathcal{C}^\lambda$  has two contact pairs at 0. For each of these, we have the following:

**Theorem 4.4.** *Let  $a^+ \in M^+$ ,  $a^- \in M^-$ , so that  $\lambda a^+ + (1-\lambda)a^- = 0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$  is a singular point of  $\mathcal{K}^\lambda$  of type  $C_{2,2}^+$ . The pair  $(\mathbf{u}, \mathbf{v})$  is a contact pair of  $\mathcal{C}^\lambda$  at 0 if and only if one of the following holds.*

(i) *The pair  $(\mathbf{u}, \mathbf{v})$  is a contact pair of  $M^+$  and of  $\mathcal{R}_0^\lambda(M^-)$  at  $a^+$  (equivalently,  $(\mathbf{u}, \mathbf{v})$  is a contact pair of  $M^-$  and of  $\mathcal{R}_0^{1-\lambda}(M^+)$  at  $a^-$ ).*

(ii) *The pair  $(\mathbf{u}, \mathbf{v})$  is not a contact pair of either  $M^+$  or  $\mathcal{R}_0^\lambda(M^-)$  at  $a^+$ , but the normal curvatures of  $M^+$  and  $\mathcal{R}_0^\lambda(M^-)$  along  $\mathbf{u}$  in the direction of  $\mathbf{v}$  are in proportion  $\frac{\lambda}{1-\lambda}$  at  $a^+$  (equivalently,  $(\mathbf{u}, \mathbf{v})$  is not a contact pair of either  $M^-$  or  $\mathcal{R}_0^{1-\lambda}(M^+)$  at  $a^-$ , but the normal curvatures of  $M^-$  and  $\mathcal{R}_0^{1-\lambda}(M^+)$  along  $\mathbf{u}$  in the direction of  $\mathbf{v}$  have the proportion  $\frac{1-\lambda}{\lambda}$  at  $a^-$ ).*

*Proof.* Let  $(\mathbf{u}, \mathbf{v})$  be a contact pair of the contact surface  $\mathcal{C}^\lambda$ . Without loss of generality we can take  $\mathbf{u} = (1, 0, 0, 0)$  and  $\mathbf{v} = (0, 0, 1, 0)$ . Then, since  $\mathbf{v}$  is a binormal direction, it follows that the hessian of the function germ

$$\mathcal{K}_2^\lambda(y, z) = \psi(y, z) + \frac{1-\lambda}{\lambda} \zeta(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}z)$$

is degenerate and  $\mathbf{u}$  is its kernel. Then  $\frac{\partial^2 \mathcal{K}_2^\lambda}{\partial y^2}(0) = 0$ , hence

$$\frac{\partial^2 \psi}{\partial y^2}(0) = -\frac{\lambda}{1-\lambda} \frac{\partial^2 \zeta}{\partial y^2}(0).$$

As in the proof of Theorem 4.3, either  $\frac{\partial^2 \psi}{\partial y^2}(0) = 0$  and  $\frac{\partial^2 \zeta}{\partial y^2}(0) = 0$  or they are not zero, but the normal curvatures of  $M^+$  and  $\mathcal{R}_0^\lambda(M^-)$  along  $\mathbf{v}$  in the direction of  $\mathbf{u}$  are proportional. Similar statement holds for  $M^-$  and  $\mathcal{R}_0^{1-\lambda}(M^+)$ .  $\square$

If  $0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$  is a singular point of type  $C_{2,2}^-$ , then  $\Delta_{\mathcal{C}^\lambda}(0) > 0$  [13]. It follows that  $\mathcal{C}^\lambda$  has no contact pairs at 0. We thus have:

**Theorem 4.5.** *Let  $a^+ \in M^+$ ,  $a^- \in M^-$ , so that  $\lambda a^+ + (1 - \lambda)a^- = 0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$  is a singular point of type  $C_{2,2}^-$ . Although  $a^+ \in M^+$  and  $a^- \in M^-$  are strongly parallel points, both of the following holds true.*

(i)  *$M^+$  and  $\mathcal{R}_0^\lambda(M^-)$  do not have any common contact pair at  $a^+$  (or equivalently,  $M^-$  and  $\mathcal{R}_0^{1-\lambda}(M^+)$  do not have any common contact pair at  $a^-$ ).*

(ii) *There is no pair  $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4$  with  $\mathbf{u} \in T_{a^+}M^+$  and  $\mathbf{v} \in N_{a^+}M^+$ , such that the normal curvature along  $\mathbf{u}$  in the  $\mathbf{v}$  direction of  $M^+$  and of  $\mathcal{R}_0^\lambda(M^-)$  are in proportion  $\frac{\lambda}{1-\lambda}$  at  $a^+$  (or equivalently, the normal curvature along  $\mathbf{u}$  in the  $\mathbf{v}$  direction of  $M^-$  and of  $\mathcal{R}_0^{1-\lambda}(M^+)$  are in proportion  $\frac{1-\lambda}{\lambda}$  at  $a^-$ ).*

**Remark 4.6.** Generically,  $\Delta_{\mathcal{C}^\lambda} \neq 0$  because singular points of  $\mathcal{C}^\lambda \subset \mathbb{R}^4$  of type  $C_{2,3}$  are not unfolded to a stable point of  $\Psi_\lambda$  ([5]).

## 5. GEOMETRY OF THE SET OF WEAKLY PARALLEL POINTS

We now extend our geometric investigations in order to describe the set of weakly parallel points of  $M$ , as this set is naturally related to the set of affine equidistants of  $M$  and its singularities.

**5.1. Grassmannian investigation of weakly parallel points.** We start by using the Grassmannian  $Gr(2, 4)$ , the space of 2-planes in  $\mathbb{R}^4$ .

First, we recall the Plücker coordinates for  $Gr(2, 4)$ . Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$  be any basis for  $\mathbb{R}^4$  (not necessarily orthonormal or orthogonal, no metric is needed or assumed here). Then,  $\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_4, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_4, \mathbf{e}_3 \wedge \mathbf{e}_4$  is a basis for  $\Lambda^2 \mathbb{R}^4$  and we denote by  $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$  the coordinates of an element  $\pi \in \Lambda^2 \mathbb{R}^4$  in the above basis.

If the bi-vector  $\pi \in \Lambda^2 \mathbb{R}^4$  with coordinates  $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$  represents an element in  $Gr(2, 4)$ , then the bi-vector  $\pi' \in \Lambda^2 \mathbb{R}^4$  with coordinates  $(kp_{12}, kp_{13}, kp_{14}, kp_{23}, kp_{24}, kp_{34})$ ,  $0 \neq k \in \mathbb{R}$ , represents the same element in  $Gr(2, 4)$ . Thus, defining the equivalence class  $[\pi] = \{\pi' \in \Lambda^2 \mathbb{R}^4 \mid \pi' = k\pi, k \in \mathbb{R}^*\}$ , it follows that  $[\pi] \in \mathbb{P}(\Lambda^2 \mathbb{R}^4)$  has homogeneous coordinates  $[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$ .

However, not every element  $[\pi] \in \mathbb{P}(\Lambda^2 \mathbb{R}^4)$  lies in  $Gr(2, 4)$ .  $\pi$  is in  $Gr(2, 4)$  iff  $\pi$  is an elementary bi-vector, i.e.  $\pi = \mathbf{u} \wedge \mathbf{v}$ , for some  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$ . Thus  $[\pi] \in Gr(2, 4)$  iff

$$\pi \wedge \pi = 0.$$

In terms of the above coordinates, this translates into the equation

$$(5.1) \quad p_{12}p_{34} + p_{23}p_{14} - p_{13}p_{24} = 0.$$

The homogeneous coordinates  $[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$  subject to constraint (5.1) are the Plücker coordinates of  $[\pi] \in Gr(2, 4)$  with respect to the basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$  of  $\mathbb{R}^4$ . It follows that  $\dim_{\mathbb{R}}(Gr(2, 4)) = 4$ .

Now, consider the Gauss map

$$G : M \rightarrow Gr(2, 4), \quad \mathbb{R}^4 \supset M \ni a \mapsto [T_a M] \in Gr(2, 4).$$

The Gauss map fails to be injective precisely for (non-diagonal) strongly parallel pairs, i.e.  $a_1 \neq a_2 \in M$ , such that  $G(a_1) = G(a_2)$ . Thus, for a residual set of embeddings  $M \subset \mathbb{R}^4$ ,  $G : M \rightarrow Gr(2, 4)$  is an immersion with transversal double points and such a  $[\pi] \in G(M)$  whose neighborhood in  $G(M)$  is not homeomorphic to  $\mathbb{R}^2$  is the common tangent plane for a (non-diagonal) 2-parallel pair  $(a_1, a_2) \in M \times M$ .

Consider also the double Gauss map:

$$G \times G : M \times M \rightarrow Gr(2, 4) \times Gr(2, 4), \quad (a_1, a_2) \mapsto ([\pi_1], [\pi_2])$$

Then,  $[\pi_1]$  and  $[\pi_2]$  are weakly parallel, iff

$$(5.2) \quad \pi_1 \wedge \pi_2 = 0.$$

And we denote

$$W = \{([\pi_1], [\pi_2]) \in Gr(2, 4) \times Gr(2, 4) \mid \pi_1 \wedge \pi_2 = 0\}.$$

In terms of the Plücker coordinates for  $Gr(2, 4)$ ,

$$(5.3) \quad [\pi_1] = [p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}], \quad p_{12}p_{34} + p_{23}p_{14} - p_{13}p_{24} = 0,$$

$$(5.4) \quad [\pi_2] = [q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}], \quad q_{12}q_{34} + q_{23}q_{14} - q_{13}q_{24} = 0,$$

condition (5.2) translates into

$$(5.5) \quad p_{12}q_{34} + p_{34}q_{12} + p_{14}q_{23} + p_{23}q_{14} - p_{13}q_{24} - p_{24}q_{13} = 0.$$

Thus, equations (5.3), (5.4) and (5.5) define coordinates for an element  $([\pi_1], [\pi_2])$  of the 7-dimensional subvariety  $W \subset (Gr(2, 4) \times Gr(2, 4))$ .

We denote by  $W_{reg}$  the set of smooth points of  $W$ , and by  $Sing(W)$  the set of singular points of  $W$ .

**Proposition 5.1.** *Away from the diagonal,  $W$  is a smooth hypersurface of  $Gr(2, 4) \times Gr(2, 4)$ .*

*Proof.* First, note that each of the equations (5.3) and (5.4) define smooth submanifolds  $Gr(2, 4) \subset \mathbb{P}(\Lambda^2 \mathbb{R}^4)$  and, similarly, equation (5.5) defines a smooth submanifold of  $\mathbb{P}(\Lambda^2 \mathbb{R}^4) \times \mathbb{P}(\Lambda^2 \mathbb{R}^4)$ . Thus,  $W$  is singular only where these three submanifolds of  $\mathbb{P}(\Lambda^2 \mathbb{R}^4) \times \mathbb{P}(\Lambda^2 \mathbb{R}^4)$  do not intersect transversally. By straightforward computation, we see that the rank of the matrix of the derivatives of equations (5.3), (5.4) and (5.5) is not maximal iff  $\forall 1 \leq i < j \leq 4, p_{ij}/q_{ij} = k \in \mathbb{R}^*$ . It follows that  $Sing(W) = \{([\pi_1], [\pi_2]) \in Gr(2, 4) \times Gr(2, 4) \mid [\pi_1] = [\pi_2]\}$ .  $\square$

Now, as  $Gr(2, 4) \times Gr(2, 4)$  fibers (trivially) over  $Gr(2, 4)$ , say, via the first projection  $Pr_1$ , this induces a sub-bundle  $W \rightarrow Gr(2, 4)$ ,  $([\pi_1], [\pi_2]) \mapsto [\pi_1]$ , which may not be trivial. Its typical fiber  $W_{[\pi_1]}$  is a 3-variety, which can locally be described as follows.

Chose a basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$  for  $\mathbb{R}^4$  such that  $[\pi_1] = [\mathbf{e}_1 \wedge \mathbf{e}_2]$ . Then,  $[\pi_1] = [1, 0, 0, 0, 0, 0]$ , and  $[\pi_2] = [q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}] \in W_{[\pi_1]}$  iff  $q_{12}q_{34} + q_{23}q_{14} - q_{13}q_{24} = 0$  and  $q_{34} = 0$ , that is,

$$[\pi_2] \in W_{[\pi_1]} \iff [\pi_2] = [q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, 0], \quad q_{23}q_{14} - q_{13}q_{24} = 0,$$

or equivalently,

$$(5.6) \quad [\pi_2] \in W_{[\pi_1]} \iff [\pi_2] = [1, \alpha, \beta, \gamma, \delta, 0], \quad \beta\gamma - \alpha\delta = 0,$$

in other words, close to  $\alpha = \beta = \gamma = \delta = 0$ ,

$$(5.7) \quad W_{[\pi_1]} = \{(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mid \alpha\delta - \beta\gamma = 0\}.$$

Thus, we have a refinement of Proposition 5.1, that is,

**Proposition 5.2.** *In a neighborhood of  $[\pi_2] = [\pi_1]$ , the 3-variety  $W_{[\pi_1]}$  is a cone.*

The following theorem, which follows from standard transversality arguments, describes how affine equidistants  $E_\lambda(M)$  are related to the intersection of  $W$  and  $G(M) \times G(M)$ .

**Theorem 5.3.** *Let  $M \subset \mathbb{R}^4$  be a generic embedding and  $(a, b)$  be a weakly parallel pair on  $M$ .*

(i) *Let  $(a, b)$  be a 1-parallel pair, so that  $(G(a), G(b)) \in W_{reg}$ . If  $\Psi_\lambda|_{M \times M} : (\mathbb{R}^2 \times \mathbb{R}^2, (a, b)) \rightarrow (\mathbb{R}^4, \lambda a + (1 - \lambda)b)$  has a stable singularity (of type  $A_k$ ,  $k = 1, 2, 3, 4$ ), then  $G(M) \times G(M)$  is transverse to  $W_{reg}$  at  $(G(a), G(b))$ .*

(ii) *Let  $(a, b)$  be a 2-parallel pair, so that  $(G(a), G(b)) \in \text{Sing}(W)$ . If  $\Psi_\lambda|_{M \times M} : (\mathbb{R}^2 \times \mathbb{R}^2, (a, b)) \rightarrow (\mathbb{R}^4, \lambda a + (1 - \lambda)b)$  has a stable singularity (of type  $C_{2,2}^\pm$ ) then  $(a, b)$  is a transversal double point of the Gauss map.*

## 5.2. Geometric description of the set of weakly parallel points.

We emphasize that, from Theorem 5.3, for generic embeddings of smooth closed surfaces in  $\mathbb{R}^4$  there are only double points of Gauss map. There are no triple, quadruple... points of the Gauss map, generically.

Therefore, we obtain the following corollary of Theorem 5.3:

**Corollary 5.4.** *For generic embeddings of smooth closed surfaces in  $\mathbb{R}^4$ , strongly parallel (nonidentical) points come only in pairs and there are only finite numbers of such pairs.*

**Remark 5.5.** An interesting question, whose answer is unknown to us, is whether there exists any embedded compact surface  $M \subset \mathbb{R}^4$  without nonidentical 2-parallel points, in other words, such that the Gauss map  $G : M \rightarrow Gr(2, 4)$  is injective.

**Notation 5.6.** For  $p \in M$ , let  $\mathcal{W}_p \subset M$  denote the set of weakly parallel points to  $p$  and let  $\mathcal{W}_p^q$  denote the germ of  $\mathcal{W}_p$  at  $q \in M$ .

**Remark 5.7.** It is easy to see that  $G(\mathcal{W}_p) \subset W$  where the latter is described in Propositions 5.1 and 5.2.

Then, the following theorem describes  $\mathcal{W}_p^q$  in all possible situations.

**Theorem 5.8.** *For a generic embedding of  $M$  into  $\mathbb{R}^4$ , cf. Definition 4.2 and Theorem 4.1, the following hold.*

- (1) *If  $q$  is 1-parallel to  $p$ , then  $\mathcal{W}_p^q$  is a germ of smooth curve.*
- (2) *If  $q$  is 2-parallel to  $p$ , then:*
  - (i) *If  $q$  is an elliptic point of  $M$ , then  $\mathcal{W}_p^q = \{q\}$ .*
  - (ii) *If  $q$  is a parabolic point of  $M$ , then  $\mathcal{W}_p^q$  is a singular curve with a cusp singularity at  $q$  which is tangent to the asymptotic direction at  $q$  (this is generic for  $q = p$ , as a generic embedding has a parabolic point, or in a 1-parameter family of embeddings for  $q \neq p$ , cf. Remark 4.6).*
  - (iii) *If  $q$  is a hyperbolic point of  $M$ , then  $\mathcal{W}_p^q$  is a singular curve with a transversal double point at  $q$  so that each branch of  $\mathcal{W}_p^q$  is a smooth curve tangent to an asymptotic direction at  $q$ .*

*Proof.* If the points  $p, q \in M$  are 1-parallel then the germs of  $M$  at  $p = (p_1, p_2, p_3, p_4)$  and at  $q = (q_1, q_2, q_3, q_4)$  can be parametrized in the following way  $F(x, y) = (p_1 + x, p_2 + y, p_3 + f_3(x, y), p_4 + f_4(x, y))$  and  $G(u, v) = (q_1 + u, q_2 + g_2(u, v), q_3 + g_3(u, v), q_4 + v)$  respectively, where  $f_3, f_4, g_2, g_4$  are smooth function-germs vanishing at  $(0, 0)$  such that  $df_3|_{(0,0)} = df_4|_{(0,0)} = dg_2|_{(0,0)} = dg_3|_{(0,0)} = 0$ . The point  $G(u, v)$  is weakly parallel to  $p$  if the Jacobian of the map

$$(5.8) \quad (x, y, u, v) \mapsto \lambda F(x, y) + (1 - \lambda)G(u, v)$$

vanishes at the point  $(0, 0, u, v)$ . The Jacobian of the map (5.8) at  $(0, 0, u, v)$  has the form  $\frac{\partial g_3}{\partial u}(u, v)$ . Generically  $d(\frac{\partial g_3}{\partial u})|_{(0,0)} \neq 0$ , therefore  $\mathcal{W}_p^q$  is a germ at  $q$  of a smooth curve.

If the points  $p, q \in M$  are 2-parallel then the germs of  $M$  at  $p = (p_1, p_2, p_3, p_4)$  and at  $q = (q_1, q_2, q_3, q_4)$  can be parametrized in the following way  $F(x, y) = (p_1 + x, p_2 + y, p_3 + f_3(x, y), p_4 + f_4(x, y))$  and  $G(u, v) = (q_1 + u, q_2 + v, q_3 + g_3(u, v), q_4 + g_4(u, v))$  respectively, where  $f_3, f_4, g_3, g_4$  are smooth function-germs vanishing at  $(0, 0)$  such that  $df_3|_{(0,0)} = df_4|_{(0,0)} = dg_3|_{(0,0)} = dg_4|_{(0,0)} = 0$ .

The point  $G(u, v)$  is weakly parallel to  $p$  if the Jacobian of the map (5.8) vanishes at  $(0, 0, u, v)$ . It is easy to see that the Jacobian of the map (5.8) at  $(0, 0, u, v)$  is  $Jac(g_3, g_4)(u, v)$ , i.e. the Jacobian of the map  $(g_3, g_4)$  at  $(u, v)$ . It is also easy to see  $d(Jac(g_3, g_4))|_{(0,0)}$  vanishes.

The Hessian of the function  $(u, v) \mapsto Jac(g_3, g_4)(u, v)$  at  $(0, 0)$  is equal to  $4\Delta_M(q)$ . Therefore if  $q$  is an elliptic point, then  $\mathcal{W}_p^q = \{q\}$ , if  $q$  is a parabolic point, then  $\mathcal{W}_p^q$  is a singular curve with a cusp singularity at  $q$  which is tangent to the asymptotic direction at  $q$ , and finally if  $q$  is a hyperbolic point, then  $\mathcal{W}_p^q$  consists of the crossing of two smooth curves at  $q$ , each one tangent to an asymptotic direction at  $q$ .

We can also interpret the above calculations in terms of singularities of projections into planes. In fact, let  $\rho_p : M \rightarrow N_p M$  be the projection of  $M$  into the 2-plane  $N_p M = \mathbb{R}^2$ , which is fixed.

Then the singular set of the projection,

$$\Sigma_{\rho_p} = \{q \in M \mid \text{there exists some } \mathbf{v} \in T_q M, \mathbf{v} \in \ker \rho_p\}$$

coincides with the set  $\mathcal{W}_p$ . Given  $q \in \mathcal{W}_p$ , we use the above local parametrizations to study  $\mathcal{W}_p^q$ .

If points  $p, q \in M$  are 1-parallel then the germs of  $M$  at  $p = (p_1, p_2, p_3, p_4)$  and at  $q = (q_1, q_2, q_3, q_4)$  can be parametrized respectively by  $F(x, y) = (p_1 + x, p_2 + y, p_3 + f_3(x, y), p_4 + f_4(x, y))$  and  $G(u, v) = (q_1 + u, q_2 + g_2(u, v), q_3 + g_3(u, v), q_4 + v)$ . The normal plane of  $M$  at  $p$  is the plane  $[(0, 0, 1, 0), (0, 0, 0, 1)]$ . Hence, the germ at  $q$  of the projection  $\rho_p : M \rightarrow N_p M$  is given by

$$\rho_p \circ G(u, v) = (q_3 + g_3(u, v), q_4 + v).$$

Thus, as above,  $\mathcal{W}_p^q$  is smooth in a neighbourhood of  $q$  if and only if  $(g_{3uu}(0), g_{3uv}(0)) \neq (0, 0)$  and this condition is satisfied for generic embeddings of  $M$ . On the other hand, if points  $p, q \in M$  are 2-parallel, the germ at  $q$  of the projection is given by

$$\rho_p \circ G(u, v) = (q_3 + g_3(u, v), q_4 + g_4(u, v)),$$

and we proceed as above.  $\square$

Because the case (2 - ii) above for  $q \neq p$  is only generic in a 1-parameter family of embeddings, according to Definition 4.2 and Theorem 4.1, we now analyze its bifurcation set.



**Proposition 5.9.** *Let  $I \ni t$  be an open interval containing 0, with  $M_t$  a generic smooth 1-parameter family of smooth surface embeddings in  $\mathbb{R}^4$  such that the points  $p_t$  and  $q_t$  in  $M_t$  are strongly parallel  $\forall t \in I$  and  $q_0$  is a parabolic point of  $M_0 \subset \mathbb{R}^4$ . Let  $\mathcal{W}_{p_t}^{q_t} \subset M_t$  denote the germ of weakly parallel points to  $p_t$  at  $q_t$ . Then,  $\mathcal{W}_{p_t}^{q_t}$  is described by the Whitney umbrella*

$$(5.9) \quad 2u^2 - 3v^3 - 2tv^2 = 0 ,$$

*such that a curve  $\mathcal{C}_{t=t_0}$  on this surface in  $\mathbb{R}^3$  has a smooth branch and an isolated point if  $t_0 < 0$ , or is a cusp if  $t_0 = 0$ , or is a looped curve with a transversal self-crossing if  $t_0 > 0$ . These three cases correspond to the point  $q_{t_0} = (u, v) = (0, 0)$  being an isolated point for  $t_0 < 0$  (elliptic case), a cuspidal point for  $t_0 = 0$  (parabolic case), or a transversal double point for  $t_0 > 0$  (hyperbolic case).*

*Proof.* Following the same notation of the proof of Theorem 5.8, with  $t$  denoting the parameter of the family of embeddings and assuming  $q_t$  is parabolic for  $t = 0$ , the germ of  $M_t$  at  $q_t$  can be put after translation to the form<sup>1</sup>  $g_t(u, v) = (u, v, g_t^3(u, v), g_t^4(u, v))$ , where  $g_t^3(u, v) = u^2 + v^3 + tv^2 + V_t(u, v)$  and  $g_t^4(u, v) = uv + W_t(u, v)$ , with  $V_t$  and  $W_t$  of third or higher order in  $(u, v)$  for all  $t$ .

The point  $p_t$  being 2-parallel to  $q_t$ , the germ of  $M_t$  at  $p_t$  is, after translation, of the general form  $f_t(x, y) = (x, y, f_t^3(x, y), f_t^4(x, y))$ , with  $f_t^3$  and  $f_t^4$  of second order in  $(u, v)$  for all  $t$ .

Thus, as before,  $g_t(u, v)$  is weakly parallel to  $p_t$  if the Jacobian of the map (5.8) vanishes at  $(0, 0, u, v)$  and this Jacobian is the same as the Jacobian of the map  $(g_t^3, g_t^4)$  at  $(u, v)$ , which is of the form  $J(u, v, t) = 2u^2 - 3v^3 - 2tv^2 + R_t(u, v)$ , where  $R_t$  is of third or higher order in  $(u, v)$  for all  $t$ . We now apply the following lemma:

**Lemma 5.10.** *The Jacobian  $J(u, v, t) = 2u^2 - 3v^3 - 2tv^2 + R_t(u, v)$ , with  $R_t$  of third or higher order in  $(u, v)$ ,  $\forall t$ , can be put for small  $t$ , by a smooth near-identity change of coordinates of the form  $(u, v, t) \mapsto (U(u, v, t), V(u, v, t), t)$ , to the normal form  $H(U, V, t) = 2U^2 - 3V^3 - 2tV^2(1 + \phi(U, V, t))$ , with  $\phi$  a smooth function satisfying  $\phi(0, 0, t) = 0$ , for small  $t$ .*

*Proof.* Start by writing  $R_t(u, v) = t(\psi_3(v, t) + u\psi_2(v, t) + 2u^2\psi_1(u, v, t))$ , where  $\psi_3$  is of order at least 3 in  $v$ ,  $\forall t$ ,  $\psi_2$  is of order at least 2 in  $v$ ,  $\forall t$ , and  $\psi_1(0, 0, t) = 0$ . Then,  $J(u, v, t) = 2u^2(1 + t\psi_1(u, v, t)) - 3v^3(1 - t\tilde{\psi}_3(v, t)) - 2tv^2(1 - u\tilde{\psi}_2(v, t))$ , where  $\tilde{\psi}_3(v, t) = \psi_3(v, t)/3v^3$ ,  $\tilde{\psi}_2(v, t) =$

<sup>1</sup>In general, the tangent plane to  $M_t$  at  $q_t$  will change with  $t$ , but we can adopt an orthonormal moving frame such that  $T_{q_t}M_t = \text{span} < (1, 0, 0, 0), (0, 1, 0, 0) >, \forall t$ .

$\psi_2(v, t)/2v^2$ . Thus, setting  $V(u, v, t) = V(v, t) = v\sqrt[3]{1 - t\tilde{\psi}_3(v, t)}$  and  $U(u, v, t) = u\sqrt{1 + t\psi_1(u, v, t)}$ , we note that  $(u, v, t) \mapsto (U, V, t)$  is a near-identity transformation for small  $t$ , therefore invertible, so that we can write  $J(u, v, t) = H(U, V, t) = 2U^2 - 3V^3 - 2tV^2(1 + \phi(U, V, t))$ , where  $\phi$  is a smooth function satisfying  $\phi(0, 0, t) = 0$ , for small  $t$ .  $\square$

It follows that, for small  $t$  and in a neighborhood of  $(U, V) = (0, 0)$ , the curve  $\mathcal{C}'_{t=t_0}$ , which is obtained as the section  $\{H(U, V, t = t_0) = 0\}$ , is a small deformation of the curve  $\mathcal{C}_{t=t_0}$ , which is obtained as the section  $\{h(u, v, t = t_0) = 0\}$ , where  $h(u, v, t) = 2u^2 - 3v^3 - 2tv^2$ . In particular, for  $t_0 = 0$  the curve  $\mathcal{C}'_{t=t_0}$  is a cusp, just as  $\mathcal{C}_{t=t_0}$ , for  $t_0 < 0$  the curve  $\mathcal{C}'_{t=t_0}$  has a smooth branch and an isolated point at  $(0, 0)$ , just as  $\mathcal{C}_{t=t_0}$ , and for  $t_0 > 0$  the curve  $\mathcal{C}'_{t=t_0}$  is a looped curve with a transversal self-crossing at  $(0, 0)$ , just like  $\mathcal{C}_{t=t_0}$ .  $\square$

**Remark 5.11.** In the same vein, when the embedding is fixed and  $q = p$ , if a smooth curve  $I \ni s \mapsto p(s) \in M$  is transversal to the smooth curve of parabolic points on  $M$  at a parabolic point  $p(0)$ , then by slightly adapting the above reasoning we can easily see that the family of germs  $\mathcal{W}_{p(s)}^{p(s)}$  is described by the Whitney umbrella (5.9).

**Remark 5.12.** As a last remark, we note that two distinct points  $q, q' \in \mathcal{W}_p$  need not be weakly parallel to each other. For instance, if  $[\pi_1] = G(p) = [\mathbf{e}_1 \wedge \mathbf{e}_2]$ , we may have that  $G(q) = [\mathbf{e}_1 \wedge \mathbf{e}_3]$  and  $G(q') = [\mathbf{e}_2 \wedge \mathbf{e}_4]$ . We also note that, if  $(p, q)$  is a strongly parallel pair ( $p \neq q$ ), the local geometry of  $p$  and  $q$  can be distinct (one elliptic, the other hyperbolic, etc), thus in general  $\mathcal{W}_p^q$  and  $\mathcal{W}_q^p$  can be of distinct types.

**5.3. Illustrations.** We now provide examples of Theorem 5.8 and Proposition 5.9, this latter in the form of Remark 5.11.

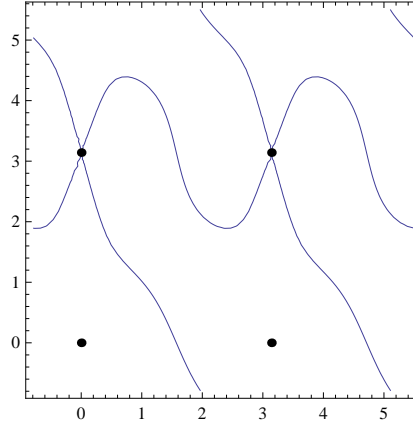
**Example 5.13.** Let us consider the following embedding of a torus into the affine space  $\mathbb{R}^4([6])$ ,  $F(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y), f_4(x, y))$ ,

$$\begin{aligned} f_1(x, y) &= \cos(x) \left( 1 - \frac{\cos(y)}{10} \right) + \frac{1}{10} \sin(x) \sin(y), \\ f_2(x, y) &= \left( 1 - \frac{\cos(y)}{10} \right) \sin(x) - \frac{1}{10} \cos(x) \sin(y), \\ f_3(x, y) &= \cos(2x) \left( 1 - \frac{2\cos(y)}{5} \right) + \frac{4}{5} \sin(2x) \sin(y), \\ f_4(x, y) &= \left( 1 - \frac{2\cos(y)}{5} \right) \sin(2x) - \frac{4}{5} \cos(2x) \sin(y). \end{aligned}$$

The curves of parabolic points on this torus are given by

$$y = \pm 2 \arctan \left( \sqrt{\frac{1}{5} (-4 + \sqrt{41})} \right).$$

Fig. 1 presents the curve of weakly parallel points on the  $x, y$ -plane to a hyperbolic point  $(\pi, \pi)$  (or elliptic point  $(0, 0)$ ). All points marked by black dots on Fig. 1 are strongly parallel. Elliptic points  $(0, 0)$  and  $(\pi, 0)$  are isolated points of the the curve. There are transversal self-intersections of the curve in hyperbolic points  $(0, \pi)$  and  $(\pi, \pi)$ .



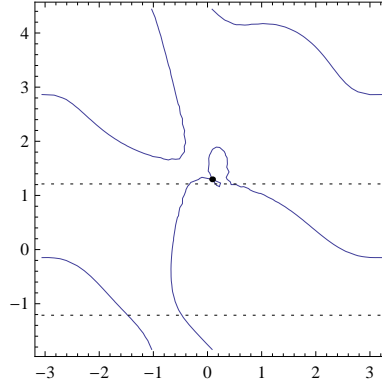
**Figure 1.** Set of weakly parallel points to an elliptic or hyperbolic point.

**Example 5.14.** Let us again consider the torus from Example 5.13. In Figures 2 to 4 we present the bifurcation of  $\mathcal{W}_p^p$  - the germ at a point  $p$  of the curve of weakly parallel points to  $p$  - when we change  $p$  from a hyperbolic point to a parabolic point and then to an elliptic point. For  $p$  we chose a point with the following coordinates on the  $(x, y)$ -plane:

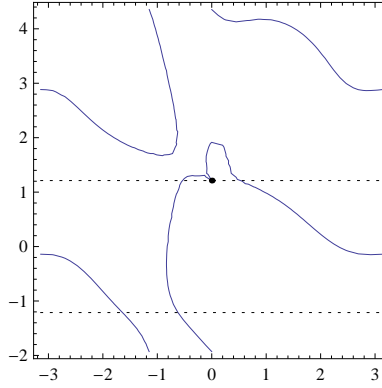
$$\left( s, 2 \arctan \left( \sqrt{\frac{1}{5} (-4 + \sqrt{41})} \right) + s \right)$$

For  $s = 0$  the point  $p$  is parabolic and at this parabolic point (marked by a black dot) the curve has a cusp singularity, cf. Fig. 3, which also shows the curve of weakly parallel points to this parabolic point.

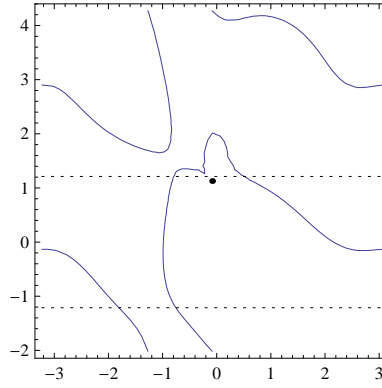
For sufficiently small positive  $s$  the point  $p$  is hyperbolic (cf. Fig. 2) and for sufficiently small negative  $s$  the point  $p$  is elliptic (cf. Fig 4). The dotted lines on Figs. 2-4 are lines of parabolic points. From the figures we see that the bifurcation of the set  $\mathcal{W}_p^p$  when we change  $s$  is diffeomorphic to the Whitney umbrella, which is presented on Fig. 5.



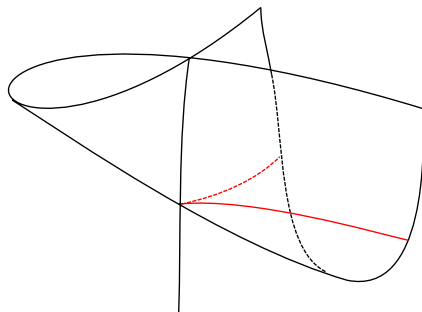
**Figure 2.** Set of weakly parallel points to a hyperbolic point ( $s=0.085$ ).



**Figure 3.** Set of weakly parallel points to a parabolic point ( $s=0$ ).



**Figure 4.** Set of weakly parallel points to an elliptic point ( $s = -0.085$ ).



**Figure 5.** The bifurcation of the germ, at a parabolic point  $p$ , of the set of weakly parallel points to  $p$ .

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