Wojciech Domitrz Geometria form różniczkowych Wykład: Metoda reperu ruchomego dla powierzchni w $\mathbb{R}^3$

$x: U \hookrightarrow \mathbb{R}^3$ zanurzenie powierzchni w $\mathbb{R}^3$

$dx = \theta_1 e_1 + \theta_2 e_2$ $\quad e_1, e_2$ - reper ortonormalny na $x(U) \subset M$ tzn. $\forall u \in U$ $e_1(u), e_2(u)$ to

baza ortonormalna $T_{x(u)}M$ tzn. $\langle e_i, e_j \rangle = \delta_{ij}$ dla $i = 1, 2$

$e_3 = e_1 \times e_2$ jest normalny do M w punktach $x(u)$ tzn. $\langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0$

$\theta_i = \langle dx, e_i \rangle$ dla $i = 1, 2, 3$ $\quad$ stąd $\theta_3 = 0$ w punktach $x(u)$

$de_i = \omega_{i1} e_1 + \omega_{i2} e_2 + \omega_{i3} e_3$ dla $i = 1, 2, 3$ $\quad \omega_{ij} = \langle de_i, e_j \rangle$

$\langle e_i, e_j \rangle = \delta_{ij} \Rightarrow 0 = d(\langle e_i, e_j \rangle) = \langle de_i | e_j \rangle + \langle e_i, de_j \rangle = \omega_{ij} + \omega_{ji} = 0 \Rightarrow \omega_{ij} = -\omega_{ji} \Rightarrow \omega_{ii} = 0$

stąd $de_1 = \omega_{12} e_2 + \omega_{13} e_3$ $\quad de_2 = -\omega_{12} e_1 + \omega_{23} e_3$ $\quad de_3 = -\omega_{13} e_1 - \omega_{23} e_2$

$$dx = \theta_1 e_1 + \theta_2 e_2 \Rightarrow 0 = d(dx) = d(\theta_1 e_1 + \theta_2 e_2) = d\theta_1 e_1 - \theta_1 \wedge de_1 + d\theta_2 e_2 - \theta_2 \wedge de_2 =$$

$$= d\theta_1 e_1 - \theta_1 \wedge (\omega_{12} e_2 + \omega_{13} e_3) + d\theta_2 e_2 - \theta_2 \wedge (-\omega_{12} e_1 + \omega_{23} e_3) =$$

$$= (d\theta_1 + \theta_2 \wedge \omega_{12}) e_1 + (d\theta_2 - \theta_1 \wedge \omega_{12}) e_2 - (\theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23}) e_3 = 0$$

$$d\theta_1 + \theta_2 \wedge \omega_{12} = 0 \qquad d\theta_2 - \theta_1 \wedge \omega_{12} = 0 \qquad -(\theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23}) = 0$$

$$\boxed{d\theta_1 = \theta_2 \wedge \omega_{21} \qquad d\theta_2 = \theta_1 \wedge \omega_{12} \qquad \theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23} = 0}$$

$$d\theta_i = \sum_k \theta_k \wedge \omega_{ki}$$

I równania strukturalne Cartana

$$de_1 = \omega_{12} e_2 + \omega_{13} e_3 \qquad 0 = d(de_1) = d(\omega_{12} e_2 + \omega_{13} e_3) = d\omega_{12} e_2 - \omega_{12}\wedge de_2 + d\omega_{13} e_3 - \omega_{13}\wedge de_3 =$$

$$= d\omega_{12} e_2 - \omega_{12}\wedge(\omega_{21} e_1 + \omega_{23} e_3) + d\omega_{13} e_3 - \omega_{13}\wedge(\omega_{31} e_1 + \omega_{32} e_2) =$$

$$= (\underbrace{\omega_{21}\wedge\omega_{21}}_{=0}) e_1 + (\underbrace{\omega_{31}\wedge\omega_{31}}_{=0}) e_1 + (d\omega_{12} - \omega_{13}\wedge\omega_{32}) e_2 + (d\omega_{13} - \omega_{12}\wedge\omega_{23}) e_3 = 0$$

$$(*)\ d\omega_{12} = \omega_{13}\wedge\omega_{32} \qquad d\omega_{13} = \omega_{12}\wedge\omega_{23}$$

$$de_2 = \omega_{21} e_1 + \omega_{23} e_3 \qquad 0 = d(de_2) = d\omega_{21} e_1 - \omega_{21}\wedge de_1 + d\omega_{23} e_3 - \omega_{23}\wedge de_3 =$$

$$= d\omega_{21} e_1 - \omega_{21}\wedge(\omega_{12} e_2 + \omega_{13} e_3) + d\omega_{23} e_3 - \omega_{23}\wedge(\omega_{31} e_1 + \omega_{32} e_2) =$$

$$= (d\omega_{21} - \omega_{23}\wedge\omega_{31}) e_1 + (\underbrace{\omega_{12}\wedge\omega_{12}}_{=0}) e_2 + (\underbrace{\omega_{32}\wedge\omega_{32}}_{=0}) e_2 + (d\omega_{23} - \omega_{21}\wedge\omega_{13}) e_3 = 0$$

$$d\omega_{21} = \omega_{23}\wedge\omega_{31} \qquad d\omega_{23} = \omega_{21}\wedge\omega_{13}$$

$$-d\omega_{12} = -\omega_{13}\wedge\omega_{32}$$

$$d\omega_{12} = \omega_{13}\wedge\omega_{32} \ (*)$$

$$d\omega_{12} = \omega_{13}\wedge\omega_{23} \qquad d\omega_{13} = \omega_{12}\wedge\omega_{23} \qquad d\omega_{23} = \omega_{21}\wedge\omega_{13}$$

$$d\omega_{ij} = \sum_k \omega_{ik}\wedge\omega_{kj}$$

II równania strukturalne Cartana

zn. $\theta_i = x^* \theta_i$, $dx = \theta_1 e_1 + \theta_2 e_2$ $\quad \theta_3 = 0$ $\quad d\theta_1 = \theta_2 \wedge \omega_{21}$ $\quad d\theta_2 = \theta_1 \wedge \omega_{12}$ $\quad \omega_{ij} = -\omega_{ji}$

$0 = d(0) = d\theta_3 = \theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23}$ $\quad 0 = \theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23}$

$d\omega_{12} = \omega_{13} \wedge \omega_{32}$ $\quad d\omega_{23} = \omega_{21} \wedge \omega_{13}$ $\quad d\omega_{13} = \omega_{12} \wedge \omega_{23}$

$$(*) \begin{cases} \omega_{13} = h_{11}\theta_1 + h_{12}\theta_2 \\ \omega_{23} = h_{12}\theta_1 + h_{22}\theta_2 \end{cases} \qquad \begin{matrix} \omega_{13}(e_1) = h_{11} & \omega_{13}(e_2) = h_{12} \\ \omega_{23}(e_1) = h_{12} & \omega_{23}(e_2) = h_{22} \end{matrix}$$

$$de_3 = \omega_{31} e_1 + \omega_{32} e_2$$

$$de_3(e_1) = \omega_{31}(e_1)e_1 + \omega_{32}(e_1)e_2 = -h_{11}e_1 - h_{12}e_2$$
$$de_3(e_2) = \omega_{31}(e_2)e_1 + \omega_{32}(e_2)e_2 = -h_{12}e_1 - h_{22}e_2$$

$$de_3 = -\begin{bmatrix} h_{11}, & h_{12} \\ h_{12}, & h_{22} \end{bmatrix}$$

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = (h_{11}\theta_1 + h_{12}\theta_2) \wedge -(h_{12}\theta_1 \wedge h_{22}\theta_2)$$

$$d\omega_{12} = -(h_{11}h_{22} - h_{12}^2)\,\theta_1 \wedge \theta_2$$

$K = \det de_3 = h_{11}h_{22} - h_{12}^2$

krzywizna Gaussa

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

$H = -\frac{1}{2} \operatorname{tr} de_3 = \dfrac{h_{11}+h_{22}}{2}$ – średnia krzywizna

$$\omega_{13}\wedge\theta_2 + \theta_1\wedge\omega_{23} = (h_{11}\theta_1 + h_{12}\theta_2)\wedge\theta_2 + \theta_1\wedge(h_{12}\theta_1 + h_{22}\theta_2) = (h_{11}+h_{22})\,\theta_1\wedge\theta_2 = 2H\,\theta_1\wedge\theta_2$$

$$\omega_{13}\wedge\theta_2 + \theta_1\wedge\omega_{23} = 2H\,\theta_1\wedge\theta_2$$

Tw. (Theoreme Egregium Gaussa)

Krzywizna Gaussa K zależy tylko od tensora metrycznego na powierzchni M tzn. jeżeli $x, x' : M \hookrightarrow \mathbb{R}^3$ są imersjami zadającymi ten sam tensor metryczny to

$$\forall\, p\in M \qquad K(p) = K'(p),$$

gdzie K i K' są krzywiznami Gaussa zadanymi przez imersje x i x' (odp.)

Dowód: $U\subset M$ otwarty $p\in U$ $\{e_1, e_2\}$ – ortonormalny reper ruchomy na U względem $\langle dx, dx\rangle$ oraz $\langle dx', dx'\rangle$

$dx = \theta_1 e_1 + \theta_2 e_2 \qquad dx' = \theta_1' e_1 + \theta_2' e_2 \qquad \theta_i(e_j) = \delta_{ij} = \theta_i'(e_j)$ dla $i,j = 1,2$

Stąd $\theta_i = \theta_i' \qquad d\theta_1 = \theta_2\wedge\omega_{21} \quad d\theta_1' = \theta_2'\wedge\omega_{21}' \Rightarrow d\theta_1 = \theta_2\wedge\omega_{21}'$

$0 = d\theta_1 - d\theta_1' = \theta_2\wedge(\omega_{21}-\omega_{21}') \qquad d\theta_2 = \theta_1\wedge\omega_{12}$ i $d\theta_2' = \theta_1'\wedge\omega_{12}' \Rightarrow d\theta_2 = \theta_1\wedge\omega_{12}'$

$0 = d\theta_2 - d\theta_2' = \theta_1\wedge(\omega_{12}-\omega_{12}')$ stąd $\theta_1\wedge(\omega_{12}-\omega_{12}') = \theta_2\wedge(\omega_{12}-\omega_{12}') = 0$

$$\left.\begin{aligned} &\omega_{12}-\omega_{12}' = a\,\theta_1 + b\,\theta_2,\ 0 = \theta_1\wedge(\omega_{12}-\omega_{12}') = b\,\theta_1\wedge\theta_2 = 0 \Rightarrow b = 0\\ &0 = \theta_2\wedge(\omega_{12}-\omega_{12}') = a\,\theta_2\wedge\theta_1 = 0 \Rightarrow a = 0\end{aligned}\right\} \Rightarrow \omega_{12}-\omega_{12}' = 0$$

$\omega_{12} = \omega_{12}'$

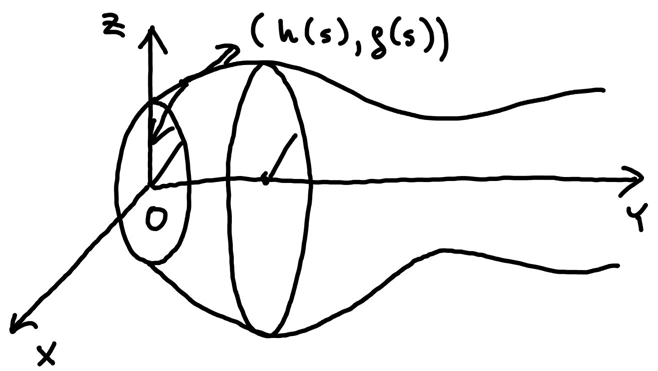
$d\omega_{12} = -K\,\theta_1\wedge\theta_2 \qquad d\omega_{12} = d\omega_{12}' = -K'\,\theta_1'\wedge\theta_2' = -K'\theta_1\wedge\theta_2 \Rightarrow K = K' \quad \square$

Przykład.

$U = \mathbb{R} \times (0, 2\pi) \qquad p(s,\varphi) = (h(s)\sin\varphi,\ h(s)\cos\varphi,\ g(s)) \qquad p: U \to \mathbb{R}^3 \qquad h(s) \neq 0$

$h, g : \mathbb{R} \to \mathbb{R}$ gładkie zał. że $\left(\frac{dh}{ds}\right)^2 + \left(\frac{dg}{ds}\right)^2 = 1$ ← parametryzacja łukowa krzywej na pł. $\{x=0\}$

$p(s, 0) = (0, h(s), g(s))$ którą obracamy wokół osi OY



$$e_1 = \frac{\partial p}{\partial s} = dp\left(\frac{\partial}{\partial s}\right) = (h'(s)\sin\varphi, h'(s)\cos\varphi, g'(s))$$

$$|e_1| = \left((h'(s))^2 + (g'(s))^2\right)^{\frac{1}{2}} = 1$$

$$e_2 = \frac{1}{h}\frac{\partial p}{\partial \varphi} = dp\left(\frac{1}{h}\frac{\partial}{\partial \varphi}\right) = \frac{1}{h(s)}\left(h(s)\cos\varphi, -h(s)\sin\varphi, 0\right) = (\cos\varphi, -\sin\varphi, 0)$$

$$|e_2| = 1 \qquad \langle e_1, e_2\rangle = h'(s)\sin\varphi\cos\varphi - h'(s)\cos\varphi\sin\varphi + 0 = 0$$

$$e_3 = e_1 \times e_2$$

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial \varphi} d\varphi = ds \otimes e_1 + h\, d\varphi \otimes e_2 \qquad \theta_1 = ds \quad \theta_2 = h\, d\varphi$$

$$0 = d\theta_1 = \theta_2 \wedge \omega_{21} = h\, d\varphi \wedge \omega_{21} \qquad \omega_{12} = h'(s)\, d\varphi$$

$$d\theta_2 = h'(s)\, ds \wedge d\varphi = \theta_1 \wedge \omega_{12} = ds \wedge h'(s)\, d\varphi$$

$$d\omega_{12} = h'' \, ds \wedge d\varphi = \frac{h''}{h} ds \wedge h \, d\varphi = \frac{h''}{h} \theta_1 \wedge \theta_2 = -K \, \theta_1 \wedge \theta_2 \qquad K = -\frac{h''}{h}$$

$h(s) = R \cos\left(\frac{s}{R}\right)$ wtedy $p : U \to S^2(R)$ – sfera o promieniu R

$$K = -\frac{-R\cos\left(\frac{s}{R}\right)\frac{1}{R^2}}{R\cos\left(\frac{s}{R}\right)} = \frac{1}{R^2}$$

$h(s) = R$ wtedy $p : U \to S^1 \times \mathbb{R}$ – walec o promieniu R

$g(s) = s$

$K = 0$

$x : M \hookrightarrow \mathbb{R}^3$

$I = \langle dx, dx \rangle \qquad dx = \theta_1 e_1 + \theta_2 e_2$

$I = \langle \theta_1 e_1 + \theta_2 e_2 , \theta_1 e_1 + \theta_2 e_2 \rangle = \theta_1^2 + \theta_2^2$

$I = \theta_1^2 + \theta_2^2$

$II = -\langle dx, de_3 \rangle = -\langle \theta_1 e_1 + \theta_2 e_2, \omega_{31} e_1 + \omega_{32} e_2 \rangle = -(\theta_1 \omega_{31} + \theta_2 \omega_{32}) =$

$= \theta_1 \omega_{13} + \theta_2 \omega_{23} = h_{11} \theta_1^2 + 2h_{12} \theta_1 \theta_2 + h_{22} \theta_2^2$

$\begin{cases} \omega_{13} = h_{11} \theta_1 + h_{12} \theta_2 \\ \omega_{23} = h_{12} \theta_1 + h_{22} \theta_2 \end{cases}$ (patrz (*) str 3)

$\gamma : (-\varepsilon, \varepsilon) \to M$ gł. krzywa łukowo sparametryzowana

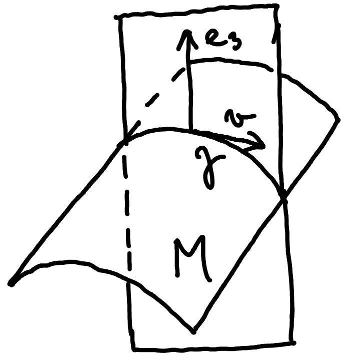
$\gamma(0) = p \qquad \gamma'(0) = v \in T_p M \qquad x(s) := x(\gamma(s)) \qquad e_3(s) = e_3(\gamma(s))$

$\langle \frac{dx}{ds}, e_3(s) \rangle = 0$ bo $e_3(s) \perp T_{\gamma(s)} M$

$\langle \frac{d^2x}{ds^2}, e_3(s) \rangle + \langle \frac{dx}{ds}, \frac{de_3}{ds} \rangle = 0$

$\langle \frac{d^2x}{ds^2}, e_3(s) \rangle = -\langle \frac{dx}{ds}, \frac{de_3}{ds} \rangle = -\langle dx(v), de_3(v) \rangle = II(v)$

$\varkappa(s) := \varkappa(\gamma(s)) \qquad \frac{d^2x}{ds^2} = \varkappa(s)\, n_\gamma(s)$



$II(v) = \langle \frac{d^2x}{ds^2}, e_3(s)\rangle = \langle \varkappa(s)\, n_\gamma(s), e_3(s)\rangle = \varkappa(s)\langle n_\gamma(s), e_3(s)\rangle$

$\varkappa_n = \varkappa\langle n_\gamma, e_3\rangle$ – krzywizna normalna powierzchni w kierunku $v = \gamma'(s)$

$II_p(v) = \varkappa_n(v) = -\langle de_3(v), v\rangle = \langle -de_3(v), v\rangle$

$(-de_3)^T = -de_3 \qquad -de_3(v) = \lambda v \qquad (x|y) = \bar{x}^T y = \sum_{i=1}^{n} \bar{x}_i y_i \qquad \overline{(-de_3)} = -de_3 \in M_2(\mathbb{R})$

$\lambda|v|^2 = \lambda(v|v) = (v, \lambda v) = (v|-de_3(v)) = \bar{v}^T(-de_3)v = \bar{v}^T\overline{(-de_3)}^T v = \overline{(-de_3(v))}^T v =$

$= (-de_3(v)|v) = (\lambda v|v) = \bar{\lambda}|v|^2 \Rightarrow \lambda = \bar{\lambda} \qquad \lambda \in \mathbb{R}$

$II(v_1) = \lambda_1 v_1 \qquad I(v_2) = \lambda_2 v_2$

$II_p(v_1) = \langle -de_3(v_1), v_1\rangle = \langle \lambda_1 v_1, v_1\rangle = \lambda_1|v_1|^2$

$II_p(v_2) = \langle -de_3(v_2), v_2\rangle = \langle \lambda_2 v_2, v_2\rangle = \lambda_2|v_2|^2$

$w_1 = \frac{v_1}{|v_1|} \qquad w_2 = \frac{v_2}{|v_2|} \qquad II(w_1) = \lambda_1 \qquad II(w_2) = \lambda_2$

$-de_3(w_1) = \lambda_1 w_1 \qquad (\cdot|\cdot)|_{\mathbb{R}^2\times\mathbb{R}^2} = \langle \cdot, \cdot\rangle$

$\lambda_1\langle w_1, w_2\rangle = \langle -de_3(w_1), w_2\rangle = \langle w_1, -de_3(w_2)\rangle = \langle w_1, \lambda_2 w_2\rangle = \lambda_2\langle w_1, w_2\rangle$

$\lambda_1 \neq \lambda_2 \Rightarrow \langle w_1, w_2\rangle = 0$

$$de_3(v_1) = \lambda v_1$$

$$de_3(v_2) = \lambda v_2$$

$$\langle -de_3(v_1), -de_3(v_2)\rangle = \langle \lambda v_1, \lambda v_2\rangle = \lambda^2 \langle v_1, v_2\rangle$$

$$[-de_3] = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \qquad \det \begin{bmatrix} h_{11}-t & h_{12} \\ h_{12} & h_{22}-t \end{bmatrix} = t^2 - (h_{11}+h_{22}) + h_{11}h_{22} - h_{12}^2$$

$$de_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\langle w_1 | w_2 \rangle = 0 \qquad w_1 \perp w_2$$

$$\Delta = (h_{11}+h_{22})^2 - 4(h_{11}h_{22} - h_{12}^2) =$$

$$= h_{11}^2 + h_{22}^2 + 2h_{11}h_{22} - 4h_{11}h_{22} + 4h_{12}^2$$

$$= (h_{11}-h_{22})^2 + 4h_{12}^2 = 0$$

$$h_{11} = h_{22} = \lambda \quad , \quad h_{12} = 0$$

$$\mathrm{II}(\alpha w_1+\beta w_2) = \langle -de_3(\alpha w_1+\beta w_2), \alpha w_1+\beta w_2\rangle = \langle \alpha\lambda_1 w_1+\beta\lambda_2 w_2, \alpha w_1+\beta w_2\rangle =$$

$$= \lambda_1\alpha^2+\lambda_2\beta^2$$

$\alpha = \cos t$ $\quad \beta = \sin t$ $\qquad \mathrm{II}(\cos t\, w_1 + \sin t\, w_2) = \lambda_1\cos^2 t + \lambda_2 \sin^2 t$

$$\frac{d\mathrm{II}}{dt} = 2\lambda_1 \cos t(-\sin t) + 2\lambda_2 \sin t\cos t = \sin 2t(\lambda_2-\lambda_1) \qquad t\in[0,2\pi)$$

$\frac{d\mathrm{II}}{dt} = 0$ dla $t = k\frac{\pi}{2}$ $\quad k=0,1,2,3$ $\qquad \frac{d^2\mathrm{II}}{dt^2} = 2\cos 2t\,(\lambda_2-\lambda_1)$

$$\frac{d^2\mathrm{II}}{dt^2}(k\pi) = 2(\lambda_2-\lambda_1) \qquad \frac{d^2\mathrm{II}}{dt^2}\left(k\pi+\frac{\pi}{2}\right) = -2(\lambda_2-\lambda_1)$$

wartości ekstremalne krzywizne normalne osiąga dla wektorów prostopadłych równym wektorom własnym $-de_3$ $\quad w_1, w_2$

$$\mathrm{II}(k\pi) = \lambda_1 = \varkappa_1 \qquad \mathrm{II}\left(k\pi+\frac{\pi}{2}\right) = \lambda_2 = \varkappa_2$$

$\varkappa_1, \varkappa_2$ – krzywizny główne

$\mathrm{span}(w_1)$, $\mathrm{span}(w_2)$ – kierunki główne $\qquad w_1 \perp w_2$