Równania strukturalne w $\mathbb{R}^n$

$U \subseteq \mathbb{R}^n$ otwarty $\quad e_i : U \to \mathbb{R}^n$ gładkie pole wektorowe dla $i = 1, \dots, n$

Def. $e_1, \dots, e_n$ jest reperem na U jeśli $\forall x \in U$ $\ e_1(x), \dots, e_n(x)$ jest bazą $T_x \mathbb{R}^n = \mathbb{R}^n$

$v, w \in \mathbb{R}^n \quad v = (v_1, \dots, v_n), \ w = (w_1, \dots, w_n) \qquad \langle v, w \rangle = \sum_{i=1}^{n} v_i \cdot w_i$ – iloczyn skalarny

Def. Reper $e_1, \dots, e_n$ jest ortonormalny jeśli $\langle e_i, e_j \rangle = \delta_{ij}$

Def. $\theta_1, \dots, \theta_n$ reper dualny jeśli $\theta_i(e_j) = \delta_{ij}$

θ_i – gładka 1-forma różniczkowa na U

$(de_i)_x : \mathbb{R}^n \to \mathbb{R}^n$ – liniowe $\qquad (de_i)_x(v) = \sum_j (\omega_{ij})_x(v)\, e_j$

$de_i = \sum_{j=1}^{n} \omega_{ij} e_j \qquad \omega_{ij}$ – gładka 1-forma różniczkowa na $U \qquad \omega_{ij} = \langle de_i, e_j \rangle$

ω_{ij} – 1-formy koneksji na U $\quad i,j = 1, \dots, n \qquad 0 = d(\langle e_i, e_j \rangle) = \langle de_i, e_j \rangle + \langle e_i, de_j \rangle = \omega_{ij} + \omega_{ji}$

Stw. $\omega_{ji} = -\omega_{ij}$ dla $i,j = 1, \dots, n$. ($\omega_{ii} = 0$ dla $i = 1, \dots, n$)

Tw. (równania strukturalne Cartana $\mathbb{R}^n$)

(I) $d\theta_i = \sum_j \theta_j \wedge \omega_{ji}$ dla $i = 1, \dots, n$

(II) $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$ dla $i, j = 1, \dots, n$

Dowód.:

$\frac{\partial}{\partial x_i} = (0, \dots, 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \dots, 0) \qquad \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ – reper ortonormalny $\qquad \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \delta_{ij}$

$\pi_i(x_1,\dots,x_n) = x_i \qquad dx_i := d\pi_i \qquad dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij} \qquad dx_1,\dots,dx_n$ – reper dualny do $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$

(*) $e_i = \sum_j \beta_{ij} \frac{\partial}{\partial x_j}$ $\qquad \beta_{ij}$ – gładkie funkcje na U

$$\delta_{ij} = \langle e_i, e_j \rangle = \left\langle \sum_k \beta_{ik} \frac{\partial}{\partial x_k}, \sum_l \beta_{jl} \frac{\partial}{\partial x_l} \right\rangle = \sum_{k,l} \beta_{ik} \beta_{jl} \left\langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right\rangle = \sum_{k,l} \beta_{ik} \beta_{jl} \delta_{kl} =$$

$= \sum_k \beta_{ik}\beta_{jk} \qquad E_n = B \cdot B^T \qquad E_n = (\delta_{ij}) \qquad B = (\beta_{ij}) \qquad B$ – macierz ortogonalna

$$B^{-1} = B^T \qquad e = B\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial x} = B^{-1} e \qquad \frac{\partial}{\partial x} = B^T e \qquad \frac{\partial}{\partial x_i} = \sum_k \beta_{ki} e_k$$

$$\theta_i\left(\frac{\partial}{\partial x_j}\right) = \theta_i\left(\sum_k \beta_{kj} e_k\right) = \sum_k \beta_{kj}\, \theta_i(e_k) = \sum_k \beta_{kj} \delta_{ik} = \beta_{ij}$$

$$\Downarrow$$

(**)
$$\theta_i = \sum_j \beta_{ij}\, dx_j$$

$$\begin{cases} de_i = \sum_j \omega_{ij} e_j = \sum_j \omega_{ij} \sum_k \beta_{jk} \frac{\partial}{\partial x_k} = \sum_k \left(\sum_j \omega_{ij}\beta_{jk}\right) \frac{\partial}{\partial x_k} \\ (*) \Rightarrow de_i = d\left(\sum_k \beta_{ik} \frac{\partial}{\partial x_k}\right) = \sum_k d\beta_{ik} \frac{\partial}{\partial x_k} \qquad \left(d\left(\frac{\partial}{\partial x_k}\right) \equiv 0 \right) \end{cases}$$

$$(***)\ d\beta_{ik} = \sum_j \omega_{ij}\beta_{jk} \qquad (**) \Rightarrow d\theta_i = d\left(\sum_j \beta_{ij}\, dx_j\right) = \sum_j d\beta_{ij} \wedge dx_j \overset{(***)}{=} \sum_j \left(\sum_k \omega_{ik}\beta_{kj}\right) \wedge dx_j$$

$$= \sum_k \omega_{ik} \wedge \left(\sum_j \beta_{kj}\, dx_j\right) = \sum_k \omega_{ik} \wedge \theta_k = \sum_k \theta_k \wedge \omega_{ki}$$

$$\Rightarrow \quad d\theta_i = \sum_k \theta_k \wedge \omega_{ki} \quad (\mathrm{I})$$

$(***)$ $$d\beta_{ik} = \sum_j \omega_{ij}\beta_{jk} \qquad d\beta = \omega\beta \qquad \omega = d\beta\,\beta^{-1} = d\beta\,\beta^T$$

$$\omega_{ij} = \sum_k d\beta_{ik}\cdot\beta_{jk} = \sum_k \beta_{jk}\,d\beta_{ik}$$

$$d\omega_{ij} = d\Big(\sum_k \beta_{jk}\,d\beta_{ik}\Big) = \sum_k d\beta_{jk}\wedge d\beta_{ik} =$$

$$\sum_k \Big(\sum_l \omega_{jl}\beta_{lk}\Big)\wedge\sum_s \omega_{is}\beta_{sk}\Big) = \sum_l\sum_s\Big(\sum_k \beta_{lk}\underbrace{\beta_{sk}}_{=(\beta^T)_{ks}=(\beta^{-1})_{ks}}\Big)\,\omega_{jl}\wedge\omega_{is}$$

$$= \sum_{l,s}\delta_{ls}\;\omega_{jl}\wedge\omega_{is} = \sum_l \omega_{jl}\wedge\omega_{il} = \sum_l \omega_{il}\wedge\omega_{lj}$$

$$\omega_{jl} = -\omega_{lj}$$

$$d\omega_{ij} = \sum_l \omega_{il}\wedge\omega_{lj} \quad (\mathrm{II}) \quad \square$$

Dowód (drugi):

$x = (x_1, \dots, x_n) \qquad dx = \sum_i \theta_i e_i \qquad dx = E_n \qquad dx(e_j) = e_j \qquad e_j = \sum_{i=1} \theta_i(e_j) e_i$

$\theta_i(e_j) = \delta_{ij} \qquad i,j = 1, \dots, n$

$de_i = \sum_j \omega_{ij} e_j \qquad 0 = d(dx) = d(\sum_i \theta_i e_i) = \sum_i (d\theta_i e_i - \theta_i de_i =$

$= \sum_i (d\theta_i e_i - \theta_i \wedge \sum_k \omega_{ik} e_k) = \sum_k d\theta_k e_k - \sum_k (\sum_i \theta_i \wedge \omega_{ik}) e_k =$

$= \sum_k (d\theta_k - \sum_i \theta_i \wedge \omega_{ik}) e_k = 0$

$d\theta_k - \sum_i \theta_i \wedge \omega_{ik} = 0 \qquad \boxed{d\theta_k = \sum_i \theta_i \wedge \omega_{ik}} \quad (I)$

$0 = d(de_i) = d(\sum_j \omega_{ij} e_j) = \sum_j d\omega_{ij} e_j - \sum_j \omega_{ij} \wedge de_j =$

$= \sum_j d\omega_{ij} e_j - \sum_j \omega_{ij} \wedge \sum_k \omega_{jk} e_k = \sum_k d\omega_{ik} e_k - \sum_k (\sum_j \omega_{ij} \wedge \omega_{jk}) e_k$

$= \sum_k (d\omega_{ik} - \sum_j (\omega_{ij} \wedge \omega_{jk})) e_k = 0 \qquad d\omega_{ik} = \sum_j \omega_{ij} \wedge \omega_{jk}$

$\boxed{d\omega_{ik} = \sum_j \omega_{ij} \wedge \omega_{jk}} \quad (II) \quad \square$

Lemat (Cartana)

V^n - n-wymiarowa przestrzeń liniowa $\alpha_1, \dots, \alpha_r \in (V^n)^*$ liniowo niezależne

Jeżeli $\beta_1, \dots, \beta_r \in (V^n)^*$ takie, że $\sum_{i=1} \beta_i \wedge \alpha_i = 0 \Rightarrow \beta_i = \sum_{j=1}^{r} a_{ij} \alpha_j \wedge a_{ij} = a_{ji} \quad i,j = 1, \dots, r$

Dowód.: Uzupełniamy $\alpha_1, \dots, \alpha_r$ do bazy $\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_n$ przestrzeni $(V^n)^*$

Wtedy $\beta_i = \sum_{j=1}^{n} a_{ij} \alpha_j$ dla $i = 1, \dots, r$ oraz $a_{ij} = a_{ji}$ dla $i,j = 1, \dots, r$

$$(*)\ 0 = \sum_{i=1}^{r} \beta_i \wedge \alpha_i = \sum_{i=1}^{r} \sum_{j=1}^{n} a_{ij} \alpha_j \wedge \alpha_i = \sum_{\substack{i,j=1 \\ j>i}}^{r} (a_{ij} - a_{ji}) \alpha_j \wedge \alpha_i + \sum_{i=1}^{r} \sum_{j=r+1}^{n} a_{ij} \alpha_j \wedge \alpha_i$$

$\alpha_i \wedge \alpha_i = 0$

$\alpha_j \wedge \alpha_i$ dla $j > i$ są liniowo niezależne.

Stąd z $(*)$ $a_{ij} - a_{ji} = 0$ dla $j > i$, $i,j = 1, \dots, r$

oraz $a_{ji} = 0$ dla $i = 1, \dots, r$ $j = r+1, \dots, n$

czyli $\beta_i = \sum_{j=1}^{r} a_{ij} \alpha_j$ oraz $a_{ij} = a_{ji}$ dla $i,j = 1, \dots, r$ $\square$

Lemat. Niech $U \subset \mathbb{R}^n$ otwarty oraz $\theta_1, \dots, \theta_n$ liniowo niezależne 1-formy różniczkowe na U takie, że istnieje zbiór ω_{ij} 1-form różniczkowych na U dla $i,j = 1,\dots,n$ taki, że $d\theta_i = \sum_{j=1}^{n} \theta_j \wedge \omega_{ji}$ dla $i = 1,\dots,n$, $\omega_{ij} = -\omega_{ji}$ dla $i,j = 1,\dots,n$. Wtedy zbiór ω_{ij} $i,j = 1,\dots,n$ jest wyznaczony jednoznacznie.

Dowód.: Niech $\overline{\omega}_{ij}$ dla $i,j = 1,\dots,n$ będzie zbiorem 1-form różniczkowych takich, że $d\theta_i = \sum_{j=1}^{n} \theta_j \wedge \overline{\omega}_{ji}$ $i = 1,\dots,n$, $\overline{\omega}_{ij} = -\overline{\omega}_{ji}$ $i,j = 1,\dots,n$

$$0 = d\theta_i - d\theta_i = \sum_{j=1}^{n} \theta_j \wedge (\omega_{ji} - \overline{\omega}_{ji}) \qquad 0 = \sum_{j=1}^{n} \theta_j \wedge (\omega_{ji} - \overline{\omega}_{ji}) \qquad i = 1,\dots,n$$

$$\omega_{ji} - \overline{\omega}_{ji} = \sum_{k=1}^{n} B^i_{jk}\theta_k \qquad B^i_{kj} = B^i_{jk} \qquad i,j,k = 1,\dots,n$$

$$\omega_{ji} - \overline{\omega}_{ji} = -\omega_{ij} + \overline{\omega}_{ij} = -(\omega_{ij} - \overline{\omega}_{ij}) = \sum_{k=1}^{n} B^j_{ik} \theta_k \qquad B^j_{ik} = B^j_{ki} \quad i,j,k = 1,\dots,n$$

$$\sum_{k=1}^{n} (-B^j_{ik})\theta_k \quad \Rightarrow \quad B^i_{jk} = -B^j_{ik}$$

$$-B^k_{ji} = -B^k_{ij} = B^i_{kj} = B^i_{jk} = -B^j_{ik} = -B^j_{ki} = B^k_{ji}$$

$$B^k_{ji} = -B^k_{ji} \quad \Rightarrow \quad B^k_{ji} = 0 \qquad i,j,k = 1,\dots,n \quad \Rightarrow \quad \omega_{ji} - \overline{\omega}_{ji} = 0$$

$\square$