Geometria wewnętrzna powierzchni.

$p: D \to \mathbb{R}^3$ parametryzacja regularna powierzchni S

$\nu: D \to \mathbb{R}^3$ pole normalne do S tzn. $\forall (u,v) \in D$ $\quad \langle \nu(u,v), p_u(u,v)\rangle = \langle \nu(u,v), p_v(u,v)\rangle = 0 \quad |\nu(u,v)| = 1$

$\forall (u,v) \in D \quad (p_u(u,v), p_v(u,v), \nu(u,v))$ – baza $\mathbb{R}^3$ $\qquad \mathbb{R}^3 = T_{p(u,v)} S \oplus \mathrm{span}(\nu(u,v))$

$$
\begin{aligned}
p_{uu} &= \Gamma^1_{11} p_u + \Gamma^2_{11} p_v + L_1 \nu \\
p_{uv} &= \Gamma^1_{12} p_u + \Gamma^2_{12} p_v + L_2 \nu \\
p_{vu} &= \Gamma^1_{21} p_u + \Gamma^2_{21} p_v + \tilde{L}_2 \nu \\
p_{vv} &= \Gamma^1_{22} p_u + \Gamma^2_{22} p_v + L_3 \nu
\end{aligned}
\quad *
$$

$$
\nu_u = a_{11} p_u + a_{21} p_v \qquad \nu_v = a_{12} p_u + a_{22} p_v \qquad \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -A
$$

A – operator Weingartena

$p_{uv} = p_{vu}$ (Tw. Schwarza) $\Rightarrow \Gamma^i_{jk} = \Gamma^i_{kj} \quad i,j,k = 1,2,3 \qquad L_2 = \tilde{L}_2$

$L_1 = \langle p_{uu}, \nu\rangle = L \qquad L_2 = \tilde{L}_2 = \langle p_{uv}, \nu \rangle = M \qquad L_3 = \langle p_{vv}, \nu\rangle = N$

$\hat{\mathrm{II}} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ – macierz II formy podsto.

Γ^i_{jk} to symbole Christoffela. Aby obliczyć symbole Christoffela

$p_{uu} = \Gamma_{12} p_u + \Gamma_{12} p_u + L_2 \nu \ / \cdot p_u \ / \cdot p_v$

$p_{uu} \cdot p_u = \Gamma_{12}\, p_u \cdot p_u + \Gamma_{12}\, p_u \cdot p_v + L_2\, \nu \cdot p_u = \Gamma_{12} \cdot E + \Gamma_{12} \cdot F$

$\hat{\mathrm{I}} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

macierz ds^2 – I-formy podstawowej

$p_u \cdot p_u = E \;/\; \frac{\partial}{\partial u} \qquad p_{uu} \cdot p_u + p_u \cdot p_{uu} = E_u \qquad 2\, p_{uu} \cdot p_u = E_u \quad \Rightarrow \quad p_{uu} \cdot p_u = \frac{1}{2} E_u$

czyli otrzymujemy

$\Gamma^1_{11} E + \Gamma^2_{11} F = \frac{1}{2} E_u$

$p_{uu} \cdot p_v = \Gamma^1_{11}\, p_u \cdot p_v + \Gamma^2_{11}\, p_v \cdot p_v + L_2\, \nu \cdot p_v = \Gamma^1_{11} F + \Gamma^2_{11} G$

$p_u \cdot p_v = F \;/\; \frac{\partial}{\partial u} \qquad p_{uu} \cdot p_v + p_u \cdot p_{uv} = F_u$

$p_u \cdot p_u = E \;/\; \frac{\partial}{\partial v} \qquad p_{uv} \cdot p_u + p_u \cdot p_{uv} = E_v \quad \Rightarrow \quad E_v = \frac{1}{2}\, p_u \cdot p_{uv}$

$p_{uu} \cdot p_v = F_u - \frac{1}{2} E_v$ czyli otrzymujemy $\Gamma^1_{11} F + \Gamma^2_{11} G = F_u - \frac{1}{2} E_v$

Stąd $\begin{cases} \Gamma^1_{11} E + \Gamma^2_{11} F = \frac{1}{2} E_u \\ \Gamma^1_{11} F + \Gamma^2_{11} G = F_u - \frac{1}{2} E_v \end{cases}$ oraz analogicznie

$$\begin{cases} \Gamma^1_{12} E + \Gamma^2_{12} F = \frac{1}{2} E_v \\ \Gamma^1_{12} F + \Gamma^2_{12} G = \frac{1}{2} G_u \end{cases} \qquad \begin{cases} \Gamma^1_{22} E + \Gamma^2_{22} F = F_v - \frac{1}{2} G_u \\ \Gamma^1_{22} F + \Gamma^2_{22} G = \frac{1}{2} G_v \end{cases}$$

czyli $\hat{I} \cdot \begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix} \qquad \hat{I} \cdot \begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} E_v \\ G_u \end{pmatrix} \qquad \hat{I} \begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix}$

$\det \hat{I} = EG - F^2 \neq 0$

Γ^i_{jk} – wyrażają się za pomocą współczynników ds^2 I-formy podstawowej

(1) $\frac{\partial}{\partial v}(p_{uu}) - \frac{\partial}{\partial u}(p_{uv}) = 0$ (2) $\frac{\partial}{\partial u}(p_{vv}) - \frac{\partial}{\partial v}(p_{uv}) = 0$ (3) $\frac{\partial}{\partial u}(\nu_v) - \frac{\partial}{\partial v}(\nu_u) = 0$

(1)
$$(\Gamma_{11}^1)_v p_u + \Gamma_{11}^1 p_{uu} + (\Gamma_{11}^2)_v p_u + \Gamma_{11}^2 p_{vv} + L_v \nu + L \nu_v$$
$$- (\Gamma_{12}^1)_u p_u + \Gamma_{12}^1 p_{uu} + (\Gamma_{12}^2)_u p_u + \Gamma_{12}^2 p_{uu} + M_u \nu + M \cdot \nu_u = 0$$

dalej podstawiamy $p_{uu}, p_{uv}, p_{vv}, \nu_u, \nu_v$ z $(*)$ i

współczynniki przy p_u, p_v, n muszą być równe 0 bo p_u, p_v, ν – baza

(obliczenia za pomocą MATHEMATICA w pliku Dowod_Tw_Egregium.pdf)
Biorąc pod uwagę, że $K = \frac{LN - M^2}{EG - F^2}$ oraz, że

$$a_{11} = \frac{FM - GL}{EG - F^2} \quad a_{12} = \frac{FN - GM}{EG - F^2} \quad a_{21} = \frac{FL - EM}{EG - F^2} \quad a_{22} = \frac{FM - EN}{EG - F^2}$$

otrzymujemy, że współczynnik przy p_v jest równy

$$(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2 + EK = 0$$

stąd $$K = -\frac{1}{E}\left((\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2\right)$$