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Mathematics of Control, Signals, and Systems

# On Local Structural Stability of Differential 1-Forms and Nonlinear Hypersurface Systems on a Manifold with Boundary*

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**Abstract.** In this paper we consider smooth differential 1-forms and smooth nonlinear control-affine systems with $(n-1)$-inputs evolving on an $n$-dimensional manifold with boundary. These systems are called hypersurface systems under the additional assumption that the drift vector field and control vector fields span the tangent space to the manifold. We locally classify all structurally stable differential 1-forms on a manifold with boundary. We give complete local classification of structurally stable hypersurface systems on a manifold with boundary under static state feedback defined by diffeomorphisms, which preserve the manifold together with its boundary.

**Key words.** Hypersurface systems, Feedback classification, Structural stability, Differential 1-forms, Singularities.

## 1. Introduction

We consider two smooth nonlinear control-affine systems of the form

$$\dot{q} = V_j(q) + \sum_{i=1}^{m} u_{j,i} W_{j,i}(q) = V_j(q) + W_j(q)u_j, \qquad j = 1, 2, \tag{1}$$

on a smooth $n$-dimensional manifold $M$ with smooth boundary $\partial M$, where $q \in M$, $\dot{q} = dq/dt$, $V_j$ is a smooth drift vector field, $W_{j,1}, \ldots, W_{j,m}$ are smooth control vector fields, $W_j = (W_{j,1}, \ldots, W_{j,m})$, and $u_j = (u_{j,1}, \ldots, u_{j,m})^T \in R^m$ are controls.

It is natural to use the group of diffeomorphism-germs

$$\Phi\colon (M, \partial M, p) \to (M, \partial M, p)$$

at $p \in \partial M$, which preserve $M$, for the local classification problem of such systems. It is obvious that these diffeomorphism-germs preserve the boundary $\partial M$ too. We denote this group by $G_M$.

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We study local classification of systems of the above type under static state feedback of the following type:

**Definition 1.1.** The two germs of systems (1) at $p_1 \in \partial M$ and $p_2 \in \partial M$ respectively are feedback $G_M$-equivalent if there exist

1. a diffeomorphism-germ

$$\Phi: (M, \partial M, p_1) \to (M, \partial M, p_2),$$

2. feedback of the form $u_1 = A(q) + u_2 B(q)$, where $A: M \to R^m$ and $B: M \to GL(R^m)$ are germs at $p_1 \in \partial M$ of a smooth mapping

such that

$$V_2 = \Phi_*(V_1 + W_1 A^T), \qquad W_2 = \Phi_*(W_1 B).$$

The problem of classification of nonlinear control-affine systems (on a manifold without boundary) was intensively studied by many authors. Planar systems with one control were classified by Jakubczyk and Respondek [JR1], [JR2]. Respondek and Zhitomirskii classified such systems on a three-dimensional manifold [RZ] and simple germs of this systems on an $n$-dimensional manifold [ZR]. Quadratic systems were studied by Bonnard [B]. Systems of constant rank were investigated using Cartan's equivalence method in [G], [GSW], and [W].

It is natural to assume that the number of controls $m = n - 1$, because it was shown [J] (see also [T] and [RZ]) that if $m < n - 1$, then there are no open orbits in the space of germs of control systems on an $n$-dimensional manifold without boundary, therefore there are no open orbits in the space of germs of control systems on an $n$-dimensional manifold with boundary.

We also assume that

$$\dim(\mathrm{span}\{V_j, W_{j,1}, \ldots, W_{j,n-1}\}(0)) = n, \qquad j = 1, 2. \tag{2}$$

Systems which satisfy condition (2) are called **hypersurface** systems (see [H]).

*Remark 1.1.* Under condition (2), for each hypersurface system (1) there exists the unique germ of a smooth differential 1-form $\alpha_j$ such that $\alpha_j(W_{j,i}) = 0$ for $i = 1, \ldots, n-1$ and $\alpha_j(V_j) = 1$ (see [RZ] and [Z2]). We say that $\alpha_j$ is a **corresponding** 1-form to the hypersurface system (1).

There is a natural equivalence relation on a space of germs of smooth differential 1-forms on a manifold with boundary.

**Definition 1.2.** Two germs of smooth differential 1-forms $\alpha_1$ at $p_1 \in \partial M$, $\alpha_2$ at $p_2 \in \partial M$ are $G_M$-equivalent if there exists a diffeomorphism-germ $\Phi$, such that

$$\Phi: (M, \partial M, p_1) \to (M, \partial M, p_2)$$

and

$$\Phi^* \alpha_2 = \alpha_1.$$

It is easy to prove

**Proposition 1.1.** *The two germs of control systems* (1) *are feedback $G_M$-equivalent if the corresponding germs of* 1*-forms* $\alpha_1$ *and* $\alpha_2$ *are* $G_M$*-equivalent.*

We define the notion of local structural stability on the manifold with boundary.

**Definition 1.3.** 1-form $\alpha$ is structurally $\partial M$-stable at $p \in \partial M$ if for any neighborhood $U$ of $p$ there is a neighborhood $V$ of $\alpha$ (in $C^\infty$ topology of 1-forms) such that if $\beta \in V$, then there is $q \in U$ such that germs of $\alpha$ at $p \in \partial M$ and $\beta$ at $q \in \partial M$ are $G_M$-equivalent.

We say that a hypersurface system is structurally $\partial M$-stable at $p \in \partial M$ if the corresponding 1-form is structurally $\partial M$-stable at $p \in \partial M$.

Classification of differential 1-forms is a classical problem (the Darboux theorem). Local classification (on a smooth manifold without boundary) of singular differential 1-forms was studied by Martinet [M], Golubitsky and Tischler [GT1], [GT2], Pelletier [P], and Zhitomirskii [Z2]. It was proved that a locally stable 1-form on $R^{2k+1}$ (resp. $R^{2k}$) is equivalent to one of the following three models:

$$dz + \sum_{i=1}^{k} x_i\, dy_i \qquad \text{(Darboux model)},$$

$$\pm z\, dz + (1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i \qquad \text{(Martinet models)}$$

$$\Bigg(\text{resp. } (1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i \qquad \text{(Darboux model)},$$

$$(1 \pm x_1^2)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i \qquad \text{(Martinet models)}\Bigg).$$

It is obvious that if 1-form $\alpha$ is structurally $\partial M$-stable at $p$, then it is also structurally stable at $p$ on a manifold without boundary.

In this paper we classified all locally structurally $\partial M$-stable smooth 1-forms on a manifold with boundary. We obtained the following result:

**Theorem 1.1.** *Any germ of a locally structurally* $\partial M$*-stable smooth* 1*-form on a manifold with boundary* $M$ *is* $G_M$*-equivalent to one and only one of the following germs at* 0 *of* 1*-forms on* $\{(x, y) \in R^{2k}\colon x_1 \geq 0\}$:

$$(1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i,$$

$$(1 - x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i$$

*if* $\dim M = 2k$ *or on* $\{(z, x, y) \in R^{2k+1}\colon z \geq 0\}$,

$$dz + dy_1 + \sum_{i=1}^{k} x_i \, dy_i,$$

$$-dz + dy_1 + \sum_{i=1}^{k} x_i \, dy_i$$

*if* $\dim M = 2k + 1$.

By Proposition 1.1 we obtain the complete classification of structurally $\partial M$-stable smooth nonlinear hypersurface systems on a manifold with boundary, which is the main result of the paper.

**Theorem 1.2.** *Any germ of a locally structurally $\partial M$-stable smooth hypersurface system on a manifold with boundary $M$ is feedback $G_M$-equivalent to one and only one of the following germs at* 0 *of hypersurface systems*

$$\dot{q} = V(q) + \sum_{i=1}^{\dim M - 1} u_i W_i(q)$$

*on* $\{q = (x, y) \in R^{2k}\colon x_1 \geq 0\}$, *where*

$$
\begin{aligned}
V &= \frac{1}{1 + x_1} \frac{\partial}{\partial y_1}, \\
W_i &= \frac{\partial}{\partial x_i} && \textit{for} \quad i = 1, \ldots, k, \\
W_j &= (1 + x_1) \frac{\partial}{\partial y_{j-k+1}} - x_{j-k+1} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+1, \ldots, 2k-1,
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{1}{1 - x_1} \frac{\partial}{\partial y_1}, \\
W_i &= \frac{\partial}{\partial x_i} && \textit{for} \quad i = 1, \ldots, k, \\
W_j &= (1 - x_1) \frac{\partial}{\partial y_{j-k+1}} - x_{j-k+1} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+1, \ldots, 2k-1
\end{aligned}
$$

*if* $\dim M = 2k$ *or on* $\{q = (z, x, y) \in R^{2k+1}\colon z \geq 0\}$, *where*

$$
\begin{aligned}
V &= \frac{1}{1 + x_1} \frac{\partial}{\partial y_1}, \\
W_1 &= (1 + x_1) \frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}, \\
W_i &= \frac{\partial}{\partial x_{i-1}} && \textit{for} \quad i = 2, \ldots, k+1, \\
W_j &= (1 + x_1) \frac{\partial}{\partial y_{j-k}} - x_{j-k} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+2, \ldots, 2k,
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{1}{1+x_1}\frac{\partial}{\partial y_1}, \\
W_1 &= (1+x_1)\frac{\partial}{\partial z} + \frac{\partial}{\partial y_1}, \\
W_i &= \frac{\partial}{\partial x_{i-1}} && \textit{for} \quad i = 2, \dots, k+1, \\
W_j &= (1+x_1)\frac{\partial}{\partial y_{j-k}} - x_{j-k}\frac{\partial}{\partial y_1} && \textit{for} \quad j = k+2, \dots, 2k
\end{aligned}
$$

*if* $\dim M = 2k+1$.

Manifolds with boundary appear naturally in control theory problems. We consider the following example [BC], [Z1].

**Example 1.1.** We consider a simple model of an electrically heated oven. It consists of a jacket, with a coil directly heating the jacket, and an interior part.

If we assume that, at an arbitrary moment $t \geq 0$, temperatures in the jacket and in the interior part are uniformly distributed and that the flow of heat through a surface is proportional to the area of the surface and to the difference of temperature between the separated media, we obtain

$$
\begin{aligned}
c_1 \frac{dT_1}{dt} &= u - (T_1 - T_2)a_1 r_1 - (T_1 - T_0)a_2 r_2, \\
c_2 \frac{dT_2}{dt} &= (T_1 - T_2)a_1 r_1,
\end{aligned}
$$

where $T_0$ denotes the outside temperature, $T_1(t)$, $T_2(t)$ denote the temperatures in the jacket and in the interior part at the moment $t \geq 0$, $u(t)$ denotes the intensity of the heat input produced by the coil at the moment $t \geq 0$, $a_1$, $a_2$ denote the area of exterior and interior surfaces of the jacket, $c_1, c_2$ denote the heat capacities of the jacket and the interior surface of the oven, and $r_1, r_2$ denote the radiation coefficients of the exterior and the interior surfaces of the jacket.

We want that the temperature in the interior part of the oven should be as close as possible to $T$ but not greater. It is natural to ask at which points on the boundary $\{(T_1, T_2)\colon T_2 = T\}$ the system is locally structurally stable on a manifold with boundary $\{(T_1, T_2)\colon T_2 \leq T\}$. From Propositions 1.1 and 2.1 and Theorem 1.2 it is locally structurally stable at points on the boundary, which satisfy $T_1 \neq T_2 = T$.

The paper is organized as follows. In Section 2 we classify 1-forms on a $2k$-dimensional manifold with boundary. We find the normal form for a non-degenerate 1-form $\alpha$ such that a kernel of a $(2k-1)$-form $\alpha \wedge (d\alpha)^{k-1}$ is transversal to the boundary. We also find the normal forms of the corresponding structurally $\partial M$-stable hypersurface systems. We prove that the classification of generic degenerate 1-forms is equivalent to the classification of smooth functions on the boundary by diffeomorphisms preserving a contact form on the boundary.

In Section 3 we classify 1-forms on a $(2k+1)$-dimensional manifold with boundary. We find the normal form for a nondegenerate 1-form $\alpha$ such that a kernel of a $2k$-form $(d\alpha)^k$ is transversal to the boundary. We also find the normal forms of the corresponding structurally $\partial M$-stable hypersurface systems. We prove that the classification of generic degenerate 1-forms is equivalent to the classification of smooth functions on the boundary by diffeomorphisms preserving a nondegenerate 1-form on the boundary. In Section 4 we prove that degenerate 1-forms are not structurally $\partial M$-stable. We also show that a nondegenerate 1-form is not structurally $\partial M$-stable if the kernel of the corresponding $(\dim M - 1)$-form is not transversal to the boundary. Then we prove the main theorems on local structural $\partial M$-stability of 1-forms and nonlinear hypersurface systems. In this paper all objects are smooth ($C^\infty$).

We denote

$$\pi\colon R^n \to \{x_1 = 0\}, \qquad \pi(x_1, x_2, \ldots, x_n) = (0, x_2, \ldots, x_n),$$
$$\imath\colon \{x_1 = 0\} \to R^n, \qquad \imath(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n).$$

We need two simple lemmas [M] in the following sections.

**Lemma 1.1.** *Let $\tau$ be a k-form on $R^n$. If $\tau$ satisfies the following conditions,* $(\partial/\partial x_1)\rfloor\tau = 0$, $(\partial/\partial x_1)\rfloor\, d\tau = 0$, *then* $\tau = \pi^*\imath^*\tau$.

**Lemma 1.2.** *Let $\tau$ be a k-form on $R^n$. If $\tau$ satisfies the following conditions,* $(\partial/\partial x_1)\rfloor\tau = 0$, $(\partial/\partial x_1)\rfloor\, d\tau = f\tau$, *then* $\tau = g\pi^*\imath^*\tau$, *where $f$, $g$ are smooth functions on $R^n$ and* $g|_{\{x_1=0\}} = 1$.

## 2. 1-Forms and Nonlinear Hypersurface Systems on an Even-Dimensional Manifold with Boundary

Let $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_k)$ be a coordinate system on $R^{2k}$. Throughout this section, $M$ denotes a germ at 0 of the following set:

$$\{(x, y) \in R^{2k}\colon x_1 \geq 0\}.$$

Let $\alpha$ be a germ of a smooth differential 1-form on $R^{2k}$ at 0. First we prove that generic nondegenerate 1-forms are structurally $\partial M$-stable and we find a normal form of these 1-forms.

**Proposition 2.1.** *If $\alpha$ satisfies the following conditions*:

1. $(d\alpha)_0^k \neq 0$,
2. $\alpha_0 \neq 0$,
3. *a germ of a smooth vector field $X$ at* 0, *which satisfies the following condition*,

$$X\rfloor(d\alpha)^k = \alpha \wedge (d\alpha)^{k-1},$$

   *is transversal to $\partial M$ at* 0,

*then* $\alpha$ *is* $G_M$*-equivalent to one and only one of the following two germs of* 1*-forms at* 0:

$$\alpha^{\pm} = (1 \pm x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i. \tag{3}$$

**Proof.** From condition 3 we conclude that

$$\imath_{\partial M}^{*}\alpha \wedge (d\imath_{\partial M}^{*}\alpha)_0^{k-1} \neq 0,$$

where $\imath_{\partial M}\colon \partial M \hookrightarrow R^{2k}$ stands for the canonical inclusion. Therefore $\imath_{\partial M}^{*}\alpha$ is a contact form on $\partial M$. By the Darboux theorem, $\alpha$ can be reduced to such a form that

$$\imath_{\partial M}^{*}\alpha = dy_1 + \sum_{i=2}^{k} x_i\, dy_i.$$

On the other hand, $X$ can be reduced to $\pm\partial/\partial x_1$ by an element of $G_M$ which is the identity on $\partial M$. From condition 3 we have (see [M])

$$\frac{\partial}{\partial x_1} \rfloor \alpha = 0, \qquad \frac{\partial}{\partial x_1} \rfloor\, d\alpha = f\alpha,$$

where $f$ denotes a function-germ on $R^{2k}$ at 0. Therefore,

$$\alpha = h\left( dy_1 + \sum_{i=2}^{k} x_i\, dy_i \right),$$

where $h$ is a function-germ on $R^{2k}$ at 0 such that $h|_{\partial M} = 1$ by Lemma 1.2. We conclude from condition 1 that $(\partial h/\partial x_1)(0) \neq 0$, hence the following map,

$$\Psi(x, y) = \left( \frac{|(\partial h/\partial x_1)(0)|}{(\partial h/\partial x_1)(0)} (h(x, y) - 1), h(x, y)x_2, \ldots, h(x, y)x_k, y_1, \ldots, y_k \right),$$

is an element of $G_M$, and finally $\Psi^{*}\alpha = (1 \pm x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i$.

Suppose the germs $\alpha^{+}$ and $\alpha^{-}$ are $G_M$-equivalent. It is easily seen that $\alpha^{-}$ is $G_M$-equivalent to $(-1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i$. Then we could find $\Phi \in G_M$ such that $\Phi_{*}(X^{-}) = X^{+}$, which is impossible because $X^{\pm}(0) = \pm(\partial/\partial x_1)(0)$. ∎

Now we consider generic degenerate 1-forms.

**Proposition 2.2.** *If* $\alpha$ *satisfies the following conditions*:

1. $(d\alpha)_0^{k} = 0$,
2. $\alpha \wedge (d\alpha)_0^{k-1} \neq 0$,
3. $S = \{(x, y) \in R^{2k}\colon (d\alpha)_{(x,y)}^{k} = 0\}$ *is a germ of a regular hypersurface at* $0 \in \partial M$,
4. *a germ of a smooth vector field* $X$ *at* 0, *which satisfies the following,*

$$X \rfloor \alpha \wedge (d\alpha)^{k-1} = 0, \qquad X(0) \neq 0,$$

*is transversal to* $\partial M$ *and* $S$ *at* 0, *then* $\alpha$ *is* $G_M$*-equivalent to a following germ of* 1*-form at* 0,

$$\alpha_f^{\pm} = \left(1 \pm \tfrac{1}{2}(x_1 - f)^2\right)\left(dy_1 + \sum_{i=2}^{k} x_i\, dy_i\right), \tag{4}$$

*where* $f$ *is a function-germ at* 0 *which does not depend on* $x_1$, $f(0) = 0$.

*Germs* $\alpha_f^{\pm}$ *and* $\alpha_g^{\pm}$ *are* $G_M$*-equivalent if and only if they have the same index* $\pm$ ($+$ *or* $-$) *and there exists a germ of a diffeomorphism*

$$\Phi\colon (R^{2k-1}, 0) \to (R^{2k-1}, 0),$$

*which preserves the contact form* $dy_1 + \sum_{i=2}^{k} x_i\, dy_i$ *and*

$$f = g \circ \Phi.$$

**Proof.** By Martinet's results [M], there exists a diffeomorphism-germ $\Phi\colon (R^{2k}, 0) \to (R^{2k}, 0)$, such that

$$\Phi^*\alpha = \left(1 \pm \frac{x_1^2}{2}\right) dy_1 + \sum_{i=2}^{k} x_i\, dy_i.$$

Hence $(\Phi^{-1})_{*0}(X)(0) = a(\partial/\partial x_1)(0)$, where $a \in R$, $a \neq 0$. Therefore $\Phi(\partial M) = \{(x, y) \in R^{2k}\colon x_1 = h(x_2, \ldots, x_n, y_1, \ldots, y_n)\}$, where $h$ is a function-germ, $h(0) = 0$. Thus, by $\Psi \circ \Phi \in G_M$, where

$$\Psi(x, y) = (\pm x_1 + h, (1 \pm \tfrac{1}{2}(\pm x_1 + h)^2)x_2, \ldots, (1 \pm \tfrac{1}{2}(\pm x_1 + h)^2)x_n, y_1, \ldots, y_n),$$

$\alpha$ can be reduced to $\alpha_f^{\pm}$.

Germs $\alpha_f^{+}$ and $\alpha_g^{-}$ are not $G_M$-equivalent, because they are not equivalent on a manifold without boundary [M].

Assume $\alpha_f^{\pm} = \Theta^*\alpha_g^{\pm}$ and $\Theta \in G_M$. $X_f^{\pm} = a(\partial/\partial x_1)$ and $X_g^{\pm} = b(\partial/\partial x_1)$, where $a, b$ are function-germs, $a(0) \neq 0$, $b(0) \neq 0$. Therefore $\Theta(x, y) = (t(x, y), \theta(x_2, \ldots, x_n, y))$, where $t$ is a function-germ on $R^{2k}$ and $\theta$ is a diffeomorphism-germ of $R^{2k-1}$.

$$\Theta(\{(x, y) \in R^{2k}\colon x_1 = f\}) = \{(x, y) \in R^{2k}\colon x_1 = g\}.$$

Hence $\theta$ preserves a germ $dy_1 + \sum_{i=2}^{k} x_i\, dy_i$ and $(x_1 - f)^2 = (t - g \circ \Theta)^2$. However,

$$t(\{(x, y) \in R^{2k}\colon x_1 = 0\}) = 0,$$

$$t(\{(x, y) \in R^{2k}\colon x_1 \geq 0\}) \geq 0,$$

because $\Theta \in G_M$. Therefore $t(x, y) = x_1$ and $f = g \circ \Phi$. ■

Let $q = (x, y) \in R^{2k}$. From Propositions 1.1 and 2.1 we obtain

**Corollary 2.1.** *The following two hypersurface systems,*

$$\dot{q} = V^{\pm}(q) + W^{\pm}(q)u,$$

*where*

$$V^{\pm} = \frac{1}{1 \pm x_1} \frac{\partial}{\partial y_1},$$

$$W_i^{\pm} = \frac{\partial}{\partial x_i} \qquad \textit{for} \quad i = 1, \ldots, k,$$

$$W_j^{\pm} = (1 \pm x_1) \frac{\partial}{\partial y_{j-k+1}} - x_{j-k+1} \frac{\partial}{\partial y_1} \qquad \textit{for} \quad j = k+1, \ldots, 2k-1,$$

*are structurally* $\partial M$*-stable at* $0 \in \partial M$. *They are not* $G_M$*-feedback equivalent.*

## 3. 1-Forms and Nonlinear Hypersurface Systems on an Odd-Dimensional Manifold with Boundary

Let $(z, x, y) = (z, x_1, \ldots, x_k, y_1, \ldots, y_k)$ be a coordinate system on $R^{2k+1}$. Throughout this section, $M$ denotes a germ at 0 of the following set:

$$\{(z, x, y) \in R^{2k+1}: z \geq 0\}.$$

Let $\alpha$ be a germ of a smooth differential 1-form on $R^{2k+1}$ at 0. First we prove that generic nondegenerate 1-forms are structurally $\partial M$-stable and we find a normal form of these 1-forms.

**Proposition 3.1.** *If* $\alpha$ *satisfies the following conditions*:

1. $\alpha \wedge (d\alpha)_0^k \neq 0$,
2. *a germ of a smooth vector field* $X$ *at* 0, *which satisfies the following condition*,

$$X \rfloor \alpha \wedge (d\alpha)^k = (d\alpha)^k,$$

   *is transversal to* $\partial M$ *at* 0,
3. $\iota_{\partial M}^* \alpha_0 \neq 0$, *where* $\iota_{\partial M}: \partial M \hookrightarrow R^{2k+1}$ *is the canonical inclusion*,

*then* $\alpha$ *is* $G_M$*-equivalent to one and only one of the two germs of* 1*-forms at* 0:

$$\alpha^{\pm} = \pm dz + dy_1 + \sum_{i=1}^{k} x_i \, dy_i. \tag{5}$$

**Proof.** $\alpha$ is a germ of a contact form on $R^{2k+1}$. Therefore there exists a diffeomorphism-germ $\Phi: (R^{2k+1}, 0) \to (R^{2k+1}, 0)$ such that $\Phi^* \alpha = dz + \sum_1^k x_i \, dy_i$ and $\Phi(\partial M) = \{(z, x, y) \in R^{2k+1}: z = f(x, y)\}$, where $f$ is a function-germ at $0 \in R^{2k}$. Then $\alpha$ is $G_M$-equivalent to

$$\pm dz + df + \sum_{i=1}^{k} x_i \, dy_i.$$

We consider a germ $\tilde{\alpha} = \alpha \mp dz$. $\iota_{\partial M}^* \tilde{\alpha}_0 \neq 0$ and $(d\iota_{\partial M}^* \tilde{\alpha})_0^k \neq 0$, because $\iota_{\partial M}^* \tilde{\alpha} = \iota_{\partial M}^* \alpha$. Hence there exists $\Psi \in G_M$ such that $\Psi(z, x, y) = (z, \psi(x, y))$ and $\iota_{\partial M} \Psi^* \tilde{\alpha} =$

$dy_1 + \sum_{i=1}^{k} x_i\, dy_i$. It is easily seen that

$$\frac{\partial}{\partial z}\Big\rfloor \tilde{\alpha} = 0, \qquad \frac{\partial}{\partial z}\Big\rfloor\, d\tilde{\alpha} = \tilde{\alpha}.$$

Thus $\Psi^* \tilde{\alpha} = dy_1 + \sum_{i=1}^{k} x_i\, dy_i$, by Lemma 1.1. Hence $\Psi^*\alpha = \alpha^{\pm}$.

We suppose for a moment that $\alpha^+$ is $G_M$-equivalent to $\alpha^-$. Hence there exists $\Theta \in G_M$ such that $\Theta_*(X^-) = X^+$. However, $X^{\pm} = \pm\partial/\partial z$, which is a contradiction. ∎

Now we consider generic degenerate 1-forms.

**Proposition 3.2.** *If $\alpha$ satisfies the following conditions*:

1. $\alpha \wedge (d\alpha)_0^k = 0$,
2. $(d\alpha)_0^k \neq 0$,
3. $S = \{(z, x, y) \in R^{2k} \colon \alpha \wedge (d\alpha)_{(z,x,y)}^k = 0\}$ *is a germ of a regular hypersurface at* $0 \in \partial M$,
4. *a germ of a smooth vector field $X$ at* 0, *which satisfies the following*,
$$X \rfloor (d\alpha)^k = 0, \qquad X(0) \neq 0,$$
*is transversal to $\partial M$ and $S$ at* 0,
5. $\imath_S^* \alpha_0 \neq 0$, *where $\imath_S$: $S \hookrightarrow R^{2k+1}$ is the canonical inclusion*,

*then $\alpha$ is $G_M$-equivalent to a following germ of* 1*-form at* 0:

$$\alpha_f^{\pm} = \pm(z - f)\, d(z - f) + (1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i, \tag{6}$$

*where $f$ is a function-germ at* 0 *which does not depend on $z$*, $f(0) = 0$.

*Germs $\alpha_f^{\pm}$ and $\alpha_g^{\pm}$ are $G_M$-equivalent if and only if they have the same index $\pm$ ($+$ or $-$) and there exists a germ of a diffeomorphism*

$$\Phi\colon (R^{2k}, 0) \to (R^{2k}, 0)$$

*which preserves the form $(1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i$ and*

$$f = g \circ \Phi.$$

**Proof.** By Martinet's results [M] there exists a diffeomorphism-germ $\Phi\colon (R^{2k+1}, 0) \to (R^{2k+1}, 0)$, such that

$$\Phi^*\alpha = \pm z\, dz + (1 + x_1)\, dy_1 + \sum_{i=2}^{k} x_i\, dy_i.$$

Hence $(\Phi^{-1})_{*0}(X)(0) = a(\partial/\partial z)(0)$, where $a \in R$, $a \neq 0$. Therefore $\Phi(\partial M) = \{(z, x, y) \in R^{2k+1} \colon z = h(x, y)\}$, where $h$ is a function-germ, $h(0) = 0$. Thus, by $\Psi \circ \Phi \in G_M$, where

$$\Psi(x, y) = (\pm z + h, x, y),$$

$\alpha$ can be reduced to $\alpha_f^{\pm}$.

Germs $\alpha_f^+$ and $\alpha_g^-$ are not $G_M$-equivalent, because they are not equivalent on a manifold without boundary [M].

Assume $\alpha_f^\pm = \Theta^* \alpha_g^\pm$ and $\Theta \in G_M$. $X_f^\pm = a(\partial/\partial z)$ and $X_g^\pm = b(\partial/\partial z)$, where $a, b$ are function-germs, $a(0) \neq 0, b(0) \neq 0$. Therefore $\Theta(x, y) = (t(z, x, y), \theta(x, y))$, where $t$ is a function-germ on $R^{2k+1}$ and $\theta$ is a diffeomorphism-germ of $R^{2k}$.

$$\Theta(\{(z, x, y) \in R^{2k+1}\colon z = f\}) = \{(z, x, y) \in R^{2k+1}\colon z = g\}.$$

Hence, $\theta$ preserves a germ $(1 + x_1)\, dy_1 + \sum_2^k x_i\, dy_i$ and $(z - f)^2 = (t - g \circ \Theta)^2$. However,

$$t(\{(x, y) \in R^{2k}\colon z = 0\}) = 0,$$

$$t(\{(x, y) \in R^{2k}\colon z \geq 0\}) \geq 0,$$

because $\Theta \in G_M$. Therefore $t(z, x, y) = z$ and $f = g \circ \Phi$. ■

Let $q = (z, x, y) \in R^{2k+1}$. From Propositions 1.1 and 3.1 we obtain

**Corollary 3.1.** *The following two hypersurface systems,*

$$\dot{q} = V^\pm(q) + W^\pm(q)u,$$

*where*

$$V^\pm = \frac{1}{1 + x_1} \frac{\partial}{\partial y_1},$$

$$W_1^\pm = (1 + x_1) \frac{\partial}{\partial z} \mp \frac{\partial}{\partial y_1},$$

$$W_i^\pm = \frac{\partial}{\partial x_{i-1}} \qquad \textit{for} \quad i = 2, \ldots, k + 1,$$

$$W_j^\pm = (1 + x_1) \frac{\partial}{\partial y_{j-k}} - x_{j-k} \frac{\partial}{\partial y_1} \qquad \textit{for} \quad j = k + 2, \ldots, 2k,$$

*are structurally $\partial M$-stable at $0 \in \partial M$. They are not $G_M$-feedback equivalent.*

## 4. Structural Stability

By the Thom Transversality Theorem [GG], [AVG], which also holds for differential forms, closed differential forms, and distribution of corank 1 [M], a codimension of an orbit of a structurally stable element is greater than the dimension of the manifold [GT2].

We prove that degenerate 1-forms are not structurally $\partial M$-stable.

**Proposition 4.1.** *Germs* (4) *and* (6) *are not $\partial M$-stable.*

**Proof.** By Propositions 2.2 and 3.2 this problem is equivalent to the problem of stability of the pair $(\beta, f)$, where $\beta$ is a nondegenerate 1-form on $R^n$ and $f$ is a function-germ on $R^n$.

We use the method described in [GT1] and [GT2]. Let $J^l(D^1(R^n))$ be the space of $l$-jets of 1-forms and let $J^l(R^n)$ be the space of $l$-jets of smooth functions on $R^n$ to $R$. Let $O^l_{(j^l\beta, j^l f)}$ be an orbit of a pair $(j^l\beta, j^l f) \in J^l(D^1(R^n)) \times J^l(R^n)$, where $j^l\beta$ is an $l$-jet of $\beta$ and $j^l f$ is an $l$-jet of $f$, under the action of the group of invertible $(l+1)$-jets $Diff_0^{l+1}(R^n)$.

Then

$$\begin{aligned}\operatorname{codim} O^l_{(j^l\beta, J^l f)} &= \dim(J^l(D^1(R^n)) \times J^l(R^n)) - \dim O^l_{(j^l\beta, j^l f)}\\ &\geq \dim(J^l(D^1(R^n)) \times J^l(R^n)) - \dim Diff_0^{l+1}(R^n)\\ &= (n+1)\binom{n+l}{n} - n\binom{n+l+1}{n}\\ &= \frac{1}{n!}l^n + w_{n-1}(l),\end{aligned}$$

where $w_{n-1}(l)$ is a polynomial of degree at most $n-1$ in $l$. Therefore $\operatorname{codim} O^l_{(j^l\beta, j^l f)} > n$ for some $l$ large enough and consequently the forms are not stable. ∎

Now we prove that nondegenerate 1-forms which do not satisfy condition 3 of Proposition 2.1 (condition 2 of Proposition 3.1 respectively) are not structurally $\partial M$-stable. First we need two lemmas.

**Lemma 4.1.** *Let $\alpha = dz + \sum_{i=1}^k x_i\, dy_i$ be a germ at 0 of the standard contact form on $R^{2k+1}$, and let $\theta = \sum_{i=1}^k x_i\, dy_i$ be a germ of a 1-form on $R^{2k}$ such that $d\theta$ is the standard symplectic form on $R^{2k}$. If $\Phi$: $(R^{2k+1}, 0) \to (R^{2k+1}, 0)$ is a germ of diffeomorphism such that $\Phi^\star\alpha = \alpha$, then*

$$\Phi(z, x, y) = (z - h(x, y), \Psi(x, y)),$$

*where $\Psi$: $(R^{2k}, 0) \to (R^{2k}, 0)$ is a germ of a symplectomorphism $\Psi^\star\, d\theta = d\theta$ and $h$ is a function-germ on $R^{2k}$ such that $\Psi^\star\theta = \theta + dh$ and $h(0) = 0$.*

**Proof.** Notice that $\Phi_\star(\partial/\partial z) = \partial/\partial z$, because $(\partial/\partial z)\rfloor\alpha \wedge (d\alpha)^k = (d\alpha)^k$. Thus $\Phi(z, x, y) = (z - h(x, y), \Psi(x, y))$. However, $d\alpha = \pi^\star\, d\theta$, where $\pi$: $R^{2k+1} \ni (z, x, y) \to (x, y) \in R^{2k}$. Therefore $\Psi^\star\, d\theta = d\theta$ and $\Phi^\star\alpha = dz - dh + \pi^\star\Psi^\star\theta = dz + \pi^\star\theta$. ∎

**Lemma 4.2.** *Let $\alpha = (1 + x_1)(dy_1 + \sum_{i=2}^k x_i\, dy_i)$ be a germ at 0 of 1-form on $R^{2k}$, and let $\theta = dy_1 + \sum_{i=2}^k x_i\, dy_i$ be a germ of the contact form on $R^{2k-1}$. If $\Phi$: $(R^{2k}, 0) \to (R^{2k}, 0)$ is a germ of diffeomorphism such that $\Phi^\star\alpha = \alpha$, then*

$$\Phi(x, y) = (g(x_2, \ldots, x_k, y)x_1 + g(x_2, \ldots, x_k, y) - 1, \Psi(x_2, \ldots, x_k, y)),$$

*where $\Psi$: $(R^{2k-1}, 0) \to (R^{2k-1}, 0)$ is a germ of the contactomorphism such that $\Psi^\star\theta = h\theta$ and $g$ is a function-germ on $R^{2k-1}$ such that $g = 1/h$ and $g(0) = 1$.*

**Proof.** Notice that $\Phi_\star((1 + x_1)(\partial/\partial x_1)) = (1 + x_1)(\partial/\partial x_1)$, because $(1 + x_1) \cdot (\partial/\partial x_1)\rfloor\alpha \wedge (d\alpha)^k = (d\alpha)^k$. Thus $\Phi(x, y) = (g(x, y)x_1 + r(x_2, \ldots, x_n, y),$

$\Psi(x_2, \dots, x_n, y))$. However,

$$\Phi^\star \alpha = (1 + g x_1 + r)\pi^\star \Psi^\star \theta = (1 + x_1)\pi^\star \theta, \tag{7}$$

where $\pi$: $R^{2k} \ni (x, y) \to (x_2, \dots, x_k, y) \in R^{2k-1}$. Thus $\Psi^\star \theta = (1/(1+r))\theta$. Let $h = 1/(1+r)$ then from (7) we have $g = 1/h$. ∎

Now we use the same method as in the proof of Proposition 4.1.

**Proposition 4.2.** *Let $\alpha$ be a germ of* 1*-form at* 0 *on* $R^{2k}$ *and let* $M$ *be a germ of the following set*:

$$\{(x, y) \in R^{2k}: x_1 \geq 0\}.$$

*If $\alpha$ satisfies the following conditions*:

1. $(d\alpha)_0^k \neq 0$,
2. $\alpha_0 \neq 0$,
3. *a germ of a smooth vector field $X$ at* 0, *which satisfies the following*,

$$X \rfloor (d\alpha)^k = \alpha \wedge (d\alpha)^{k-1},$$

*is tangent to $\partial M$ at* 0,

*then $\alpha$ is not structurally $\partial M$-stable at* 0.

**Proof.** From (1) and (2), there exists a diffeomorphism-germ $\Phi$: $(R^{2k}, 0) \to (R^{2k}, 0)$ [M] such that

$$\Phi^\star \alpha = (1 + x_1)\left( dy_1 + \sum_{i=2}^{n} x_i \, dy_i \right).$$

$\Phi$ does not have to preserve $\partial M$. Let

$$\Phi^{-1}(\partial M) = \{(x, y) \in R^{2k}: f(x, y) = 0\},$$

where $f$ is a function-germ on $R^{2k}$ at 0, $f(0) = 0$. From (3), we have $(L_X f)|_0 = (\partial f/\partial x_1)|_0 = 0$. By genericity we may assume that $(L_X^2 f)|_0 = (\partial^2 f/\partial x_1^2)|_0 \neq 0$. By the Malgrange Preparation Theorem [AVG], [GG] we get

$$\Phi^{-1}(\partial M) = \{(x, y) \in R^{2k}: x_1^2 + p(x, y)x_1 + q(x, y) = 0\},$$

where $p, q$ are function-germs on $R^{2k}$ at 0, which do not depend on $x_1$ and $p(0) = q(0) = 0$.

Now we consider the action on $\Phi^{-1}(\partial M)$ by the group of diffeomorphism-germs which preserve $\Phi^\star \alpha$. By Lemma 4.2 we reduce the above action to the following action of contactomorphism-germs on $R^{2k-1}$ on smooth mapping-germs $R^{2k-1} \to R^2$:

$$\Psi * (p, q) = (h(p \circ \Psi) + 2(1-h), h^2(q \circ \Psi) + h(1-h)(p \circ \Psi) + (1-h)^2), \tag{8}$$

where $(p, q)$: $R^{2k-1} \to R^2$ is a smooth mapping-germ and $\Psi$ is a contactomorphism-germ such that

$$\Psi^\star\left(dx_1 + \sum_{i=2}^{k} x_i\, dy_i\right) = h\left(dx_1 + \sum_{i=2}^{k} x_i\, dy_i\right),$$

where $h$ is a function-germ on $R^{2k-1}$ at 0, $h(0) = 1$. So we have to classify a germ of $(2k-2)$-dimensional distribution $\beta = 0$ and the mapping-germ $(p, q)$ under the above action of the group of diffeomorphism-germs.

Now we use the method described in [GT1] and [GT2].

Let $J^l(Ph^{2k-2}(R^{2k-1}))$ be the space of $l$-jets of smooth $(2k-2)$-dimensional distributions on $R^{2k-1}$, let $J^l(R^{2k-1}, R^2)$ be the space of $l$-jets of smooth mappings $R^{2k-1} \to R^2$, and let $O^l_{(j^l(\beta=0), j^l(p,q))}$ be an orbit of $(j^l(\beta = 0), j^l(p, q)) \in J^l(Ph^{2k-2}(R^{2k-1})) \times J^l(R^{2k-1}, R^2)$, where $j^l(\beta = 0)$ is an $l$-jet of a smooth $(2k-2)$-dimensional distribution on $R^{2k-1}$ and $j^l(p, q)$ is an $l$-jet of a mapping $(p, q)$, under the action by pullback on $j^l(\beta = 0)$ and the action defined by (8) on $j^l(p, q)$ of the group of invertible $(l+1)$-jets $\mathit{Diff}_0^{l+1}(R^{2k-1})$.

Then

$$\dim J^l(Ph^{2k-2}(R^{2k-1})) = (2k-2)\binom{2k-1+l}{2k-1},$$

$$\dim O^l_{(j^l_\omega, j^l(p,q))} \le \dim \mathit{Diff}_0^{l+1}(R^{2k-1}) = (2k-1)\binom{2k-1+l+1}{2k-1},$$

and

$$\begin{aligned}
\operatorname{codim} O^l_{(j^l(\beta=0), j^l(p,q))} &= \dim J^l(Ph^{2k-2}(R^{2k-1})) \times J^l(R^{2k-1}, R^2) \\
&\quad - \dim O^l_{(j^l(\beta=0), j^l(p,q))} \\
&\ge (2k-2)\binom{2k-1+l}{2k-1} + 2\binom{2k-1+l}{2k-1} \\
&\quad - (2k-1)\binom{2k+l}{2k-1} \\
&= \frac{1}{(2k-1)!}(2k-2+2-2k+1)l^{2k-1} + w_{2k-2}(l) \\
&= \frac{1}{(2k-1)!} l^{2k-1} + w_{2k-2}(l),
\end{aligned}$$

where $w_{2k-2}(l)$ is a polynomial of degree at most $2k-2$ in $l$. Therefore $\operatorname{codim} O^l_{(j^l(\beta=0), j^l(p,q))} > 2k-1$ for some $l$ large enough and $\alpha$ is not $\partial M$-stable. ■

**Proposition 4.3.** *Let $\alpha$ be a germ of 1-form at 0 on $R^{2k+1}$ and let $M$ be a germ of the following set*:

$$\{(z, x, y) \in R^{2k+1} \colon z \ge 0\}.$$

*If $\alpha$ satisfies the following conditions*:

1. $\alpha \wedge (d\alpha)_0^k \neq 0$,
2. *a germ of a smooth vector field* $X$ *at* $0$, *which satisfies the following*,
$$X \rfloor \alpha \wedge (d\alpha)^k = (d\alpha)^k,$$
*is tangent to* $\partial M$ *at* $0$,
3. $\imath_{\partial M}^{*}\alpha_0 \neq 0$, *where* $\imath_{\partial M}\colon \partial M \hookrightarrow R^{2k+1}$ *is the canonical inclusion*,

*then* $\alpha$ *is not structurally* $\partial M$*-stable at* $0$.

**Proof.** $\alpha$ is a germ of contact form. By the Darboux theorem there exists a diffeomorphism-germ $\Phi$: $(R^{2k+1}, 0) \to (R^{2k+1}, 0)$ such that

$$\Phi^{\star}\alpha = dz + \sum_{i=1}^{n} x_i\, dy_i.$$

$\Phi$ does not have to preserve $\partial M$. Let

$$\Phi^{-1}(\partial M) = \{(z, x, y) \in R^{2k+1}\colon f(z, x, y) = 0\},$$

where $f$ is a function-germ on $R^{2k+1}$ at 0, $f(0) = 0$. From (2), we have $(L_X f)|_0 = (\partial f/\partial z)|_0 = 0$. By genericity we may assume that $(L_X^2 f)|_0 = (\partial^2 f/\partial z^2)|_0 \neq 0$. By the Malgrange Preparation Theorem [AVG], [GG] we get

$$\Phi^{-1}(\partial M) = \{(z, x, y) \in R^{2k+1}\colon z^2 + p(x, y)z + q(x, y) = 0\},$$

where $p$, $q$ are function-germs on $R^{2k+1}$ at 0, which do not depend on $z$ and $p(0) = q(0) = 0$. Now we consider the action on $\Phi^{-1}(\partial M)$ by the group of diffeomorphism-germs which preserve $\Phi^{\star}\alpha$. By Lemma 4.1 we reduce the above action to the following action of symplectomorphism-germs on $R^{2k}$ on smooth mapping-germs $R^{2k} \to R^2$:

$$\Psi * (p, q) = (p \circ \Psi - 2h, q \circ \Psi - h(p \circ \Psi) + h^2), \tag{9}$$

where $(p, q)$: $R^{2k} \to R^2$ is a smooth mapping-germ and $\Psi$ is a symplectomorphism-germ such that

$$\Psi^{\star}\omega = \omega = \sum_{i=1}^{k} dx_i \wedge dy_i$$

and

$$\Psi^{\star}\left(\sum_{i=1}^{k} x_i\, dy_i\right) = \sum_{i=1}^{k} x_i\, dy_i + dh,$$

$h$ is a function-germ on $R^{2k}$ at 0, $h(0) = 0$. So we have to classify a germ of the closed form $\omega$ and the mapping-germ $(p, q)$ under the above action of the group of diffeomorphism-germs.

Now we use the method described in [GT1] and [GT2]. Let $J^l(CD^2(R^{2k}))$ be the space of $l$-jets of smooth closed 2-forms on $R^{2k}$, let $J^l(R^{2k}, R^2)$ be the space of $l$-jets of smooth mappings $R^{2n} \to R^2$ and let $O^l_{(j^l\omega, j^l(p,q))}$ be an orbit of $(j^l\omega, j^l(p, q)) \in J^l(CD^2(R^{2k})) \times J^l(R^{2k}, R^2)$, where $j^l\omega$ is an $l$-jet of a closed 2-

form $\omega$ and $j^l(p,q)$ is an $l$-jet of a mapping $(p,q)$, under the action by pullback on $j^l\omega$ and the action defined by (9) on $j^l(p,q)$ of the group of invertible $(l+1)$-jets $Diff_0^{l+1}(R^{2k})$.

Then

$$\dim J^l(CD^2(R^{2k})) = \sum_{i=0}^{2k-2}(-1)^i J^{l-i}(D^{2+i}) = \sum_{i=0}^{2k-2}(-1)^i\binom{2k}{2+i}\binom{2k+l-i}{2k}$$

(see [GT2] for details), where $J^l(D^i)$ is a space of $l$-jets of smooth differential $i$-forms,

$$\dim O^l_{(j^l\omega, j^l(p,q))} \le \dim Diff_0^{l+1}(R^{2k}) = 2k\binom{2k+l+1}{2k},$$

and

$$\begin{aligned}\operatorname{codim} O^l_{(j^l\omega,j^l(p,q))} &= \dim J^l(CD^2(R^{2k}))\times J^l(R^{2k},R^2) - \dim O^l_{(j^l\omega,j^l(p,q))}\\ &\ge \sum_{i=0}^{2k-2}(-1)^i\binom{2k}{2+i}\binom{2k+l-i}{2k} + 2\binom{2k+l}{2k}\\ &\quad - 2k\binom{2k+l+1}{2k}\\ &= \frac{1}{(2k)!}\left(\sum_{i=0}^{2k-2}(-1)^i\binom{2k}{2+i} + 2 - 2k\right)l^{2k} + w_{2k-1}(l)\\ &= \frac{1}{(2k)!}(2k-1+2-2k)l^{2k} + w_{2k-1}(l)\\ &= \frac{1}{(2k)!}l^{2k} + w_{2k-1}(l),\end{aligned}$$

where $w_{2k-1}(l)$ is a polynomial of degree at most $2k-1$ in $l$. Therefore $\operatorname{codim} O^l_{(j^l\omega,j^l(p,q))} > 2k$ for some $l$ large enough and $\alpha$ is not $\partial M$-stable. ■

Now we obtain:

**Theorem 4.1.** *If $\alpha$ is a germ of locally structurally $\partial M$-stable* 1*-form on a manifold $M$ with boundary, then $\alpha$ satisfies the conditions of Proposition* 2.1 (*Proposition* 3.1 *respectively*) *and $\alpha$ is $G_M$-equivalent to one and only one of the following germs at* 0 *of* 1*-forms on* $\{(x,y)\in R^{2k}\colon x_1 \ge 0\}(\{(z,x,y)\in R^{2k+1}\colon z\ge 0\}$ *respectively*):

$$(1\pm x_1)\,dy_1 + \sum_{i=2}^{k} x_i\,dy_i \qquad \textit{if} \quad \dim M = 2k,$$

$$\pm dz + dy_1 + \sum_{i=1}^{k} x_i\,dy_i \qquad \textit{if} \quad \dim M = 2k+1.$$

**Proof.** First we assume that $\dim M = 2k$. If $\alpha$ is structurally $\partial M$-stable at $p\in\partial M$, then it is structurally stable (on $R^{2k}$-manifold without boundary) at $p$.

Then it must be equivalent to the Darboux model or one of the Martinet models [M], [GT2], [Z2].

If $\alpha$ is equivalent to the Darboux model, then it satisfies the following conditions: $(d\alpha)_p^k \neq 0$, $\alpha_p \neq 0$. If $\ker \alpha \wedge (d\alpha)^{k-1}$ is transversal to $\partial M$ at $p$, then $\alpha$ is $G_M$-equivalent to $(1 \pm x_1)\, dy_1 + \sum_{i=2}^k x_i\, dy_i$ (by Proposition 2.1). If the last condition is not satisfied, then $\alpha$ is not locally structurally $\partial M$-stable by Proposition 4.2.

If $\alpha$ is equivalent to one of the Martinet models, then it satisfies the following conditions [M]:

1. $(d\alpha)_p^k = 0$,
2. $\alpha \wedge (d\alpha)_p^{k-1} \neq 0$,
3. $S = \{(x, y) \in R^{2k}\colon (d\alpha)_{(x,y)}^k = 0\}$ is a germ of a regular hypersurface at $p \in \partial M$,
4. a germ of a smooth vector field $X$ at $p$, which satisfies the following,

$$X \rfloor \alpha \wedge (d\alpha)^{k-1} = 0, \qquad X(p) \neq 0,$$

is transversal to $S$ at $p$,

and by Proposition 4.1 it is not structurally $\partial M$-stable at $p \in \partial M$.

Therefore if $\alpha$ is locally $\partial M$-stable it must satisfy assumptions of Proposition 2.1.

Now we assume that $\dim M = 2k + 1$.

If $\alpha$ is structurally $\partial M$-stable at $p \in \partial M$, then it is structurally stable (on $R^{2k+1}$-manifold without boundary) at $p$. Then it must be equivalent to the Darboux model or one of the Martinet models [M], [GT2], [Z2].

If $\alpha$ is equivalent to the Darboux model, then it satisfies the following conditions: $\alpha \wedge (d\alpha)_p^k \neq 0$. If $\ker(d\alpha)^k$ is transversal to $\partial M$ at $p$, then $\alpha$ is $G_M$-equivalent to $\pm dz + dy_1 + \sum_{i=1}^k x_i\, dy_i$ (by Proposition 3.1). If the last condition is not satisfied, then $\alpha$ is not locally structurally $\partial M$-stable by Proposition 4.3.

If $\alpha$ is equivalent to one of the Martinet models, then it satisfies the following conditions [M]:

1. $\alpha \wedge (d\alpha)_p^k = 0$,
2. $(d\alpha)_p^k \neq 0$,
3. $S = \{(z, x, y) \in R^{2k+1}\colon \alpha \wedge (d\alpha)_{(z,x,y)}^k = 0\}$ is a germ of a regular hypersurface at $p$,
4. a germ of a smooth vector field $X$ at $p$, which satisfies the following,

$$X \rfloor (d\alpha)^k = 0, \qquad X(p) \neq 0,$$

is transversal to $S$ at $p$,
5. $\imath_S^* \alpha_p \neq 0$, where $\imath_S\colon S \hookrightarrow R^{2k+1}$ is the canonical inclusion,

and by Proposition 4.1 it is not structurally $\partial M$-stable at $p \in \partial M$.

Therefore if $\alpha$ is locally $\partial M$-stable it must satisfy assumptions of Proposition 3.1. ∎

From the above theorem and Corollaries 2.1 and 3.1 we obtain the complete classification of locally structurally $\partial M$-stable hypersurface systems on a manifold with boundary.

**Theorem 4.2.** *Any germ of a locally structurally $\partial M$-stable smooth hypersurface system on a manifold with boundary $M$ is feedback $G_M$-equivalent to one and only one of the following germs at* 0 *of hypersurface systems*

$$\dot{q} = V(q) + \sum_{i=1}^{\dim M - 1} u_i W_i(q)$$

*on* $\{q = (x, y) \in R^{2k} \colon x_1 \geq 0\}$, *where*

$$\begin{aligned} V &= \frac{1}{1+x_1} \frac{\partial}{\partial y_1}, \\ W_i &= \frac{\partial}{\partial x_i} && \textit{for} \quad i = 1, \ldots, k, \\ W_j &= (1+x_1) \frac{\partial}{\partial y_{j-k+1}} - x_{j-k+1} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+1, \ldots, 2k-1, \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{1-x_1} \frac{\partial}{\partial y_1}, \\ W_i &= \frac{\partial}{\partial x_i} && \textit{for} \quad i = 1, \ldots, k, \\ W_j &= (1-x_1) \frac{\partial}{\partial y_{j-k+1}} - x_{j-k+1} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+1, \ldots, 2k-1 \end{aligned}$$

*if* $\dim M = 2k$ *or on* $\{q = (z, x, y) \in R^{2k+1} \colon z \geq 0\}$, *where*

$$\begin{aligned} V &= \frac{1}{1+x_1} \frac{\partial}{\partial y_1}, \\ W_1 &= (1+x_1) \frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}, \\ W_i &= \frac{\partial}{\partial x_{i-1}} && \textit{for} \quad i = 2, \ldots, k+1, \\ W_j &= (1+x_1) \frac{\partial}{\partial y_{j-k}} - x_{j-k} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+2, \ldots, 2k, \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{1+x_1} \frac{\partial}{\partial y_1}, \\ W_1 &= (1+x_1) \frac{\partial}{\partial z} + \frac{\partial}{\partial y_1}, \\ W_i &= \frac{\partial}{\partial x_{i-1}} && \textit{for} \quad i = 2, \ldots, k+1, \\ W_j &= (1+x_1) \frac{\partial}{\partial y_{j-k}} - x_{j-k} \frac{\partial}{\partial y_1} && \textit{for} \quad j = k+2, \ldots, 2k \end{aligned}$$

*if* $\dim M = 2k + 1$.

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