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Symplectic S_μ Singularities

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Abstract. We study the local symplectic algebra of the 1-dimensional isolated complete intersection singularity of type S_μ. We use the method of algebraic restrictions to classify symplectic S_μ singularities. We distinguish these symplectic singularities by discrete symplectic invariants. We also give their geometric description.

1. Introduction

In this paper we study the symplectic classification of the 1-dimensional complete intersection singularity of type S_μ in the symplectic space $(\mathbb{R}^{2n}, \omega)$. We recall that ω is a symplectic form if ω is a smooth nondegenerate closed 2-form, and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if Φ is a diffeomorphism and $\Phi^*\omega = \omega$.

Definition 1.1. Let N_1, N_2 be germs of subsets of symplectic space $(\mathbb{R}^{2n}, \omega)$. N_1, N_2 are **symplectically equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$ such that $\Phi(N_1) = N_2$.

The problem of symplectic classification of singular curves was introduced by V. I. Arnold in [**A1**]. Arnold proved that the A_{2k} singularity of a planar curve (the orbit with respect to the standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to the symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve to the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these new symplectic invariants in terms of the local algebra's interaction with the symplectic structure and he proposed to call this interaction the **local symplectic algebra**.

In [**IJ1**] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [**IJ1**] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [**DR**]. The orbit of the action of all diffeomorphism-germs agrees with the volume-preserving orbit or splits into two volume-preserving

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orbits (in the case $\mathbb{K} = \mathbb{R}$) for germs which satisfy a special weak form of quasi-homogeneity e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights $\lambda_i \geq 0$ and $\sum_i \lambda_i > 0$.

A symplectic singularity is stably simple if it is simple and remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space. In [**K**] P. A. Kolgushkin classified stably simple symplectic singularities of parameterized curves (in the $\mathbb{C}$-analytic category). All stably simple symplectic singularities of curves are quasi-homogeneous too.

In [**DJZ2**] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets. The algebraic restriction is an equivalence class of the following relation on the space of differential k-forms:

Differential k-forms ω_1 and ω_2 have the same **algebraic restriction** to a subset N if $\omega_1 - \omega_2 = \alpha + d\beta$, where α is a k-form vanishing on N and β is a $(k-1)$-form vanishing on N.

In [**DJZ2**] a generalization of the Darboux-Givental theorem ([**AG**]) to germs of arbitrary subsets of the symplectic space was obtained. This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart of the algebraic restriction ([**DJZ2**], [**DJZ1**]). The dimension of the space of algebraic restrictions of closed 2-forms to a 1-dimensional quasi-homogeneous isolated complete intersection singularity C is equal to the multiplicity of C ([**DJZ2**]). In [**D**] it was proved that the space of algebraic restrictions of closed 2-forms to a 1-dimensional (singular) analytic variety is finite-dimensional.

In [**DJZ2**] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular a complete symplectic classification of the 1-dimensional S_5 singularity was obtained. Most of different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.

In this paper we obtain the complete symplectic classification of the isolated complete intersection singularities S_μ for $\mu > 5$ using the method of algebraic restrictions (Theorem 4.1). The $S_\mu, \mu \geq 5$ are the first singularities appearing in the classification of simple 1-dimensional isolated complete intersection singularities in the space of dimension greater than 2 obtained by Giusti ([**G**], [**AVG**]). Isolated complete intersection singularities were intensively studied by many authors (e. g. see [**L**]), because of their interesting geometric, topological and algebraic properties. In this paper we study their symplectic invariants. The group of symplectomorphism-germs is not a geometric subgroup in the sense of Damon. Therefore symplectic classification problems are interesting and require new methods. We calculate discrete symplectic invariants for symplectic S_μ singularities (Theorems 4.6 and 4.4) and we present their geometric descriptions (Theorem 4.9).

In [**DT**] following ideas from [**A1**] and [**D**] new discrete symplectic invariants - the Lagrangian tangency orders were introduced and used to distinguish symplectic singularities of simple planar curves of type $A - D - E$, symplectic S_5 and T_7 singularities.

In this paper using Lagrangian tangency orders we are able to give detailed classification of the S_μ singularity for $\mu > 5$ (Theorem 4.6) and to present a geometric description of its symplectic orbits (Theorem 4.9).

The paper is organized as follows. In Section 2 we recall the method of algebraic restrictions. In Section 3 we present discrete symplectic invariants. Symplectic classification of the S_μ singularity is studied in Section 4.

2. The method of algebraic restrictions

In this section we present basic facts on the method of algebraic restrictions. The proofs of all results of this section can be found in [**DJZ2**].

Given a germ of a non-singular manifold M denote by $\Lambda^p(M)$ the space of all germs at 0 of differential p-forms on M. Given a subset $N \subset M$ introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M) : \quad \omega(x) = 0 \text{ for any } x \in N\};$$
$$\mathcal{A}^p_0(N, M) = \{\alpha + d\beta : \quad \alpha \in \Lambda^p_N(M),\ \beta \in \Lambda^{p-1}_N(M).\}$$

The relation $\omega(x) = 0$ means that the p-form ω annihilates any p-tuple of vectors in T_xM, i.e. all coefficients of ω in some (and then any) local coordinate system vanish at the point x.

Definition 2.1. Let N be the germ of a subset of M and let $\omega \in \Lambda^p(M)$. The **algebraic restriction** of ω to N is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.

Notation. The algebraic restriction of the germ of a p-form ω on M to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. Writing $[\omega]_N = 0$ (or saying that ω has zero algebraic restriction to N) we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in \mathcal{A}^p_0(N, M)$.

Let M and $\widetilde{M}$ be non-singular equal-dimensional manifolds and let $\Phi : \widetilde{M} \to M$ be a local diffeomorphism. Let N be a subset of M. It is clear that $\Phi^*\mathcal{A}^p_0(N, M) = \mathcal{A}^p_0(\Phi^{-1}(N), \widetilde{M})$. Therefore the action of the group of diffeomorphisms can be defined as follows: $\Phi^*([\omega]_N) = [\Phi^*\omega]_{\Phi^{-1}(N)}$, where ω is an arbitrary p-form on M.

Definition 2.2. Two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_{\widetilde{N}}$ are called **diffeomorphic** if there exists the germ of a diffeomorphism $\Phi : \widetilde{M} \to M$ such that $\Phi(\widetilde{N}) = N$ and $\Phi^*([\omega]_N) = [\widetilde{\omega}]_{\widetilde{N}}$.

Remark 2.3. The above definition does not depend on the choice of ω and $\widetilde{\omega}$ since a local diffeomorphism maps forms with zero algebraic restriction to N to forms with zero algebraic restrictions to $\tilde{N}$. If $M = \widetilde{M}$ and $N = \widetilde{N}$ then the definition of diffeomorphic algebraic restrictions reduces to the following one: two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_N$ are diffeomorphic if there exists a local symmetry Φ of N (i.e. a local diffeomorphism preserving N) such that $[\Phi^*\omega]_N = [\widetilde{\omega}]_N$.

Definition 2.4. A subset N of $\mathbb{R}^m$ is quasi-homogeneous if there exists a coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{R}^m$ and positive numbers $\lambda_1, \cdots, \lambda_n$ such that for any point $(y_1, \cdots, y_m) \in \mathbb{R}^m$ and any $t > 0$ if $(y_1, \cdots, y_m)$ belongs to N then the point $(t^{\lambda_1}y_1, \cdots, t^{\lambda_m}y_m)$ belongs to N.

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.

THEOREM 2.5 (Theorem A in [**DJZ2**]). *Let N be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

Two germs of quasi-homogeneous subsets N_1, N_2 of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectomorphic if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.

Theorem 2.5 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of the algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of the zero algebraic restriction is explained by the following theorem.

THEOREM 2.6 (Theorem **B** in [**DJZ2**]). *The germ of a quasi-homogeneous set N of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N.*

PROPOSITION 2.7 (Lemma 2.20 in [**DJZ2**]). *Let $N \subset \mathbb{R}^m$. Let $W \subseteq T_0\mathbb{R}^m$ be the tangent space to some (and then any) non-singular submanifold containing N of minimal dimension within such submanifolds. If ω is the germ of a p-form with the zero algebraic restriction to N then $\omega|_W = 0$.*

The following result shows that the method of algebraic restrictions is a very powerful tool in symplectic classification of singular curves.

THEOREM 2.8 (Theorem 2 in [**D**]). *Let C be the germ of a $\mathbb{K}$-analytic curve (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Then the space of algebraic restrictions of germs of closed 2-forms to C is a finite dimensional vector space.*

By a **$\mathbb{K}$-analytic curve** we understand a subset of $\mathbb{K}^m$ which is locally diffeomorphic to a 1-dimensional (possibly singular) $\mathbb{K}$-analytic subvariety of $\mathbb{K}^m$. Germs of $\mathbb{C}$-analytic parameterized curves can be identified with germs of irreducible $\mathbb{C}$-analytic curves.

We now recall basic properties of algebraic restrictions which are useful for a description of this subset ([**DJZ2**]).

First we can reduce the dimension of the manifold we consider due to the following propositions.

If the germ of a set $N \subset \mathbb{R}^m$ is contained in a non-singular submanifold $M \subset \mathbb{R}^m$ then the classification of the algebraic restrictions to N of p-forms on $\mathbb{R}^m$ reduces to the classification of the algebraic restrictions to N of p-forms on M. At first note that the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ can be identified:

PROPOSITION 2.9. *Let N be the germ at 0 of a subset of $\mathbb{R}^m$ contained in a non-singular submanifold $M \subset \mathbb{R}^m$ and let ω_1, ω_2 be p-forms on $\mathbb{R}^m$. Then $[\omega_1]_N = [\omega_2]_N$ if and only if $\left[\omega_1|_{TM}\right]_N = \left[\omega_2|_{TM}\right]_N$.*

The following, less obvious statement, means that the *orbits* of the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ also can be identified.

PROPOSITION 2.10. *Let N_1, N_2 be germs of subsets of $\mathbb{R}^m$ contained in equal-dimensional non-singular submanifolds M_1, M_2 respectively. Let ω_1, ω_2 be two germs of p-forms. The algebraic restrictions $[\omega_1]_{N_1}$ and $[\omega_2]_{N_2}$ are diffeomorphic if and only if the algebraic restrictions $\left[\omega_1|_{TM_1}\right]_{N_1}$ and $\left[\omega_2|_{TM_2}\right]_{N_2}$ are diffeomorphic.*

To calculate the space of algebraic restrictions of 2-forms we will use the following obvious properties.

PROPOSITION 2.11. *If $\omega \in \mathcal{A}_0^k(N, \mathbb{R}^{2n})$ then $d\omega \in \mathcal{A}_0^{k+1}(N, \mathbb{R}^{2n})$ and $\omega \wedge \alpha \in \mathcal{A}_0^{k+p}(N, \mathbb{R}^{2n})$ for any p-form α on $\mathbb{R}^{2n}$.*

The next step of our calculation is the description of the subspace of algebraic restrictions of closed 2-forms. The following proposition is very useful for this step.

PROPOSITION 2.12. *Let $a_1, \dots, a_k$ be a basis of the space of algebraic restrictions of 2-forms to N satisfying the following conditions*

(1) $da_1 = \cdots = da_j = 0$,
(2) *the algebraic restrictions $da_{j+1}, \dots, da_k$ are linearly independent.*

Then $a_1, \dots, a_j$ is a basis of the space of algebraic restrictions of closed 2-forms to N.

Then we need to determine which algebraic restrictions of closed 2-forms are realizable by symplectic forms. This is possible due to the following fact.

PROPOSITION 2.13. *Let $N \subset \mathbb{R}^{2n}$. Let r be the minimal dimension of non-singular submanifolds of $\mathbb{R}^{2n}$ containing N. Let M be one of such r-dimensional submanifolds. The algebraic restriction $[\theta]_N$ of the germ of a closed 2-form θ is realizable by the germ of a symplectic form on $\mathbb{R}^{2n}$ if and only if $rank(\theta|_{T_0M}) \geq 2r - 2n$.*

Let us fix the following notations:

- $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$: the vector space consisting of the algebraic restrictions of germs of all 2-forms on $\mathbb{R}^{2n}$ to the germ of a subset $N \subset \mathbb{R}^{2n}$;
- $\left[Z^2(\mathbb{R}^{2n})\right]_N$: the subspace of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all closed 2-forms on $\mathbb{R}^{2n}$ to N;
- $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$: the open set in $\left[Z^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all symplectic 2-forms on $\mathbb{R}^{2n}$ to N.

3. Discrete symplectic invariants.

We can use some discrete symplectic invariants to characterize symplectic singularity classes. They show how far a curve N is from the closest non-singular Lagrangian submanifold.

The first invariant is a symplectic multiplicity ([**DJZ2**]) introduced in [**IJ1**] as a symplectic defect of a curve.

Let N be the germ of a subvariety of $(\mathbb{R}^{2n}, \omega)$.

DEFINITION 3.1. The **symplectic multiplicity** $\mu_{sympl}(N)$ of N is the codimension of the symplectic orbit of N in the orbit of N with respect to the action of the group of diffeomorphism-germs.

To make the definition of the symplectic multiplicity precise we present some explanations (see [**DJZ2**] for details). Throughout the paper *by a variety in* $\mathbb{R}^{2n}$ *we mean the zero set of a* k*-generated ideal having the property of zeros,* $k \geq 1$. Denote by $\mathrm{Var}(k, 2n)$ the space of all germs at 0 of varieties described by k-generated ideals. We associate with the germ $N \in \mathrm{Var}(k, 2n)$ the map-germ $H : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^k, 0)$ whose k components are generators of the ideal of function-germs vanishing on N. We denote by (N) the orbit of N with respect to the group of diffeomorphism-germs. Then the orbit (N) can be identified with the V-orbit of H (see [**AVG**]). Recall from [**AVG**] that the V-equivalence of two map germs $H, \tilde{H} : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^k, 0)$ means the existence of a diffeomorphism-germ Φ and a germ M of a map from $\mathbb{R}^{2n}$ to the manifold of non-singular $k \times k$ matrices such that $\tilde{H} = M \cdot H(\Phi)$. The symplectic V-equivalence is defined in the same way as the V-equivalence; the only difference is that we require that Φ is a symplectomorphism-germ. The symplectic orbit of N can be identified with the symplectic V-orbit of H.

The codimension of the symplectic orbit of N in (N) is the codimension of the symplectic V-orbit of H in the V-orbit of H.

The second invariant is the index of isotropy [**DJZ2**].

Definition 3.2. The **index of isotropy** $\iota(N)$ of N is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M containing N.

They can be described in terms of algebraic restrictions.

Proposition 3.3 ([**DJZ2**]). *The symplectic multiplicity of the germ of a quasi-homogeneous variety* N *in a symplectic space is equal to the codimension of the orbit of the algebraic restriction* $[\omega]_N$ *with respect to the group of local diffeomorphisms preserving* N *in the space of the algebraic restrictions of closed* 2*-forms to* N*.*

Proposition 3.4 ([**DJZ2**]). *The index of isotropy of the germ of a quasi-homogeneous variety* N *in a symplectic space* $(\mathbb{R}^{2n}, \omega)$ *is equal to the maximal order of vanishing of closed* 2*-forms representing the algebraic restriction* $[\omega]_N$*.*

One more discrete symplectic invariant was introduced in [**D**] following ideas from [**A1**]. It is defined specifically for a parameterized curve. This is the maximal tangency order of a curve $f : \mathbb{R} \to M$ with a smooth Lagrangian submanifold. If $H_1 = ... = H_n = 0$ define a smooth submanifold L in the symplectic space then the tangency order of a curve $f : \mathbb{R} \to M$ to L is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \cdots, H_n \circ f$. We denote the tangency order of f with L by $t(f, L)$.

Definition 3.5. The **Lagrangian tangency order** $Lt(f)$ **of a curve** f is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds L of the symplectic space.

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of the algebraic restrictions.

Proposition 3.6 ([**D**]). *Let* f *be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed* 2*-form vanishing at* 0*. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve* f *is the maximum of the order of vanishing on* f *over all* 1*-forms* α *such that* $[\omega]_f = [d\alpha]_f$*.*

In [**DT**] the above invariant was generalized for germs of curves and multi-germs of curves which may be parameterized analytically since the Lagrangian tangency order is the same for every 'good' analytic parameterization of a curve.

Consider a multi-germ $(f_i)_{i\in\{1,\cdots,r\}}$ of analytically parameterized curves f_i. We have r-tuples $(t(f_1,L),\cdots,t(f_r,L))$ for any smooth submanifold L in the symplectic space.

Definition 3.7. For any $I\subseteq\{1,\cdots,r\}$ we define **the tangency order of the multi-germ** $(f_i)_{i\in I}$ to L:

$$t[(f_i)_{i\in\ I},L]=\min_{i\in\ I}t(f_i,L).$$

Definition 3.8. The **Lagrangian tangency order** $Lt((f_i)_{i\in\ I})$ **of a multi-germ** $(f_i)_{i\in I}$ is the maximum of $t[(f_i)_{i\in\ I},L]$ over all smooth Lagrangian submanifolds L of the symplectic space.

For multi-germs one can also define relative invariants according to selected branches or collections of branches [**DT**].

Definition 3.9. For fixed $j\in I$ the **Lagrangian tangency order related to** f_j of a multi-germ $(f_i)_{i\in I}$ denoted by $Lt[(f_i)_{i\in I}:f_j]$ is the maximum of $t[(f_i)_{i\in I\setminus\{j\}},L]$ over all smooth Lagrangian submanifolds L of the symplectic space for which $t(f_j,L)=Lt(f_j)$.

These invariants have geometric interpretation. If a branch f_i is contained in a smooth Lagrangian submanifold then $Lt(f_i)=\infty$. If all curves f_i for $i\in I$ are contained in the same non-singular Lagrangian submanifold then $Lt((f_i)_{i\in\ I})=\infty$. (In the analytic category "if" can be replaced by "if and only if").

We may use these invariants for distinguishing symplectic singularities.

4. Symplectic S_μ-singularities

Denote by (S_μ) (for $\mu>5$) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n},\omega)$ which are diffeomorphic to

$$S_\mu=\{x\in\mathbb{R}^{2n\geq4}\ :x_1^2-x_2^2-x_3^{\mu-3}=x_2x_3=x_{\geq4}=0\}. \tag{4.1}$$

The $S_\mu,\ \mu\geq5$ are simple 1-dimensional isolated complete intersection singularities in the space of dimension greater than 2 ([**G**]). Let $N\in(S_\mu)$. Then N is the union of two 1-dimensional components invariant under the action of local diffeomorphisms preserving N: C_1 – diffeomorphic to the A_1 singularity and C_2 – diffeomorphic to the $A_{\mu-4}$ singularity. Here N is quasi-homogeneous with weights $w(x_1)=w(x_2)=\mu-3$, $w(x_3)=2$ when μ is an even number, or $w(x_1)=w(x_2)=(\mu-3)/2,\ w(x_3)=1$ when μ is an odd number. In our paper we often use the notation $r=\mu-3$.

We will use the method of algebraic restrictions to obtain a complete classification of symplectic singularities in (S_μ) presented in the following theorem.

Theorem 4.1. *Any submanifold of the symplectic space $(\mathbb{R}^{2n},\sum_{i=1}^n dp_i\wedge dq_i)$ which is diffeomorphic to S_μ is symplectically equivalent to one and only one of the normal forms $S_\mu^{i,j}$, $S_{\mu\ l}^{i,j}$ listed below. The parameters c_i of the normal forms are moduli.*

$S_\mu^{0,0}:\ \ p_1^2-p_2^2-q_1^r=0,\quad p_2q_1=0,\quad q_2=c_1q_1-c_2p_1,\quad p_{\geq3}=q_{\geq3}=0;$

$S_{\mu\ 2}^{k,0}(1 \leq k \leq \mu - 5) : p_2^2 - p_1^2 - q_1^r = 0, \quad p_1 q_1 = 0, \ q_2 = c_3 p_1 + \frac{c_{4+k}}{k+1} q_1^{k+1},$
$p_{\geq 3} = q_{\geq 3} = 0, \ c_{4+k} \neq 0;$
$S_{\mu\ 2}^{\mu-4,0} : p_2^2 - p_1^2 - q_1^r = 0, \quad p_1 q_1 = 0, \ q_2 = c_3 p_1 + \frac{c_\mu}{r} q_1^r, \ p_{\geq 3} = q_{\geq 3} = 0, \ c_3 c_\mu = 0;$
$S_{\mu\ r}^{1+k,0}(1 \leq k \leq \mu - 6) : p_1^2 - q_1^2 - q_2^r = 0, \quad q_1 q_2 = 0, \ p_2 = p_1 q_2^k (c_{4+k} + c_{5+k} q_2),$
$p_{\geq 3} = q_{\geq 3} = 0, \quad c_{4+k} \neq 0;$
$S_{\mu\ r}^{\mu-4,0} : p_1^2 - q_1^2 - q_2^r = 0, \quad q_1 q_2 = 0, \ p_2 = c_{\mu-1} p_1 q_2^{r-2}, \ p_{\geq 3} = q_{\geq 3} = 0;$
$S_\mu^{3,1} : p_1^2 - p_2^2 - p_3^r = 0, \ p_2 p_3 = 0, \ q_1 = \frac{1}{2} p_3^2, \ q_2 = -c_4 p_1 p_3, \ p_{\geq 4} = q_{\geq 3} = 0;$
$S_\mu^{2+k,1}(2 \leq k \leq \mu - 4) : p_1^2 - p_2^2 - p_3^r = 0, \ p_2 p_3 = 0, \ q_1 = \frac{c_{4+k}}{k+1} p_3^{k+1}, \ q_2 = -p_1 p_3,$
$p_{\geq 4} = q_{\geq 3} = 0, \ (c_{4+k} \neq 0 \text{ for } 2 \leq k \leq \mu - 5);$
$S_\mu^{3+k,k}(2 \leq k \leq \mu - 4) : p_1^2 - p_2^2 - p_3^r = 0, \ p_2 p_3 = 0, \ q_1 = \frac{1}{k+1} p_3^{k+1}, \ p_{\geq 4} = q_{\geq 2} = 0;$
$S_\mu^{\mu,\infty} : p_1^2 - p_2^2 - p_3^r = 0, \ p_2 p_3 = 0, \quad p_{\geq 4} = q_{\geq 1} = 0.$
(Here we wrote r for $\mu - 3$).

In Section 4.1 we calculate the manifolds $[\mathrm{Symp}(\mathbb{R}^{2n})]_{S_\mu}$ and classify their algebraic restrictions. This allows us to decompose S_μ into symplectic singularity classes. In Section 4.2 we transfer the normal forms for the algebraic restrictions to the symplectic normal forms to obtain a proof of Theorem 4.1. In Section 4.3 we use the Lagrangian tangency orders to distinguish more symplectic singularity classes. In Section 4.4 we propose a geometric description of these singularities which confirms this more detailed classification. Some of the proofs are presented in Section 4.5.

4.1. Algebraic restrictions and their classification. One has the relations for the S_μ-singularities

$$[d(x_2 x_3)]_{S_\mu} = [x_2 dx_3 + x_3 dx_2]_{S_\mu} = 0, \tag{4.2}$$

$$[d(x_1^2 - x_2^2 - x_3^{\mu-3})]_{S_\mu} = [2x_1 dx_1 - 2x_2 dx_2 - (\mu - 3) x_3^{\mu-4} dx_3]_{S_\mu} = 0. \tag{4.3}$$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 1.

TABLE 1. Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = S_\mu$

	relations	proof
1.	$[x_2 dx_2 \wedge dx_3]_N = 0$	(4.2)$\wedge dx_2$
2.	$[x_3 dx_2 \wedge dx_3]_N = 0$	(4.2)$\wedge dx_3$
3.	$[x_1 dx_1 \wedge dx_2]_N = 0$	(4.3)$\wedge dx_2$ and row 2.
4.	$[x_1 dx_1 \wedge dx_3]_N = 0$	(4.3)$\wedge dx_3$ and row 1.
5.	$[x_3 dx_1 \wedge dx_2]_N = [x_2 dx_3 \wedge dx_1]_N$	(4.2)$\wedge dx_1$
6.	$[2x_2 dx_1 \wedge dx_2]_N = (\mu - 3)[x_3^{\mu-4} dx_3 \wedge dx_1]_N$	(4.3)$\wedge dx_1$
7.	$[x_1^2 dx_2 \wedge dx_3]_N = 0$	rows 1. and 2. and $[x_1^2]_N = [x_2^2 + x_3^{\mu-3}]_N$
8.	$[x_3^2 dx_1 \wedge dx_2]_N = 0$	(4.2)$\wedge x_3 dx_1$ and $[x_2 x_3]_N = 0$
9.	$[x_2^2 dx_1 \wedge dx_2]_N = 0$	(4.3)$\wedge x_2 dx_1$ and $[x_2 x_3]_N = 0$

Table 1 and Proposition 2.11 easily imply the following proposition:

PROPOSITION 4.2. *The space* $[\Lambda^2(\mathbb{R}^{2n})]_{S_\mu}$ *is a* $\mu+1$*-dimensional vector space spanned by the algebraic restrictions to* S_μ *of the 2-forms*

$$\theta_1 = dx_1 \wedge dx_3, \quad \theta_2 = dx_2 \wedge dx_3, \quad \theta_3 = dx_1 \wedge dx_2,$$

$$\sigma_1 = x_3 dx_1 \wedge dx_2, \quad \sigma_2 = x_1 dx_2 \wedge dx_3,$$

$$\theta_{4+k} = x_3^k dx_1 \wedge dx_3, \quad \textit{for } 1 \le k \le \mu - 4.$$

Proposition 4.2 and results of Section 2 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{S_\mu}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{S_\mu}$.

PROPOSITION 4.3. *The space* $[Z^2(\mathbb{R}^{2n})]_{S_\mu}$ *has dimension* μ. *It is spanned by the algebraic restrictions to* S_μ *of the 2-forms*

$$\theta_1, \theta_2, \theta_3, \quad \theta_4 = \sigma_1 - \sigma_2, \quad \theta_{4+k} = x_3^k dx_1 \wedge dx_3, \quad \textit{for } \ 1 \le k \le \mu - 4.$$

If $n \ge 3$ *then* $[\mathrm{Symp}(\mathbb{R}^{2n})]_{S_\mu} = [Z^2(\mathbb{R}^{2n})]_{S_\mu}$. *The manifold* $[\mathrm{Symp}(\mathbb{R}^4)]_{S_\mu}$ *is an open part of the* μ*-space* $[Z^2(\mathbb{R}^4)]_{S_\mu}$ *consisting of the algebraic restrictions of the form* $[c_1\theta_1 + \cdots + c_\mu\theta_\mu]_{S_\mu}$ *such that* $(c_1, c_2, c_3) \ne (0,0,0)$.

THEOREM 4.4.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{S_\mu}$ *can be brought by a symmetry of* S_μ *to one of the normal forms* $[S_\mu]^{i,j}$ *given in the second column of Table 2.*

(ii) The singularity classes corresponding to the normal forms are disjoint.

(iii) The parameters c_i *of the normal forms* $[S_\mu]^{i,j}$ *are moduli.*

(iv) The codimension in $[Z^2(\mathbb{R}^{2n})]_{S_\mu}$ *of the singularity class corresponding to the normal form* $[S_\mu]^{i,j}$ *is equal to* i *and the index of isotropness is equal to* j.

TABLE 2. Classification of symplectic S_μ singularities:
cod – codimension of the classes; μ^{sym}– symplectic multiplicity;
ind – index of isotropy.

Symplectic class	Normal forms for algebraic restrictions	cod	μ^{sym}	ind
$(S_\mu)^{0,0}$ $(2n \ge 4)$	$[S_\mu]^{0,0} : [\theta_1 + c_2\theta_2 + c_2\theta_3]_{S_\mu}$	0	2	0
$(S_\mu)_2^{k,0}$ $(2n \ge 4)$ for $1 \le k \le \mu - 5$	$[S_\mu]_2^{k,0} : [\theta_2 + c_3\theta_3 + c_{4+k}\theta_{4+k}]_{S_\mu}$ $c_{4+k} \ne 0$	k	$k+2$	0
$(S_\mu)_2^{\mu-4,0}$ $(2n \ge 4)$	$[S_\mu]_2^{\mu-4,0} : [\theta_2 + c_3\theta_3 + c_\mu\theta_\mu]_{S_\mu}, \quad c_3c_\mu = 0$	$\mu - 4$	$\mu - 3$	0
$(S_\mu)_r^{1+k,0}$ $(2n \ge 4)$ for $1 \le k \le \mu - 6$	$[S_\mu]_r^{1+k,0} : [\theta_3 + c_{4+k}\theta_{4+k} + c_{5+k}\theta_{5+k}]_{S_\mu}$ $c_{4+k} \ne 0$	$k+1$	$k+3$	0
$(S_\mu)_r^{\mu-4,0}$ $(2n \ge 4)$	$[S_\mu]_r^{\mu-4,0} : [\theta_3 + c_{\mu-1}\theta_{\mu-1}]_{S_\mu}$	$\mu - 4$	$\mu - 3$	0
$(S_\mu)^{3,1}$ $(2n \ge 6)$	$[S_\mu]^{3,1} : [c_4\theta_4 + \theta_5]_{S_\mu}$	3	4	1
$(S_\mu)^{2+k,1}$ $(2n \ge 6)$ for $2 \le k \le \mu - 4$	$[S_\mu]^{2+k,1} : [\theta_4 + c_{4+k}\theta_{4+k}]_{S_\mu}$ $c_{4+k} \ne 0$ for $2 \le k \le \mu - 5$	$k+2$	$k+3$	1
$(S_\mu)^{3+k,k}$ $(2n \ge 6)$ for $2 \le k \le \mu - 4$	$[S_\mu]^{3+k,k} : [\theta_{4+k}]_{S_\mu}$ for $2 \le k \le \mu - 4$	$k+3$	$k+3$	k
$(S_\mu)^{\mu,\infty}$ $(2n \ge 6)$	$[S_\mu]^{\mu,\infty} : [0]_{S_\mu}$	μ	μ	∞

The proof of Theorem 4.4 is presented in Section 4.5.

In the first column of Table 2 by $(S_\mu)^{i,j}$ we denote a subclass of (S_μ) consisting of $N \in (S_\mu)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[S_\mu]^{i,j}$ where i is the codimension of the class and j denotes index of isotropy of the class. Classes $(S_\mu)_2^{i,0}$ and $(S_\mu)_r^{i,0}$ can be distinguished geometrically (see Section 4.4) and by relative Lagrangian tangency order $L_{2:1}$ defined in Section 4.3 (Remark 4.8). The classes $(S_\mu)_2^{i,0}$ have $L_{2:1} = \frac{2}{\lambda_\mu}$ and the classes $(S_\mu)_r^{i,0}$ have $L_{2:1} = \frac{r}{\lambda_\mu}$ where $\lambda_\mu = 1$ for even μ and $\lambda_\mu = 2$ for odd μ.

Theorem 2.5, Theorem 4.4 and Proposition 4.3 imply the following statement.

Proposition 4.5. *The classes $(S_\mu)^{i,j}$ are symplectic singularity classes, i.e. they are closed with respect to the action of the group of symplectomorphisms. The class (S_μ) is the disjoint union of the classes $(S_\mu)^{i,j}$. The classes $(S_\mu)^{0,0}$, $(S_\mu)_2^{i,0}$ and $(S_\mu)_r^{i,0}$ for $1 \leq i \leq \mu - 4$ are non-empty for any dimension $2n \geq 4$ of the symplectic space; the classes $(S_\mu)^{i,1}$ for $3 \leq i \leq \mu - 2$ and $(S_\mu)^{i,i-3}$ for $5 \leq i \leq \mu - 1$ and $(S_\mu)^{\mu,\infty}$ are empty if $n = 2$ and not empty if $n \geq 3$.*

4.2. Symplectic normal forms. Proof of Theorem 4.1. Let us transfer the normal forms $[S_\mu]^{i,j}$ to symplectic normal forms using Theorem 2.12, i.e. realizing the algorithm in Section 2. Fix a family $\omega^{i,j}$ of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[S_\mu]^{i,j}$ of algebraic restrictions. We can fix, for example

$\omega^{0,0} = \theta_1 + c_2\theta_2 + c_3\theta_3 + dx_2 \wedge dx_4 + \sum_{i=3}^n dx_{2i-1} \wedge dx_{2i}$;

$\omega_2^{k,0} = \theta_2 + c_3\theta_3 + c_{4+k}\theta_{4+k} + dx_1 \wedge dx_4 + \sum_{i=3}^n dx_{2i-1} \wedge dx_{2i},\ c_{4+k} \neq 0,\ 1 \leq k \leq \mu - 5$;

$\omega_2^{\mu-4,0} = \theta_2 + c_3\theta_3 + c_\mu\theta_\mu + dx_1 \wedge dx_4 + \sum_{i=3}^n dx_{2i-1} \wedge dx_{2i},\quad c_3 c_\mu = 0$;

$\omega_r^{1+k,0} = \theta_3 + c_{4+k}\theta_{4+k} + c_{5+k}\theta_{5+k} + \sum_{i=2}^n dx_{2i-1} \wedge dx_{2i},\ c_{4+k} \neq 0,\ 1 \leq k \leq \mu - 6$;

$\omega_r^{\mu-4,0} = \theta_3 + c_{\mu-1}\theta_{\mu-1} + dx_4 \wedge dx_3 + \sum_{i=3}^n dx_{2i-1} \wedge dx_{2i}$;

$\omega^{3,1} = c_4\theta_4 + \theta_5 + \sum_{i=1}^3 dx_i \wedge dx_{i+3} + \sum_{i=4}^n dx_{2i-1} \wedge dx_{2i}$;

$\omega^{2+k,1} = \theta_4 + c_{4+k}\theta_{4+k} + \sum_{i=1}^3 dx_i \wedge dx_{i+3} + \sum_{i=4}^n dx_{2i-1} \wedge dx_{2i},\ 2 \leq k \leq \mu - 4$;

$\omega^{3+k,k} = \theta_{4+k} + \sum_{i=1}^3 dx_i \wedge dx_{i+3} + \sum_{i=4}^n dx_{2i-1} \wedge dx_{2i},\ 2 \leq k \leq \mu - 4$;

$\omega^{\mu,\infty} = \sum_{i=1}^3 dx_i \wedge dx_{i+3} + \sum_{i=4}^n dx_{2i-1} \wedge dx_{2i}$.

Let $\omega = \sum_{i=1}^m dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix a family $\Phi^{i,j}$ of local diffeomorphisms which bring the family of symplectic forms $\omega^{i,j}$ to the symplectic form ω: $(\Phi^{i,j})^*\omega^{i,j} = \omega$. Consider the families $S_\mu^{i,j} = (\Phi^{i,j})^{-1}(S_\mu)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega)$ which is diffeomorphic to S_μ is symplectically equivalent to one and only one of the normal forms $S_\mu^{i,j}$ presented in Theorem 4.1. By Theorem 4.4 we obtain that the parameters c_i of the normal forms are moduli.

4.3. Distinguishing symplectic classes of S_μ by Lagrangian tangency orders. Lagrangian tangency orders will be used to obtain a more detailed classification of (S_μ). A curve $N \in (S_\mu)$ may be described as a union of two invariant components C_1 and C_2. The curve C_1 is diffeomorphic to the A_1 singularity and consists of two parameterized branches B_{1+} and B_{1-}. The curve C_2 is diffeomorphic to the $A_{\mu-4}$ singularity and consists of one parameterized branch if μ is even and consists of two branches B_{2+} and B_{2-} if μ is odd. The parametrization of these branches is given in the second column of Table 3 or Table 4. To distinguish the classes of this singularity completely we need following three invariants:

- $Lt(N) = Lt(C_1, C_2)$
- $L_1 = Lt(C_1) = \max_L(\min\{t(B_{1+}, L), t(B_{1-}, L)\})$
- $L_2 = Lt(C_2)$

where L is a smooth Lagrangian submanifold of the symplectic space.

Considering the triples $(Lt(N), L_1, L_2)$ we obtain a detailed classification of symplectic singularities of S_μ. Some subclasses (see Table 3 and 4) have a natural geometric interpretation (Table 5).

THEOREM 4.6. *A stratified submanifold $N \in (S_\mu)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 3 or 4. The parameters c_i are moduli. The Lagrangian tangency orders of the curve are presented in the fifth, sixth and seventh columns of these tables and the codimension of the classes is given in the fourth column.*

TABLE 3. Lagrangian tangency orders for symplectic classes of S_μ singularity (μ even)

Class	Parametrization of branches $B_{1\pm}$ and C_2	Conditions for subclasses	cod	$Lt(N)$	L_1	L_2
$(S_\mu)^{0,0}$	$(t, 0, \pm t, -c_3 t, 0, \cdots)$	$c_3 \neq 0$	0	1	1	r
$2n \geq 4$	$(t^r, t^2, 0, c_2 t^2 - c_3 t^r, 0, \cdots)$	$c_3 = 0$	1	2	∞	r
$(S_\mu)_2^{k,0}$	$(t, 0, \pm t, c_3 t, 0, \cdots)$	$c_{4+k} \cdot c_3 \neq 0$	k	1	1	$r+2k$
$2n \geq 4$	$(0, t^2, t^r, \frac{c_{4+k}}{k+1} t^{2+2k}, 0, \cdots)$	$c_3 = 0, c_{4+k} \neq 0$	$k+1$	2	∞	$r+2k$
$(S_\mu)_2^{\mu-4,0}$	$(t, 0, \pm t, c_3 t, 0, \cdots)$	$c_3 \neq 0$	$\mu-4$	1	1	∞
$2n \geq 4$	$(0, t^2, t^r, \frac{c_\mu}{r} t^{2r}, 0, \cdots)$	$c_3 = 0$	$\mu-3$	2	∞	∞
$(S_\mu)_r^{1+k,0}$	$(t, \pm t, 0, 0, \cdots)$	$c_{k+4} \neq 0$	$k+1$	1	1	$r+2k$
$2n \geq 4$	$(t^r, 0, (c_{4+k} + c_{5+k} t^2) t^{r+2k}, t^2, 0, \cdots)$	$1 \leq k \leq \mu - 6$				
$(S_\mu)_r^{\mu-4,0}$	$(t, \pm t, 0, 0, \cdots)$	$c_{\mu-1} \neq 0$	$\mu-4$	1	1	$3r-4$
$2n \geq 4$	$(t^r, 0, c_{\mu-1} t^{3r-4}, t^2, 0, \cdots)$	$c_{\mu-1} = 0$	$\mu-3$	1	1	∞
$(S_\mu)^{3,1}$	$(t, 0, \pm t, 0, 0, \cdots)$		3	$r+2$	∞	$r+2$
$2n \geq 6$	$(t^r, \frac{1}{2} t^4, 0, -c_4 t^{r+2}, t^2, 0, \cdots)$					
$(S_\mu)^{2+k,1}$	$(t, 0, \pm t, 0, 0, 0, \cdots)$	$c_{4+k} \neq 0$	$k+2$	$r+2$	∞	$r+2k$
$2n \geq 6$	$(t^r, \frac{c_{4+k} t^{2(k+1)}}{k+1}, 0, -t^{r+2}, t^2, 0, \cdots)$	$2 \leq k \leq \mu - 5$				
		$k = \mu - 4$	$\mu-2$	$r+2$	∞	∞
$(S_\mu)^{3+k,k}$	$(t, 0, \pm t, 0, 0, 0, \cdots)$	$2 \leq k \leq \mu - 5$	$k+3$	$r+2k$	∞	$r+2k$
$2n \geq 6$	$(t^r, \frac{t^{2(k+1)}}{k+1}, 0, 0, t^2, 0, 0, \cdots)$	$k = \mu - 4$	$\mu-1$	$3r-2$	∞	∞
$(S_\mu)^{\mu,\infty}$	$(t, 0, \pm t, 0, 0, \cdots)$		μ	∞	∞	∞
$2n \geq 6$	$(t^r, 0, 0, 0, t^2, 0, \cdots)$					

TABLE 4. Lagrangian tangency orders for symplectic classes of S_μ singularity (μ odd)

Class	Parametrization of branches $B_{1\pm}$ and $B_{2\pm}$	Conditions for subclasses	cod	$Lt(N)$	L_1	L_2
$(S_\mu)^{0,0}$	$(t,0,\pm t,-c_3t,0,\cdots)$	$c_3\neq 0$	0	1	1	$\frac{r}{2}$
$2n\geq 4$	$(\pm t^{\frac{r}{2}},t,0,c_2t^2\mp c_3t^{\frac{r}{2}},0,\cdots)$	$c_3=0$	1	1	∞	$\frac{r}{2}$
$(S_\mu)_2^{k,0}$	$(t,0,\pm t,c_3t,0,\cdots)$	$c_{4+k}\cdot c_3\neq 0$	k	1	1	$\frac{r}{2}+k$
$2n\geq 4$	$(0,t,\pm t^{\frac{r}{2}},\frac{c_{4+k}}{k+1}t^{1+k},0,\cdots)$	$c_3=0,c_{4+k}\neq 0$	$k+1$	1	∞	$\frac{r}{2}+k$
$(S_\mu)_2^{\mu-4,0}$	$(t,0,\pm t,c_3t,0,\cdots)$	$c_3\neq 0$	$\mu-4$	1	1	∞
$2n\geq 4$	$(0,t,t^{\frac{r}{2}},\frac{c_\mu}{r}t^r,0,\cdots)$	$c_3=0$	$\mu-3$	1	∞	∞
$(S_\mu)_r^{1+k,0}$	$(t,\pm t,0,0,\cdots)$	$c_{k+4}\neq 0$	$k+1$	1	1	$\frac{r}{2}+k$
$2n\geq 4$	$(\pm t^{\frac{r}{2}},0,\pm(c_{4+k}+c_{5+k}t)t^{\frac{r}{2}+k},t,0,\cdots)$	$1\leq k\leq \mu-6$				
$(S_\mu)_r^{\mu-4,0}$	$(t,\pm t,0,0,\cdots)$	$c_{\mu-1}\neq 0$	$\mu-4$	1	1	$\frac{3r}{2}-2$
$2n\geq 4$	$(\pm t^{\frac{r}{2}},0,\pm c_{\mu-1}t^{\frac{3r-4}{2}},t,0,\cdots)$	$c_{\mu-1}=0$	$\mu-3$	1	1	∞
$(S_\mu)^{3,1}$	$(t,0,\pm t,0,0,\cdots)$		3	$\frac{r}{2}+1$	∞	$\frac{r}{2}+1$
$2n\geq 6$	$(\pm t^{\frac{r}{2}},\frac{1}{2}t^2,0,\mp c_4t^{\frac{r+2}{2}},t,0,\cdots)$					
$(S_\mu)^{2+k,1}$	$(t,0,\pm t,0,0,0,\cdots)$	$c_{4+k}\neq 0$	$k+2$	$\frac{r}{2}+1$	∞	$\frac{r}{2}+k$
$2n\geq 6$	$(\pm t^{\frac{r}{2}},\frac{c_{4+k}t^{k+1}}{k+1},0,\mp t^{\frac{r+2}{2}},t,0,\cdots)$	$2\leq k\leq \mu-5$				
		$k=\mu-4$	$\mu-2$	$\frac{r}{2}+1$	∞	∞
$(S_\mu)^{3+k,k}$	$(t,0,\pm t,0,0,0,\cdots)$	$2\leq k\leq \mu-5$	$k+3$	$\frac{r}{2}+k$	∞	$\frac{r}{2}+k$
$2n\geq 6$	$(\pm t^{\frac{r}{2}},\frac{t^{k+1}}{k+1},0,0,t,0,0,\cdots)$	$k=\mu-4$	$\mu-1$	$\frac{3}{2}r-1$	∞	∞
$(S_\mu)^{\mu,\infty}$	$(t,0,\pm t,0,0,\cdots)$		μ	∞	∞	∞
$2n\geq 6$	$(\pm t^{\frac{r}{2}},0,0,0,t,0,\cdots)$					

REMARK 4.7. The numbers L_1 and L_2 can be easily calculated knowing Lagrangian tangency orders for the A_1 and $A_{\mu-4}$ singularities (see Table 2 in [**DT**]) or by applying directly the definition of the Lagrangian tangency order and finding a Lagrangian submanifold nearest to the components. Next we calculate $Lt(N)$ from the definition knowing that it can not be greater than $\min(L_1,L_2)$.

We can compute L_1 using the algebraic restrictions $[\omega^{i,j}]_{C_1}$ where the space $[Z^2(\mathbb{R}^{2n})]_{C_1}$ is spanned only by the algebraic restriction to C_1 of the 2-form θ_3. For example for the class $(S_\mu)^{0,0}$ we have $[\theta_1+c_2\theta_2+c_3\theta_3]_{C_1}=[c_3\theta_3]_{C_1}$ and thus $L_1=1$ when $c_3\neq 0$ and $L_1=\infty$ when $c_3=0$.

We can compute L_2 using the algebraic restrictions $[\omega^{i,j}]_{C_2}$ where the space $[Z^2(\mathbb{R}^{2n})]_{C_2}$ is spanned only by the algebraic restrictions to C_2 of the 2-forms θ_1, θ_{4+k} for $k=1,2,\ldots,\theta_{\mu-1}$. For example for the class $(S_\mu)^{0,0}$ we have $[\theta_1+c_2\theta_2+c_3\theta_3]_{C_2}=[\theta_1]_{C_2}$ and thus $L_2=\mu-3$ if μ is an even number and $L_2=\frac{\mu-3}{2}$ if μ is an odd number.

$Lt(N)\leq 1=\min(L_1,L_2)$ when $c_3\neq 0$. Applying the definition of $Lt(N)$ we find the smooth Lagrangian submanifold L described by the conditions $p_i=0,\ i\in\{1,\ldots,n\}$ and we get $Lt(N)\geq t(N,L)=1$ in this case.

If $c_3=0$ then $Lt(N)\leq L_2=\min(L_1,L_2)$, but applying the definition of $Lt(N)$ we have $t(N,L)\leq 2$ (resp. $t(N,L)\leq 1$) for all Lagrangian submanifolds L. For L

described by the conditions $q_i = 0,\ i \in \{1, \dots, n\}$ we get $Lt(N) = t(N, L) = 2$ if μ is even and $Lt(N) = t(N, L) = 1$ if μ is odd.

REMARK 4.8. We are not able to distinguish some classes $(S_\mu)_2^{i,0}$ and $(S_\mu)_r^{i,0}$ by the triples $(Lt(N), L_1, L_2)$ but we can do this using relative Lagrangian tangency orders.

We define $L_{2:1} = Lt[C_2 : B_{1\pm}] = \max(Lt[C_2 : B_{1+}], Lt[C_2 : B_{1-}])$.

Since branches B_{1+} and B_{1-} are smooth curves then $Lt(B_{1+}) = Lt(B_{1-}) = \infty$ and $L_{2:1} = \max\limits_L(t(C_2, L))$ where L is a smooth Lagrangian submanifold containing B_{1+} or B_{1-}.

Considering such smooth Lagrangian submanifolds we obtain $L_{2:1} = \frac{2}{\lambda_\mu}$ for the classes $(S_\mu)_2^{i,0}$ and $L_{2:1} = \frac{\mu-3}{\lambda_\mu}$ for the classes $(S_\mu)_r^{i,0}$ ($\lambda_\mu = 1$ for even μ and $\lambda_\mu = 2$ for odd μ).

4.4. Geometric conditions for the classes $(S_\mu)^{i,j}$. The classes $(S_\mu)^{i,j}$ can be distinguished geometrically, without using any local coordinate system.

Let $N \in (S_\mu)$. Then N is the union of two singular 1-dimensional irreducible components diffeomorphic to the A_1 and $A_{\mu-4}$ singularities. In local coordinates they have the form

$$\mathcal{C}_1 = \{x_1^2 - x_2^2 = 0,\ x_{\geq 3} = 0\},$$
$$\mathcal{C}_2 = \{x_1^2 - x_3^{\mu-3} = 0,\ x_2 = x_{\geq 4} = 0\}.$$

Denote by ℓ_{1+}, ℓ_{1-} the tangent lines at 0 to the branches $\mathcal{B}_{1+}$ and $\mathcal{B}_{1-}$ respectively. These lines span a 2-space P_1. Denote by ℓ_2 the tangent line at 0 to the component $\mathcal{C}_2$ and let P_2 be the 2-space tangent at 0 to the component $\mathcal{C}_2$. Define the line $\ell_3 = P_1 \cap P_2$. The lines $\ell_{1\pm}, \ell_2$ span a 3-space $W = W(N)$. Equivalently W is the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. The classes $(S_\mu)^{i,j}$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form. For $N = S_\mu =$(4.1) it is easy to calculate

$$\ell_{1\pm} = \mathrm{span}(\partial/\partial x_1 \pm \partial/\partial x_2),\ \ell_2 = \mathrm{span}(\partial/\partial x_3),\ \ell_3 = \mathrm{span}(\partial/\partial x_1). \tag{4.4}$$

THEOREM 4.9. *A stratified submanifold $N \in (S_\mu)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(S_\mu)^{i,j}$ if and only if the couple (N, ω) satisfies the corresponding conditions in the last column of Table* 5.

PROOF OF THEOREM 4.9. The conditions on the pair (ω, N) in the last column of Table 5 are disjoint. It suffices to prove that these conditions in the row of $(S_\mu)^{i,j}$, are satisfied for any $N \in (S_\mu)^{i,j}$. This is a corollary of the following claims:

1. Each of the conditions in the last column of Table 5 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs (ω, N);

2. Each of these conditions depends only on the algebraic restriction $[\omega]_N$;

3. Take the simplest 2-forms $\omega^{i,j}$ representing the normal forms $[S_\mu]^{i,j}$ for the algebraic restrictions. The pair $(\omega = \omega^{i,j}, S_\mu)$ satisfies the condition in the last column of Table 5, the row of $(S_\mu)^{i,j}$.

The first statement is obvious, the second one follows from Lemma 2.7.

TABLE 5. Geometric interpretation of singularity classes of S_μ: W is the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n\geq 4},\omega)$ containing $N\in(S_\mu)$, $\lambda_\mu=1$ for even μ and $\lambda_\mu=2$ for odd μ.

Class	Normal form	Geometric conditions
$(S_\mu)^{0,0}$		$\omega\vert_{\ell_2+\ell_3}\neq 0$
	$[S_\mu]^{0,0}_{L_1=1}:[\theta_1+c_2\theta_2+c_3\theta_3]_{S_\mu}$ $c_3\neq 0$	$\omega\vert_{\ell_{1+}+\ell_{1-}}\neq 0$ and none of the components is contained in a Lagrangian submanifold
	$[S_\mu]^{0,0}_{L_1=\infty}:[\theta_1+c_2\theta_2]_{S_\mu}$	$\omega\vert_{\ell_{1+}+\ell_{1-}}=0$ (so component C_1 is contained in a Lagrangian submanifold)
$(S_\mu)^{i,0}_2$		$\omega\vert_{\ell_2+\ell_3}=0$ but $\omega\vert_{\ell_{1\pm}+\ell_2}\neq 0$
$(S_\mu)^{k,0}_2$	$[S_\mu]^{k,0}_{2,L_1=1}:[\theta_2+c_3\theta_3+c_{4+k}\theta_{4+k}]_{S_\mu}$ $c_3\cdot c_{4+k}\neq 0$ for $1\leq k\leq\mu-5$	$\omega\vert_{\ell_{1+}+\ell_{1-}}\neq 0$ and $L_2=\frac{r+2k}{\lambda_\mu}$
	$[S_\mu]^{k,0}_{2,L_1=\infty}:[\theta_2+c_{4+k}\theta_{4+k}]_{S_\mu}$ $c_{4+k}\neq 0$ for $1\leq k\leq\mu-5$	$\omega\vert_{\ell_{1+}+\ell_{1-}}=0$ (so component C_1 is contained in a Lagrangian submanifold) and $L_2=\frac{r+2k}{\lambda_\mu}$
$(S_\mu)^{\mu-4,0}_2$	$[S_\mu]^{\mu-4,0}_2:[\theta_2+c_3\theta_3]_{S_\mu}$, $c_3\neq 0$	$\omega\vert_{\ell_{1+}+\ell_{1-}}\neq 0$ and component C_2 is contained in a Lagrangian submanifold
	$[S_\mu]^{\mu-3,0}_2:[\theta_2+c_\mu\theta_\mu]_{S_\mu}$	$\omega\vert_{\ell_{1+}+\ell_{1-}}=0$, both components are contained in Lagrangian submanifolds
$(S_\mu)^{i,0}_r$		$\omega\vert_{\ell_2+\ell_3}=0$ and $\omega\vert_{\ell_{1\pm}+\ell_2}=0$ but $\omega\vert_{\ell_{1+}+\ell_{1-}}\neq 0$
$(S_\mu)^{1+k,0}_r$	$[S_\mu]^{1+k,0}_r$: $[\theta_3+c_{4+k}\theta_{4+k}+c_{5+k}\theta_{5+k}]_{S_\mu}$ $c_{4+k}\neq 0$ for $1\leq k\leq\mu-6$	none of the components is contained in a Lagrangian submanifold and $L_2=\frac{r+2k}{\lambda_\mu}$
$(S_\mu)^{\mu-4,0}_r$	$[S_\mu]^{\mu-4,0}_r:[\theta_3+c_{\mu-1}\theta_{\mu-1}]_{S_\mu}$ $c_{\mu-1}\neq 0$	none of the components is contained in a Lagrangian submanifold and $L_2=\frac{3r-4}{\lambda_\mu}$
	$[S_\mu]^{\mu-3,0}_r:[\theta_3]_{S_\mu}$	component C_2 is contained in Lagrangian submanifolds
		$\omega\vert_W=0$ and component C_1 is contained in a Lagrangian submanifold
$(S_\mu)^{3,1}$	$[S_\mu]^{3,1}:[c_4\theta_4+\theta_5]_{S_\mu}$	$L_2=\frac{r+2}{\lambda_\mu}$ and $Lt(N)=\frac{r+2}{\lambda_\mu}$
$(S_\mu)^{2+k,1}$	$[S_\mu]^{2+k,1}:[\theta_4+c_{4+k}\theta_{4+k}]_{S_\mu}$ $c_{4+k}\neq 0$ and $2\leq k\leq\mu-5$	$L_2=\frac{r+2k}{\lambda_\mu}$ and $Lt(N)=\frac{r+2}{\lambda_\mu}$
	$[S_\mu]^{\mu-2,1}:[\theta_4+c\theta_\mu]_{S_\mu}$	both components are contained in Lagrangian submanifolds and $Lt(N)=\frac{r+2}{\lambda_\mu}$
$(S_\mu)^{3+k,k}$	$[S_\mu]^{3+k,k}:[\theta_{4+k}]_{S_\mu}$ $2\leq k\leq\mu-5$	$L_2=\frac{r+2k}{\lambda_\mu}$ and $Lt(N)=\frac{r+2k}{\lambda_\mu}$
	$[S_\mu]^{\mu-1,\mu-4}:[\theta_\mu]_{S_\mu}$	both components are contained in Lagrangian submanifolds and $Lt(N)=\frac{3r-2}{\lambda_\mu}$
$(S_\mu)^{\mu,\infty}$	$[S_\mu]^{\mu,\infty}:[0]_{S_\mu}$	both components are contained in the same Lagrangian submanifold

To prove the third statement we note that in the case $N = S_\mu = (4.1)$ one has $\ell_{1\pm} = \mathrm{span}(\partial/\partial x_1 \pm \partial/\partial x_2)$, $\ell_2 = \mathrm{span}(\partial/\partial x_3)$, $\ell_3 = \mathrm{span}(\partial/\partial x_1)$ and $W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. By simple calculation and observation of the Lagrangian tangency orders we obtain that the conditions in the last column of Table 5, the row of $(S_\mu)^{i,j}$ are satisfied. □

4.5. Proof of Theorem 4.4.

PROOF. In our proof we use vector fields tangent to $N \in S_\mu$. Any vector fields tangent to $N \in S_\mu$ may be described as $V = g_1 E + g_2 \mathcal{H}$ where E is the Euler vector field and $\mathcal{H}$ is a Hamiltonian vector field and g_1, g_2 are functions. It was shown in [**DT**] (Prop. 6.13) that the action of a Hamiltonian vector field on any 1-dimensional complete intersection is trivial.

The germ of a vector field tangent to S_μ of non trivial action on algebraic restrictions of closed 2-forms to S_μ may be described as a linear combination germs of the following vector fields: $X_0 = E$, $X_1 = x_1 E$, $X_2 = x_2 E$, $X_3 = x_3 E$, $X_{l+2} = x_3^l E$ for $1 < l < \mu - 3$, where E is the Euler vector field $E = \sum_{i=1}^3 \lambda_i x_i \partial/\partial x_i$ and the λ_i are the weights for x_i.

PROPOSITION 4.10. *When μ is an even number then the infinitesimal action of germs of quasi-homogeneous vector fields tangent to N on the basis of the vector space of the algebraic restrictions of closed 2-forms to N is presented in Table 6.*

TABLE 6. Infinitesimal actions on algebraic restrictions of closed 2-forms to S_μ. $E = (\mu-3)x_1\partial/\partial x_1 + (\mu-3)x_2\partial/\partial x_2 + 2x_3\partial/\partial x_3$

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_{4+k}]$ *for* $0<k<r$
$X_0 = E$	$(r+2)[\theta_1]$	$(r+2)[\theta_2]$	$2r[\theta_3]$	$(2r+2)[\theta_4]$	$(r+2(k+1))[\theta_{4+k}]$
$X_1 = x_1 E$	$[0]$	$-(r+2)[\theta_4]$	$[0]$	$[0]$	$[0]$
$X_2 = x_2 E$	$-r[\theta_4]$	$[0]$	$\frac{-3r^2}{2}[\theta_\mu]$	$[0]$	$[0]$
$X_3 = x_3 E$	$(r+4)[\theta_5]$	$[0]$	$r[\theta_4]$	$[0]$	$(r+2(k+2))[\theta_{5+k}]$
$X_{l+2} = x_3^l E$ $l < r-k$	$(r+2l+2)[\theta_{4+l}]$	$[0]$	$[0]$	$[0]$	$(r+2(k+l+1))[\theta_{4+k+l}]$
$X_{l+2} = x_3^l E$ $r-k \le l \le r-1$	$(r+2l+2)[\theta_{4+l}]$	$[0]$	$[0]$	$[0]$	$[0]$

REMARK 4.11. When μ is odd we obtain a very similar table, we only have to divide by 2 all coefficients in Table 6. The next part of the proof is written for even μ. In the case of odd μ we repeat the same scheme.

Let $\mathcal{A} = [\sum_{l=1}^{\mu} c_l \theta_l]_{S_\mu}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 4.4 follows from the following lemmas.

LEMMA 4.12. *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{l=1}^{\mu} c_l \theta_l]_{S_\mu}$ can be reduced by a symmetry of S_μ to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{S_\mu}$.*

Proof of Lemma 4.12. We use the homotopy method to prove that $\mathcal{A}$ is diffeomorphic to $[\theta_1+\widetilde{c}_2\theta_2+\widetilde{c}_3\theta_3]_{S_\mu}$. Let $\mathcal{B}_t=[c_1\theta_1+c_2\theta_2+c_3\theta_3+(1-t)\sum_{l=4}^{\mu}c_l\theta_l]_{S_\mu}$ for $t\in[0;1]$. Then $\mathcal{B}_0=\mathcal{A}$ and $\mathcal{B}_1=[c_1\theta_1+c_2\theta_2+c_3\theta_3]_{S_\mu}$. We prove that there exists a family $\Phi_t\in Symm(S_\mu)$, $t\in[0;1]$ such that

$$\Phi_t^*\mathcal{B}_t=\mathcal{B}_0,\ \Phi_0=id. \tag{4.5}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt}=V_t(\Phi_t)$. Then differentiating (4.5) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t=[\sum_{l=4}^{\mu}c_l\theta_l]. \tag{4.6}$$

We are looking for V_t in the form $V_t=\sum_{k=1}^{\mu-2}b_k(t)X_k$ where the $b_k(t)$ for $k=1,\ldots,\mu-2$ are smooth functions $b_k:[0;1]\to\mathbb{R}$. Then by Proposition 4.10 equation (4.6) has a form

(4.7)
$$\begin{bmatrix}
-(r+2)c_2 & -rc_1 & rc_3 & 0 & 0 & 0 & 0\\
0 & 0 & (r+4)c_1 & 0 & 0 & 0 & 0\\
0 & 0 & (1-t)(r+6)c_5 & (r+6)c_1 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & 0 & 0 & 0\\
0 & 0 & (1-t)(r+2k)c_{k+2} & \cdots & (r+2k)c_1 & 0 & 0\\
0 & 0 & \vdots & \cdots & \vdots & \ddots & 0\\
0 & -\frac{3r^2}{2}c_3 & 3(1-t)rc_{\mu-1} & \cdots & 3(1-t)rc_{\mu-k+1} & \cdots & 3rc_1
\end{bmatrix}
\begin{bmatrix} b_1(t)\\ b_2(t)\\ b_3(t)\\ \vdots\\ b_{k+1}(t)\\ \vdots\\ b_{\mu-2}(t)\end{bmatrix}
=
\begin{bmatrix} c_4\\ c_5\\ c_6\\ \vdots\\ c_{k+3}\\ \vdots\\ c_\mu\end{bmatrix}$$

If $c_1\neq 0$ we can solve (4.7).
We obtain $b_3(t)=\frac{c_5}{c_1(r+4)}$ and we may choose any b_1.
Other functions b_k are determined by that choice.
Let $b_1(t)=0$. This implies $b_2(t)=\frac{rc_3b_3(t)-c_4}{rc_1}=\frac{c_3c_5}{(r+4)c_1^2}-\frac{c_4}{rc_1}$.
Next $b_4(t)=\frac{c_6}{(r+6)c_1}-\frac{(1-t)}{c_1}c_5b_3(t)$, $\quad b_5(t)=\frac{c_7}{(r+8)c_1}-\frac{(1-t)}{c_1}(c_6b_3(t)+c_5b_4(t))$,
consequently $b_{k+1}(t)=\frac{c_{k+3}}{(r+2k)c_1}-\frac{(1-t)}{c_1}\sum_{l=3}^{k}c_{k+5-l}b_l(t)$ for $k<\mu-3$,
and eventually $b_{\mu-2}(t)=\frac{c_\mu}{3rc_1}+\frac{r}{2c_1}c_3b_2(t)-\frac{(1-t)}{c_1}\sum_{l=3}^{\mu-3}c_{\mu+2-l}b_l(t)$.

Diffeomorphisms Φ_t may be obtained as a flow of the vector field V_t. The family Φ_t preserves S_μ, because V_t is tangent to S_μ and $\Phi_t^*\mathcal{B}_t=\mathcal{A}$. Using the homotopy arguments we have $\mathcal{A}$ diffeomorphic to $\mathcal{B}_1=[c_1\theta_1+c_2\theta_2+c_3\theta_3]_{S_\mu}$. By the condition $c_1\neq 0$ we have a diffeomorphism $\Psi\in Symm(S_\mu)$ of the form

$$\Psi:(x_1,x_2,x_3)\mapsto(c_1^{-\frac{r}{r+2}}x_1,c_1^{-\frac{r}{r+2}}x_2,c_1^{-\frac{2}{r+2}}x_3), \tag{4.8}$$

and we obtain

$$\Psi^*(\mathcal{B}_1)=[\theta_1+\frac{c_2}{c_1}\theta_2+c_3c_1^{-\frac{2r}{r+2}}\theta_3]_{S_\mu}=[\theta_1+\widetilde{c}_2\theta_2+\widetilde{c}_3\theta_3]_{S_\mu}.$$

□

Lemma 4.13. *If $c_1=0$ and $c_2\neq 0$ and $c_{4+k}\neq 0$ and $c_l=0$ for $5\leq l<4+k$, then the algebraic restriction $\mathcal{A}=[\sum_{l=1}^{\mu}c_l\theta_l]_{S_\mu}$ can be reduced by a symmetry of S_μ to an algebraic restriction $[\theta_2+\widetilde{c}_3\theta_3++\widetilde{c}_{4+k}\theta_{4+k}]_{S_\mu}$.*

PROOF OF LEMMA 4.13. If $c_1=0$ and $c_2\neq 0$ and $c_{4+k}\neq 0$ and $c_l=0$ for $5\le l<4+k$, then $\mathcal{A}=[c_2\theta_2+c_3\theta_3+c_4\theta_4+\sum_{l=4+k}^{\mu}c_l\theta_l]_{S_\mu}$.
Let $\mathcal{B}_t=[c_2\theta_2+c_3\theta_3+(1-t)c_4\theta_4+c_{4+k}\theta_{4+k}+(1-t)\sum_{l=5+k}^{\mu}c_l\theta_l]_{S_\mu}$ for $t\in[0;1]$. Then $\mathcal{B}_0=\mathcal{A}$ and $\mathcal{B}_1=[c_2\theta_2+c_3\theta_3+c_{4+k}\theta_{4+k}]_{S_\mu}$. We prove that there exists a family $\Phi_t\in Symm(S_\mu)$, $t\in[0;1]$ such that

$$\Phi_t^*\mathcal{B}_t=\mathcal{B}_0,\ \Phi_0=id. \tag{4.9}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt}=V_t(\Phi_t)$. Then differentiating (4.9) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t=[c_4\theta_4+\sum_{l=5+k}^{\mu}c_l\theta_l]_{S_\mu}. \tag{4.10}$$

We are looking for V_t in the form $V_t=\sum_{k=1}^{\mu-2}b_k(t)X_k$ where the $b_k(t)$ are smooth functions $b_k:[0;1]\to\mathbb{R}$ for $k=1,\dots,\mu-2$. Then by Proposition 4.10 equation (4.10) has a form

$$\begin{bmatrix}-(r+2)c_2 & 0 & rc_3 & 0 & \cdots & \cdots & 0\\ 0 & 0 & (r+2k+4)c_{k+4} & 0 & \cdots & \cdots & 0\\ 0 & 0 & \vdots & \ddots & 0 & \cdots & 0\\ 0 & -\frac{3r^2}{2}c_3 & 3(1-t)rc_{\mu-1} & \cdots & 3rc_{k+4} & 0\cdots & 0\end{bmatrix}\begin{bmatrix}b_1(t)\\ b_2(t)\\ b_3(t)\\ \vdots\\ b_{\mu-2}(t)\end{bmatrix}=\begin{bmatrix}c_4\\ c_{k+5}\\ \vdots\\ c_\mu\end{bmatrix} \tag{4.11}$$

If $c_2\neq 0$ we can solve (4.11). Diffeomorphisms Φ_t may be obtained as a flow of the vector field V_t. The family Φ_t preserves S_μ, because V_t is tangent to S_μ and $\Phi_t^*\mathcal{B}_t=\mathcal{A}$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1=[c_2\theta_2+c_3\theta_3+c_{4+k}\theta_{4+k}]_{S_\mu}$. By the condition $c_2\neq 0$ we have a diffeomorphism $\Psi\in Symm(S_\mu)$ of the form

$$\Psi:(x_1,x_2,x_3)\mapsto(c_2^{-\frac{r}{r+2}}x_1,c_2^{-\frac{r}{r+2}}x_2,c_2^{-\frac{2}{r+2}}x_3), \tag{4.12}$$

and we obtain

$$\Psi^*(\mathcal{B}_1)=[\theta_2+c_3c_2^{-\frac{2r}{r+2}}\theta_3+c_{4+k}c_2^{-(1+\frac{2k}{r+2})}\theta_{4+k}]_{S_\mu}=[\theta_2+\widetilde{c}_3\theta_3+\widetilde{c}_{4+k}\theta_{4+k}]_{S_\mu}.$$

□

LEMMA 4.14. *If* $c_1=0$ *and* $c_2\neq 0$ *and* $c_{4+k}=0$ *for* $k\in\{1,\dots,\mu-5\}$, *then the algebraic restriction* $\mathcal{A}=[\sum_{l=1}^{\mu}c_l\theta_l]_{S_\mu}$ *can be reduced by a symmetry of* S_μ *to an algebraic restriction* $[\theta_2+\widetilde{c}_3\theta_3+\widetilde{c}_\mu\theta_\mu]_{S_\mu}$ *where* $\widetilde{c}_3\widetilde{c}_\mu=0$.

PROOF OF LEMMA 4.14. We use methods similar to those in the proof of the previous lemma. Now $\mathcal{A}=[c_2\theta_2+c_3\theta_3+c_4\theta_4+c_\mu\theta_\mu]_{S_\mu}$.
When $c_3\neq 0$ let $\mathcal{B}_t=[c_2\theta_2+c_3\theta_3+(1-t)c_4\theta_4+(1-t)c_\mu\theta_\mu]_{S_\mu}$ for $t\in[0;1]$. Then $\mathcal{B}_0=\mathcal{A}$ and $\mathcal{B}_1=[c_2\theta_2+c_3\theta_3]_{S_\mu}$. We prove that there exists a family $\Phi_t\in Symm(S_\mu)$, $t\in[0;1]$ such that

$$\Phi_t^*\mathcal{B}_t=\mathcal{B}_0,\ \Phi_0=id. \tag{4.13}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt}=V_t(\Phi_t)$. Then differentiating (4.13) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t=[c_4\theta_4+c_\mu\theta_\mu]_{S_\mu}. \tag{4.14}$$

We are looking for V_t in the form $V_t = \sum_{k=1}^{3} b_k X_k$ where the $b_k \in \mathbb{R}$. Then by Proposition 4.10 equation (4.14) has a form

$$\begin{bmatrix} -(r+2)c_2 & 0 & rc_3 \\ 0 & -\frac{3r^2}{2}c_3 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_\mu \end{bmatrix} \tag{4.15}$$

If $c_3 \neq 0$ we can solve (4.15) and Φ_t may be obtained as a flow of the vector field V_t. The family Φ_t preserves S_μ, because V_t is tangent to S_μ and $\Phi_t^* \mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_2\theta_2 + c_3\theta_3]_{S_\mu}$. By the condition $c_2 \neq 0$ we have a diffeomorphism $\Psi \in Symm(S_\mu)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_2^{-\frac{r}{r+2}} x_1, c_2^{-\frac{r}{r+2}} x_2, c_2^{-\frac{2}{r+2}} x_3), \tag{4.16}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [\theta_2 + c_3 c_2^{-\frac{2r}{r+2}} \theta_3]_{S_\mu} = [\theta_2 + \widetilde{c}_3\theta_3]_{S_\mu}.$$

In the case $c_3 = 0$ we take $\mathcal{B}_t = [c_2\theta_2 + (1-t)c_4\theta_4 + c_\mu\theta_\mu]_{S_\mu}$ for $t \in [0;1]$ and we can solve only the first equation of (4.15). Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_2\theta_2 + c_\mu\theta_\mu]_{S_\mu}$. Using the diffeomorphism (4.16) we obtain

$$\Psi^*(\mathcal{B}_1) = [\theta_2 + c_\mu c_2^{-\frac{3r}{r+2}} \theta_\mu]_{S_\mu} = [\theta_2 + \widetilde{c}_\mu\theta_\mu]_{S_\mu}.$$

□

LEMMA 4.15. *If $c_1 = 0$ and $c_2 = 0$ and $c_3c_{4+k} \neq 0$ and $c_l = 0$ for $5 \leq l < 4+k$, then the algebraic restriction $\mathcal{A} = [\sum_{l=1}^{\mu} c_l\theta_l]_{S_\mu}$ can be reduced by a symmetry of S_μ to an algebraic restriction $[\theta_3 + \widetilde{c}_{4+k}\theta_{4+k} + +\widetilde{c}_{5+k}\theta_{5+k}]_{S_\mu}$.*

PROOF OF LEMMA 4.15. If $c_1 = 0, c_2 = 0$ and $c_3 \neq 0$ and $c_{4+k} \neq 0$ and $c_l = 0$ for $5 \leq l < 4+k$, then $\mathcal{A} = [c_3\theta_3 + c_4\theta_4 + \sum_{l=4+k}^{\mu} c_l\theta_l]_{S_\mu}$.
Let $\mathcal{B}_t = [c_3\theta_3 + (1-t)c_4\theta_4 + c_{4+k}\theta_{4+k} + \sum_{l=5+k}^{\mu} \widetilde{c}_l(t)\theta_l]_{S_\mu}$ for $t \in [0;1]$ where the $\widetilde{c}_l(t)$ are smooth functions $\widetilde{c}_l(t) : [0;1] \to \mathbb{R}$ such that $\widetilde{c}_l(0) = c_l$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_3\theta_3 + c_{4+k}\theta_{4+k} + \sum_{l=5+k}^{\mu} \widetilde{c}_l(1)\theta_l]_{S_\mu}$.

Let Φ_t, $t \in [0;1]$, be the flow of the vector field $V = \frac{c_4}{rc_3} X_3$. We show that there exist functions $\widetilde{c}_l$ such that

$$\Phi_t^*\mathcal{B}_t = \mathcal{B}_0, \ \Phi_0 = id. \tag{4.17}$$

Then differentiating (4.17) we obtain

$$\mathcal{L}_V \mathcal{B}_t = [c_4\theta_4 - \sum_{l=5+k}^{\mu} \frac{d\widetilde{c}_l}{dt}\theta_l]_{S_\mu}. \tag{4.18}$$

We can find the $\widetilde{c}_l$ as solutions of the system of first order linear ODEs defined by (4.18) with the initial data $\widetilde{c}_l(0) = c_l$ for $l = 5+k, \ldots, \mu$. This implies that $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_3\theta_3 + c_{4+k}\theta_{4+k} + \sum_{l=5+k}^{\mu} \widetilde{c}_l(1)\theta_l]_{S_\mu}$ are diffeomorphic. Denote $\hat{c}_l = \widetilde{c}_l(1)$ for $l = 5+k, \ldots, \mu$.

Next let $\mathcal{C}_t = [c_3\theta_3 + c_{4+k}\theta_{4+k} + \hat{c}_{5+k}\theta_{5+k} + (1-t)\sum_{l=6+k}^{\mu} \hat{c}_l\theta_l]_{S_\mu}$ for $t \in [0;1]$. Then $\mathcal{C}_0 = \mathcal{B}_1$ and $\mathcal{C}_1 = [c_3\theta_3 + c_{4+k}\theta_{4+k} + \hat{c}_{5+k}\theta_{5+k}]_{S_\mu}$.

We prove that there exists a family $\Upsilon_t \in Symm(S_\mu)$, $t \in [0;1]$ such that

$$\Upsilon_t^*\mathcal{C}_t = \mathcal{C}_0, \ \Upsilon_0 = id. \tag{4.19}$$

Let V_t be a vector field defined by $\frac{d\Upsilon_t}{dt} = V_t(\Upsilon_t)$. Then differentiating (4.19) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t = [\sum_{l=6+k}^{\mu} \hat{c}_l\theta_l]_{S_\mu}. \tag{4.20}$$

We are looking for V_t in the form $V_t = \sum_{k=4}^{\mu-2} b_k(t)X_k$ where the $b_k(t)$ are smooth functions $b_k : [0;1] \to \mathbb{R}$ for $k = 4,\dots,\mu-2$. Then by Proposition 4.10 equation (4.20) has a form

$$\begin{bmatrix} (r+2k+6)c_{k+4} & 0 & 0 & \cdots & \cdots & 0 \\ (r+2k+8)\hat{c}_{k+5} & (r+2k+8)c_{k+4} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \cdots & 0 \\ 3r(1-t)\hat{c}_{\mu-1} & 3r\hat{c}_{\mu-2}(1-t) & \cdots & 3rc_{k+4} & 0\cdots & 0 \end{bmatrix} \begin{bmatrix} b_4(t) \\ b_5(t) \\ \vdots \\ b_{\mu-2}(t) \end{bmatrix} = \begin{bmatrix} \hat{c}_{k+6} \\ \hat{c}_{k+7} \\ \vdots \\ \hat{c}_\mu \end{bmatrix} \tag{4.21}$$

If $c_{4+k} \neq 0$ we can solve (4.21) and Υ_t may be obtained as a flow of the vector field V_t. The family Υ_t preserves S_μ, because V_t is tangent to S_μ and $\Upsilon_t^*\mathcal{C}_t = \mathcal{C}_0 = \mathcal{B}_1$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1$ and $\mathcal{B}_1$ is diffeomorphic to $\mathcal{C}_1$. By the condition $c_3 \neq 0$ we have a diffeomorphism $\Psi \in Symm(S_\mu)$ of the form

$$\Psi : (x_1,x_2,x_3) \mapsto (|c_3|^{-\frac{1}{2}}x_1, |c_3|^{-\frac{1}{2}}x_2, |c_3|^{-\frac{1}{r}}x_3), \tag{4.22}$$

and we obtain

$$\Psi^*(\mathcal{C}_1) = [\frac{c_3}{|c_3|}\theta_3 + \widetilde{c}_{4+k}\theta_{4+k} + \widetilde{c}_{5+k}\theta_{5+k}]_{S_\mu} = [sgn(c_3)\theta_3 + \widetilde{c}_{4+k}\theta_{4+k} + \widetilde{c}_{5+k}\theta_{5+k}]_{S_\mu}.$$

By the following symmetry of S_μ: $(x_1,x_2,x_3) \mapsto (-x_1,x_2,x_3)$, we have that $[-\theta_3 + \widetilde{c}_{4+k}\theta_{4+k} + \widetilde{c}_{5+k}\theta_{5+k}]_{S_\mu}$ is diffeomorphic to $[\theta_3 - \widetilde{c}_{4+k}\theta_{4+k} - \widetilde{c}_{5+k}\theta_{5+k}]_{S_\mu}$. □

LEMMA 4.16. *If $c_1 = 0$ and $c_2 = 0$ and $c_3 \neq 0$ and $c_l = 0$ for $5 \leq l < \mu - 1$, then the algebraic restriction $\mathcal{A} = [\sum_{l=1}^{\mu} c_l\theta_l]_{S_\mu}$ can be reduced by a symmetry of S_μ to an algebraic restriction $[\theta_3 + \widetilde{c}_{\mu-1}\theta_{\mu-1}]_{S_\mu}$.*

PROOF OF LEMMA 4.16. The proof of this lemma is very similar to the previous case. It suffices to notice that if $c_3 \neq 0$ we can solve the equation

$$\begin{bmatrix} 0 & rc_3 \\ -\frac{3r^2}{2}c_3 & 3rc_{\mu-1} \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_\mu \end{bmatrix} \tag{4.23}$$

□

LEMMA 4.17. *If $c_1 = c_2 = c_3 = 0$ and $c_{4+k} \neq 0$ and $c_l = 0$ for $5 \leq l < 4+k$, then the algebraic restriction $\mathcal{A} = [\sum_{l=1}^{\mu} c_l\theta_l]_{S_\mu}$ can be reduced by a symmetry of S_μ to an algebraic restriction $[c_4\theta_4 + c_{4+k}\theta_{4+k}]_{S_\mu}$.*

PROOF OF LEMMA 4.17. We use similar methods as above to prove this lemma. In this case $\mathcal{A} = [c_4\theta_4 + \sum_{l=4+k}^{\mu} c_l\theta_l]_{S_\mu}$. Let $\mathcal{B}_t = [c_4\theta_4 + c_{4+k}\theta_{4+k} + (1-t)\sum_{l=5+k}^{\mu} c_l\theta_l]_{S_\mu}$ for $t \in [0;1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_4\theta_4 + c_{4+k}\theta_{4+k}]_{S_\mu}$. We prove that there exists a family $\Phi_t \in Symm(S_\mu)$, $t \in [0;1]$ such that

$$\Phi_t^*\mathcal{B}_t = \mathcal{B}_0,\ \Phi_0 = id. \tag{4.24}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt} = V_t(\Phi_t)$. Then differentiating (4.24) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t = [\sum_{l=5+k}^{\mu} c_l\theta_l]_{S_\mu}. \tag{4.25}$$

We are looking for V_t in the form $V_t = \sum_{k=3}^{\mu-2} b_k(t)X_k$ where the $b_k(t)$ are smooth functions $b_k : [0;1] \to \mathbb{R}$ for $k = 3,\dots,\mu-2$. Then by Proposition 4.10 equation (4.25) has a form

$$\begin{bmatrix} (r+2k+4)c_{k+4} & 0 & 0 & \cdots & \cdots & 0 \\ (r+2k+6)c_{k+5}(1-t) & (r+2k+6)c_{k+4} & 0 & \cdots & \cdots & 0 \\ \vdots \quad \ddots & & \ddots & \ddots & 0 & \cdots \; 0 \\ 3rc_{\mu-1}(1-t) & 3rc_{\mu-2}(1-t) & \cdots & 3rc_{k+4} & 0\cdots & 0 \end{bmatrix} \begin{bmatrix} b_3(t) \\ \vdots \\ b_{\mu-2}(t) \end{bmatrix} = \begin{bmatrix} c_{k+5} \\ c_{k+6} \\ \vdots \\ c_\mu \end{bmatrix} \tag{4.26}$$

If $c_{4+k} \neq 0$ we can solve (4.26) and Φ_t may be obtained as a flow of the vector field V_t. The family Φ_t preserves S_μ, because V_t is tangent to S_μ and $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_4\theta_4 + c_{4+k}\theta_{4+k}]_{S_\mu}$.

When $c_4 \neq 0$ we have a diffeomorphism $\Psi \in Symm(S_\mu)$ of the form

$$\Psi : (x_1,x_2,x_3) \mapsto (|c_4|^{-\frac{r}{2r+2}}x_1, |c_4|^{-\frac{r}{2r+2}}x_2, |c_4|^{-\frac{2}{2r+2}}x_3), \tag{4.27}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [sgn(c_4)\theta_4 + c_{4+k}|c_4|^{-(\frac{2k+r+2}{2r+2})}\theta_{4+k}]_{S_\mu} = [\pm\theta_4 + \widetilde{c}_{4+k}\theta_{4+k}]_{S_\mu}.$$

By the following symmetry of S_μ: $(x_1,x_2,x_3) \mapsto (-x_1,x_2,x_3)$, we have that $[-\theta_4 + \widetilde{c}_{4+k}\theta_{4+k}]_{S_\mu}$ is diffeomorphic to $[\theta_4 - \widetilde{c}_{4+k}\theta_{4+k}]_{S_\mu}$.

When $c_{4+k} \neq 0$ then we may use a diffeomorphism $\Psi_1 \in Symm(S_\mu)$ of the form

$$\Psi_1 : (x_1,x_2,x_3) \mapsto (c_{4+k}^{-\frac{r}{2k+r+2}}x_1, c_{4+k}^{-\frac{r}{2k+r+2}}x_2, c_{4+k}^{-\frac{2}{2k+r+2}}x_3), \tag{4.28}$$

and we obtain

$$\Psi_1^*(\mathcal{B}_1) = [c_4c_{4+k}^{-(\frac{2r+2}{2k+r+2})}\theta_4 + \theta_{4+k}]_{S_\mu} = [\widetilde{c}_4\theta_4 + \theta_{4+k}]_{S_\mu}.$$

□

Statement (ii) of Theorem 4.4 follows from Theorem 4.9.

(iii) Now we prove that the parameters c_i are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[\theta_1 + c_2\theta_2 + c_3\theta_3]_{S_\mu}$. From Table 6 we see that the tangent space to the orbit of $[\theta_1 + c_2\theta_2 + c_3\theta_3]_{S_\mu}$ at $[\theta_1 + c_2\theta_2 + c_3\theta_3]_{S_\mu}$ is spanned by the linearly independent algebraic restrictions $[r\theta_1 + rc_2\theta_2 + 2c_3\theta_3]_{S_\mu}$, $[\theta_4]_{S_\mu}, [\theta_5]_{S_\mu}, \dots, [\theta_\mu]_{S_\mu}$. Hence the algebraic restrictions $[\theta_2]_{S_\mu}$ and $[\theta_3]_{S_\mu}$ do not belong to it. Therefore the parameters c_2 and c_3 are independent moduli in the normal form $[\theta_1+c_2\theta_2+c_3\theta_3]_{S_\mu}$.

Statement (iv) of Theorem 4.4 follows from the conditions in the proof of part (i) (codimension) and from Theorem 2.6 and Propositions 3.4 and 2.7 (index of isotropy).

□

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