Proceedings of the Edinburgh Mathematical Society (2012) **55**, 657–683
DOI:10.1017/S0013091510001124

SYMPLECTIC T_7, T_8 SINGULARITIES AND LAGRANGIAN TANGENCY ORDERS

WOJCIECH DOMITRZ AND ŻANETA TRĘBSKA

Faculty of Mathematics and Information Science, Warsaw University of Technology, Plac Politechniki 1, 00661 Warsaw, Poland
(domitrz@mini.pw.edu.pl; ztrebska@mini.pw.edu.pl)

(Received 1 September 2010)

Abstract We study the local symplectic algebra of curves. We use the method of algebraic restrictions to classify symplectic T_7, T_8 singularities. We define discrete symplectic invariants (the Lagrangian tangency orders) and compare them with the index of isotropy. We use these invariants to distinguish symplectic singularities of classical T_7 singularity. We also give the geometric description of symplectic classes of the singularity.

Keywords: symplectic manifold; curves; local symplectic algebra; algebraic restrictions; relative Darboux theorem; singularities

2010 *Mathematics subject classification:* Primary 53D05
Secondary 14H20; 58K50; 58A10

1. Introduction

In this paper we study the symplectic classification of singular curves under the following equivalence.

Definition 1.1. Let N_1 and N_2 be germs of subsets of symplectic space $(\mathbb{R}^{2n}, \omega)$. N_1 and N_2 are *symplectically equivalent* if there exists a symplectomorphism germ

$$\Phi\colon (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$$

such that $\Phi(N_1) = N_2$.

We recall that ω is a symplectic form if ω is a smooth non-degenerate closed 2-form, and $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if Φ is diffeomorphism and $\Phi^*\omega = \omega$.

Symplectic classification of curves was first studied by Arnold. In [**2**] he discovered new symplectic invariants of singular curves. He proved that the A_{2k} singularity of a planar curve (the orbit with respect to standard $\mathcal{A}$-equivalence of parametrized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to symplectic equivalence of parametrized curves). He distinguished different symplectic singularities by different orders of tangency of the parametrized curve to the *nearest* smooth Lagrangian submanifold. He posed the problem of expressing these invariants in terms of the local algebra's interaction with the symplectic structure and he proposed calling this interaction the 'local symplectic algebra'.

In [**12**, **13**] Ishikawa and Janeczko classified symplectic singularities of curves in the two-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous.

We recall that a subset N of $\mathbb{R}^m$ is *quasi-homogeneous* if there exist a coordinate system $(x_1, \dots, x_m)$ on $\mathbb{R}^m$ and positive numbers $w_1, \dots, w_m$ (called weights) such that, for any point $(y_1, \dots, y_m) \in \mathbb{R}^m$ and any $t \in \mathbb{R}$, if $(y_1, \dots, y_m)$ belongs to N, then a point $(t^{w_1} y_1, \dots, t^{w_m} y_m)$ belongs to N.

A symplectic form on a two-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [**12**] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [**9**]. The orbit of action of all diffeomorphism germs agrees with volume-preserving orbit or splits into two volume-preserving orbits (in the case $\mathbb{K} = \mathbb{R}$) for germs which satisfy a special weak form of quasi-homogeneity, e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights $w_i \geqslant 0$ and $\sum_i w_i > 0$.

Symplectic singularity is stably simple if it is simple, and remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space. In [**14**] Kolgushkin classified the stably simple symplectic singularities of parametrized curves (in the $\mathbb{C}$-analytic category). All stably simple symplectic singularities of curves are also quasi-homogeneous.

In [**8**] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential k-forms.

Differential k-forms ω_1 and ω_2 have the same *algebraic restriction* to a subset N if $\omega_1 - \omega_2 = \alpha + \mathrm{d}\beta$, where α is a k-form vanishing on N and β is a $(k-1)$-form vanishing on N.

The generalization of the Darboux–Givental Theorem [**3**] to germs of arbitrary subsets of the symplectic space was obtained in [**8**] (see also [**17**]). This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart from the algebraic restriction [**7**, **8**]. The dimension of the space of algebraic restrictions of closed 2-forms to a one-dimensional quasi-homogeneous isolated complete intersection singularity C is equal to the multiplicity of C [**8**]. In [**6**] it was proved that the space of algebraic restrictions of closed 2-forms to a one-dimensional (singular) analytic variety is finite dimensional. In [**8**] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular, the complete symplectic classification of classical A–D–E singularities of planar curves and the S_5 singularity were obtained. Most of the different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.

In this paper, following ideas from [**2**,**6**], we use new discrete symplectic invariants: the Lagrangian tangency orders (see § 2.1). Although this invariant has a similar definition to the index of isotropy, its nature is different. Since the Lagrangian tangency order takes into account the weights of quasi-homogeneity of curves, it allows us to distinguish more symplectic classes in many cases. For example, using the Lagrangian tangency order, we are able to the distinguish classes E_6^3 and $E_6^{4,\pm}$ of classical planar singularity E_6, which cannot be distinguished by the isotropy index or by the symplectic multiplicity. We also present other examples of singularities which can be distinguished only by the Lagrangian tangency order. On the other hand, there are singularities for which symplectic classes can be distinguished by the index of isotropy but not by the Lagrangian tangency order, for example, the parametric curve with semigroup $(3, 7, 11)$ and T_8 singularity. These examples show that there are no simple relations between the Lagrangian tangency order and the index of isotropy, even for the case of parametric curves.

We also obtain the complete symplectic classification of the classical isolated complete intersection singularity T_7 using the method of algebraic restrictions (Theorem 3.1). We calculate discrete symplectic invariants for this classification (Theorems 3.3) and we present geometric descriptions of its symplectic orbits (Theorem 3.5).

The paper is organized as follows. In § 2 we present known discrete symplectic invariants and introduce the Lagrangian tangency orders. We also compare the Lagrangian tangency order and the index of isotropy. Symplectic classification of T_7 singularity is studied in § 3. In § 4 we recall the method of algebraic restrictions and use it to classify T_7 symplectic singularities.

2. Discrete symplectic invariants

We define discrete symplectic invariants to distinguish symplectic singularity classes. The first one is the symplectic multiplicity [**8**] introduced in [**12**] as a symplectic defect of a curve.

Let N be a germ of a subset of $(\mathbb{R}^{2n}, \omega)$.

Definition 2.1. The *symplectic multiplicity* $\mu_{sympl}(N)$ of N is the codimension of a symplectic orbit of N in an orbit of N with respect to the action of the group of local diffeomorphisms.

The second one is the index of isotropy [**8**].

Definition 2.2. The *index of isotropy* $\operatorname{ind}(N)$ of N is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M containing N.

This invariant has geometrical interpretation. An equivalent definition is as follows: the index of isotropy of N is the maximal order of tangency between non-singular submanifolds containing N and non-singular isotropic submanifolds of the same dimension. The index of isotropy is equal to 0 if N is not contained in any non-singular submanifold which is tangent to some isotropic submanifold of the same dimension. If N is contained in a non-singular Lagrangian submanifold, then the index of isotropy is ∞.

Remark 2.3. If N consists of invariant components C_i we can calculate the index of isotropy for each component $\operatorname{ind}(C_i)$ as the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M containing C_i.

The symplectic multiplicity and the index of isotropy can be described in terms of algebraic restrictions (Propositions 4.6 and 4.7).

2.1. Lagrangian tangency order

There is one more discrete symplectic invariant, introduced in [**6**] (following ideas from [**2**]), which is defined specifically for a parametrized curve. This is the maximal tangency order of a curve $f\colon \mathbb{R} \to M$ to a smooth Lagrangian submanifold. If $H_1 = \cdots = H_n = 0$ define a smooth submanifold L in the symplectic space, then the tangency order of a curve $f\colon \mathbb{R} \to M$ to L is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \ldots, H_n \circ f$. We denote the tangency order of f to L by $t(f, L)$.

Definition 2.4. The *Lagrangian tangency order $Lt(f)$ of a curve f* is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds L of the symplectic space.

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of algebraic restrictions (Proposition 4.8).

We can generalize this invariant for curves which may be parametrized analytically. Lagrangian tangency order is the same for every 'good' analytic parametrization of a curve [**16**]. Considering only such parametrizations, we can choose one and calculate the invariant for it. It is easy to show that this invariant does not depend on chosen parametrization.

Proposition 2.5. *Let $f\colon \mathbb{R} \to M$ and $g\colon \mathbb{R} \to M$ be good analytic parametrizations of the same curve. Then $Lt(f) = Lt(g)$.*

Proof. There exists a diffeomorphism $\theta\colon \mathbb{R} \to \mathbb{R}$ such that $g(s) = f(\theta(s))$ and $\mathrm{d}\theta/\mathrm{d}s|_0 \neq 0$. Let $H_1 = \cdots = H_n = 0$ define a smooth submanifold L in the symplectic space. If $\mathrm{d}^l(H_i \circ f)/\mathrm{d}t^l|_0 = 0$ for $l = 1, \ldots, k$, then

$$\left.\frac{\mathrm{d}^{k+1}(H_i \circ g)}{\mathrm{d}s^{k+1}}\right|_0 = \left.\frac{\mathrm{d}^{k+1}(H_i \circ f \circ \theta)}{\mathrm{d}s^{k+1}}\right|_0 = \left.\frac{\mathrm{d}^{k+1}(H_i \circ f)}{\mathrm{d}t^{k+1}}\right|_0 \cdot \left.\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^{k+1}\right|_0,$$

so the orders of vanishing at 0 of functions $H_i \circ f$ and $H_i \circ g$ are equal, and hence $t(f, L) = t(g, L)$, which implies that $Lt(f) = Lt(g)$. $\square$

We can generalize Lagrangian tangency order for sets containing parametric curves. Let N be a subset of a symplectic space $(\mathbb{R}^{2n}, \omega)$.

Definition 2.6. The *tangency order of the germ of a subset N to the germ of a submanifold L* $t[N, L]$ is equal to the minimum of $t(f, L)$ over all parametrized curve-germs f such that $\operatorname{Im} f \subseteq N$.

Definition 2.7. The *Lagrangian tangency order of N*, $Lt(N)$, is equal to the maximum of $t[N, L]$ over all smooth Lagrangian submanifold-germs L of the symplectic space.

Table 1. *Comparison of symplectic invariants of A_k singularity.*

normal form	parametrization	$Lt(N)$	ind
$A_k^{0\leqslant i\leqslant k-1}$ (k even)	$C\colon (t^2, t^{k+1+2i}, t^{k+1}, 0, \ldots, 0)$	$k+1+2i$	i
A_k^k (k even)	$C\colon (t^2, 0, t^{k+1}, 0, \ldots, 0)$	∞	∞
$A_k^{0\leqslant i\leqslant k-1}$ (k odd)	$B_\pm\colon (t, \pm t^{(k+1)/2+i}, \pm t^{(k+1)/2}, 0, \ldots, 0)$	$\frac{1}{2}(k+1)+i$	i
A_k^k, (k odd)	$B_\pm\colon (t, 0, \pm t^{(k+1)/2}, 0, \ldots, 0)$	∞	∞

In this paper we consider N which are singular analytic curves. They may be identified with a multi-germ of parametric curves. We define invariants which are special cases of the above definition.

Consider a multi-germ $(f_i)_{i\in\{1,\ldots,r\}}$ of analytically parametrized curves f_i. For any smooth submanifold L in the symplectic space we have r-tuples $(t(f_1, L), \ldots, t(f_r, L))$.

Definition 2.8. For any $I \subseteq \{1, \ldots, r\}$ we define *the tangency order of the multi-germ* $(f_i)_{i\in I}$ to L:

$$t[(f_i)_{i\in I}, L] = \min_{i\in I} t(f_i, L).$$

Definition 2.9. The *Lagrangian tangency order* $Lt((f_i)_{i\in I})$ *of a multi-germ* $(f_i)_{i\in I}$ is the maximum of $t[(f_i)_{i\in I}, L]$ over all smooth Lagrangian submanifolds L of the symplectic space.

For multi-germs we can also define relative invariants according to selected branches or collections of branches.

Definition 2.10. Let $S \subseteq I \subseteq \{1, \ldots, r\}$. For $i \in S$ let us fix numbers $t_i \leqslant Lt(f_i)$. The *relative Lagrangian tangency order* $Lt[(f_i)_{i\in I} : (S, (t_i)_{i\in S})]$ of a multi-germ $(f_i)_{i\in I}$ related to S and $(t_i)_{i\in S}$ is the maximum of $t[(f_i)_{i\in I\setminus S}, L]$ over all smooth Lagrangian submanifolds L of the symplectic space for which $t(f_i, L) = t_i$, if such submanifolds exist, or $-\infty$ if there are no such submanifolds.

We can also define special relative invariants according to selected branches of the multi-germ.

Definition 2.11. For fixed $j \in I$ the *Lagrangian tangency order related to* f_j of a multi-germ $(f_i)_{i\in I}$ denoted by $Lt[(f_i)_{i\in I} : f_j]$ is the maximum of $t[(f_i)_{i\in I\setminus\{j\}}, L]$ over all smooth Lagrangian submanifolds L of the symplectic space for which $t(f_j, L) = Lt(f_j)$,

These invariants have geometric interpretations. If $Lt(f_i) = \infty$, then a branch f_i is included in a smooth Lagrangian submanifold. If $Lt((f_i)_{i\in I}) = \infty$, then there exists a Lagrangian submanifold containing all the curves f_i for $i \in I$.

We may use these invariants to distinguish symplectic singularities.

2.2. Comparison of the Lagrangian tangency order and the index of isotropy

Definitions of the Lagrangian tangency order and the index of isotropy are similar. They show how far a variety N is from the nearest non-singular Lagrangian submanifold.

Table 2. *Symplectic invariants of D_k singularity.*
(The branch C_1 has a form $(t,0,0,0,\dots,0)$. If k is odd, then C_2 has a form $(t^{k-2}, f(t), t^2, 0,\dots,0)$ and $\lambda_k = 1$. If k is even, then C_2 consists of two branches: $B_\pm : (\pm t^{(k-2)/2}, f(t), t, 0,\dots,0)$ and $\lambda_k = \frac{1}{2}$.)

normal form	$f(t)$	$Lt(N)$	$Lt(C_2)$	ind	ind_2
D_k^0	$t^{2\lambda_k}$	$2\lambda_k$	$(k-2)\lambda_k$	0	0
D_k^1	$bt^{k\lambda_k} + \frac{1}{2}t^{4\lambda_k}$	$k\lambda_k$	$k\lambda_k$	1	1
D_k^i $(1 < i < k-3)$	$bt^{k\lambda_k} + \frac{1}{i+1}t^{2(i+1)\lambda_k}$, $b \neq 0$	$k\lambda_k$	$(k-2+2i)\lambda_k$	1	i
	$\frac{1}{i+1}t^{2(i+1)\lambda_k}$	$(k-2+2i)\lambda_k$	$(k-2+2i)\lambda_k$	i	i
$D_k^{k-3,\pm}$	$(\pm 1)^k t^{k\lambda_k} + \frac{b}{k-2}t^{2(k-2)\lambda_k}$	$k\lambda_k$	∞	1	∞
D_k^{k-2}	$\frac{1}{k-2}t^{2(k-2)\lambda_k}$	$(3k-8)\lambda_k$	∞	$k-3$	∞
D_k^{k-1}	$\frac{1}{k-1}t^{2(k-1)\lambda_k}$	$(3k-6)\lambda_k$	∞	$k-2$	∞
D_k^k	0	∞	∞	∞	∞

The index of isotropy of a quasi-homogeneous set N is ∞ if and only if the Lagrangian tangency order of N is ∞. Studying classical singularities, we have found examples of all possible interactions between these invariants.

Example 2.12. For some singularities the index of isotropy distinguishes the same symplectic classes that can be distinguished by the Lagrangian tangency order. It is observed, for example, for planar curves: the classical A_k and D_k singularities (Tables 1 and 2) and for S_μ singularities studied in [**10**].

A complete symplectic classification of classical A–D–E singularities of planar curves was obtained using a method of algebraic restriction in [**8**]. Below, we compare the Lagrangian tangency order and the index of isotropy for these singularities. A curve N may be described as a parametrized curve or as a union of parametrized components C_i preserved by local diffeomorphisms in the symplectic space $(\mathbb{R}^{2n},\ \omega_0 = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i)$, $n \geqslant 2$. For calculating the Lagrangian tangency orders, we give their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \dots, p_n, q_n)$.

Denote by (A_k) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$A_k = \{x \in \mathbb{R}^{2n \geqslant 4} \colon x_1^{k+1} - x_2^2 = x_{\geqslant 3} = 0\}. \tag{2.1}$$

A curve $N \in (A_k)$ can be described as a parametrized singular curve C for k even, or as a pair of two smooth parametrized branches B_+ and B_- if k is odd. We denote $Lt(C)$ or $Lt(B_+, B_-)$, respectively, by $Lt(N)$.

Table 3. *Symplectic invariants of E_6 singularity.*

normal form	parametrization	$Lt(N)$	ind	μ^{symp}
E_6^0	$(t^4, t^3, t^3, 0, \dots, 0)$	4	0	0
$E_6^{1,\pm}$	$(t^4, \pm\frac{1}{2}t^6 + bt^7, t^3, 0, \dots, 0)$	7	1	2
E_6^2	$(t^4, t^7 + \frac{1}{3}bt^9, t^3, 0, \dots, 0)$	8	1	3
E_6^3	$(t^4, \frac{1}{3}t^9 + \frac{1}{2}bt^{10}, t^3, 0, \dots, 0)$	**10**	**2**	**4**
$E_6^{4,\pm}$	$(t^4, \pm\frac{1}{2}t^{10}, t^3, 0, \dots, 0)$	**11**	**2**	**4**
E_6^5	$(t^4, \frac{1}{3}t^{13}, t^3, 0, \dots, 0)$	14	3	5
E_6^6	$(t^4, 0, t^3, 0, \dots, 0)$	∞	∞	6

Denote by (D_k) for $k \geqslant 4$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$D_k = \{x \in \mathbb{R}^{2n\geqslant 4} \colon x_1^2 x_2 - x_2^{k-1} = x_{\geqslant 3} = 0\}. \tag{2.2}$$

A curve $N \in (D_k)$ consists of two invariant components: C_1 (smooth) and C_2 (singular diffeomorphic to A_{k-3}). C_2 may consist of one or two branches, depending on k. To distinguish the symplectic classes completely we need two invariants: $Lt(N)$ (the Lagrangian tangency order of N) and $Lt(C_2)$ (the Lagrangian tangency order of the singular component C_2). Equivalently, we can use the index of isotropy of N, ind, and the index of isotropy of C_2, ind_2.

Example 2.13. There are also symplectic singularities distinguished by the Lagrangian tangency order but not by the index of isotropy. The simplest example is planar singularity E_6 (Table 3). We also observe such a 'greater sensitivity' of the Lagrangian tangency order for E_7 and E_8 singularities and for parametric curves with the semigroups $(3, 4, 5)$, $(3, 5, 7)$ and $(3, 7, 8)$ studied in [**6**].

Denote by (E_6) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$E_6 = \{x \in \mathbb{R}^{2n\geqslant 4} \colon x_1^3 - x_2^4 = x_{\geqslant 3} = 0\}. \tag{2.3}$$

As can be seen in Table 3, we are able, by the Lagrangian tangency order, to distinguish the classes E_6^3 and $E_6^{4,\pm}$ which cannot be distinguished by the index of isotropy or by the symplectic multiplicity.

Example 2.14. Some symplectic singularities can be distinguished by the index of isotropy but not by the Lagrangian tangency order. We observe such a situation for a parametric quasi-homogeneous curve-germ with semigroup $(3, 7, 11)$ listed as a stably simple singularity of curves in [**1**]. Another example is the T_8 singularity presented below (see the rows for $(T_8)^4$ and $(T_8)^{6,2}$ in Table 6).

The germ of a curve $f \colon (\mathbb{R}, 0) \to (\mathbb{R}^{2n}, 0)$ with semigroup $(3, 7, 11)$ is diffeomorphic to the curve $t \to (t^3, t^7, t^{11}, 0, \dots, 0)$. Among symplectic singularities of this curve-germ

Table 4. *Symplectic invariants for some symplectic classes of the curve with semigroup* $(3, 7, 11)$.

Class	Normal form of f	$Lt(f)$	ind
1	$t \to (t^3, t^{10}, t^7, 0, t^{11}, 0, \ldots, 0)$	10	1
2	$t \to (t^3, t^{11}, t^7, 0, t^{11}, 0, \ldots, 0)$	11	0
3	$t \to (t^3, t^{10} + ct^{11}, t^7, 0, t^{11}, 0, \ldots, 0)$, $c \neq 0$	10	0

in the symplectic space $(\mathbb{R}^{2n},\ \omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i)$ with the canonical coordinates $(p_1, q_1, \ldots, p_n, q_n)$ we have, for example, the classes represented by the normal forms given in Table 4.

Symplectic classes (1) and (3) have the same Lagrangian tangency order (equal to 10) but have different indices of isotropy (1 and 0, respectively). Symplectic classes (2) and (3) have the same index of isotropy (equal to 0) but have different Lagrangian tangency orders (11 and 10, respectively). We also observe that the Lagrangian tangency order for class (1) is less than that for class (2) but the inverse inequality is satisfied for the indices of isotropy.

Another example is T_8 singularity. Denote by (T_8) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$T_8 = \{x \in \mathbb{R}^{2n \geqslant 4}\colon x_1^2 + x_2^3 - x_3^4 = x_2 x_3 = x_{\geqslant 4} = 0\}. \tag{2.4}$$

This is the classical one-dimensional isolated complete intersection singularity T_8 [**5**, **11**].

Let $N \in (T_8)$. N is quasi-homogeneous with weights $w(x_1) = 6$, $w(x_2) = 4$, $w(x_3) = 3$. A curve N consists of two invariant singular components: C_1 (diffeomorphic to the A_2 singularity) and C_2 (diffeomorphic to the A_3 singularity), which is a union of two smooth branches B_+ and B_-. In local coordinates they have the form

$$\mathcal{C}_1 = \{x_1^2 + x_2^3 = 0,\ x_3 = x_{\geqslant 4} = 0\},$$
$$\mathcal{B}_\pm = \{x_1 \pm x_3^2 = 0,\ x_2 = x_{\geqslant 4} = 0\}.$$

Using the method of algebraic restrictions, one can obtain, in the same way as presented in the last two sections for the case of the T_7 singularity, the following complete classification of symplectic T_8 singularities.

Theorem 2.15. *Any stratified submanifold of the symplectic space* $(\mathbb{R}^{2n},\ \omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i)$ *which is diffeomorphic to* T_8 *is symplectically equivalent to one and only one of the normal forms* $(T_8)^i$, $i = 0, 1, \ldots, 8$. *The parameters* c, c_1, c_2, c_3 *of the normal forms are moduli:*

$$T_8^0\colon p_1^2 + p_2^3 - q_1^4 = 0, \qquad p_2 q_1 = 0,\ q_2 = c_1 q_1 - c_2 p_1 p_{\geqslant 3} = q_{\geqslant 3} = 0,$$
$$c_1 \cdot c_2 \neq 0;$$

$$T_8^{1_2}\colon p_1^2 + p_2^3 - q_1^4 = 0, \qquad p_2 q_1 = 0,\ q_2 = c_1 q_1 - c_2 p_1 - c_3 p_1 p_2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0,$$
$$c_1 \cdot c_2 = 0;$$

$$T_8^{1_3}: p_1^2 + q_1^3 - q_2^4 = 0, \quad q_1q_2 = 0,\ p_2 = c_1q_1 + c_2p_1q_2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0, \quad c_1 \cdot c_2 \neq 0;$$

$$T_8^{2_3}: p_1^2 + q_1^3 - q_2^4 = 0, \quad q_1q_2 = 0,\ p_2 = c_1q_1 + c_2p_1q_2 + c_3p_1q_2^2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0, \quad c_1 \cdot c_2 = 0;$$

$$T_8^{2_{>3}}: p_2^2 + p_1^3 - q_1^4 = 0, \quad p_1q_1 = 0,\ q_2 = \tfrac{1}{2}c_1q_1^2 + \tfrac{1}{2}c_2p_1^2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0, \quad c_1 \neq 0;$$

$$T_8^{3,0}: p_2^2 + p_1^3 - q_1^4 = 0, \quad p_1q_1 = 0,\ q_2 = \tfrac{1}{2}c_1p_1^2 + \tfrac{1}{3}c_2q_1^3,\ p_{\geqslant 3} = q_{\geqslant 3} = 0, \quad (c_1, c_2) \neq (0,0);$$

$$T_8^{5,0}: p_2^2 + p_1^3 - q_1^4 = 0, \quad p_1q_1 = 0,\ q_2 = \tfrac{1}{4}cq_1^4,\ p_{\geqslant 3} = q_{\geqslant 3} = 0;$$

$$T_8^{3,1}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_1 = \tfrac{1}{2}p_3^2 + \tfrac{1}{2}c_2p_2^2,\ q_2 = -c_1p_1p_3,\ p_{\geqslant 4} = q_{\geqslant 3} = 0;$$

$$T_8^{4}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_1 = \tfrac{1}{2}c_1p_2^2 + \tfrac{1}{3}c_2p_3^3,\ q_2 = -p_1p_3,\ p_{\geqslant 4} = q_{\geqslant 3} = 0, \quad (c_1, c_2) \neq (0,0);$$

$$T_8^{6,1}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_1 = \tfrac{1}{4}cp_3^4,\ q_2 = -p_1p_3,\ p_{\geqslant 4} = q_{\geqslant 3} = 0;$$

$$T_8^{5,1}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_1 = \tfrac{1}{2}p_2^2 + \tfrac{1}{3}cp_3^3,\ p_{\geqslant 4} = q_{\geqslant 2} = 0;$$

$$T_8^{6,2}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_1 = \tfrac{1}{3}p_3^3,\ p_{\geqslant 4} = q_{\geqslant 2} = 0;$$

$$T_8^{7}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_1 = \tfrac{1}{4}p_3^4,\ p_{\geqslant 4} = q_{\geqslant 2} = 0;$$

$$T_8^{8}: p_1^2 + p_2^3 - p_3^4 = 0, \quad p_2p_3 = 0,\ q_{\geqslant 1} = p_{\geqslant 4} = 0.$$

Lagrangian tangency orders and indices of isotropy were used to obtain a detailed classification of (T_8). A curve $N \in (T_8)$ may be described as a union of three parametrical branches C_1, B_+, B_-. Their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \dots, p_n, q_n)$ is presented in the second column of Tables 5 and 6. To distinguish the classes of this singularity, we need the following three invariants:

(i) $Lt(N) = Lt(C_1, B_+, B_-) = \max_L(\min\{t(C_1, L), t(B_+, L), t(B_-, L)\})$;

(ii) $L_1 = Lt(C_1) = \max_L(t(C_1, L))$;

(iii) $L_2 = Lt(C_2) = \max_L(\min\{t(B_+, L), t(B_-, L)\})$;

here L is a smooth Lagrangian submanifold of the symplectic space.

Branches B_+ and B_- are diffeomorphic and are not preserved by all symmetries of T_8, so we can use neither $Lt(B_+)$ nor $Lt(B_-)$ as invariants. Considering the triples (Lt, L_1, L_2), we obtain a more detailed classification of symplectic singularities of T_8 than the classification given in Theorem 2.15. Some subclasses appear to have a natural geometric interpretation.

We also calculate the index of isotropy of $N \in (T_8)$, denoted by ind, and the indices of isotropy of components C_1 and C_2, denoted by ind_1 and ind_2, respectively. In Tables 5 and 6 we present a comparison of the invariants.

Table 5. *Symplectic invariants for symplectic classes of T_8 singularity when $\omega|_W \neq 0$.* (W is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n\geqslant 4},\omega)$ containing $N \in (T_8)$.)

class	parametrization	conditions	Lt	L_1	L_2	ind	ind$_1$	ind$_2$
$(T_8)^0$	$(t^3,0,-t^2,-c_2t^3,0,\dots)$	$c_1\cdot c_2\neq 0$	2	3	2	0	0	0
	$(\pm t^2,t,0,c_1t\mp c_2t^2,0,\dots)$							
$(T_8)^1_2$	$(t^3,0,-t^2,-c_2t^3+c_3t^5,0,\dots)$	$c_1=0,\ c_2\neq 0$	2	3	2	0	0	0
	$(\pm t^2,t,0,c_1t\mp c_2t^2,0,\dots)$	$c_2=0,\ c_3\neq 0$	2	5	2	0	1	0
		$c_2=c_3=0$	2	∞	2	0	∞	0
$(T_8)^1_3$	$(t^3,-t^2,-c_1t^2,0,0,\dots)$	$c_1\cdot c_2\neq 0$	2	3	3	0	0	1
	$(\pm t^2,0,\pm c_2t^3,t,0,\dots)$							
$(T_8)^2_3$	$(t^3,-t^2,-c_1t^2,0,0,\dots)$	$c_1=0,\ c_2\neq 0$	2	3	3	0	0	1
	$(\pm t^2,0,\pm c_2t^3\pm c_3t^4,t,0,\dots)$	$c_2=0,\ c_3\neq 0$	2	3	4	0	0	2
		$c_2=0,\ c_3=0$	2	3	∞	0	0	∞
$(T_8)^2_{>3}$	$(-t^2,0,t^3,\frac{1}{2}c_2t^4,0,\dots)$	$c_1\cdot c_2\neq 0$	2	5	3	0	1	1
	$(0,t,\pm t^2,\frac{1}{2}c_1t^2,0,\dots)$	$c_1\neq 0, c_2=0$	2	∞	3	0	∞	1
$(T_8)^{3,0}$	$(-t^2,0,t^3,\frac{1}{2}c_1t^4,0,\dots)$	$c_1\cdot c_2\neq 0$	2	5	4	0	1	2
	$(0,t,\pm t^2,\frac{1}{3}c_2t^3,0,\dots)$	$c_1\neq 0, c_2=0$	2	5	∞	0	1	∞
		$c_1=0,\ c_2\neq 0$	2	∞	4	0	∞	2
$(T_8)^{5,0}$	$(-t^2,0,t^3,0,0,\dots)$		2	∞	∞	0	∞	∞
	$(0,t,\pm t^2,\frac{1}{4}ct^4,0,\dots)$							

Remark 2.16. We note that considering the pairs (L_1,L_2) gives the same classification as considering the pairs (ind$_1$,ind$_2$). To distinguish classes $(T_8)^0$ and $(T_8)^1_2$ for $c_2\neq 0$, $c_1=0$ we may use Lagrangian tangency order related to component C_1. We have $Lt[C_2:C_1]=1$ for class $(T_8)^0$ but $Lt[C_2:C_1]=2$ for class $(T_8)^1_2$ if $c_2\neq 0$, $c_1=0$. In similar way, we can distinguish classes $(T_8)^1_3$ and $(T_8)^2_3$ for $c_2\neq 0$, $c_1=0$.

Remark 2.17. We can see from Table 6 that the Lagrangian tangency order, Lt, distinguishes different classes from the index of isotropy, ind. For example, the class $(T_8)^4$ in the case $c_1=0$, $c_2\neq 0$ and the class $(T_8)^{6,2}$ are distinguished by the index of isotropy, ind, but are not distinguished by the Lagrangian tangency order. We can distinguish these classes using the relative Lagrangian tangency order: for the class $(T_8)^4$ in the case $c_1=0$, $c_2\neq 0$ we have $Lt[C_2:C_1]=3$, and for the class $(T_8)^{6,2}$ we have $Lt[C_2:C_1]=4$.

The index of isotropy for the classes $(T_8)^{3,1}$, $(T_8)^4$, $(T_8)^{6,1}$, $(T_8)^{5,1}$ is less than that for the class $(T_8)^{6,2}$ but the analogical inequality does not hold for the Lagrangian tangency order.

We are not able to distinguish all symplectic classes using the Lagrangian tangency orders or the indices of isotropy, but we can do so by checking geometric conditions formulated analogously to the T_7 singularity (see § 3.2).

Table 6. *Lagrangian invariants for symplectic classes of T_8 singularity when $\omega|_W = 0$.* (W is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n\geqslant 6}, \omega)$ containing $N \in (T_8)$.)

class	parametrization	conditions	Lt	L_1	L_2	ind	ind_1	ind_2
$(T_8)^{3,1}$	$(t^3, \frac{1}{2}c_2t^4, -t^2, 0, 0, 0, \dots)$	$c_2 \neq 0$	3	5	3	1	1	1
	$(\pm t^2, \frac{1}{2}t^2, 0, \mp c_1t^3, t, 0, \dots)$	$c_2 = 0$	3	∞	3	1	∞	1
$(T_8)^4$	$(t^3, \frac{1}{2}c_1t^4, -t^2, 0, 0, 0, \dots)$	$c_1 \cdot c_2 \neq 0$	4	5	4	1	1	2
	$(\pm t^2, \frac{1}{3}c_2t^3, 0, \mp t^3, t, 0, \dots)$	$c_1 = 0$, $c_2 \neq 0$	**4**	$\boldsymbol{\infty}$	**4**	**1**	$\boldsymbol{\infty}$	**2**
		$c_1 \neq 0, c_2 = 0$	5	5	∞	1	1	∞
$(T_8)^{6,1}$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$		5	∞	∞	1	∞	∞
	$(\pm t^2, \frac{1}{4}ct^4, 0, \mp t^3, t, 0, \dots)$							
$(T_8)^{5,1}$	$(t^3, \frac{1}{2}t^4, -t^2, 0, 0, 0, \dots)$	$c \neq 0$	4	5	4	1	1	2
	$(\pm t^2, \frac{1}{3}ct^3, 0, 0, t, 0, \dots)$	$c = 0$	5	5	∞	1	1	∞
$(T_8)^{6,2}$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$		**4**	$\boldsymbol{\infty}$	**4**	**2**	$\boldsymbol{\infty}$	**2**
	$(\pm t^2, \frac{1}{3}t^3, 0, 0, t, 0, \dots)$							
$(T_8)^7$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$		7	∞	∞	3	∞	∞
	$(\pm t^2, \frac{1}{4}t^4, 0, 0, t, 0, \dots)$							
$(T_8)^8$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$		∞	∞	∞	∞	∞	∞
	$(\pm t^2, 0, 0, 0, t, 0, \dots)$							

3. Symplectic T_7-singularities

Denote by (T_7) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$T_7 = \{x \in \mathbb{R}^{2n\geqslant 4} \colon x_1^2 + x_2^3 + x_3^3 = x_2x_3 = x_{\geqslant 4} = 0\}. \tag{3.1}$$

This is the classical one-dimensional isolated complete intersection singularity T_7 [**5**, **11**]. N is quasi-homogeneous with weights $w(x_1) = 3, w(x_2) = w(x_3) = 2$.

We use the method of algebraic restrictions to obtain the complete classification of symplectic singularities of (T_7) presented in the following theorem.

Theorem 3.1. *Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i)$ which is diffeomorphic to T_7 is symplectically equivalent to one and only one of the normal forms T_7^i, $i = 0, 1, \dots, 7$ (respectively, $i = 0, 1, 2, 4$). The parameters c, c_1, c_2 of the normal forms are moduli:*

$T_7^0 \colon p_1^2 + p_2^3 + q_2^3 = 0,\ p_2q_2 = 0,\ q_1 = c_1q_2 + c_2p_2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0,\ c_1 \cdot c_2 \neq 0;$

$T_7^1 \colon p_1^2 + p_2^3 + q_1^3 = 0,\ p_2q_1 = 0,\ q_2 = c_1q_1 - c_2p_1p_2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0;$

$T_7^2 \colon p_1^2 + p_2^3 + q_2^3 = 0,\ p_2q_2 = 0,\ q_1 = \frac{1}{2}c_1q_2^2 + \frac{1}{2}c_2p_2^2,\ p_{\geqslant 3} = q_{\geqslant 3} = 0,\ (c_1, c_2) \neq (0, 0);$

$T_7^4 \colon p_1^2 + p_2^3 + q_2^3 = 0,\ p_2q_2 = 0,\ q_1 = \frac{1}{3}cq_2^3,\ p_{\geqslant 3} = q_{\geqslant 3} = 0;$

T_7^3: $p_1^2+p_2^3+p_3^3=0,\ p_2p_3=0,\ q_1=\frac{1}{2}c_1p_2^2+\frac{1}{2}p_3^2,\ q_2=-c_2p_1p_3,\ p_{\geqslant 4}=q_{\geqslant 3}=0;$
T_7^5: $p_1^2+p_2^3+p_3^3=0,\ p_2p_3=0,\ q_1=\frac{1}{3}cp_3^3,\ q_2=-p_1p_3,\ p_{\geqslant 4}=q_{\geqslant 3}=0;$
T_7^6: $p_1^2+p_2^3+p_3^3=0,\ p_2p_3=0,\ q_1=\frac{1}{3}p_3^3,\ p_{\geqslant 4}=q_{\geqslant 2}=0;$
T_7^7: $p_1^2+p_2^3+p_3^3=0,\ p_2p_3=0,\ q_{\geqslant 1}=p_{\geqslant 4}=0.$

In § 3.1 we use the Lagrangian tangency orders to distinguish more symplectic singularity classes. In § 3.2 we propose a geometric description of these singularities that confirms this more detailed classification. Some of the proofs are presented in § 4.

3.1. Distinguishing symplectic classes of T_7 by Lagrangian tangency orders and the indices of isotropy

A curve $N \in (T_7)$ can be described as a union of two parametrical branches B_1 and B_2. Their parametrization is given in the second column of Table 7. To distinguish the classes of this singularity we need the following three invariants:

(i) $Lt(N)=Lt(B_1,B_2)=\max_L(\min\{t(B_1,L),t(B_2,L)\})$;

(ii) $L_{\mathrm{n}}=\max\{Lt(B_1),Lt(B_2)\}=\max\{\max_L t(B_1,L),\max_L t(B_2,L)\}$;

(iii) $L_{\mathrm{f}}=\min\{Lt(B_1),Lt(B_2)\}=\min\{\max_L t(B_1,L),\max_L t(B_2,L)\}$.

Here L is a smooth Lagrangian submanifold of the symplectic space.

Branches B_1 and B_2 are diffeomorphic and are not preserved by all symmetries of T_7, so neither $Lt(B_1)$ nor $Lt(B_2)$ can be used as invariants. The new invariants are defined instead: L_{n}, which describes the Lagrangian tangency order of the *nearest* branch, and L_{f}, which represents the Lagrangian tangency order of the *farthest* branch. Considering the triples $(Lt(N),L_{\mathrm{n}},L_{\mathrm{f}})$, we obtain a more detailed classification of symplectic singularities of T_7 than the classification given in Table 11. Some subclasses appear to have a natural geometric interpretation (Tables 8 and 9).

Remark 3.2. We can define the indices of isotropy for branches analogously to the Lagrangian tangency orders and use them to characterize singularities of T_7. We use the following invariants:

(i) $\mathrm{ind_n}=\max\{\mathrm{ind}(B_1),\mathrm{ind}(B_2)\}$;

(ii) $\mathrm{ind_f}=\min\{\mathrm{ind}(B_1),\mathrm{ind}(B_2)\}$.

Here $\mathrm{ind}(B_1)$, $\mathrm{ind}(B_2)$ denote the indices of isotropy for individual branches. They can be calculated by knowing their dependence on the Lagrangian tangency orders $Lt(B_1)$, $Lt(B_2)$ for the A_2 singularity (Table 1).

Theorem 3.3. *A stratified submanifold $N \in (T_7)$ of a symplectic space $(\mathbb{R}^{2n},\omega)$ with the canonical coordinates $(p_1,q_1,\ldots,p_n,q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 7. The parameters c, c_1, c_2 are moduli. The indices of isotropy are presented in the fourth, fifth and sixth columns of Table 7 and the Lagrangian tangency orders of the curve are presented in the seventh, eighth and ninth columns of the table.*

Table 7. *The Lagrangian tangency orders and the indices of isotropy for symplectic classes of T_7 singularity.*

class	parametrization of branches	conditions for subclasses	ind	$\mathrm{ind_n}$	$\mathrm{ind_f}$	$Lt(N)$	L_n	L_f
$(T_7)^0$	$(t^3, -c_2t^2, -t^2, 0, 0, \dots)$	$c_1 \cdot c_2 \neq 0$	0	0	0	2	3	3
$2n \geqslant 4$	$(t^3, -c_1t^2, 0, -t^2, 0, \dots)$							
		$c_1 \cdot c_2 \neq 0$	0	1	0	2	5	3
$(T_7)^1$	$(t^3, -t^2, 0, -c_1t^2, 0, \dots)$	$c_1 = 0,\, c_2 \neq 0$	0	1	0	3	5	3
$2n \geqslant 4$	$(t^3, 0, -t^2, c_2t^5, 0, \dots)$	$c_1 \neq 0, c_2 = 0$	0	∞	0	2	∞	3
		$c_1 = 0, c_2 = 0$	0	∞	0	3	∞	3
$(T_7)^2$	$(t^3, \frac{1}{2}c_1^2t^4, 0, -t^2, 0, \dots)$	$c_1 \cdot c_2 \neq 0$	0	1	1	2	5	5
$2n \geqslant 4$	$(t^3, \frac{1}{2}c_2^2t^4, -t^2, 0, 0, \dots)$	$c_1 \cdot c_2 = 0,$	0	∞	1	2	∞	5
		$(c_1, c_2) \neq (0, 0)$						
$(T_7)^3$	$(t^3, \frac{1}{2}t^4, 0, c_2t^5, -t^2, 0, \dots)$	$c_1 \neq 0$	1	1	1	5	5	5
$2n \geqslant 6$	$(t^3, \frac{1}{2}c_1t^4, -t^2, 0, 0, 0, \dots)$	$c_1 = 0$	1	∞	1	5	∞	5
$(T_7)^4$	$(t^3, \frac{1}{3}ct^6, 0, -t^2, 0, \dots)$		0	∞	∞	2	∞	∞
$2n \geqslant 4$	$(t^3, 0, -t^2, 0, 0, \dots)$							
$(T_7)^5$	$(t^3, -\frac{1}{3}ct^6, 0, t^5, -t^2, 0, \dots)$		1	∞	∞	5	∞	∞
$2n \geqslant 6$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$							
$(T_7)^6$	$(t^3, -\frac{1}{3}t^6, 0, 0, -t^2, 0, \dots)$		2	∞	∞	7	∞	∞
$2n \geqslant 6$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$							
$(T_7)^7$	$(t^3, 0, 0, 0, -t^2, 0, \dots)$		∞	∞	∞	∞	∞	∞
$2n \geqslant 6$	$(t^3, 0, -t^2, 0, 0, 0, \dots)$							

The comparison of invariants presented in Table 7 shows that the Lagrangian tangency orders distinguish more symplectic classes than the indices of isotropy. The method of calculating these invariants is described in § 4.4.

3.2. Geometric conditions for the classes $(T_7)^i$

The classes $(T_7)^i$ can be distinguished geometrically, without using any local coordinate system.

Let $N \in (T_7)$. Then N is the union of two branches: singular one-dimensional irreducible components diffeomorphic to the A_2 singularity. In local coordinates they have the form

$$\mathcal{B}_1 = \{x_1^2 + x_3^3 = 0,\ x_2 = x_{\geqslant 4} = 0\},$$
$$\mathcal{B}_2 = \{x_1^2 + x_2^3 = 0,\ x_{\geqslant 3} = 0\}.$$

Denote by ℓ_1, ℓ_2 the tangent lines at 0 to the branches $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. These lines span a 2-space P_1. Let P_2 be 2-space tangent at 0 to the branch $\mathcal{B}_1$ and P_3 be 2-space tangent at 0 to the branch $\mathcal{B}_2$. Define the line $\ell_3 = P_2 \cap P_3$. The lines ℓ_1, ℓ_2, ℓ_3 span a 3-space $W = W(N)$. Equivalently, W is the tangent space at 0 to some (and then any) non-singular 3-manifold containing N.

The classes $(T_7)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form. For $N = T_7 = (3.1)$ it is easy to calculate

$$\ell_1 = \operatorname{span}(\partial/\partial x_3), \qquad \ell_2 = \operatorname{span}(\partial/\partial x_2), \qquad \ell_3 = \operatorname{span}(\partial/\partial x_1). \tag{3.2}$$

3.2.1. *Geometric conditions for the class* $[0]_{T_7}$

The geometric distinction of the class $(T_7)^7$ follows from Theorem 4.4: $N \in (T_7)^7$ if and only if N is contained in a non-singular Lagrangian submanifold. The following theorem gives a simple way to check the latter condition without using algebraic restrictions. Given a 2-form σ on a non-singular submanifold M of $\mathbb{R}^{2n}$ such that $\sigma(0) = 0$ and a vector $v \in T_0M$, we denote by $\mathcal{L}_v\sigma$ the value at 0 of the Lie derivative of σ along a vector field V on M such that $v = V(0)$. The assumption $\sigma(0) = 0$ implies that the choice of V is irrelevant.

Proposition 3.4. *Let $N \in (T_7)$ be a stratified submanifold of a symplectic space $(\mathbb{R}^{2n}, \omega)$. Let M^3 be any non-singular submanifold containing N and let σ be the restriction of ω to TM^3. Let $v_i \in \ell_i$ be non-zero vectors. If the symplectic form ω has zero algebraic restriction to N, then the following conditions are satisfied:*

(I) $\sigma(0) = 0$;

(II) $\mathcal{L}_{v_3}\sigma(v_i, v_j) = 0$ *for* $i, j \in \{1, 2\}$;

(III) $\mathcal{L}_{v_i}\sigma(v_3, v_i) = 0$ *for* $i \in \{1, 2\}$;

(IV) $\mathcal{L}_{v_i}\sigma(v_3, v_j) = \mathcal{L}_{v_j}\sigma(v_3, v_i)$ *for* $i \neq j \in \{1, 2\}$.

Theorem 3.5. *A stratified submanifold $N \in (T_7)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(T_7)^i$ if and only if the couple (N, ω) satisfies corresponding conditions in the last column of Table 8 or Table 9.*

The proofs of the theorems of this section are presented in § 4.5.

4. Proofs

4.1. The method of algebraic restrictions

In this section we present basic facts about the method of algebraic restrictions, which is a very powerful tool for the symplectic classification. The details of the method and proofs of all results of this section can be found in [**8**].

Given a germ of a non-singular manifold M, denote by $\Lambda^p(M)$ the space of all germs at 0 of differential p-forms on M. Given a subset $N \subset M$, introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M) \colon \omega(x) = 0 \text{ for any } x \in N\};$$
$$\mathcal{A}^p_0(N, M) = \{\alpha + \mathrm{d}\beta \colon \alpha \in \Lambda^p_N(M),\ \beta \in \Lambda^{p-1}_N(M)\}.$$

Table 8. *Geometric interpretation of singularity classes of T_7 when $\omega|_W \neq 0$.*
(W is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n\geqslant 4}, \omega)$ containing $N \in (T_7)$.)

class	normal form	geometric conditions
$(T_7)^0$	$[T_7]^0\colon [\theta_1 + c_1\theta_2 + c_2\theta_3]_{T_7}$ $c_1 \cdot c_2 \neq 0$	$\omega\|_{\ell_i+\ell_j} \neq 0\ \forall i,j \in \{1,2,3\}$ so 2-spaces tangent to branches are not isotropic
$(T_7)^1$		$\exists i \neq j \in \{1,2\} \omega\|_{\ell_i+\ell_3} = 0$ and $\omega\|_{\ell_j+\ell_3} \neq 0$ (exactly one branch has tangent 2-space isotropic)
	$[T_7]^1_{2,5}\colon [c_1\theta_1 + \theta_2 + c_2\theta_5]_{T_7}$ $c_1 \cdot c_2 \neq 0$	$\omega\|_{\ell_1+\ell_2} \neq 0$ and no branch is contained in a Lagrangian submanifold
	$[T_7]^1_{3,5}\colon [\theta_2 + c_2\theta_5]_{T_7}$, $c_2 \neq 0$	$\omega\|_{\ell_1+\ell_2} = 0$ and no branch is contained in a Lagrangian submanifold
	$[T_7]^1_{2,\infty}\colon [c_1\theta_1 + \theta_2]_{T_7}$, $c_1 \neq 0$	$\omega\|_{\ell_1+\ell_2} \neq 0$ and exactly one branch is contained in a Lagrangian submanifold
	$[T_7]^1_{3,\infty}\colon [\theta_2]_{T_7}$	$\omega\|_{\ell_1+\ell_2} = 0$ and exactly one branch is contained in a Lagrangian submanifold
$(T_7)^2$		$\omega\|_{\ell_1+\ell_2} \neq 0$, $\omega\|_{\ell_i+\ell_3} = 0\ \forall i \in \{1,2\}$
	$[T_7]^2_5\colon [\theta_1 + c_1\theta_4 + c_2\theta_5]_{T_7}$ $c_1 \cdot c_2 \neq 0$	no branch is contained in a Lagrangian submanifold
	$[T_7]^2_\infty\colon [\theta_1 + c_1\theta_4 + c_2\theta_5]_{T_7}$ $c_1 \cdot c_2 = 0, c_1 + c_2 \neq 0$	exactly one branch is contained in a Lagrangian submanifold
$(T_7)^4$	$[T_7]^4\colon [\theta_1 + c\theta_7]_{T_7}$	$\omega\|_{\ell_1+\ell_2} \neq 0$, $\omega\|_{\ell_i+\ell_3} = 0\ \forall i \in \{1,2\}$, and branches are contained in different Lagrangian submanifolds

Definition 4.1. Let N be the germ of a subset of M and let $\omega \in \Lambda^p(M)$. The *algebraic restriction* of ω to N is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\tilde{\omega}$ if $\omega - \tilde{\omega} \in \mathcal{A}_0^p(N, M)$.

Notation. The algebraic restriction of the germ of a p-form ω on M to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. By writing $[\omega]_N = 0$ (or saying that ω has zero algebraic restriction to N), we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in A_0^p(N, M)$.

Definition 4.2. Two algebraic restrictions $[\omega]_N$ and $[\tilde{\omega}]_{\tilde{N}}$ are called *diffeomorphic* if there exists the germ of a diffeomorphism $\Phi\colon \tilde{M} \to M$ such that $\Phi(\tilde{N}) = N$ and $\Phi^*([\omega]_N) = [\tilde{\omega}]_{\tilde{N}}$.

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.

Table 9. *Geometric interpretation of singularity classes of T_7 when $\omega|_W = 0$.*
(W is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n\geqslant 6}, \omega)$ containing $N \in (T_7)$; (I)–(IV) are the conditions of Proposition 3.4.)

class	normal form	geometric conditions
$(T_7)^3$	$[T_7]^3_5 \colon [\theta_4 + c_1\theta_5 + c_2\theta_6]_{T_7}$ $c_1 \neq 0$	(III) is not satisfied and no branch is contained in a Lagrangian submanifold
	$[T_7]^3_\infty \colon [\theta_4 + c_2\theta_6]_{T_7}$	(III) is not satisfied and exactly one branch is contained in a Lagrangian submanifold
$(T_7)^5$	$[T_7]^5 \colon [\theta_6 + c\theta_7]_{T_7}$	(III) is satisfied but (II) is not and branches are contained in different Lagrangian submanifolds
$(T_7)^6$	$[T_7]^6 \colon [\theta_7]_{T_7}$	(I)–(IV) are satisfied and branches are contained in different Lagrangian submanifolds
$(T_7)^7$	$[T_7]^7 \colon [0]_{T_7}$	(I)–(IV) are satisfied and N is contained in a Lagrangian submanifold

Theorem 4.3 (Theorem A in [8]). *Let N be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

Two germs of quasi-homogeneous subsets N_1, N_2 of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectically equivalent if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.

Theorem 4.3 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of zero algebraic restriction is explained by the following theorem.

Theorem 4.4 (Theorem B in [8]). *The germ of a quasi-homogeneous set N of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N.*

The following result shows that the method of algebraic restrictions is very powerful tool in symplectic classification of singular curves.

Theorem 4.5 (Theorem 2 in [6]). *Let C be the germ of a $\mathbb{K}$-analytic curve (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Then the space of algebraic restrictions of germs of closed 2-forms to C is a finite-dimensional vector space.*

By a $\mathbb{K}$-*analytic curve* we mean a subset of $\mathbb{K}^m$ which is locally diffeomorphic to a one-dimensional (possibly singular) $\mathbb{K}$-analytic subvariety of $\mathbb{K}^m$. Germs of $\mathbb{C}$-analytic parametrized curves can be identified with germs of irreducible $\mathbb{C}$-analytic curves.

Table 10. *Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = T_7$.*

	relations	proof
1	$[x_2\,\mathrm{d}x_2 \wedge \mathrm{d}x_3]_N = 0$	$(4.1) \wedge \mathrm{d}x_2$
2	$[x_3\,\mathrm{d}x_2 \wedge \mathrm{d}x_3]_N = 0$	$(4.1) \wedge \mathrm{d}x_3$
3	$[x_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2]_N = [x_2\,\mathrm{d}xy_3 \wedge \mathrm{d}x_1]_N$	$(4.1) \wedge \mathrm{d}x_1$
4	$[x_1\,\mathrm{d}x_1 \wedge \mathrm{d}x_2]_N = 0$	$(4.2) \wedge \mathrm{d}x_2$ and row 2
5	$[x_1\,\mathrm{d}x_1 \wedge \mathrm{d}x_3]_N = 0$	$(4.2) \wedge \mathrm{d}x_3$ and row 1
6	$[x_2^2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2]_N = [x_3^2\,\mathrm{d}x_3 \wedge \mathrm{d}xy_1]_N$	$(4.2) \wedge \mathrm{d}x_1$
7	$[x_1^2\,\mathrm{d}x_2 \wedge \mathrm{d}x_3]_N = 0$	rows 1 and 2 and $[x_1^2]_N = [-x_2^3 - x_3^3]_N$
8	$[x_3^2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2]_N = 0$	$(4.1) \wedge x_3\,\mathrm{d}x_1$ and $[x_2x_3]_N = 0$

In the remainder of this paper we use the following notation:

- $[\Lambda^2(\mathbb{R}^{2n})]_N$ is the vector space consisting of algebraic restrictions of germs of all 2-forms on $\mathbb{R}^{2n}$ to the germ of a subset $N \subset \mathbb{R}^{2n}$;
- $[Z^2(\mathbb{R}^{2n})]_N$ is the subspace of $[\Lambda^2(\mathbb{R}^{2n})]_N$ consisting of algebraic restrictions of germs of all closed 2-forms on $\mathbb{R}^{2n}$ to N;
- $[\mathrm{Symp}(\mathbb{R}^{2n})]_N$ is the open set in $[Z^2(\mathbb{R}^{2n})]_N$ consisting of algebraic restrictions of germs of all symplectic 2-forms on $\mathbb{R}^{2n}$ to N.

For calculating discrete invariants we use the following propositions.

Proposition 4.6 (Domitrz *et al.* [8]). *The symplectic multiplicity of the germ of a quasi-homogeneous subset N in a symplectic space is equal to the codimension of the orbit of the algebraic restriction $[\omega]_N$ with respect to the group of local diffeomorphisms preserving N in the space of algebraic restrictions of closed 2-forms to N.*

Proposition 4.7 (Domitrz *et al.* [8]). *The index of isotropy of the germ of a quasi-homogeneous subset N in a symplectic space $(\mathbb{R}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_N$.*

Proposition 4.8 (Domitrz [6]). *Let f be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at 0. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve f is the maximum of the order of vanishing on f over all 1-forms α such that $[\omega]_f = [\mathrm{d}\alpha]_f$*

4.2. Algebraic restrictions to T_7 and their classification

One has the following relations for (T_7)-singularities:

$$[\mathrm{d}(x_2x_3)]_{T_7} = [x_2\,\mathrm{d}x_3 + x_3\,\mathrm{d}x_2]_{T_7} = 0, \tag{4.1}$$

$$[\mathrm{d}(x_1^2 + x_2^3 + x_3^3)]_{T_7} = [2x_1\,\mathrm{d}x_1 + 3x_2^2\,\mathrm{d}x_2 + 3x_3^2\,\mathrm{d}x_3]_{T_7} = 0. \tag{4.2}$$

Multiplying these relations by suitable 1-forms, we obtain the relations in Table 10.

Using the method of algebraic restrictions and Table 10, we obtain the following proposition.

Proposition 4.9. $[\Lambda^2(\mathbb{R}^{2n})]_{T_7}$ *is an eight-dimensional vector space spanned by the algebraic restrictions to* T_7 *of the* 2*-forms:*

$$\begin{aligned}
&\theta_1 = \mathrm{d}x_2 \wedge \mathrm{d}x_3, && \theta_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_3, && \theta_3 = \mathrm{d}x_1 \wedge \mathrm{d}x_2,\\
&\theta_4 = x_3\mathrm{d}x_1 \wedge \mathrm{d}x_3, && \theta_5 = x_2\mathrm{d}x_1 \wedge \mathrm{d}x_2, &&\\
&\sigma_1 = x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2, && \sigma_2 = x_1\mathrm{d}x_2 \wedge \mathrm{d}x_3, &&\\
&\theta_7 = x_3^2\mathrm{d}x_1 \wedge \mathrm{d}x_3. && &&
\end{aligned}$$

Proposition 4.9 and the results of § 4.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{T_7}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{T_7}$.

Theorem 4.10. $[Z^2(\mathbb{R}^{2n})]_{T_7}$ *is a seven-dimensional vector space spanned by the algebraic restrictions to* T_7 *of the quasi-homogeneous* 2*-forms* θ_i*:*

$$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 = \sigma_1 - \sigma_2, \theta_7.$$

If $n \geqslant 3$*, then* $[\mathrm{Symp}(\mathbb{R}^{2n})]_{T_7} = [Z^2(\mathbb{R}^{2n})]_{T_7}$*. The manifold* $[\mathrm{Symp}(\mathbb{R}^4)]_{T_7}$ *is an open part of the* 7*-space* $[Z^2(\mathbb{R}^4)]_{T_7}$ *consisting of algebraic restrictions of the form* $[c_1\theta_1 + \cdots + c_7\theta_7]_{T_7}$ *such that* $(c_1, c_2, c_3) \neq (0, 0, 0)$*.*

Theorem 4.11.

(i) *Any algebraic restriction in* $[Z^2(\mathbb{R}^{2n})]_{T_7}$ *can be brought by a symmetry of* T_7 *to one of the normal forms* $[T_7]^i$ *given in the second column of Table 11.*

(ii) *The codimension in* $[Z^2(\mathbb{R}^{2n})]_{T_7}$ *of the singularity class corresponding to the normal form* $[T_7]^i$ *is equal to* i*.*

(iii) *The singularity classes corresponding to the normal forms are disjoint.*

(iv) *The parameters* c*,* c_1*,* c_2 *of the normal forms* $[T_7]^0$*,* $[T_7]^1$*,* $[T_7]^2$*,* $[T_7]^3$*,* $[T_7]^4$*,* $[T_7]^5$ *are moduli.*

The proof of Theorem 4.11 is presented in § 4.6.

In the first column of Table 11, we denote by $(T_7)^i$ a subclass of (T_7) consisting of $N \in (T_7)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[T_7]^i$. Theorems 4.3 and 4.11 and Proposition 4.10 imply the following statement, which explains why the given stratification of (T_7) is natural.

Theorem 4.12. *Fix* $i \in \{0, 1, \ldots, 7\}$*. All stratified submanifolds* $N \in (T_7)^i$ *have the same*

(a) *symplectic multiplicity and*

(b) *index of isotropy given in Table 11 by* $(T_7)^i$*.*

Table 11. *Classification of symplectic T_7 singularities.*

('cod' denotes the codimension of the classes; μ^{sym} denotes the symplectic multiplicity; 'ind' denotes the index of isotropy.)

symplectic class	normal forms for algebraic restrictions	cod	μ^{sym}	ind
$(T_7)^0$ $(2n \geqslant 4)$	$[T_7]^0\colon [\theta_1 + c_1\theta_2 + c_2\theta_3]_{T_7}$, $c_1 \cdot c_2 \neq 0$	0	2	0
$(T_7)^1$ $(2n \geqslant 4)$	$[T_7]^1\colon [c_1\theta_1 + \theta_2 + c_2\theta_5]_{T_7}$	1	3	0
$(T_7)^2$ $(2n \geqslant 4)$	$[T_7]^2\colon [\theta_1 + c_1\theta_4 + c_2\theta_5]_{T_7}$, $(c_1, c_2) \neq (0,0)$	2	4	0
$(T_7)^3$ $(2n \geqslant 6)$	$[T_7]^3\colon [\theta_4 + c_1\theta_5 + c_2\theta_6]_{T_7}$	3	5	1
$(T_7)^4$ $(2n \geqslant 4)$	$[T_7]^4\colon [\theta_1 + c\theta_7]_{T_7}$	4	5	0
$(T_7)^5$ $(2n \geqslant 6)$	$[T_7]^5\colon [\theta_6 + c\theta_7]_{T_7}$	5	6	1
$(T_7)^6$ $(2n \geqslant 6)$	$[T_7]^6\colon [\theta_7]_{T_7}$	6	6	2
$(T_7)^7$ $(2n \geqslant 6)$	$[T_7]^7\colon [0]_{T_7}$	7	7	∞

Proof. Part (a) follows from Proposition 4.6 and Theorem 4.11 and the fact that the codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ of the orbit of an algebraic restriction $a \in [T_7]^i$ is equal to the sum of the number of moduli in the normal form $[T_7]^i$ and the codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ of the class of algebraic restrictions defined by this normal form.

Part (b) follows from Theorem 4.4 and Proposition 4.7. □

Proposition 4.13. *The classes $(T_7)^i$ are symplectic singularity classes, i.e. they are closed with respect to the action of the group of symplectomorphisms. The class (T_7) is the disjoint union of the classes $(T_7)^i$, $i \in \{0, 1, \ldots, 7\}$. The classes $(T_7)^0$, $(T_7)^1$, $(T_7)^2$, $(T_7)^4$ are non-empty for any dimension $2n \geqslant 4$ of the symplectic space; the classes $(T_7)^3$, $(T_7)^5$, $(T_7)^6$, $(T_7)^7$ are empty if $n = 2$ and not empty if $n \geqslant 3$.*

4.3. Symplectic normal forms

Let us transfer the normal forms $[T_7]^i$ to symplectic normal forms. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[T_7]^i$ of algebraic restrictions. We can fix, for example,

$$\begin{aligned}
\omega^0 &= \theta_1 + c_1\theta_2 + c_2\theta_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n}, \quad c_1 \cdot c_2 \neq 0,\\
\omega^1 &= c_1\theta_1 + \theta_2 + c_2\theta_5 + \mathrm{d}x_2 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n},\\
\omega^2 &= \theta_1 + c_1\theta_4 + c_2\theta_5 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n}, \quad (c_1, c_2) \neq (0,0),\\
\omega^3 &= \theta_4 + c_1\theta_5 + c_2\theta_6 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_5 + \mathrm{d}x_3 \wedge \mathrm{d}x_6\\
&\qquad + \mathrm{d}x_7 \wedge \mathrm{d}x_8 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n},\\
\omega^4 &= \theta_1 + c\theta_7 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n},\\
\omega^5 &= \theta_6 + c\theta_7 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_5 + \mathrm{d}x_3 \wedge \mathrm{d}x_6 + \mathrm{d}x_7 \wedge \mathrm{d}x_8 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n},\\
\omega^6 &= \theta_7 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_5 + \mathrm{d}x_3 \wedge \mathrm{d}x_6 + \mathrm{d}x_7 \wedge \mathrm{d}x_8 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n},\\
\omega^7 &= \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_5 + \mathrm{d}x_3 \wedge \mathrm{d}x_6 + \mathrm{d}x_7 \wedge \mathrm{d}x_8 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n}.
\end{aligned}$$

Let $\omega = \sum_{i=1}^{m} \mathrm{d}p_i \wedge \mathrm{d}q_i$, where $(p_1, q_1, \dots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}$, $n \geqslant 3$ (respectively, $n = 2$). Fix, for $i = 0, 1, \dots, 7$ (respectively, for $i = 0, 1, 2, 4$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω: $(\Phi^i)^*\omega^i = \omega$. Consider the families $T_7^i = (\Phi^i)^{-1}(T_7)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega)$ which is diffeomorphic to T_7 is symplectically equivalent to one and only one of the normal forms T_7^i, $i = 0, 1, \dots, 7$ (respectively, $i = 0, 1, 2, 4$) presented in Theorem 3.1. By Theorem 4.11 we obtain that parameters c, c_1, c_2 of the normal forms are moduli.

4.4. Proof of Theorem 3.3

The numbers $\mathrm{ind}(B_1)$ and $\mathrm{ind}(B_2)$ are computed using Proposition 4.7 for branches B_1 and B_2. The space $[Z^2(\mathbb{R}^{2n})]_{B_1}$ is spanned only by the algebraic restrictions to B_1 of the 2-forms θ_2, θ_4. The space $[Z^2(\mathbb{R}^{2n})]_{B_2}$ is spanned only by the algebraic restrictions to B_2 of the 2-forms θ_3, θ_5. Branches are curves of type A_2, and from Table 1 we know the interaction between the index of isotropy and the Lagrangian tangency order. Knowing $\mathrm{ind}(B_1)$ and $\mathrm{ind}(B_2)$, we obtain $Lt(B_1) = 3 + \mathrm{ind}(B_1)$ and $Lt(B_2) = 3 + \mathrm{ind}(B_2)$. Then L_{f} is the minimum of these numbers and L_{n} is their maximum. Next we calculate $Lt(N)$ by definition, finding the nearest Lagrangian submanifold to the branches, knowing that it cannot be greater than L_{f}.

As an example we calculate the invariants for the class $(T_7)^1$.

We have $[\omega^1]_{B_1} = [c_1\theta_1 + \theta_2 + c_2\theta_5]_{B_1} = [\theta_2]_{B_1}$ and thus $\mathrm{ind}(B_1) = 0$ and $Lt(B_1) = 3$. $[\omega^1]_{B_2} = [c_1\theta_1 + \theta_2 + c_2\theta_5]_{B_2} = [c_2\theta_5]_{B_2}$ and thus $\mathrm{ind}(B_2) = 1$ and $Lt(B_2) = 5$ if $c_2 \neq 0$ and $\mathrm{ind}(B_2) = \infty$ and $Lt(B_2) = \infty$ if $c_2 = 0$.

Finally, for the class $(T_7)^1$ we have $L_{\mathrm{n}} = 5$ if $c_2 \neq 0$ and $L_{\mathrm{n}} = \infty$ if $c_2 = 0$ and $L_{\mathrm{f}} = 3$ so $Lt(N) \leqslant 3$.

For the smooth Lagrangian submanifolds L defined by the conditions $p_1 = 0$, $q_2 = 0$ and $p_i = 0$, $i > 2$, we get $t[N, L] = 3$ if $c_1 = 0$; thus, $Lt(N) = 3$ in this case. But, if $c_1 \neq 0$, then $t[N, L] = 2$ and it cannot be greater for any other smooth Lagrangian submanifold, so $Lt(N) = 2$ in this case.

4.5. Proof of Theorem 3.5

Proof of Proposition 3.4. Any 2-form σ which has zero algebraic restriction to T_7 can be expressed in the following form:

$$\sigma = H_1\alpha + H_2\beta + \mathrm{d}H_1 \wedge \gamma + \mathrm{d}H_2 \wedge \delta,$$

where $H_1 = x_1^2 + x_2^3 + x_3^3$, $H_2 = x_2x_3$ and α, β are 2-forms on TM^3 and $\gamma = \gamma_1\,\mathrm{d}x_1 + \gamma_2\,\mathrm{d}x_2 + \gamma_3\,\mathrm{d}x_3$ and $\delta = \delta_1\,\mathrm{d}x_1 + \delta_2\,\mathrm{d}x_2 + \delta_3\,\mathrm{d}x_3$ are 1-forms on TM^3. Since

$$H_1(0) = H_2(0) = 0, \qquad \mathrm{d}H_1|_0 = \mathrm{d}H_2|_0 = 0, \tag{4.3}$$

we obtain the following equality:

$$\mathcal{L}_v\sigma = \mathrm{d}(V \rfloor \sigma)|_0 + (V \rfloor \mathrm{d}\sigma)|_0 = \mathrm{d}(V \rfloor \sigma)|_0.$$

Equation (4.3) also implies that

$$\mathrm{d}(V \rfloor \sigma)|_0 = \mathrm{d}(V \rfloor \mathrm{d}H_1)|_0 \wedge \gamma|_0 + \mathrm{d}(V \rfloor \mathrm{d}H_2)|_0 \wedge \delta|_0.$$

By simple calculation we get

$$\begin{aligned}
\mathcal{L}_{v_1}\sigma &= \mathrm{d}x_2 \wedge \delta|_0 = \delta_3|_0\, \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \delta_1|_0\, \mathrm{d}x_1 \wedge \mathrm{d}x_2,\\
\mathcal{L}_{v_2}\sigma &= \mathrm{d}x_3 \wedge \delta|_0 = \delta_1|_0\, \mathrm{d}x_3 \wedge \mathrm{d}x_1 - \delta_2|_0\, \mathrm{d}x_2 \wedge \mathrm{d}x_3,\\
\mathcal{L}_{v_3}\sigma &= 2\,\mathrm{d}x_1 \wedge \gamma|_0 = 2\gamma_2|_0\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\gamma_3|_0\, \mathrm{d}x_3 \wedge \mathrm{d}x_1.
\end{aligned}$$

Finally, we obtain

$$\mathcal{L}_{v_1}\sigma(v_3, v_1) = 0, \qquad \mathcal{L}_{v_2}\sigma(v_3, v_2) = 0, \qquad \mathcal{L}_{v_3}\sigma(v_1, v_2) = 0,$$
$$\mathcal{L}_{v_1}\sigma(v_3, v_2) = -\delta_1|_0 = \mathcal{L}_{v_2}\sigma(v_3, v_1).$$

$\square$

Proof of Theorem 3.5. The conditions on the pair (ω, N) in the last columns of Tables 8 and 9 are disjoint. It suffices to prove that these conditions in the row of $(T_7)^i$ are satisfied for any $N \in (T_7)^i$. This is a corollary of the following claims.

1. Each of the conditions in the last column of Tables 8 and 9 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs (ω, N).

2. Each of these conditions depends only on the algebraic restriction $[\omega]_N$.

3. Take the simplest 2-forms ω^i representing the normal forms $[T_7]^i$ for algebraic restrictions: ω^0, ω^1, ω^2, ω^3, ω^4, ω^5, ω^6, ω^7. The pair $(\omega = \omega^i, T_7)$ satisfies the condition in the last column of Table 8 or Table 9 (the row of $(T_7)^i$).

To prove the third statement we note that in the case $N = T_7 = (3.1)$ one has

$$W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$$

and

$$\begin{aligned}
v_1 &\in \ell_1 = \mathrm{span}(\partial/\partial x_3),\\
v_2 &\in \ell_2 = \mathrm{span}(\partial/\partial x_2),\\
v_3 &\in \ell_3 = \mathrm{span}(\partial/\partial x_1).
\end{aligned}$$

By simple calculation and observation of Lagrangian tangency orders, we obtain that following statements are true.

(T^0) $\omega^0|_{\ell_1+\ell_2} \neq 0$, $\omega^0|_{\ell_1+\ell_3} \neq 0$, $\omega^0|_{\ell_2+\ell_3} \neq 0$, $L_{\mathrm{n}} < \infty$ and $L_{\mathrm{f}} < \infty$; hence, no branch is contained in a smooth Lagrangian submanifold.

(T^1) For any c_1 and c_2, $\omega^1|_{\ell_1+\ell_3} = 0$ and $\omega^1|_{\ell_2+\ell_3} \neq 0$ or $\omega^1|_{\ell_1+\ell_3} \neq 0$ and $\omega^1|_{\ell_2+\ell_3} = 0$. If $c_2 = 0$, then $L_{\mathrm{n}} = \infty$ and $L_{\mathrm{f}} < \infty$; hence, exactly one branch is contained in some smooth Lagrangian submanifold. For $c_2 \neq 0$, $L_{\mathrm{n}} < \infty$ and $L_{\mathrm{f}} < \infty$, so no branch is contained in a smooth Lagrangian submanifold. $\omega^1|_{\ell_1+\ell_2} = 0$ if and only if $c_1 = 0$.

(T^2) For any c_1 and c_2, $\omega^2|_{\ell_1+\ell_2} \neq 0$, $\omega^2|_{\ell_1+\ell_3} = 0$ and $\omega^2|_{\ell_2+\ell_3} = 0$. If $c_1 \cdot c_2 \neq 0$, then $L_{\mathrm{n}} < \infty$ and $L_{\mathrm{f}} < \infty$ so no branch is contained in a Lagrangian submanifold. If $c_1 = 0$ and $c_2 \neq 0$ or $c_1 \neq 0$ and $c_2 = 0$, then $L_{\mathrm{n}} = \infty$ and $L_{\mathrm{f}} < \infty$; hence, exactly one branch is contained in some smooth Lagrangian submanifold.

(T^3) The Lie derivative of $\omega^3 = \theta_4 + c_1\theta_5 + c_2\theta_6$ along a vector field $V = \partial/\partial x_3$ is not equal to 0, so condition (III) of Proposition 3.4 is not satisfied. If $c_1 \neq 0$, then $L_{\mathrm{n}} < \infty$ and $L_{\mathrm{f}} < \infty$; hence, no branch is contained in a Lagrangian submanifold. If $c_1 = 0$, then $L_{\mathrm{n}} = \infty$ and $L_{\mathrm{f}} < \infty$; hence, only one branch is contained in some Lagrangian submanifold.

(T^4) For any c, $\omega^4|_{\ell_1+\ell_2} \neq 0$, $\omega^4|_{\ell_1+\ell_3} = 0$ and $\omega^4|_{\ell_2+\ell_3} = 0$. Both branches are contained in different Lagrangian submanifolds since $L_{\mathrm{n}} = L_{\mathrm{f}} = \infty$ and $Lt(N) < \infty$.

(T^5) We can calculate the Lie derivatives of $\omega^5 = \theta_6 + c\theta_7$ along vector fields $V_1 = \partial/\partial x_3$, $V_2 = \partial/\partial x_2$ and $V_3 = \partial/\partial x_3$: $\mathcal{L}_{V_1}\omega^5(V_3, V_1) = 0$ and $\mathcal{L}_{V_2}\omega^5(V_3, V_2) = 0$, so condition (III) of Proposition 3.4 is satisfied, but the Lie derivative $\mathcal{L}_{V_3}\omega^5(V_1, V_2)$ is not equal to 0, so condition (II) of Proposition 3.4 is not satisfied. We have $Lt(N) < \infty$ and $L_{\mathrm{n}} = L_{\mathrm{f}} = \infty$; hence, branches are contained in different Lagrangian submanifolds.

(T^6) The Lie derivatives of $\omega^6 = \theta_7$, $\mathcal{L}_{V_i}\omega^6(V_j, V_k) = 0$ for $i, j, k \in \{1, 2, 3\}$, so conditions (II)–(IV) of Proposition 3.4 are satisfied. We have $Lt(N) < \infty$ and $L_{\mathrm{n}} = L_{\mathrm{f}} = \infty$; hence, branches are contained in different Lagrangian submanifolds.

(T^7) For $\omega^7 = 0$ we have $\mathcal{L}_{V_i}\omega^7(V_j, V_k) = 0$ for $i, j, k \in \{1, 2, 3\}$, so conditions (II)–(IV) of Proposition 3.4 are satisfied. The condition $Lt(N) = \infty$ implies the curve N is contained in a smooth Lagrangian submanifold.

□

4.6. Proof of Theorem 4.11

In our proof we use vector fields tangent to $N \in (T_7)$. A Hamiltonian vector field is an example of such a vector field. We recall by [**4**] a suitable definition and facts.

Definition 4.14. Let $H = \{H_1 = \cdots = H_p = 0\} \subset \mathbb{R}^n$ be a complete intersection. Consider a set of $p+1$ integers $1 \leqslant i_1 < \cdots < i_{p+1} \leqslant n$. A Hamiltonian vector field $X_H(i_1, \ldots, i_{p+1})$ on a complete intersection H is the determinant obtained by expansion

Table 12. *Infinitesimal actions on algebraic restrictions of closed 2-forms to T_7.* (E is defined as in (4.5).)

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_5]$	$[\theta_6]$	$[\theta_7]$
$X_0 = E$	$4[\theta_1]$	$5[\theta_2]$	$5[\theta_3]$	$7[\theta_4]$	$7[\theta_5]$	$7[\theta_6]$	$9[\theta_7]$
$X_1 = x_3E$	$[0]$	$7[\theta_4]$	$3[\theta_6]$	$9[\theta_7]$	$[0]$	$[0]$	$[0]$
$X_2 = x_2E$	$[0]$	$-3[\theta_6]$	$7[\theta_5]$	$[0]$	$-9[\theta_7]$	$[0]$	$[0]$
$X_3 = x_1E$	$-4[\theta_6]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_4 = x_2^2E$	$[0]$	$[0]$	$-9[\theta_7]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_5 = x_3^2E$	$[0]$	$9[\theta_7]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$

with respect to the first row of the symbolic $(p+1)\times(p+1)$ matrix

$$X_H(i_1,\dots,i_{p+1}) = \det\begin{bmatrix} \partial/\partial x_{i_1} & \cdots & \partial/\partial x_{i_{p+1}} \\ \partial H_1/\partial x_{i_1} & \cdots & \partial H_1/\partial x_{i_{p+1}} \\ \vdots & \ddots & \vdots \\ \partial H_p/\partial x_{i_1} & \cdots & \partial H_p/\partial x_{i_{p+1}} \end{bmatrix}. \tag{4.4}$$

Theorem 4.15 (Wahl [15]). *Let $H = \{H_1 = \cdots = H_p = 0\} \subset \mathbb{R}^n$ be a positive-dimensional complete intersection with an isolated singularity. If $H_1,\dots,H_p$ are quasi-homogeneous with positive weights $\lambda_1,\dots,\lambda_n$, then the module of vector fields tangent to H is generated by the Euler vector field*

$$E = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}$$

and the Hamiltonian fields $X_H(i_1,\dots,i_{p+1})$, where the numbers $i_1,\dots,i_{p+1}$ run through all possible sets $1 \leqslant i_1 < \cdots < i_{p+1} \leqslant n$.

Proposition 4.16. *Let $H = \{H_1 = \cdots = H_{n-1} = 0\} \subset \mathbb{R}^n$ be a one-dimensional complete intersection. If X_H is the Hamiltonian vector field on H, then $[\mathcal{L}_{X_H}(\alpha)]_H = [0]_H$ for any closed 2-form α.*

Proof. Note that $X_H \rfloor \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \mathrm{d}H_1 \wedge \cdots \wedge \mathrm{d}H_{n-1}$. This implies that, for $i<j$,

$$\begin{aligned} X_H \rfloor \mathrm{d}x_i \wedge \mathrm{d}x_j &= (-1)^{i+j+1}\left(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{n-2}}}\right) \rfloor (\mathrm{d}H_1 \wedge \cdots \wedge \mathrm{d}H_{n-1}) \\ &= \sum_{k=1}^{n-1} (-1)^{k+i+j}\left(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{n-2}}}\right) \rfloor (\mathrm{d}H_{l_{1,k}} \wedge \cdots \wedge \mathrm{d}H_{l_{n-2,k}})\,\mathrm{d}H_k \\ &= \sum_{k=1}^{n-1} f_k\,\mathrm{d}H_k, \end{aligned}$$

where $(i_1,\dots,i_{n-2})=(1,\dots,i-1,i+1,\dots,j-1,j+1,\dots,n)$, and for $k\in\{1,\dots,n-1\}$ we take a sequence $(l_{1,k},\dots,l_{n-2,k})=(1,\dots,k-1,k+1,\dots,n-1)$.

Thus,

$$[X_H \rfloor \mathrm{d}x_i\wedge \mathrm{d}x_j]_{H=0}=\left[\sum_{k=1}^{n-1} f_k\,\mathrm{d}H_k\right]_H=[0]_H.$$

If

$$\alpha=\sum_{i<j} g_{i,j}\,\mathrm{d}x_i\wedge\mathrm{d}x_j$$

is a closed 2-form, then $[\mathcal{L}_{X_H}\alpha]_H=[\mathrm{d}(X_H\rfloor\alpha)]_H$. It implies that

$$[\mathcal{L}_{X_H}\alpha]_H=\sum_{i<j} g_{i,j}[\mathrm{d}(X_H\rfloor \mathrm{d}x_i\wedge\mathrm{d}x_j)]_H+[\mathrm{d}g_{i,j}\wedge(X_H\rfloor\mathrm{d}x_i\wedge\mathrm{d}x_j)]_H=[0]_H.$$

□

The germ of a vector field tangent to T_7 of non-trivial action on algebraic restriction of closed 2-forms to T_7 may be described as a linear combination of germs of vector fields: $X_0=E$, $X_1=x_3E$, $X_2=x_2E$, $X_3=x_1E$, $X_4=x_2^2E$, $X_5=x_3^2E$, where E is the Euler vector field

$$E=3x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+2x_3\frac{\partial}{\partial x_3}. \tag{4.5}$$

Proposition 4.17. *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N\in(T_7)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to N is presented in Table 12.*

Let $\mathcal{A}=[c_1\theta_1+c_2\theta_2+c_3\theta_3+c_4\theta_4+c_5\theta_5+c_6\theta_6+c_7\theta_7]_{T_7}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 4.11 follows from the following lemmas.

Lemma 4.18. *If $c_1\cdot c_2\cdot c_3\neq 0$, then the algebraic restriction $\mathcal{A}=[\sum_{k=1}^7 c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_1+\tilde{c}_2\theta_2+\tilde{c}_3\theta_3]_{T_7}$.*

Proof of Lemma 4.18. We use the homotopy method to prove that $\mathcal{A}$ is diffeomorphic to $[\theta_1+\tilde{c}_2\theta_2+\tilde{c}_3\theta_3]_{T_7}$.

Let

$$\mathcal{B}_t=[c_1\theta_1+c_2\theta_2+c_3\theta_3+(1-t)c_4\theta_4+(1-t)c_5\theta_5+(1-t)c_6\theta_6+(1-t)c_7\theta_7]_{T_7}$$

for $t\in[0;1]$. Then $\mathcal{B}_0=\mathcal{A}$ and $\mathcal{B}_1=[c_1\theta_1+c_2\theta_2+c_3\theta_3]_{T_7}$. We prove that there exists a family $\Phi_t\in\mathrm{Symm}(T_7), t\in[0;1]$ such that

$$\Phi_t^*\mathcal{B}_t=\mathcal{B}_0,\qquad \Phi_0=\mathrm{id}. \tag{4.6}$$

Let V_t be a vector field defined by $\mathrm{d}\Phi_t/\mathrm{d}t=V_t(\Phi_t)$. Then, by differentiating (4.6), we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t=c_4\theta_4+c_5\theta_5+c_6\theta_6+c_7\theta_7. \tag{4.7}$$

We are looking for V_t in the form

$$V_t = \sum_{k=1}^{5} b_k(t) X_k,$$

where $b_k(t)$ for $k = 1, \ldots, 5$ are smooth functions $b_k\colon [0;1] \to \mathbb{R}$. Then, by Proposition 4.17, (4.7) has the form

$$\begin{bmatrix} 7c_2 & 0 & 0 & 0 & 0 \\ 0 & 7c_3 & 0 & 0 & 0 \\ 3c_3 & -3c_2 & -4c_1 & 0 & 0 \\ 9c_4(1-t) & -9c_5(1-t) & 0 & -9c_3 & 9c_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix}. \tag{4.8}$$

If $c_1 \cdot c_2 \cdot c_3 \neq 0$, we can solve (4.8), and Φ_t may be obtained as a flow of vector field V_t. The family Φ_t preserves T_7, because V_t is tangent to T_7 and $\Phi_t^* \mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments, we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{T_7}$. By the condition $c_1 \neq 0$ we have a diffeomorphism $\Psi \in \mathrm{Symm}(T_7)$ of the form

$$\Psi\colon (x_1, x_2, x_3) \mapsto (|c_1|^{-3/4} x_1, |c_1|^{-1/2} x_2, |c_1|^{-1/2} x_3) \tag{4.9}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = \left[\frac{c_1}{|c_1|}\theta_1 + c_2|c_1|^{-5/4}\theta_2 + c_3|c_1|^{-5/4}\theta_3 \right]_{T_7} = [\pm\theta_1 + \tilde{c}_2\theta_2 + \tilde{c}_3\theta_3]_{T_7}.$$

By the symmetry of $T_7\colon (x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we have that $[\theta_1 + \tilde{c}_2\theta_2 + \tilde{c}_3\theta_3]_{T_7}$ and $[-\theta_1 + \tilde{c}_3\theta_2 + \tilde{c}_2\theta_3]_{T_7}$ are diffeomorphic. □

Lemma 4.19. *If $c_2 \cdot c_3 = 0$ and $c_2 + c_3 \neq 0$, then the algebraic restriction of the form $[\sum_{k=1}^{7} c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\tilde{c}_1\theta_1 + \theta_2 + \tilde{c}_5\theta_5]_{T_7}$.*

Proof of Lemma 4.19. We use methods similar to those used above to prove that if $c_2 \cdot c_3 = 0$ and $c_2 + c_3 \neq 0$, then $\mathcal{A}$ is diffeomorphic to $[\tilde{c}_1\theta_1 + \theta_2 + \tilde{c}_5\theta_5]_{T_7}$. If $c_3 = 0$, then $c_2 \neq 0$ and $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7}$, Let

$$\mathcal{B}_t = [c_1\theta_1 + c_2\theta_2 + (1-t)c_4\theta_4 + c_5\theta_5 + (1-t)c_6\theta_6 + (1-t)c_7\theta_7]_{T_7} \quad \text{for } t \in [0;1].$$

Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_5\theta_5]_{T_7}$. We prove that there exists a family $\Phi_t \in \mathrm{Symm}(T_7)$, $t \in [0;1]$, such that

$$\Phi_t^* \mathcal{B}_t = \mathcal{B}_0, \qquad \Phi_0 = \mathrm{id}. \tag{4.10}$$

Let V_t be a vector field defined by $\mathrm{d}\Phi_t/\mathrm{d}t = V_t(\Phi_t)$. Then, by differentiating (4.10), we obtain

$$\mathcal{L}_{V_t} \mathcal{B}_t = c_4\theta_4 + c_6\theta_6 + c_7\theta_7. \tag{4.11}$$

We are looking for V_t in the form $V_t = b_1(t)X_1 + b_2(t)X_2 + b_4(t)X_4 + b_5(t)X_5$, where $b_k(t)$ for $k = 1, 2, 4, 5$ are smooth functions $b_k\colon [0;1] \to \mathbb{R}$. Then, by Proposition 4.17, (4.11) has the form

$$\begin{bmatrix} 7c_2 & 0 & 0 & 0 \\ 0 & -3c_2 & -4c_1 & 0 \\ 9c_4(1-t) & -9c_5 & 0 & 9c_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_6 \\ c_7 \end{bmatrix}. \tag{4.12}$$

If $c_2 \neq 0$, we can solve (4.12), and Φ_t may be obtained as a flow of vector field V_t. The family Φ_t preserves T_7, because V_t is tangent to T_7 and $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_1\theta_1 + c_2\theta_2 + c_5\theta_5]_{T_7}$. By the condition $c_2 \neq 0$, we have a diffeomorphism $\Psi \in \mathrm{Symm}(T_7)$ of the form

$$\Psi\colon (x_1, x_2, x_3) \mapsto (c_2^{-3/5}x_1, c_2^{-2/5}x_2, c_2^{-2/5}x_3) \tag{4.13}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [c_1c_2^{-4/5}\theta_1 + \theta_2 + c_5c_2^{-7/5}\theta_5]_{T_7} = [\tilde{c}_1\theta_1 + \theta_2 + \tilde{c}_5\theta_5]_{T_7}.$$

If $c_2 = 0$, then $c_3 \neq 0$ and by the diffeomorphism $\Theta \in \mathrm{Symm}(T_7)$ of the form $(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we obtain

$$\Theta^*[c_1\theta_1 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{T_7} = [-c_1\theta_1 + c_3\theta_2 + c_4\theta_5 + c_5\theta_4 - c_6\theta_6 - c_7\theta_7]_{T_7}$$

and we may now use the homotopy method. □

Lemma 4.20. *If $c_2 = c_3 = 0$, $c_1 \neq 0$ and $(c_4, c_5) \neq (0, 0)$, then the algebraic restriction of the form $[\sum_{k=1}^7 c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_1 + \tilde{c}_4\theta_4 + \tilde{c}_5\theta_5]_{T_7}$.*

Lemma 4.21. *If $c_1 \neq 0$ and $c_2 = c_3 = c_4 = c_5 = 0$, then the algebraic restriction of the form $[\sum_{k=1}^7 c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_1 + \tilde{c}_7\theta_7]_{T_7}$.*

Lemma 4.22. *If $c_1 = c_2 = c_3 = 0$ and $(c_4, c_5) \neq (0, 0)$, then the algebraic restriction of the form $[\sum_{k=1}^7 c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_4 + \tilde{c}_5\theta_5 + \tilde{c}_6\theta_6]_{T_7}$.*

Lemma 4.23. *If $c_1 = \cdots = c_5 = 0$ and $c_6 \neq 0$, then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^7 c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_6 + \tilde{c}_7\theta_7]_{T_7}$.*

Lemma 4.24. *If $c_1 = \cdots = c_6 = 0$ and $c_7 \neq 0$, then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^7 c_k\theta_k]_{T_7}$ can be reduced by a symmetry of T_7 to an algebraic restriction $[\theta_7]_{T_7}$.*

The proofs of Lemmas 4.20–4.24 are similar and are based on Table 12.

Statement (ii) of Theorem 4.11 follows from conditions in the proof of part (i), and statement (iii) follows from Theorem 3.5, which was proved in § 4.5.

Now we prove that the parameters c, c_1, c_2 are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$. From Table 12 we see that the tangent space to the orbit of $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$ at $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$ is spanned by the linearly independent algebraic restrictions $[4c_1\theta_1 + 5\theta_2 + 5c_2\theta_3]_{T_7}$, $[\theta_4]_{T_7}$, $[\theta_5]_{T_7}$, $[\theta_6]_{T_7}$ and $[\theta_7]_{T_7}$. Hence, the algebraic restrictions $[\theta_1]_{T_7}$ and $[\theta_3]_{T_7}$ do not belong to it. Therefore, the parameters c_1 and c_2 are independent moduli in the normal form $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}$.

Acknowledgements. The work of W.D. was supported by Polish MNiSW Grant NN201 397237. The authors thank the referee for many valuable suggestions.

References

1. V. I. Arnold, Simple singularities of curves, *Proc. Steklov Inst. Math.* **226** (1999), 20–28.
2. V. I. Arnold, First step of local symplectic algebra, in *Differential topology, infinite-dimensional Lie algebras, and applications: D. B. Fuchs' 60th anniversary collection*, American Mathematical Society Translations, Series 2, Volume 194, pp. 1–8 (American Mathematical Society, Providence, RI, 1999).
3. V. I. Arnold and A. B. Givental, Symplectic geometry, in *Dynamical systems IV*, Encyclopedia of Matematical Sciences, Volume 4, pp. 1–138 (Springer, 2001).
4. V. I. Arnold, V. V. Goryunov, O. V. Lyashko and V. A. Vasil'ev, Singualrity theory, II, Classification and applications, in *Dynamical systems VIII*, Encyclopedia of Matematical Sciences, Volume 39, pp. 33–34 (Springer, 1993).
5. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps*, Volume 1 (Birkhäuser, Boston, MA, 1985).
6. W. Domitrz, Local symplectic algebra of quasi-homogeneous curves, *Fund. Math.* **204** (2009), 57–86.
7. W. Domitrz, S. Janeczko and M. Zhitomirskii, Relative Poincare lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety, *Illinois J. Math.* **48** (2004), 803–835.
8. W. Domitrz, S. Janeczko and M. Zhitomirskii, Symplectic singularities of varietes: the method of algebraic restrictions, *J. Reine Angew. Math.* **618** (2008), 197–235.
9. W. Domitrz and J. H. Rieger, Volume preserving subgroups of $\mathcal{A}$ and $\mathcal{K}$ and singularities in unimodular geometry, *Math. Annalen* **345** (2009), 783–817.
10. W. Domitrz and Z. Trebska, Symplectic S_μ singularities, in *Proce. of 11th Intl Workshop on Real and Complex Singularities*, Contemporary Mathematics (American Mathematical Society, Providence, RI), in press.
11. M. Giusti, Classification des singularités isolées d'intersections complètes simples, *C. R. Acad. Sci. Paris Sér. A* **284** (1977), 167–170.
12. G. Ishikawa and S. Janeczko, Symplectic bifurcations of plane curves and isotropic liftings, *Q. J. Math.* **54** (2003), 73–102.
13. G. Ishikawa and S. Janeczko, Symplectic singularities of isotropic mappings, *Banach Center Publ.* **65** (2004), 85–106.
14. P. A. Kolgushkin, Classification of simple multigerms of curves in a space endowed with a symplectic structure, *St Petersburg Math. J.* **15** (2004), 103–126.
15. J. M. Wahl, Derivations, automorphisms and deformations of quasi-homogeneous singularities, in *Singularities*, Part 2, Proceedings of Symposia in Pure Mathematics, Volume 40, pp. 613–624 (American Mathematical Society, Providence, RI, 1983).
16. C. T. C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, Volume 63 (Cambridge University Press, 2004).
17. M. Zhitomirskii, Relative Darboux theorem for singular manifolds and local contact algebra, *Can. J. Math.* **57** (2005), 1314–1340.