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RELATIVE POINCARE LEMMA, CONTRACTIBILITY, QUASI-HOMOGENEITY AND VECTOR FIELDS TANGENT TO A SINGULAR VARIETY

W. DOMITRZ, S. JANECZKO, AND M. ZHITOMIRSKII

ABSTRACT. We study the interplay between the properties of the germ of a singular variety $N \subset \mathbb{R}^n$ given in the title and the algebra of vector fields tangent to N. The Poincare lemma property means that any closed differential $(p+1)$-form vanishing at any point of N is a differential of a p-form which also vanishes at any point of N. In particular, we show that the classical quasi-homogeneity is not a necessary condition for the Poincare lemma property; it can be replaced by quasi-homogeneity with respect to a smooth submanifold of $\mathbb{R}^n$ or a chain of smooth submanifolds. We prove that N is quasi-homogeneous if and only if there exists a vector field $V, V(0) = 0$, which is tangent to N and has positive eigenvalues. We also generalize this theorem to quasi-homogeneity with respect to a smooth submanifold of $\mathbb{R}^n$.

1. Introduction

Let N be the germ at 0 of a singular variety in $\mathbb{R}^n$. We study the interplay between the properties of N given in the title and the algebra of vector fields tangent to N.

We work with germs in either the analytic category or the C^∞ category. By the Poincare lemma property we mean the following property of N: Any closed differential $(p+1)$-form vanishing at any point of N is a differential of a p-form which also vanishes at any point of N.

The proof of the classical (global) Poincare lemma uses contraction to a point; see, for example, [5]. This method also can be applied to singular varieties $N \subset \mathbb{R}^n$. The main corollary of the results in [18] is as follows (Theorem 2.3): If $\mathbb{R}^n$ is analytically contractible to 0 along N then N has the Poincare lemma property. The analytic contraction of $\mathbb{R}^n$ to 0 along N is an analytic

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family of maps $F_t : \mathbb{R}^n \to \mathbb{R}^n$ such that F_1 is the identity map, $F_0(\mathbb{R}^n) = 0$, and $F_t(N) \subset N$ for all t. In Section 2 we show (Theorem 2.6) that the Poincare lemma property holds under weaker assumptions: It is enough to require that $\mathbb{R}^n$ is analytically contractible to N along N (i.e., $F_0(\mathbb{R}^n) \subset N$ instead of $F_0(\mathbb{R}^n) = 0$). Also, the analytic contraction can be replaced by the piece-wise analytic contraction with respect to the parameter t. This result remains true in the C^∞ category.

The Poincare lemma property of N can be expressed as the triviality of the de Rham cohomology groups of N. Such cohomology groups were constructed in [12], [13]. See also [18], [4], [14] and Section 3 of the present paper. In Section 3 we present a reduction theorem (Theorem 3.1), which helps to study the cohomology groups and to distinguish the cases when they are trivial.

Checking if there exists a smooth or analytic contraction to 0 along N is problematic. The simplest case where this is so is the case when N is quasi-homogeneous. This means that in some local coordinate system N contains, along with any point $(x_1, \dots, x_n)$, the curve $(t^{\lambda_1} x_1, \dots, t^{\lambda_n} x_n)$, where $\lambda_1, \dots, \lambda_n$ are positive numbers, called weights. Is this the only case of smooth (analytic) contractibility? This question was studied in many papers, in an attempt to give a positive answer for a wide class of singular varieties N. In [16] it was proved that analytic contractibility and quasi-homogeneity are the same property if N is a singular plain curve, with an algebraically isolated singularity. Moreover, in [16] it was shown that these properties are equivalent to the Poincare lemma property. Later this result was generalized in [19], where the same was proved in the case when N is a singular hypersurface with an algebraically isolated singularity.

In Section 4 we show that the classical quasi-homogeneity is not a necessary condition for contractibility (and consequently for the Poincare lemma property). We give a definition of quasi-homogeneity of N with respect to a smooth submanifold $S \subset \mathbb{R}^n$, which may be regarded as the classical quasi-homogeneity with some of the weights allowed to be 0. The classical quasi-homogeneity is the quasi-homogeneity with respect to $S = \{0\}$. We prove (Theorem 4.7) that if N is quasi-homogeneous with respect to S and S is contained in N then $\mathbb{R}^n$ is contractible to N along N (and so, by our results in Section 2, N has the Poincare lemma property). We give an example of an analytic singular set N which is quasi-homogeneous with respect to a certain smooth submanifold S in some coordinate system, but not quasi-homogeneous, i.e., not quasi-homogeneous with respect to $S = \{0\}$, in *any* coordinate system. Theorem 4.11 generalizes Theorem 4.7. We define quasi-homogeneity with respect to a chain of smooth submanifolds $S_1 \subset S_2 \subset \cdots \subset S_r$ and show that the quasi-homogeneity of N with respect to the chain implies piece-wise smooth contractibility of $\mathbb{R}^n$ to S_1 along N. If $S_1 \subset N$ then this implies contractibility to N and consequently the Poincare lemma property. In the general case, when S_1 is not contained in N, our reduction theorem (Theorem 3.1) reduces the cohomology groups of $N \subset \mathbb{R}^n$ to the cohomology groups of $(N \cap S_1) \subset S_1$ (Theorem 4.13).

The quasi-homogeneity or its generalizations (quasi-homogeneity with respect to a smooth submanifold or a chain of smooth submanifolds) remain the main tools to check the (piece-wise) smooth or analytic contractibility and the Poincare lemma property. According to A. Givental', positive quasi-homogeneity should be regarded as an analytic analog of contractibility; see [10].

How can one check if N is quasi-homogeneous? Assume that the set of non-singular points of N is dense and that the ideal $I(N)$ of functions vanishing on N is p-generated ($p < \infty$) and can be identified with N (this is always so for analytic varieties). Then the simplest way to prove that N is quasi-homogeneous is to prove that the ideal $I(N)$ is quasi-homogeneous, i.e., there exist (a) a local coordinate system x and (b) a tuple of generators $H_1, \dots, H_p$ such that in the coordinate system x each of the generators $H_1, \dots, H_p$ is quasi-homogeneous with the same weights. How can one check that (a) and (b) exist or prove that they do not exist if one works with arbitrary generators and an arbitrary local coordinate system? One should not expect an algorithm, but it is important to give an answer in terms of some canonical object.

It is clear that the quasi-homogeneity of N (or the ideal $I(N)$) is related to the following property of the algebra of all smooth or analytic vector fields V tangent to N (or to the ideal $I(N)$, which means that $V(f) \in I(N)$ for any $f \in I(N)$). If N is quasi-homogeneous then one of these vector fields must have positive eigenvalues. In fact, it follows from the definition of quasi-homogeneity of N that in suitable coordinates the Euler vector field $E = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_n x_n \frac{\partial}{\partial x_n}$, where $\lambda_1, \dots, \lambda_n$ are the weights, is tangent to N. If we change the coordinate system then E will be transformed to a vector field of another form, but the new vector field has the same eigenvalues $\lambda_1, \dots, \lambda_n$. Therefore the quasi-homogeneity of N implies the existence of a smooth (analytic) vector field V which is tangent to N and has positive eigenvalues at the singular point 0. Is this also a sufficient property for quasi-homogeneity?

We answer this question in Section 5, which contains the main contribution of the present paper. Theorem 5.1 gives a positive answer: *N is quasi-homogeneous if and only if there exists a smooth (analytic) vector field V, $V(0) = 0$, which is tangent to N and has positive eigenvalues at the singular point* 0. Theorem 5.2 generalizes Theorem 5.1 from the classical quasi-homogeneity (i.e., quasi-homogeneity with respect to $\{0\}$) to the quasi-homogeneity with respect to a smooth submanifold $S \subset \mathbb{R}^n$. A necessary and sufficient condition for such quasi-homogeneity is the existence of a vector field V which vanishes at any point of S and has at any point of S the same positive eigenvalues corresponding to directions transversal to S.

Theorem 5.7 is a reformulation of Theorem 5.1 in terms of the ideal $I(N)$, but it also contains an additional statement on the degree of quasi-homogeneity of the ideal. The tangency of a vector field V to the ideal $I(N)$ implies that $V(H) = R(\cdot)H$, where $H = (H_1, \dots, H_p)^t$ is any tuple of generators of $I(N)$ and $R(\cdot)$ is a matrix function. It is easy to see that the eigenvalues $d_1, \dots, d_p$ of the matrix $R(0)$ do not depend on the choice of generators, i.e., they are the invariants of a vector field V tangent to $I(N)$. Theorem 5.7 states that if V has positive eigenvalues then there exists a coordinate system and a tuple of generators $\hat{H}_1, \dots, \hat{H}_p$ such that in this coordinate system $\hat{H}_i$ is quasi-homogeneous of degree d_i.

Theorem 5.8 generalizes, in the same way, Theorem 5.2. Theorems 5.1, 5.2, 5.7, and 5.8 imply Theorem 5.10, which looks obvious, but in fact is not. Theorem 5.10 states that if N can be identified with $I(N)$ and the set of non-singular points of N is dense then the quasi-homogeneity (with respect to a smooth submanifold) of N and the quasi-homogeneity (with respect to a smooth submanifold) of $I(N)$ is the same property.

Like all other results in this paper, Theorems 5.1, 5.2, 5.7, and 5.8 hold in either the analytic or the C^∞ category. In the analytic category the particular case $p = 1$ of Theorem 5.7 can be compared with the distinguished Saito theorem [19] stating that a function germ H with algebraically isolated singularity is quasi-homogeneous if and only if it belongs to the ideal generated by its partial derivatives. Recently the Saito theorem was generalized in [20] (see also the references in [20]) to complete intersection singularities. The relation between the results in [20] and Theorem 5.7 is yet to be understood.

We prove the results of Section 5 in Section 6. (The theorems in Sections 2–4 are proved right after their formulations.) The proof of the results of Section 5 consists of several steps; therefore we divide Section 6 into several subsections.

In the Appendix we compare the Poincare lemma property used in this paper with a different version of this property studied in [9], [10]: The property of an analytic set N that any closed $(p+1)$-form with vanishing pullback to the regular part of N is a differential of a p-form satisfying the same condition. The corresponding de Rham complex is a priori different from the complex in Section 3. The conditions (certain types of contractibility) given in Sections 2 and 3 are sufficient for exactness of both complexes. Nevertheless, we do not know if the cohomology groups of the two complexes are isomorphic for any analytic varieties.

Note that in the present paper we work with germs of smooth or analytic differential forms defined on the whole neighborhood of $0 \in \mathbb{R}^n$. There are also definitions of de Rham complexes of a singular variety N based on differential forms of certain functional categories defined on N *only*. A survey of results and references can be found in the paper [6] and the book [15]; see also the recent work [7].

2. Relative Poincare lemma and contractibility

CONVENTION. We work in either the C^∞ or the analytic category. All objects (varieties, maps, differential forms, etc.) are germs at 0.

DEFINITION 2.1. We say that a set $N \subset \mathbb{R}^n$ has the Poincare lemma property if any closed differential $(p+1)$-form vanishing at any point of N is a differential of a p-form which also vanishes at any point of N.

By saying that a differential p-form ω vanishes at a point $x \in \mathbb{R}^n$ we mean that the algebraic form $\omega|_x$ annihilates any tuple of p-vectors in $T_x\mathbb{R}^n$. This of course implies that the pullback of ω to the regular part of N (the set of points near which N is a smooth submanifold of $\mathbb{R}^n$) is zero, but the inverse is not true. For example, the 1-form dx_1 on $\mathbb{R}^2(x_1, x_2)$ has zero pullback to the line $x_1 = 0$, but it does not vanish at points of this line.

DEFINITION 2.2. Let N be a subset in $\mathbb{R}^n$. We say that $\mathbb{R}^n$ is smoothly (analytically) contractible to 0 along N if there exists a family F_t of smooth (analytic) maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ depending smoothly (analytically) on $t \in [0,1]$ such that F_1 is the identity map, $F_0(\mathbb{R}^n) = \{0\}$, and $F_t(N) \subset N$ for all $t \in [0,1]$. The family F_t is called a smooth (analytic) contraction.

THEOREM 2.3 (Main corollary of the results in [18]). *Let $N \subset \mathbb{R}^n$. If $\mathbb{R}^n$ is smoothly (analytically) contractible to* 0 *along N then N has the Poincare lemma property.*

This theorem holds globally, not only locally, and one can replace $\mathbb{R}^n$ by a smooth manifold. Though in [18] only the holomorphic case was considered, the proof remains the same in the analytic and the C^∞ categories. It is similar to one of the proofs of the classical Poincare lemma (see, for example, [5]). The key point is the following lemma.

LEMMA 2.4. *Let $F_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in [a,b]$, be a family of maps depending smoothly (analytically) on t such that $F_t(N) \subset N$ for any $t \in [a,b]$. Let ω be a closed differential $(p+1)$-form on $\mathbb{R}^n$ vanishing at any point of N. Then $F_b^*\omega - F_a^*\omega = d\alpha$, where α is a differential p-form vanishing at any point of N.*

Theorem 2.3 follows from Lemma 2.4 applied to a contraction F_t of $\mathbb{R}^n$ to 0 along N because $F_1^*\omega = \omega$ and $F_0^*\omega = 0$.

Proof of Lemma 2.4. We have $F_b^*\omega - F_a^*\omega = \int_a^b (F_t^*\omega)' dt$, where the derivative is taken with respect to t. It is well known that

$$(F_t^*\omega)' = F_t^*(L_{V_t}\omega), \qquad L_{V_t}\omega = V_t \rfloor d\omega + d(V_t \rfloor \omega), \qquad V_t = \frac{dF_t}{dt}.$$

Here L_{V_t} is the Lie derivative along the vector field V_t. Since ω is closed we get

$$F_b^*\omega - F_a^*\omega = d\alpha, \qquad \alpha = \int_a^b F_t^*(V_t\rfloor\omega)dt.$$

Since the $(p+1)$-form ω vanishes at any point of N, so does the form $V_t\rfloor\omega$ and, due to assumption that $F_t(N) \subset N$, the form $F_t^*(V_t\rfloor\omega)$ also vanishes at any point of N. Consequently, the p-form α vanishes at any point of N. □

Theorem 2.3 can be generalized. The Poincare lemma property of a subset $N \subset \mathbb{R}^n$ holds with weaker assumptions: It suffices to have contractibility along N to N (not necessarily to 0), and the smooth (analytic) contractibility can be replaced by piece-wise smooth (analytic) contractibility with respect to t.

Definition 2.5 (cf. [18]). Let N and Y be subsets of $\mathbb{R}^n$. We say that $\mathbb{R}^n$ is piece-wise smoothly (analytically) contractible to Y along N if there exists a family F_t, $t \in [0,1]$, of smooth (analytic) maps $\mathbb{R}^n \to \mathbb{R}^n$ which is piece-wise smooth (analytic) in t such that F_1 is the identity map, $F_0(\mathbb{R}^n) \subset Y$ and $F_t(N) \subset N$ for all $t \in [0,1]$.

In the present section this definition will be used with $Y = N$ and in the next section, on de Rham cohomology groups, with Y being a smooth submanifold of $\mathbb{R}^n$.

Theorem 2.6. *Let $N \subset \mathbb{R}^n$. If $\mathbb{R}^n$ is piece-wise smoothly (analytically) contractible to N along N then N has the Poincare lemma property.*

Like Theorem 2.3 , this theorem also holds globally and $\mathbb{R}^n$ can be replaced by a smooth manifold.

Proof. Let F_t be the contraction of $\mathbb{R}^n$ to N along N. The proof is also based on Lemma 2.4. Fix points $0 = t_0 < t_1 < \cdots < t_r = 1$ such that F_t is smooth (analytic) in t when $t \in [t_i, t_{i+1}]$. Take any closed differential $(p+1)$-form ω on $\mathbb{R}^n$ vanishing at any point of N. Applying Lemma 2.4 r times (with F_t restricted to $[t_{r-1}, t_r] = [t_{r-1}, 1]$, then to $[t_{r-2}, t_{r-1}]$, and so on until we reach the interval $[t_0, t_1] = [0, t_1]$), we obtain that $F_1^*\omega - F_0^*\omega = d\alpha_1 + \cdots + d\alpha_r$, where the α_i are differential p-forms vanishing at any point of N. It remains to note that $F_1^*\omega = \omega$ and $F_0^*\omega = 0$. The latter is true because $F_0(\mathbb{R}^n) \subset N$ and ω vanishes at any point of N. □

3. De Rham cohomology of a singular set

The Poincare lemma property can be expressed in terms of the de Rham cohomology of a singular set: A set N has the Poincare lemma property if and only if the de Rham cohomology groups are all trivial.

The de Rham cohomology were defined in [12], [13]. See also [18], [4], [14]. They are related with the following objects (recall that we work throughout the paper with germs):

$\Omega^p(\mathbb{R}^n)$: The space of smooth (analytic) differential p-forms (functions when $p = 0$) on $\mathbb{R}^n$.

$K^p_N(\mathbb{R}^n)$: The subspace of $\Omega^p(\mathbb{R}^n)$ consisting of p-forms of the form $\alpha + d\beta$, where α and β are p-forms and $(p-1)$-forms, respectively, vanishing at any point of the set N.

$\Omega^p_N(\mathbb{R}^n)$: The factor-space $\Omega^p(\mathbb{R}^n)/K^p_N(\mathbb{R}^n)$.

Note that if $\omega \in K^p_N(\mathbb{R}^n)$ then $d\omega \in K^{p+1}_N(\mathbb{R}^n)$. Therefore the operator $d_p : \Omega^p_N(\mathbb{R}^n) \to \Omega^{p+1}_N(\mathbb{R}^n)$, $d_p(\omega) = d\omega$, is well-defined, and one has the complex (called the Grauert-Grothendieck complex)

$$\Omega^0_N(\mathbb{R}^n) \xrightarrow{d_0} \Omega^1_N(\mathbb{R}^n) \xrightarrow{d_1} \Omega^2_N(\mathbb{R}^n) \xrightarrow{d_2} \Omega^3_N(\mathbb{R}^n) \xrightarrow{d_3} \cdots$$

This complex is the factor-complex of the classical de Rham complex. The classical de Rham sequence is exact because we work with germs. Therefore the Grauert-Grothendieck sequence is exact if and only if the sequence $K^1_N(\mathbb{R}^n) \to K^2_N(\mathbb{R}^n) \to K^3_N(\mathbb{R}^n) \to \cdots$ is exact. It is easy to see that exactness of the sequence $K^{p-1}_N(\mathbb{R}^n) \to K^p_N(\mathbb{R}^n) \to K^{p+1}_N(\mathbb{R}^n)$ is the same condition as the Poincare lemma property for closed p-forms vanishing at any point of N. It follows that exactness of the sequence $\Omega^{p-1}_N(\mathbb{R}^n) \to \Omega^p_N(\mathbb{R}^n) \to \Omega^{p+1}_N(\mathbb{R}^n)$ is the same condition as the Poincare lemma property for closed $(p+1)$-forms vanishing at any point of N. Therefore the set N has the Poincare lemma property if and only if the Grauert-Grothendieck sequence is exact.

In general, the Grauert-Grothendieck sequence defines the cohomology groups

$$H^p_N(\mathbb{R}^n) = \mathrm{Kernel}(d_p)/\,\mathrm{Image}(d_{p-1}),$$

which are invariants of N. If N does not have the Poincare lemma property then at least one of the cohomology groups is not trivial. Namely, the group $H^p_N(\mathbb{R}^n)$ is trivial if and only if N has Poincare lemma property for closed $(p+1)$-forms vanishing at any point of N.

The following theorem allows us to reduce the study of the cohomology groups $H^p_N(\mathbb{R}^n)$ to the study of the cohomology groups $H^p_{N\cap S}(S)$, where S is a smooth submanifold of $\mathbb{R}^n$, provided that one has a type of contraction of $\mathbb{R}^n$ to S along N; see Definition 2.5.

THEOREM 3.1. *Let $N \subset \mathbb{R}^n$, and let S be a smooth submanifold of $\mathbb{R}^n$. If $\mathbb{R}^n$ is piece-wise smoothly (analytically) contractible to S along N then the cohomology group $H^p_N(\mathbb{R}^n)$ is isomorphic to the cohomology group $H^p_{N\cap S}(S)$, for any p.*

Note that Theorem 2.3 is a simple particular case of Theorem 3.1: It can be obtained from Theorem 3.1 by taking $S = \{0\}$. Theorem 3.1 holds only locally; a global analog of Theorem 3.1 requires additional assumptions.

Proof. Let $i : S \to \mathbb{R}^n$ be the natural embedding. It is clear that

$$i^*(K^p_N(\mathbb{R}^n)) \subset K^p_{N\cap S}(S).$$

Therefore the map $i^* : \Omega^p_N(\mathbb{R}^n) \to \Omega^p_{N\cap S}(S)$ is well-defined. Since $d \circ i^* = i^* \circ d$, i^* induces the map $i^* : H^p_N(\mathbb{R}^n) \to H^p_{S\cap N}(S)$. Any germ of a differential form on S can be obtained as the pullback i^* of a germ of a differential form on $\mathbb{R}^n$. Therefore the map i^* is a surjective homomorphism. It remains to prove that it is injective, i.e., that it has trivial kernel. Analyzing this condition one can reduce Theorem 3.1 to the following lemma.

LEMMA 3.2. *Let S be a smooth submanifold of $\mathbb{R}^n$ such that $\mathbb{R}^n$ is piece-wise smoothly (analytically) contractible to S along a subset $N \subset \mathbb{R}^n$. Let i be the natural embedding $S \to \mathbb{R}^n$. Let ω be a closed $(p+1)$-form on $\mathbb{R}^n$ vanishing at any point of N. If $i^*\omega$ is a differential of a p-form on S vanishing at any point of $N \cap S$ then ω is a differential of a p-form on $\mathbb{R}^n$ vanishing at any point of N.*

Proof of Lemma 3.2. Let F_t be a piece-wise smooth (analytic) contraction of $\mathbb{R}^n$ to S along N. We use Lemma 2.4 in the same way as in the proof of Theorem 2.6. This gives us the relation $\omega = F_0^*\omega + d\alpha$, where α is a p-form vanishing at any point of N. To prove the lemma we have to show that $F_0^*\omega$ is a differential of a p-form vanishing at points of N. □

Since $F_0(\mathbb{R}^n) \subset S$, one has $F_0 = i \circ F_0$ and consequently $F_0^*\omega = F_0^* \circ i^*\omega$. The p-form $i^*\omega$ vanishes at any point of the set $N \cap S$. By the assumption of the lemma $i^*\omega = d\alpha$, where α is a p-form on S vanishing at any point of the set $S \cap N$. We obtain $F_0^*\omega = d(F_0^*\alpha)$ (on the left hand side of this relation we consider F_0 as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and on the right hand side as a map from $\mathbb{R}^n$ to S). Since F_0 takes N to $N \cap S$ and α vanishes at any point of the set $N \cap S$, $F_0^*\alpha$ is a p-form on $\mathbb{R}^n$ vanishing at any point of N. □

In the next section we will present some corollaries to Theorem 3.1. An immediate corollary is as follows.

COROLLARY 3.3. *If $\mathbb{R}^n$ is contractible to a smooth k-dimensional submanifold along N then for any $p \geq k$ the cohomology group $H^p_N(\mathbb{R}^n)$ is trivial and the Poincare lemma property holds for closed $(p+1)$-forms vanishing at any point of N.*

4. Contractibility and quasi-homogeneity

In general, it is hard to determine if there exists a piece-wise smooth or analytic contraction. Nevertheless, in certain cases the existence of a smooth (analytic) contraction is obvious.

EXAMPLE 4.1. Let us prove, using Theorem 2.3, that the germ N at 0 of a smooth submanifold of $\mathbb{R}^n$ has the Poincare lemma property. Take a local coordinate system $x_1, \ldots, x_n$ such that $N = \{x_1 = \cdots = x_s = 0\}$. Then the family $F_t : (x_1, \ldots, x_n) \to (tx_1, \ldots, tx_n)$ is an analytic contraction of $\mathbb{R}^n$ to 0 along N. Consequently, the germ of any smooth submanifold of $\mathbb{R}^n$ has the Poincare lemma property.

If N is a singular variety then the existence of a smooth (analytic) contraction of $\mathbb{R}^n$ to 0 or N along N depends on the singularity of N.

EXAMPLE 4.2 (from [18]). Let N be the singular plane curve given by the equation $H(x_1, x_2) = x_1^4 + ax_1x_2^4 + x_2^5 = 0$, $a \in \mathbb{R}$. It follows from results in [18] that N does not have the Poincare lemma property if $a \neq 0$. For example, the closed 2-form $H(x_1, x_2)dx_1 \wedge dx_2$ vanishes at any point of N, but if $a \neq 0$ it is not a differential of any 1-form vanishing at any point of N. Therefore there is no smooth or analytic contraction of $\mathbb{R}^2$ to 0 (and, by Theorem 2.6, to N) along N if $a \neq 0$. Such a contraction exists if $a = 0$; it is given by the maps $(x_1, x_2) \to (t^5x_1, t^4x_2)$. Therefore, if $a = 0$ then N has the Poincare lemma property.

In fact, Example 4.1 is based on the *local homogeneity* of any smooth submanifold $N \subset \mathbb{R}^n$ (in suitable coordinates the map $x \to tx$ takes N to itself), and Example 4.2 with $a = 0$ is based on the local quasi-homogeneity: In suitable coordinates the map $(x_1, x_2) \to (t^5x_1, t^4x_2)$ takes N to itself. The quasi-homogeneity generalizes homogeneity. The close relation between the analytic contractibility to 0 along N and the quasi-homogeneity of N was shown in [16], [17], [19], [10]. We will use the following definition of quasi-homogeneity of a priori an arbitrary set.

DEFINITION 4.3. A subset $N \subset \mathbb{R}^n$ is called quasi-homogeneous if there exists a local coordinate system $x_1, \ldots, x_n$ and *positive* numbers $\lambda_1, \ldots, \lambda_n$ (called weights) such that for all t the map $F_t : (x_1, \ldots x_n) \to (t^{\lambda_1}x_1, \ldots, t^{\lambda_n}x_n)$ takes any point $p \in N$ to a point $F_t(p) \in N$ provided that p and $F_t(p)$ are sufficiently close to 0.

If N is quasi-homogeneous then the family F_t in this definition is a smooth (analytic) contraction of $\mathbb{R}^n$ to 0 along N. Therefore Theorem 2.3 implies the following corollary.

THEOREM 4.4. *Any quasi-homogeneous germ of a subset of* $\mathbb{R}^n$ *has the Poincare lemma property.*

EXAMPLE 4.5. (a) The image of any smooth curve in $\mathbb{R}^n$ of the form $s \to (s^{i_1}, \ldots, s^{i_n})$ has the Poincare lemma property because it is a quasi-homogeneous set with the weights $i_1, \ldots, i_n$.

(b) The image of any smooth plane curve germ of the form $(s^2, o(s^2))$, except for infinitely degenerate curves whose Taylor series is RL-equivalent to $(s^2, 0)$, has the Poincare lemma property. The same is true for any plane curve of the form $(s^3, s^4 + o(s^4))$ or $(s^3, s^5 + o(s^5))$, because any such curve is RL-equivalent to one of the curves $(s^2, s^{2k+1}),\ (s^3, s^4),\ (s^3, s^5)$; see [8].

In Section 2 we showed that the Poincare lemma property also holds under a weaker type of contractibility. This suggests that the quasi-homogeneity is not a necessary condition for the Poincare lemma property. We will show that if a subset $N \subset \mathbb{R}^n$ contains a smooth submanifold S of $\mathbb{R}^n$ then N has the Poincare lemma property provided that N is quasi-homogeneous with respect to S according to the definition given below. We will also give an example showing that in general the quasi-homogeneity with respect to S does not imply the classical quasi-homogeneity (in any coordinate system), and therefore the classical quasi-homogeneity is not a necessary condition for the Poincare lemma property.

DEFINITION 4.6. Let N be a subset of $\mathbb{R}^n$, and let S be a smooth submanifold of $\mathbb{R}^n$ of codimension k. We say that N is quasi-homogeneous with respect to S if there exists a local coordinate system (x, y) of $\mathbb{R}^n$, $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_{n-k})$, and *positive* numbers $\lambda_1, \ldots, \lambda_k$ such that S is given by the equations $x = 0$ and such that for all t the map $F_t : (x_1, \ldots x_k, y_1, \ldots, y_{n-k}) \to (t^{\lambda_1} x_1, \ldots, t^{\lambda_n} x_k, y_1, \ldots, y_{n-k})$ takes any point $p \in N$ to a point $F_t(p) \in N$ provided that p and $F_t(p)$ are sufficiently close to 0.

This definition generalizes the definition of the classical quasi-homogeneity, which is the quasi-homogeneity with respect to $S = \{0\}$. The quasi-homogeneity with respect to S can be understood as the classical quasi-homogeneity with some of the weights allowed to be 0.

If N is quasi-homogeneous with respect to S then $\mathbb{R}^n$ is smoothly (analytically) contractible to N along N provided that $S \subset N$. Therefore Theorem 2.6 implies the following corollary.

THEOREM 4.7. *Let* $N \subset \mathbb{R}^n$. *Assume that* $S \subset N$, *where* S *is a smooth submanifold of* $\mathbb{R}^n$. *If* N *is quasi-homogeneous with respect to* S *then* N *has the Poincare lemma property.*

REMARK 4.8. Theorem 4.7 can also be obtained as a corollary of Theorem 3.1. In fact, the quasi-homogeneity of N with respect to S implies, by Theorem 3.1, that the cohomology groups $H_N^p(\mathbb{R}^n)$ are isomorphic to the cohomology groups $H_{N\cap S}^p(S)$. Since $S \subset N$, $N \cap S = S$ and $H_{N\cap S}^p(S) = H_S^p(S) = \{0\}$.

EXAMPLE 4.9. Let

$$N = \{(x_1, x_2, y) \in \mathbb{R}^3 : H(x_1, x_2, y) = (x_1^2 - x_2^2)^2 + y x_1^2 x_2^2 = 0\}.$$

The set N is quasi-homogeneous with respect to the curve $S : \{x_1 = x_2 = 0\}$ with weights $(1, 1)$: If $(x_1, x_2, y) \in N$ then $(tx_1, tx_2, y) \in N$. Since $S \subset N$, by Theorem 4.7 the set N has the Poincare lemma property. In the next section we will show that N is not quasi-homogeneous (in any coordinate system) in the classical sense; see Example 5.11.

The quasi-homogeneity of N with respect to a smooth submanifold of $\mathbb{R}^n$ contained in N is also not a necessary condition for the Poincare lemma property, as we will show in Example 4.12 below. It can be weakened by replacing the quasi-homogeneity with respect to S by the quasi-homogeneity with respect to a chain of smooth submanifolds $S_1 \subset S_2 \subset \cdots \subset S_r$ such that $S_1 \subset N$.

DEFINITION 4.10. A subset $N \subset \mathbb{R}^n$ is called quasi-homogeneous with respect to the chain $S_1 \subset S_2 \subset \cdots \subset S_r$ of smooth submanifolds of $\mathbb{R}^n$ if N is quasi-homogeneous with respect to S_r, and the intersection $N \cap S_i$ is quasi-homogeneous with respect to S_{i-1}, $i = r, r-1, \ldots, 2$. The weights of the quasi-homogeneity of $N \cap S_i$ with respect to S_{i-1} are allowed to depend on i.

It is easy to see that the quasi-homogeneity of N with respect to a chain $S_1 \subset S_2 \subset \cdots \subset S_r$ implies the existence of a contraction of $\mathbb{R}^n$ to S_1 along N, but now this contraction is piece-wise smooth (analytic). If $S_1 \subset N$ then we have a piece-wise smooth (analytic) contraction to N along N. Using again Theorem 2.6 we obtain the following corollary.

THEOREM 4.11. *Let $N \subset \mathbb{R}^n$. If N is quasi-homogeneous with respect to a chain $S_1 \subset S_2 \subset \cdots \subset S_r$ of smooth submanifolds of $\mathbb{R}^n$ and $S_1 \subset N$ then N has the Poincare lemma property.*

EXAMPLE 4.12. Let N be the subvariety of $\mathbb{R}^{14}(x_1, \ldots, x_8, y_1, \ldots, y_4, z_1, z_2)$ given as the common zero level of the 6 functions

$$H_1(x,y,z) = (x_1^2 - x_2^2)^2 + y_1 x_1^2 x_2^2, \quad H_2(x,y,z) = (x_3^2 - x_4^2)^2 + y_2 x_3^2 x_4^2,$$
$$H_3(x,y,z) = (x_5^2 - x_6^2)^2 + y_3 x_5^2 x_6^2, \quad H_4(x,y,z) = (x_7^2 - x_8^2)^2 + y_4 x_7^2 x_8^2,$$
$$G_1(x,y,z) = (y_1^2 - y_2^2)^2 + z_1 y_1^2 y_2^2, \quad G_2(x,y,z) = (y_3^2 - y_4^2)^2 + z_2 y_3^2 y_4^2.$$

Let us show that N has the Poincare lemma property using Theorem 4.11. Let S_1 be the smooth 2-dimensional submanifold of $\mathbb{R}^{14}$ given by the equations $x = y = 0$ and let S_2 be the smooth 6-dimensional submanifold given by the equations $x = 0$. It is clear that N is quasi-homogeneous with respect to S_2 with weights $1, 1, 1, 1, 1, 1, 1, 1$ and the set $N \cap S_2 = \{G_1(x, y, z) = G_2(x, y, z) = x = 0\}$ is quasi-homogeneous with respect to S_1 with weights $1, 1, 1, 1$. Therefore N is quasi-homogeneous with respect to the chain $S_1 \subset S_2$. Since S_1 is a subset of N, by Theorem 4.11 N has the Poincare lemma property.

One can show that in this example N is not quasi-homogeneous with respect to any single smooth submanifold of $\mathbb{R}^{14}$ contained in N. In fact, any smooth submanifold of $\mathbb{R}^{14}$ contained in N is either the plane S_1 or a non-singular curve in this plane. It is easy to see that N is not quasi-homogeneous with respect to any of such submanifolds.

The assumption $S_1 \subset N$ in Theorem 4.11 cannot be removed; see Example 4.14. In the general case, without the assumption $S_1 \subset N$, we have the following corollary to Theorem 3.1.

THEOREM 4.13. *Let N be the germ at 0 of a subset of $\mathbb{R}^n$. If N is quasi-homogeneous with respect to a chain $S_1 \subset S_2 \subset \cdots \subset S_r$ of smooth submanifolds of $\mathbb{R}^n$ then the cohomology groups $H^p_N(\mathbb{R}^n)$ and $H^p_{N \cap S_1}(S_1)$ are isomorphic.*

Theorem 4.13 generalizes Theorem 4.11. In fact, if $S_1 \subset N$ then

$$H^p_{N \cap S_1}(S_1) = H^p_{S_1}(S_1) = \{0\}.$$

Therefore N has the Poincare lemma property.

EXAMPLE 4.14. Let N_a be the subvariety of $\mathbb{R}^{14}(x_1, \ldots, x_8, y_1, \ldots, y_4, z_1, z_2)$ defined as the common zero level of the functions $H_1, H_2, H_3, H_4, G_1, G_2$ given in Example 4.12 and the function

$$F(x, y, z) = z_1^4 + a z_1 z_2^4 + z_2^5.$$

The set N_a is quasi-homogeneous with respect to a chain of submanifolds $S_1 \subset S_2$, where S_1 is a smooth 2-dimensional submanifold of $\mathbb{R}^{14}$ given by the equations $x = y = 0$ and S_2 is a smooth 6-dimensional submanifold given by the equations $x = 0$ (see Example 4.12). Let C_a be a subset of $\mathbb{R}^2(z_1, z_2)$ given by the equation $F(0, 0, z) = 0$. Then the cohomology groups $H^p_{N_a}(\mathbb{R}^{14})$ and $H^p_{N_a \cap S_1}(S_1) = H^p_{C_a}(\mathbb{R}^2)$ are isomorphic by Theorem 4.13. If $p > 2$ then any p-form on $\mathbb{R}^2$ is zero and therefore $H^p_{C_a}(\mathbb{R}^2) = 0$ for $p > 1$; see Section 3. By Example 4.2, $H^1_{C_a}(\mathbb{R}^2) = 0$ if and only if $a = 0$. Therefore if $p > 1$ then $H^p_{N_a}(\mathbb{R}^{14})$ is trivial and $H^1_{N_a}(\mathbb{R}^{14})$ is trivial if and only if $a = 0$. Consequently N_a has the Poincare lemma property if and only if $a = 0$, and if $a \neq 0$ then the Poincare lemma property holds for closed p-forms if and only if $p \neq 2$.

5. Quasi-homogeneity and vector fields tangent to a singular variety

In this section we present our main results relating the quasi-homogeneity of a variety $N \subset \mathbb{R}^n$ with the algebra of smooth (analytic) vector fields tangent to N. We will work with germs of subsets $N \subset \mathbb{R}^n$ satisfying the following conditions:

(a) $N = \{H_1 = \cdots = H_p = 0\}$, where $(H_1, \dots, H_p)$ is a tuple of generators of the ideal of all smooth (analytic) function germs vanishing at any point of N.

(b) The set of non-singular points of N (the points near which N has the structure of a smooth submanifold of $\mathbb{R}^n$) is dense in N.

By saying that a vector field V on $\mathbb{R}^n$ is tangent to a set $N \subset \mathbb{R}^n$ we mean that V is tangent to N at any non-singular point of N.

THEOREM 5.1. *The germ N at 0 of a subset of $\mathbb{R}^n$ satisfying the assumptions* (a) *and* (b) *is quasi-homogeneous if and only if there exists a vector field V, $V(0) = 0$, which is tangent to N and whose eigenvalues at 0 are positive real numbers.*

A generalization of this theorem to the quasi-homogeneity with respect to a smooth submanifold $S \subset \mathbb{R}^n$ is as follows. Note that if V is a vector field which vanishes at any point of S then at any point $x \in S$ it has zero eigenvalues corresponding to directions in T_xS, i.e., V always has $(\dim S)$ zero eigenvalues. The other $(\operatorname{codim} S)$ eigenvalues corresponding to directions in $T_0\mathbb{R}^n$ transversal to S are, in general, arbitrary.

THEOREM 5.2. *Let S be a smooth submanifold of $\mathbb{R}^n$. The germ N at 0 of a subset of $\mathbb{R}^n$ satisfying the assumptions* (a) *and* (b) *is quasi-homogeneous with respect to S if and only if there exists a vector field V which is tangent to N, vanishes at any point $x \in S$, and whose eigenvalues at $x \in S$ corresponding to directions transversal to S do not depend on x and are positive real numbers.*

The implication from quasi-homogeneity to the existence of a vector with the required properties is simple. In fact, if N is quasi-homogeneous with respect to S (the classical quasi-homogeneity is the case $S = \{0\}$) then in some coordinate system S is given by the equations $x_1 = \cdots = x_k = 0$ and there exists a tuple $(\lambda_1, \dots, \lambda_k)$ of positive numbers such that for any non-singular point $a = (x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in N$ one has $(t^{\lambda_1}x_1, \dots, t^{\lambda_n}x_k, y_1, \dots, y_{n-k}) \in N$ provided that t is close to 1. Differentiating this inclusion with respect to t at $t = 1$, we obtain that the vector

$$E_\lambda = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_k x_k \frac{\partial}{\partial x_k}$$

is tangent to N (belongs to the space T_aN). Consider now E_λ as a vector field in $\mathbb{R}^n$. It is tangent to N, vanishes at any point of S, and has at any point of S the same positive eigenvalues $\lambda_1, \ldots, \lambda_k$ corresponding to the directions transversal to S (all positive eigenvalues if $S = \{0\}$ and consequently $k = n$).

To prove the difficult part of Theorems 5.1 and 5.2 we will work with quasi-homogeneous function germs and quasi-homogeneous ideals in the ring of function germs.

Definition 5.3. Let S be a smooth submanifold of $\mathbb{R}^n$ of codimension k.

(a) A function germ $f : \mathbb{R}^n \to \mathbb{R}$ is called quasi-homogeneous with respect to S in a coordinate system $(x_1, \ldots, x_k, y_1, \ldots y_{n-k})$ if S is given by the equations $x_1 = \cdots = x_k = 0$ and there exist *positive* numbers $\lambda_1, \ldots, \lambda_k$ (called weights) and a real number d (called the degree) such that $f(t^{\lambda_1}x_1, \ldots, t^{\lambda_k}x_k, y_1, \ldots, y_{n-k}) = t^d f(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ for all t such that the points $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ and $(t^{\lambda_1}x_1, \ldots, t^{\lambda_k}x_k, y_1, \ldots, y_{n-k})$ are sufficiently close to 0.

(b) A p-generated ideal in the ring of function germs is quasi-homogeneous with respect to S if there exists a tuple of generators $H_1, \ldots, H_p$ and a local coordinate system in which these generators are quasi-homogeneous with respect to S with the same weights $\lambda_1, \ldots, \lambda_k$. The degrees $d_1, \ldots, d_p$ of quasi-homogeneity of $H_1, \ldots, H_p$ may be different. The numbers $\lambda_1, \ldots, \lambda_k$ are called the weights of the quasi-homogeneity of the ideal, and $(d_1, \ldots, d_p)$ is called the tuple of degrees of quasi-homogeneity.

The usual quasi-homogeneity corresponds to the case $S = \{0\}$. In this case $k = n$. In Section 6.1 we will present two other, equivalent, definitions of quasi-homogeneity of function germs.

Example 5.4. Consider the ideal I in the ring of function germs $H(x_1, x_2, x_3)$ generated by the functions

$$H_1 = x_1^2 - x_2^3, \quad H_2 = x_2^5 - x_3^{14} + x_1^2 x_2 - x_2^4.$$

It is easy to see that H_1 and H_2 are not quasi-homogeneous with the same weights in the given coordinate system. Moreover, one can show that they are not quasi-homogeneous with the same weights in any coordinate system. On the other hand, one can choose other generators of the ideal I, namely $\hat{H}_1 = H_1, \hat{H}_2 = H_2 - x_2 H_1 = x_2^5 - x_3^{14}$. Now we see that the ideal I is quasi-homogeneous with weights $(3, 2, 5/7)$. Consequently the set $N = \{H_1 = H_2 = 0\}$ is quasi-homogeneous with the same weights.

Definition 5.5. Let I be an ideal in the ring of function germs. A vector field V is tangent to I if $V(f) \in I$ for any function $f \in I$.

Let I be a p-generated ideal in the ring of function germs and let $H = (H_1, \dots, H_p)^t$ (where the superscript t denotes the transpose) be a tuple of generators. Then the tangency of a vector field V to I means a relation of the form

$$V(H) = R(\cdot)H, \tag{5.1}$$

where $R(\cdot)$ is a $p \times p$ matrix function. The matrix $R(\cdot)$ depends of course on the choice of generators, but the eigenvalues of the matrix $R(0)$ do not. In fact, when replacing the tuple H by $\tilde{H} = T^{-1}(\cdot)H$, where $T(\cdot)$ is a non-degenerate $p \times p$ matrix, the matrix $R(0)$ is replaced by $T^{-1}(0)R(0)T(0)$ (the whole matrix $R(\cdot)$ changes in a more complicated way; see Section 6.5).

NOTATION. Let V be a vector field tangent to a p-generated ideal I in the ring of function germs. The invariants of V defined above (the eigenvalues of the matrix $R(0)$ in (5.1)) will be denoted by $d_1(V, I), \dots, d_p(V, I)$.

EXAMPLE 5.6. Let I be the ideal in Example 5.4. We showed that the Euler vector field E_λ, $\lambda = (3, 2, 5/7)$, is tangent to the ideal $I(N)$. The invariants $d_1(E_\lambda, I), d_2(E_\lambda, I)$ are equal to 6 and 10.

THEOREM 5.7. *Let I be a p-generated ideal in the ring of function germs on $\mathbb{R}^n$. Let $(\lambda_1, \dots, \lambda_n)$ be a tuple of positive numbers. The following conditions are equivalent:*

(i) *The ideal I is quasi-homogeneous with weights $(\lambda_1, \dots, \lambda_n)$.*
(ii) *There exists a vector field $V, V(0) = 0$, which is tangent to I and has at 0 eigenvalues $\lambda_1, \dots, \lambda_n$.*

Moreover, if V is a vector field satisfying (ii) *then*

$$d_1(V, I), \dots, d_p(V, I)$$

are positive real numbers and the ideal I is quasi-homogeneous with the tuple of degrees $d_1(V, I), \dots, d_p(V, I)$.

Like Theorem 5.1, Theorem 5.7 can be generalized to the case of quasi-homogeneity with respect to a smooth submanifold $S \subset \mathbb{R}^n$.

THEOREM 5.8. *Let I be a p-generated ideal in the ring of function germs on $\mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a smooth submanifold of codimension k. Let $(\lambda_1, \dots, \lambda_k)$ be a tuple of positive numbers. The following conditions are equivalent:*

(i) *The ideal I is quasi-homogeneous with respect to S with weights $(\lambda_1, \dots, \lambda_k)$.*
(ii) *There exists a vector field V which is tangent to I, vanishes at any point of S, and has the same eigenvalues $\lambda_1, \dots, \lambda_k$ at any point of S in the directions transversal to S.*

Moreover, if V is a vector field satisfying (ii) *then the invariants*

$$d_1(V, I), \ldots, d_p(V, I)$$

are non-negative real numbers and the ideal I is quasi-homogeneous with the tuple of degrees $d_1(V, I), \ldots, d_p(V, I)$.

REMARK 5.9. One can ask why the invariants $d_1(V, I), \ldots, d_p(V, I)$ are positive numbers in the case of Theorem 5.7 and non-negative numbers in the case of Theorem 5.8. The answer is as follows. If V satisfies (ii) in Theorem 5.8 then the invariants $d_1(V, I), \ldots, d_p(V, I)$ are all positive if and only if S belongs to the zero set N of the ideal I. Theorem 5.8 does not require this assumption. But if $S = \{0\}$ (as in Theorem 5.7) then this is of course so.

The implication (i) $\Longrightarrow$ (ii) in Theorems 5.7 and 5.8 is obvious: (i) implies (ii) with V being the Euler vector field E_λ, and (5.1) holds with the $R(\cdot)$ being constant and diagonal: $R(\cdot) = \mathrm{diag}(d_1, \ldots, d_p)$. The implication (ii) $\Longrightarrow$ (i) will be proved in Sections 6.1–6.5.

The simple part of Theorems 5.1 and 5.2 (the quasi-homogeneity of N implies the existence of a vector field V with the required properties) was proved above in this section. The difficult part of these theorems (the existence of a vector field V with the given properties implies the quasi-homogeneity) is a corollary of Theorems 5.7 and 5.8. In fact, under the assumptions (a) and (b) on the set N in the beginning of the present section any vector field V tangent to N is also tangent to the ideal $I = I(N)$ consisting of function germs vanishing at any point of N and the quasi-homogeneity of the ideal $I(N)$ implies the quasi-homogeneity of N.

Theorem 5.8 implies one more result, which looks trivial, but in fact is not. As we have just noticed, under assumptions (a) and (b) on N the quasi-homogeneity of the ideal $I(N)$ implies the quasi-homogeneity of N. This statement is clear. The inverse statement is also true, but it is not trivial.

THEOREM 5.10. *Let N be the germ of a subset of $\mathbb{R}^n$ satisfying assumptions* (a) *and* (b)*. Let $I(N)$ be the ideal of smooth (analytic) function germs vanishing at any point of N. Then the quasi-homogeneity of N with respect to a smooth submanifold $S \subset \mathbb{R}^n$ and the quasi-homogeneity of the ideal $I(N)$ with respect to S is the same property.*

This theorem follows from Theorem 5.8: If N is quasi-homogeneous and satisfies (a) and (b) then the Euler vector field E_λ is tangent to the ideal $I(N)$ and by Theorem 5.8 this ideal is quasi-homogeneous.

We emphasize once again that the classical quasi-homogeneity is a particular case of the quasi-homogeneity with respect to a smooth submanifold S: it corresponds to the case $S = \{0\}$.

We end this section with an example of an analytic variety N having the Poincare lemma property which is not quasi-homogeneous in the classical sense. The latter will be proved using Theorem 5.7.

EXAMPLE 5.11. In Example 4.9 we showed that the germ

$$N = \{(x_1, x_2, y) \in \mathbb{R}^3 : H(x_1, x_2, y) = (x_1^2 - x_2^2)^2 + y x_1^2 x_2^2 = 0\}$$

is quasi-homogeneous with respect to the smooth submanifold $\{x_1 = x_2 = 0\}$, and since $S \subset N$, N has the Poincare lemma property by Theorem 4.7. Let us prove that N is not quasi-homogeneous in the classical sense, i.e., with respect to $\{0\}$. Assume that N is quasi-homogeneous. The ideal $I(N)$ is generated by the function $H = H(x_1, x_2, y)$. By Theorem 5.7 it suffices to check that there is no vector field $V, V(0) = 0$, which is tangent to N and has positive eigenvalues at 0. Assume that such a vector field V exists. Then $V(H) = Q(x_1, x_2, y)H$, where Q is a function. By the last statement of Theorem 5.7, $Q(0,0,0) > 0$. Therefore $X = V/Q$ is a smooth vector field, also with positive eigenvalues, and $X(H) = H$. Let

$$X = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + g \frac{\partial}{\partial y},$$

where f_1, f_2, g are smooth (analytic) function germs at 0.

The relation $X(H) = H$ takes the form

$$\begin{aligned}(4x_1(x_1^2 - x_2^2) + 2y x_1 x_2^2) f_1 + (-4x_2(x_1^2 - x_2^2) + 2y x_1^2 x_2) f_2 + x_1^2 x_2^2 g \\ = (x_1^2 - x_2^2)^2 + y x_1^2 x_2^2.\end{aligned}$$

Calculating the coefficients of the terms $x_1^3 y, x_2^3 y, x_1^4$, x_2^4, $x_1^4 y, x_2^4 y$, we obtain

$$\begin{aligned}\frac{\partial f_1}{\partial y}(0,0,0) &= \frac{\partial f_2}{\partial y}(0,0,0) = 0;\\ \frac{\partial f_1}{\partial x_1}(0,0,0) &= \frac{\partial f_2}{\partial x_2}(0,0,0) = 1/4;\\ \frac{\partial^2 f_1}{\partial x_1 \partial y}(0,0,0) &= \frac{\partial^2 f_2}{\partial x_2 \partial y}(0,0,0) = 0.\end{aligned}$$

Calculating now the coefficient of the term $y x_1^2 x_2^2$, taking into account these relations, we get

$$\frac{\partial g}{\partial y}(0,0,0) = 0.$$

It follows that the matrix of linearization of X has zero column. This contradicts the condition that all eigenvalues of X are positive.

6. Proof of Theorems 5.7 and 5.8

Theorem 5.7 is a particular case of Theorem 5.8; therefore we will prove Theorem 5.8. As we explained in the previous section, the implication (i) $\Longrightarrow$ (ii) in Theorem 5.8 is obvious, so we will prove the implication (ii) $\Longrightarrow$ (i) and the statement on the invariants $d_1(V, I), \ldots, d_p(V, I)$. Since the proof consists of several steps, we divide this section onto several subsections. Throughout the proof we work with quasi-homogeneous functions and also quasi-homogeneous vector fields with respect to a smooth submanifold $S \subset \mathbb{R}^n$. We need three equivalent definitions of quasi-homogeneity, which are given in Section 6.1.

To prove the implication (ii) $\Longrightarrow$ (i) we have to find two objects, (a) a coordinate system and (b) a tuple of generators of the ideal I, such that each of the generators is quasi-homogeneous with weights $\lambda_1, \ldots, \lambda_k$ in the chosen coordinate system. The coordinate system will be chosen in Section 6.2. It is a coordinate system in which the vector field V has the classical resonant normal form if $S = \{0\}$ and generalized resonant normal form if $S \neq \{0\}$. The advantage of this coordinate system, used throughout the proof, is that in this coordinate system V is quasi-homogeneous (with respect to S and with weights $\lambda_1, \ldots, \lambda_k$) of degree 0.

The choice of generators of the ideal is a more difficult task. We will work in the coordinate system chosen in Section 6.2. Take any tuple $H = (H_1, \ldots, H_p)^t$ of generators of the ideal. Then we have the system of equations (5.1) with some matrix $R(\cdot)$. In Section 6.4 we describe a certain normal form for the matrix $R(\cdot)$, which we call the resonant normal form. We will prove that if $R(\cdot)$ has the resonant normal form then (5.1) implies that the generators $H_1, \ldots, H_p$ are quasi-homogeneous with respect to S with the same weights $\lambda_1, \ldots, \lambda_k$. The proof requires several statements on the spectrum of operator $H \to V(H) - R(\cdot)H$, which are collected and proved in Section 6.3. To complete the proof of the implication (ii) $\Longrightarrow$ (i) we have to reduce the matrix $R(\cdot)$ in (5.1) to the resonant normal form by changing the tuple of generators H to another tuple of generators $\hat{H}$. The two tuples are related via a non-degenerate matrix $T(\cdot)$: $H = T(\cdot)\hat{H}$. The change of generators takes the matrix $R(\cdot)$ in (5.1) to a certain matrix $T_{\#}R$. The map $(T, R) \to T_{\#}R$ is an action of the group of non-degenerate matrix functions in the space of all matrix functions. In Section 6.5 we prove that any orbit of this action contains a matrix having the resonant normal form. This completes the proof of the implication (ii) $\Longrightarrow$ (i). Simultaneously, in Section 6.5 we prove the statement of Theorem 5.8 on the invariants $d_1(V, I), \ldots, d_p(V, I)$.

The proof of several statements in Sections 6.1–6.5 consists of two steps. First we give a proof on the level of formal series with respect to $x_1, \ldots, x_k$ whose coefficients are smooth (analytic) functions of $y_1, \ldots, y_{n-k}$, assuming that the coordinates are such that S is given by the equations $x_1 = \cdots =$

$x_k = 0$. (If $S = \{0\}$ then these are the usual formal series.) Then we use certain results that allow us to pass to the analytic and the C^∞ category. These results are collected in Section 6.6.

6.1. Quasi-homogeneous functions and vector fields. In this section we give three equivalent definitions of quasi-homogeneous functions and vector fields with respect to a smooth submanifold $S \subset \mathbb{R}^n$ of codimension k. Each of them will be used throughout the proof of Theorem 5.8. The classical quasi-homogeneity corresponds to the case $S = \{0\}$, $k = n$.

Let S be a smooth submanifold of $\mathbb{R}^n$ of codimension k. Fix a local coordinate system in which S is given by the equations $x_1 = \cdots = x_k = 0$:

$$\mathbb{R}^n = \mathbb{R}^n(x, y),\ x = (x_1, \dots, x_k),\ y = (y_1, \dots, y_{n-k}),\ S = \{x = 0\}.$$

Fix *positive numbers* $\lambda_1, \dots, \lambda_k$ and the Euler vector field

$$E_\lambda = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_k x_k \frac{\partial}{\partial x_k}.$$

In Section 5 a definition of quasi-homogeneity of a function germ with respect to S was given; the following proposition gives two more equivalent definitions.

PROPOSITION 6.1. *Let $d \in \mathbb{R}$. The following conditions on a function germ $f(x, y)$ are equivalent:*

(i) $f(t^{\lambda_1} x_1, \cdots, t^{\lambda_k} x_k, y_1, \dots y_{n-k}) = t^d \cdot f(x, y), \quad t \geq 0.$

(ii) $E_\lambda(f) = d \cdot f$.

(iii) $f(x) = \sum_{\alpha:(\lambda,\alpha)=d} a_\alpha(y) x^\alpha$, *where $a_\alpha(y)$ are functions on S.*

Here $\alpha = (\alpha_1, \dots, \alpha_k), \quad \alpha_i \in \{0\} \cup \mathbb{N}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \quad (\lambda, \alpha) = \lambda_1 \alpha_1 + \cdots + \lambda_k \alpha_k$.

Proof. To obtain the implication (i) $\Longrightarrow$ (ii) we differentiate (i) with respect to t at $t = 1$. The implication (iii) $\Longrightarrow$ (i) is obvious. It remains to prove (ii) $\Longrightarrow$ (iii). This implication is clear on the level of formal series with respect to x (the coefficients are smooth or analytic functions of y) and consequently it holds in the analytic category. The assumption $\lambda_i > 0$ is required in Proposition 6.1 only to prove the implication (ii) $\Longrightarrow$ (iii) in the C^∞-category; this assumption follows from the same implication on the level of formal series with respect to x and Proposition 6.16 (see Section 6.6). $\square$

The numbers $\lambda_1, \dots, \lambda_k$ are called weights, and the number d the degree (of quasi-homogeneity with respect to S).

NOTATION. The space of all function germs $f(x, y)$ which are quasi-homogeneous with respect to S with positive weights $\lambda_1, \dots, \lambda_k$ will be denoted by $QH_{\lambda,S}(\mathbb{R}^n)$. The subspace consisting of quasi-homogeneous function germs of a fixed degree d will be denoted by $QH_{\lambda,S}^{(d)}(\mathbb{R}^n)$.

It is worth making the following observations, which follow immediately from Proposition 6.1. A function $f \in QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$ is a polynomial with respect to x, for any d. The space $QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$ is a finitely generated module over the ring of function germs of the form $g(y)$ (functions on $\mathbb{R}^n$ that do not depend on x). If $S = \{0\}$ then $QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$ is a finite dimensional vector space. If $d \notin \{(\lambda, \alpha),\ \ \alpha_i \in \{0\} \cup \mathbb{N}\}$ then $QH^{(d)}_{\lambda,S}(\mathbb{R}^n) = \{0\}$. Consequently, the degree of quasi-homogeneity with respect to S of a function germ $f(x,y) \not\equiv 0$ is always a non-negative number. The space $QH^{(0)}_{\lambda,S}(\mathbb{R}^n)$ consists of function germs of the form $g(y)$. If $S = \{0\}$ then it consists only of constant functions. Note also that by Proposition 6.1, if $d \neq 0$ and $f(x,y) \in H^{(d)}_{\lambda,S}(\mathbb{R}^n)$ then $f(0,y) \equiv 0$, i.e., $f(x,y)$ vanishes at any point of S. Finally, the space $QH_{\lambda,S}(\mathbb{R}^n)$ is a ring: If $f_1 \in QH^{(d_1)}_{\lambda,S}(\mathbb{R}^n)$, $f_2 \in QH^{(d_2)}_{\lambda,S}(\mathbb{R}^n)$, then $f_1 f_2 \in QH^{(d_1+d_2)}_{\lambda,S}(\mathbb{R}^n)$.

Now we give three equivalent definitions of quasi-homogeneous vector fields.

PROPOSITION 6.2. *Let $d \in \mathbb{R}$. The following conditions on the germ V of a vector field on $\mathbb{R}^n$ are equivalent:*

(i) *If $f \in QH^{(r)}_{\lambda,S}(\mathbb{R}^n)$ then $V(f) \in QH^{(d+r)}_{\lambda,S}(\mathbb{R}^n)$ (for any $r \in \mathbb{R}$).*
(ii) $[E_\lambda, V] = d \cdot V$.
(iii) $V = \sum_{i=1}^k X_i(x,y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n-k} Y_j(x,y)\frac{\partial}{\partial y_j}$, $X_i \in QH^{(d+\lambda_i)}_{\lambda,S}(\mathbb{R}^n)$, $Y_j \in QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$.

Proof. Let $V = \sum_{i=1}^k X_i(x,y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n-k} Y_j(x,y)\frac{\partial}{\partial y_j}$. Condition (ii) means that $E_\lambda(X_i) = (d + \lambda_i)X_i$ and $E_\lambda(Y_j) = d \cdot Y_j$. Therefore by Proposition 6.1, (ii) and (iii) are equivalent. The implication (iii) $\Longrightarrow$ (i) also follows from Proposition 6.1. Taking in (i) $f(x,y) = x_i \in QH^{(\lambda_i)}_{\lambda,S}(\mathbb{R}^n)$ and $f(x,y) = y_j \in QH^{(0)}_{\lambda,S}(\mathbb{R}^n)$ we get (i) $\Longrightarrow$ (iii). □

DEFINITION AND NOTATION. The germ of a vector field V satisfying condition (i) (and consequently conditions (ii) and (iii)) of Proposition 6.2 with positive $\lambda_1, \ldots, \lambda_k$ is called quasi-homogeneous with respect to S. The numbers $\lambda_1, \ldots, \lambda_k$ are called weights, and the number d the degree (of quasi-homogeneity). The space of all quasi-homogeneous with respect to S germs of vector fields with weights $\lambda_1, \ldots, \lambda_k$ is denoted by $QH_{\lambda,S}(\mathbb{R}^n)$. The subspace consisting of quasi-homogeneous germs of vector fields of a fixed degree d will be denoted by $QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$.

Note that we use the same notations as for quasi-homogeneous functions. One can easily check that $QH_{\lambda,S}(\mathbb{R}^n)$ (the space of quasi-homogeneous germs of vector fields with respect to S) is a Lie algebra: If $V_1 \in QH^{(d_1)}_{\lambda,S}(\mathbb{R}^n), V_2 \in QH^{(d_2)}_{\lambda,S}(\mathbb{R}^n)$ then $[V_1, V_2] \in QH^{(d_1+d_2)}_{\lambda,S}(\mathbb{R}^n)$.

6.2. The choice of a coordinate system. In this section we fix a coordinate system in which suitable generators of the ideal are quasi-homogeneous. Assume first that $S = \{0\}$. Then V is a vector field on $\mathbb{R}^k$ with positive eigenvalues $\lambda_1, \dots, \lambda_k$ at the singular point 0. Consider the classical resonant normal form

$$E_\lambda + \sum_{i=1}^{k-1} \delta_i x_{i+1} \frac{\partial}{\partial x_i} + \sum_{i=1}^{k} \sum_{|\alpha| \geq 2} a_{i,\alpha} x^\alpha \frac{\partial}{\partial x_i},$$
$$(\lambda_i \neq \lambda_{i+1}) \implies \delta_i = 0, \quad (\lambda, \alpha) \neq \lambda_i \implies a_{i,\alpha} = 0.$$

The resonant normal form is polynomial because $\lambda_1, \dots, \lambda_k > 0$. It is well known (see [1]) that in this case the vector field V can be reduced to the resonant normal form by a change of coordinates in either the analytic or the C^∞ category. The advantage of a coordinate system in which V has the resonant normal form is that *in such a coordinate system the vector field V is quasi-homogeneous of degree* 0 *with weights $\lambda_1, \dots, \lambda_k$.*

Consider now the general case of Theorem 5.8 when the vector field V vanishes at any point of the smooth submanifold $S = \{x = 0\}$ of codimension k and has at any point of S the same positive eigenvalues $\lambda_1, \dots, \lambda_k$ corresponding to directions transversal to S. In this case, as was shown in [21], the vector field V can be reduced, also in either the analytic or the C^∞ category, to the normal form

$$E_\lambda + \sum_{i,j=1}^{k} \delta_{ij}(y) x_i \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} \sum_{|\alpha| \geq 2} a_{i,\alpha}(y) x^\alpha \frac{\partial}{\partial x_i}, \tag{6.1}$$

where the functions $\delta_{ij}(y)$ and $a_{i,\alpha}(y)$ satisfy the following conditions:

$$(\lambda_i \neq \lambda_j) \implies \delta_{ij}(y) \equiv 0; \tag{6.2}$$

$$\text{for any } y \text{ the matrix } \{\delta_{ij}(y)\} \text{ is nilpotent;} \tag{6.3}$$

$$(\lambda, \alpha) \neq \lambda_i \implies a_{i,\alpha}(y) \equiv 0. \tag{6.4}$$

As in the case $S = \{0\}$ we will call this normal form resonant. Since $\lambda_1, \dots, \lambda_k > 0$, the resonant normal form is polynomial with respect to $x_1, \dots, x_k$. Conditions (6.2)–(6.4) and Proposition 6.2 imply the following statement.

PROPOSITION 6.3. *Assume that a coordinate system is chosen so that the vector field V has the resonant normal form* (6.1). *Then:*

(i) *V is quasi-homogeneous with respect to S of degree* 0, *with weights $\lambda_1, \dots, \lambda_k$: $V \in QH^{(0)}_{\lambda,S}(\mathbb{R}^n)$.*

(ii) *V is tangent to the foliation $y = const$. This means that V can be treated as a family of vector fields V_y on $\mathbb{R}^k$ parameterized by $y = (y_1, \dots, y_{n-k})$, i.e., by a point of S.*

(iii) *Any of the vector fields* V_y *is quasi-homogeneous of degree* 0 *with weights* $\lambda_1, \dots, \lambda_k$; *moreover* $V_y = E_\lambda + N_y$, *where* N_y *is a nilpotent quasi-homogeneous vector field of degree* 0 *with weights* $\lambda_1, \dots, \lambda_k$, $[E_\lambda, N_y] = 0$.

Here by a nilpotent vector field we mean a vector field whose eigenvalues at the singular point 0 are all equal to zero.

EXAMPLE 6.4. Let $n = 5, k = 3, \lambda_1 = \lambda_2 = 1, \lambda_3 = 2$. Then in suitable coordinates the vector field V has the resonant normal form

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3} + \sum_{i,j=1}^{2} \delta_{ij}(y_1, y_2) x_i \frac{\partial}{\partial x_j} + \\ + (a(y_1, y_2) x_1^2 + b(y_1, y_2) x_1 x_2 + c(y_1, y_2) x_2^2) \frac{\partial}{\partial x_3},$$

where the 2×2 matrix $\delta(y_1, y_2) = \{\delta_{ij}(y_1, y_2)\}$ has zero eigenvalues for any y_1, y_2. This matrix can be replaced by any matrix of the form

$$T^{-1}(y_1, y_2) \delta(y_1, y_2) T(y_1, y_2),$$

where $T(y_1, y_2)$ is a 2×2 matrix, with $\det T(0, 0) \neq 0$, depending smoothly (analytically) on y_1, y_2. Note that in general this transformation does not allow us to reduce the matrix $\delta(y_1, y_2)$ even to triangular form. Take, for example,

$$\delta(y_1, y_2) = \begin{pmatrix} y_1^2 y_2 & y_1 y_2^2 \\ -y_1^3 & -y_1^2 y_2 \end{pmatrix}.$$

Then trace $\delta(y_1, y_2) \equiv \det \delta(y_1, y_2) \equiv 0$, but the relation $\delta(y_1, y_2) v(y_1, y_2) = 0$, where $v(y_1, y_2)$ is a vector depending smoothly (analytically) on y_1, y_2, implies $v(0) = 0$. Therefore the matrix $\delta(y_1, y_2)$ cannot be reduced to triangular form.

6.3. The operator $H \to V(H) - R(y)H$.

CONVENTION. In this and the next sections we work in a fixed coordinate system (x, y) such that $S = \{x = 0\}$ and the vector field V has the resonant normal form, i.e., form (6.1) with the functional coefficients satisfying conditions (6.2), (6.3) and (6.4).

Denote by $(QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^p$ the space of tuples

$$H(x, y) = (H_1(x, y), \dots, H_p(x, y))^t$$

such that $H_i(x,y) \in QH^{(r)}_{\lambda,S}(\mathbb{R}^n)$. Let $R(y)$ be any $p\times p$ matrix function which depends only on y. Consider the linear operator

$$L^p_{V,r,R(y)} : (QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^p \to (QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^p,$$
$$L^p_{V,r,R(y)}(H(x,y)) = V(H(x,y)) - R(y)H(x,y).$$

By Proposition 6.3, $V \in QH^{(0)}_{\lambda,S}(\mathbb{R}^n)$. Therefore, using Proposition 6.2, it is easy to see that this linear operator is well-defined. The following result will be used throughout the next section.

PROPOSITION 6.5. *If r is not an eigenvalue of the matrix $R(0)$ then the linear operator $L^p_{V,r,R(y)}$ is an isomorphism.*

Proof. The vector field V has form (6.1). Therefore it can be treated as a family of vector fields $V_y \in H_{\lambda,\{0\}}(\mathbb{R}^k)$ parameterized by $y = (y_1,\dots,y_{n-k})$, i.e., by a point of S; see Section 6.2. A vector function $H(x,y) \in (QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^p$ can be treated as a family of vector functions $H_y \in (QH^{(r)}_{\lambda,\{0\}}(\mathbb{R}^k))^p$, also parametrized by y. Therefore the operator $L^p_{V,r,R(y)}$ can be treated as a family of linear operators in the finite-dimensional vector space $(QH^{(r)}_{\lambda,\{0\}}(\mathbb{R}^k))^p$. This family depends smoothly (analytically) on the parameter y. Therefore it suffices to prove that 0 is not an eigenvalue of the linear operator corresponding to $y=0$, i.e., the operator

$$(6.5)\quad (QH^{(r)}_{\lambda,\{0\}}(\mathbb{R}^k))^p \to (QH^{(r)}_{\lambda,\{0\}}(\mathbb{R}^k))^p, \quad H(x) \to V_0(H(x)) - R(0)H(x).$$

The vector field V_0 has the form $E_\lambda + N$, where N is a quasi-homogeneous degree 0 nilpotent vector field; see Proposition 6.3. Therefore the operator (6.5) is the sum of the operator $H(x) \to E_\lambda(H(x)) - R(0)H(x)$ and the nilpotent operator $H(x) \to N(H(x))$. By Proposition 6.2 $[E_\lambda, N] = 0$ and $E_\lambda(H(x)) = rH(x)$ for any $H(x) \in (QH^{(r)}_{\lambda,\{0\}}(\mathbb{R}^k))^p$. Therefore these two operators commute and the spectrum of the first operator consists of the numbers $r-\lambda$, where λ is an eigenvalue of the matrix $R(0)$. It follows that the spectrum of the operator (6.5) consists of the same numbers. By the assumption of Proposition 6.5 none of these numbers is equal to 0. $\square$

Proposition 6.5 implies the following corollary.

PROPOSITION 6.6. *Let $R(y)$ be a $p\times p$ matrix such that the matrix $R(0)$ has the only eigenvalue d. If $H(x,y) = (H_1(x,y),\dots,H_p(x,y))^t$ is a tuple of functions such that $V(H(x,y)) - R(y)H(x,y) \in QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$ then $H_1(x,y),\dots, H_p(x,y) \in QH^{(d)}_{\lambda,S}(\mathbb{R}^n)$.*

Proposition 6.6 easily follows from Proposition 6.5 on the level of formal series with respect to x (the coefficients are smooth or analytic functions of y)

and consequently in the analytic category. The transition from formal series to the C^∞ category is possible due to Proposition 6.16 (see Section 6.6).

We need one more result generalizing Proposition 6.5.

Let $(QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^{p_1\times p_2}$ be the space of all $p_1 \times p_2$ matrices whose entries are functions in the space $QH^{(r)}_{\lambda,S}(\mathbb{R}^n)$. Let $A(y)$ be a $p_1 \times p_1$ matrix and let $B(y)$ be a $p_2 \times p_2$ matrix. Consider the operator

$$\begin{aligned} &(QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^{p_1\times p_2} \to (QH^{(r)}_{\lambda,S}(\mathbb{R}^n))^{p_1\times p_2}, \\ &U(x,y) \to -V(U(x,y)) + A(y)U(x,y) + U(x,y)B(y). \end{aligned} \tag{6.6}$$

PROPOSITION 6.7. *The linear operator* (6.6) *is an isomorphism unless there exist an eigenvalue a of the matrix $A(0)$ and an eigenvalue b of the matrix $B(0)$ such that $a+b=r$.*

Proof. Fix the operator

$$(QH^{(r)}_{\lambda,0}(\mathbb{R}^k))^{p_1\times p_2} \ni U(x) \to -V_0(U(x)) + A(0)U(x) + U(x)B(0) \tag{6.7}$$

in the finite-dimensional space $(QH^{(r)}_{\lambda,0}(\mathbb{R}^k))^{p_1\times p_2}$, where $V_0 = E_\lambda + N$, N is quasi-homogeneous degree 0 nilpotent vector field on $\mathbb{R}^k$. Arguing exactly as in the proof of Proposition 6.5 we reduce Proposition 6.7 to the statement that the linear operator (6.7) is non-singular. The operator $U(x) \to -N(U(x))$ is nilpotent. It commutes with the operator $U(x) \to -E_\lambda(U(x)) + A(0)U(x) + U(x)B(0)$ since $[E_\lambda, N] = 0$. Therefore it suffices to prove that the operator $U(x) \to -E_\lambda(U(x)) + A(0)U(x) + U(x)B(0)$ is non-singular. By Proposition 6.2 we have $-E_\lambda(U(x)) = -rU(x)$ for any $U(x) \in (QH^{(r)}_{\lambda,0}(\mathbb{R}^k))^{p_1\times p_2}$. It is easy to show that the eigenvalues of the operator $U(x) \to A(0)U(x) + U(x)B(0)$ have the form $a+b$, where a is an eigenvalue of the matrix $A(0)$ and b is an eigenvalue of the matrix $B(0)$. Therefore the spectrum of the operator $U(x) \to -E_\lambda(U(x)) + A(0)U(x) + U(x)B(0)$ consists of numbers $-r+a+b$, where a is an eigenvalue of the matrix $A(0)$ and b is an eigenvalue of the matrix $B(0)$. By the assumption of Proposition 6.7 none of these numbers is equal to 0. $\square$

6.4. Normal form for the equation $V(H(x,y)) = R(x,y)H(x,y)$. In this equation $R(x,y)$ is a $p\times p$ matrix function and $H(x,y) = (H_1(x,y),\ldots, H_p(x,y))^t$ is a tuple of functions. We study this equation in a fixed coordinate system (x,y) such that $S = \{x = 0\}$ and the vector field V has resonant normal form (6.1). Let $d_1,\ldots,d_p$ be the eigenvalues of the matrix $R(0,0)$. We will prove that the equation $V(H(x,y)) = R(x,y)H(x,y)$ implies that the functions $H_1,\ldots,H_p$ are quasi-homogeneous with respect to S with the same weights $\lambda_1,\ldots,\lambda_k$ and degrees $d_1,\ldots,d_p$ *provided* that the matrix $R(x,y)$ has the following normal form.

DEFINITION 6.8. We say that the matrix $R(x,y) = \{R_{ij}(x,y)\}_{i,j=1,\dots,p}$ has the resonant normal form if the following conditions hold:

(a) The matrix $R(0,0)$ is a below-triangular matrix with real diagonal entries $d_1 \le d_2 \le \cdots \le d_p$.

(b) The function $R_{ij}(x,y)$ belongs to the space $H^{(d_i-d_j)}_{\lambda,S}(\mathbb{R}^n)$.

The motivation for this definition is as follows. Let $H(x,y) = (H_1(x,y),\dots, H_p(x,y))^t$, $H_1(x,y) \in QH^{(d_1)}_{\lambda,S}(\mathbb{R}^n),\dots,H_p(x,y) \in QH^{(d_p)}_{\lambda,S}(\mathbb{R}^n)$. Let $\hat{H}(x,y) = R(x,y)H(x,y)$. If $R(x,y)$ has the resonant normal form then, as it is easy to check,

$$\hat{H}_1(x,y) \in QH^{(d_1)}_{\lambda,S}(\mathbb{R}^n),\dots,\hat{H}_p(x,y) \in QH^{(d_p)}_{\lambda,S}(\mathbb{R}^n).$$

Assume that the matrix $R(x,y)$ has resonant normal form. The requirement $d_1 \le d_2 \le \cdots \le d_p$ implies that if $i \le j$ then the function $R_{ij}(x,y)$ depends on y only. Moreover, if $d_i \ne d_j$ then $R_{ij}(x,y) \equiv 0$. Consequently, if the numbers $d_1,\dots,d_p$ are different then the matrix $R(x,y)$ is lower-triangular. In general, the square blocks in $R(x,y)$ corresponding to equal numbers in the sequence $d_1 \le d_2 \le \cdots \le d_p$ are matrices which depend on y only.

EXAMPLE 6.9. Assume that $R(x,y)$ is a 4×4 matrix and the diagonal entries of the matrix $R(0,0)$ are $d_1 = d_2 = 2, d_3 = d_4 = 3$. If $R(x,y)$ has resonant normal form then

$$R(x,y) = \begin{pmatrix} R_{11}(y) & R_{12}(y) & 0 & 0 \\ R_{21}(y) & R_{22}(y) & 0 & 0 \\ R_{31}(x,y) & R_{32}(x,y) & R_{33}(y) & R_{34}(y) \\ R_{41}(x,y) & R_{42}(x,y) & R_{43}(y) & R_{44}(y) \end{pmatrix},$$

where $R_{31}(x,y), R_{32}(x,y), R_{41}(x,y), R_{42}(x,y) \in QH^{(1)}_{\lambda,S}(\mathbb{R}^n)$.

PROPOSITION 6.10. *Let $H(x,y) = (H_1(x,y),\dots,H_p(x,y))^t$. If the matrix $R = R(x,y)$ in the equation $V(H(x,y)) = R(x,y)H(x,y)$ has the resonant normal form then $H_i(x,y) \in QH^{(d_i)}_{\lambda,S}(\mathbb{R}^n)$, where $d_1,\dots,d_p$ are the diagonal entries of the matrix $R(0,0)$.*

Proof. We will give a proof for the case when $R(x,y)$ is the matrix in Example 6.9; the proof in the general case is the same modulo a change of notation. The equation $V(H(x,y)) = R(x,y)H(x,y)$ is a system of four equations. Take the first two. They have the form

$$V(H_1(x,y)) = R_{11}(y)H_1(x,y) + R_{12}(y)H_2(x,y),$$
$$V(H_2(x,y)) = R_{21}(y)H_1(x,y) + R_{22}(y)H_2(x,y).$$

The eigenvalues of the 2×2 matrix $\{R_{ij}(0)\}, i,j=1,2$, are its diagonal entries, each of which is equal to $d_1 = d_2 = 2$. Now we use Proposition 6.6, which implies that $H_1(x,y), H_2(x,y) \in QH^{(2)}_{\lambda,S}(\mathbb{R}^n)$.

Consider the next two equations of the system $V(H(x,y)) = R(x,y)H(x,y)$ corresponding to $d_3 = d_4 = 3$. They have the form

$$\begin{aligned} V(H_3(x,y)) &= R_{31}(x,y)H_1(x,y) + R_{32}(x,y)H_2(x,y) \\ &\quad + R_{33}(y)H_3(x,y) + R_{34}(y)H_4(x,y), \\ V(H_4(x,y)) &= R_{41}(x,y)H_1(x,y) + R_{42}(x,y)H_2(x,y) \\ &\quad + R_{43}(y)H_3(x,y) + R_{44}(y)H_4(x,y). \end{aligned}$$

Since, as we have proved, $H_1(x,y), H_2(x,y) \in QH^{(2)}_{\lambda,S}(\mathbb{R}^n)$ and the functions $R_{31}(x,y), R_{32}(x,y), R_{41}(x,y), R_{42}(x,y)$ belong to the space $QH^{(1)}_{\lambda,S}(\mathbb{R}^n)$, we have

$$\begin{aligned} &V(H_3(x,y)) - R_{33}(y)H_3(x,y) - R_{34}(y)H_4(x,y) \in QH^{(3)}_{\lambda,S}(\mathbb{R}^n), \\ &V(H_4(x,y)) - R_{43}(y)H_3(x,y) - R_{44}(y)H_4(x,y) \in QH^{(3)}_{\lambda,S}(\mathbb{R}^n). \end{aligned}$$

The eigenvalues of the 2×2 matrix $\{R_{ij}(0)\}, i,j=3,4$, are its diagonal entries, each of which is equal to $d_3 = d_4 = 3$. Using again Proposition 6.6 we obtain that $H_3(x,y), H_4(x,y) \in QH^{(3)}_{\lambda,S}(\mathbb{R}^n)$. □

6.5. Reduction of the matrix $R(x,y)$ to the resonant normal form. As in previous sections we work in a coordinate system such that the vector field V has resonant normal form. To complete the proof of the implication (ii) $\Longrightarrow$ (i) in Theorem 5.8 we have to show that the matrix $R(x,y)$ in the equation $V(H(x,y)) = R(x,y)H(x,y)$ can be reduced to the resonant normal form defined in Section 6.4 by changing the tuple $H = (H_1, \dots, H_p)^t$ of generators of the ideal I to another tuple of generators $\hat{H}$. Then the implication (ii) $\Longrightarrow$ (i) follows from Proposition 6.10.

Let $H(x,y) = T(x,y)\hat{H}(x,y)$, where $T(x,y)$ is a non-degenerate $p\times p$ matrix. Then $V(\hat{H}(x,y)) = \hat{R}(x,y)\hat{H}(x,y)$, where

$$\hat{R}(x,y) = T(x,y)_{\#}R(x,y) = T^{-1}(x,y)\left(R(x,y)T(x,y) - V(T(x,y))\right).$$

The map $(T(x,y), R(x,y)) \to T(x,y)_{\#}R(x,y)$ is an action of the group of non-degenerate matrices $T(x,y)$ in the space of all matrices $R(x,y)$.

Proposition 6.11. *Let I be a p-generated ideal in the ring of functions. If $R(x,y)$ is a $p\times p$ matrix such that $V(H(x,y)) = R(x,y)H(x,y)$, where $H(x,y) = (H_1(x,y), \dots, H_p(x,y))^t$ is a tuple of generators of the ideal I, then the eigenvalues of the matrix $R(0,0)$ are non-negative real numbers.*

Proposition 6.12. *Let $R(x,y)$ be a $p\times p$ matrix such that the eigenvalues of the matrix $R(0,0)$ are real numbers. There exists a non-degenerate matrix*

$T(x,y)$ such that the matrix $T(x,y)_{\#}R(x,y)$ has the resonant normal form defined in Section 6.4.

These two propositions complete the proof of Theorem 5.8 including the statement about the relation between the invariants $d_1(V,I),\dots,d_p(V,I)$ and the tuple of degrees of quasi-homogeneity of the ideal I.

Proof of Proposition 6.12. Note first that for any $d_1\ \dots,d_p$ there exists a non-degenerate matrix $T(y)$ such that the matrix $T^{-1}(y)R(y)T(y)$ has the resonant normal form.

EXAMPLE 6.13. Any 5×5 matrix $R(y)$ such that the eigenvalues of the matrix $R(0)$ are equal to $2,2,5,5,5$ can, by transformations $R(y) \to T^{-1}(y)R(y)T(y)$, be taken to the form

$$\begin{pmatrix} E(y) & 0 \\ 0 & F(y) \end{pmatrix}, \tag{6.8}$$

where

$$E(y) = \begin{pmatrix} E_{11}(y) & E_{12}(y) \\ E_{21}(y) & E_{22}(y) \end{pmatrix}, \quad F(y) = \begin{pmatrix} F_{11}(y) & F_{12}(y) & F_{13}(y) \\ F_{21}(y) & F_{22}(y) & F_{23}(y) \\ F_{31}(y) & F_{32}(y) & F_{33}(y) \end{pmatrix},$$
$$E_{11}(0) = E_{22}(0) = 2,\ F_{11}(0) = F_{22}(0) = F_{33}(0) = 5,$$
$$E_{12}(0) = F_{12}(0) = F_{13}(0) = F_{23}(0) = 0.$$

Therefore, to prove Proposition 6.12 we may assume that the matrix $R(0,y)$ has resonant normal form. In what follows $d_1 \le d_2 \le \cdots \le d_p$ are the diagonal entries (and the eigenvalues) of the matrix $R(0,0)$.

Express the Taylor series with respect to x of the matrix $R(x,y)$ in the form $R(0,y) + R^{(r_1)}(x,y) + R^{(r_2)}(x,y) + \cdots$, where $R^{(r_i)}(x,y) \in QH^{(r_i)}_{\lambda,S}(\mathbb{R}^n)$ is the quasi-homogeneous part of $R(x,y)$ of degree r_i, and $0 < r_1 < r_2 < \cdots$. Assume that the matrices $R^{(r_i)}(x,y)$ have the resonant normal form if $i \le m-1$. If we show that $R^{(r_m)}(x,y)$ can be taken to the resonant normal form then Proposition 6.12 will be proved on the level of formal series with respect to x. One can pass from formal series to the analytic and the C^∞ category using Proposition 6.15 (see Section 6.6).

Let $r = r_m$. To prove that $R^{(r)}(x,y)$ can be taken to the resonant normal form by a transformation $R(x,y) \to T(x,y)_{\#}R(x,y)$ we seek matrices of the form $T(x,y) = I + T^{(r)}(x,y)$, where I is the identity matrix and $T^{(r)}(x,y)$ is a matrix whose entries belong to the space $QH^{(r)}_{\lambda,S}(\mathbb{R}^n)$. Then the matrix $T^{-1}(x,y)$ has the form $I - T^{(r)}(x,y)$ modulo quasi-homogeneous terms of degree $> r$. It follows that the matrices $R(x,y)$ and $T_{\#}R(x,y)$ have the same quasi-homogeneous terms of degrees $0, r_1, \dots, r_{m-1}$, and the quasi-homogeneous part of degree $r_m = r$ of the matrix $T_\# R(x,y)$ is equal to

$$\tilde{R}^{(r)}(x,y) = R^{(r)}(x,y) - V(T^{(r)}(x,y)) + R(0,y)T^{(r)}(x,y) - T^{(r)}(x,y)R(0,y).$$

We have to find $T^{(r)}(x,y)$ such that the matrix $\tilde{R}^{(r)}(x,y) = \{\tilde{R}^{(r)}_{ij}(x,y)\}$ has the resonant normal form. This means that

$$\tilde{R}^{(r)}_{ij}(x,y) \in QH^{(d_i - d_j)}_{\lambda,S}(\mathbb{R}^n), \quad i,j = 1,\dots,p. \tag{6.9}$$

We will prove that the required matrix $T^{(r)}(x,y)$ exists in the case when $R(0,y)$ is a matrix of the form (6.8). The proof in the general case is the same modulo a change of notation.

Let

$$R^{(r)}(x,y) = \begin{pmatrix} W_1(x,y) & W_2(x,y) \\ W_3(x,y) & W_4(x,y) \end{pmatrix}, \; T^{(r)}(x,y) = \begin{pmatrix} U_1(x,y) & U_2(x,y) \\ U_3(x,y) & U_4(x,y) \end{pmatrix},$$

where

$$W_1(x,y), U_1(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{2\times 2}, \; W_2(x,y), U_2(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{2\times 3},$$
$$W_3(x,y), U_3(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{3\times 2}, \; W_4(x,y), U_4(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{3\times 3}.$$

The requirement (6.9) can be expressed as follows:

(6.10)
$$W_1(x,y) - V(U_1(x,y)) + E(y)U_1(x,y) - U_1(x,y)E(y) \in QH^{(0)}_{S,\lambda}(\mathbb{R}^n),$$
(6.11)
$$W_2(x,y) - V(U_2(x,y)) + E(y)U_2(x,y) - U_2(x,y)F(y) \in QH^{(-3)}_{S,\lambda}(\mathbb{R}^n) = \{0\},$$
(6.12)
$$W_3(x,y) - V(U_3(x,y)) + F(y)U_3(x,y) - U_3(x,y)E(y) \in QH^{(3)}_{S,\lambda}(\mathbb{R}^n),$$
(6.13)
$$W_4(x,y) - V(U_4(x,y)) + F(y)U_4(x,y) - U_4(x,y)F(y) \in QH^{(0)}_{S,\lambda}(\mathbb{R}^n).$$

The existence of $U_1(x,y), \dots, U_4(x,y)$ satisfying (6.10)–(6.13) is a corollary of Proposition 6.7. Consider, for example, condition (6.12). If $r = 3$ then (6.12) holds for any $U_3(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{3\times 2}$, and if $r \neq 3$ then (6.12) is equivalent to the equation $W_4(x,y) - V(U_3(x,y)) + F(y)U_3(x,y) - U_3(x,y)E(y) = 0$. This equation has a solution $U_3(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{3\times 2}$ by Proposition 6.7 with $p_1 = 3, p_2 = 2, A(y) = F(y), B(y) = -E(y)$ because the sum of any eigenvalue of the matrix $F(0)$ and any eigenvalue of the matrix $-E(0)$ is equal to $5 - 2 = 3 \neq r$. □

Proof of Proposition 6.11. It suffices to prove that all eigenvalues of the matrix $R(0,0)$ are real. Then they are non-negative by Proposition 6.12 and 6.10. Assume, to get contradiction, that at least one of the eigenvalues of the matrix $R(0,0)$ is not real. Then there is no loss of generality to assume that the matrix $R(0,y)$ has the form

$$\begin{pmatrix} E(y) & 0 \\ 0 & F(y) \end{pmatrix},$$

where $E(0)$ is an $s \times s$ matrix with no real eigenvalues, $s > 0$, and $F(0)$ is a $(p-s) \times (p-s)$ matrix with real eigenvalues.

Lemma 6.14. *There exists a non-degenerate $p \times p$ matrix $T(x,y)$ such that*

$$T(x,y)_{\#}R(x,y) = \begin{pmatrix} E(x,y) & 0 \\ 0 & F(x,y) \end{pmatrix}, \quad E(0,y) = E(y), F(0,y) = F(y).$$

Lemma 6.14 implies that there exists a tuple $\hat{H}(x,y) = (\hat{H}_1(x,y), \dots, \hat{H}_p(x,y))^t$ of generators of the ideal such that

$$V(\hat{H}_1(x,y), \dots \hat{H}_s(x,y))^t = E(x,y)(\hat{H}_1(x,y), \dots \hat{H}_s(x,y))^t.$$

Since the ideal is p-generated, $(H_1(x,y), \dots, H_s(x,y)) \not\equiv 0$. By Proposition 6.16 (see Section 6.6) the Taylor series of the tuple $(\hat{H}_1(x,y), \dots, \hat{H}_s(x,y))$ with respect to x is not zero. (In the analytic category this is obvious, but we need Proposition 6.16 in the C^∞ category.) Therefore

$$\begin{aligned}(\hat{H}_1(x,y), &\dots, \hat{H}_s(x,y)) \\ &= (\hat{H}_1(x,y), \dots, \hat{H}_s(x,y))^{(r_1)} + (\hat{H}_1(x,y), \dots, \hat{H}_s(x,y))^{(r_2)} + \cdots,\end{aligned}$$

where the superscript denotes the degree of quasi-homogeneity, $r_1 < r_2 < \cdots$, and $(\hat{H}_1(x,y), \dots, \hat{H}_s(x,y))^{(r_1)} \not\equiv 0$. The equation

$$V(\hat{H}_1, \dots \hat{H}_s)^t = E(x,y)(\hat{H}_1, \dots \hat{H}_s)^t$$

implies

$$V(\hat{H}_1^{(r_1)}(x,y), \dots \hat{H}_s^{(r_1)}(x,y))^t = E(0,y)(\hat{H}_1^{(r_1)}(x,y), \dots \hat{H}_s^{(r_1)}(x,y))^t.$$

Since the matrix $E(0,0)$ has no real eigenvalues, by Proposition 6.7 we have $(\hat{H}_1(x,y), \dots, \hat{H}_s(x,y))^{(r_1)} \equiv 0$ and we get a contradiction.

It remains to prove Lemma 6.14. The proof is similar to the proof of Proposition 6.12. We seek a matrix of the form

$$T(x,y) = \begin{pmatrix} I & U_1(x,y) \\ U_2(x,y) & I \end{pmatrix},$$

where $U_1(x,y)$ is an $s \times (p-s)$ matrix and $U_2(x,y)$ is a $(p-s) \times s$ matrix. Arguing exactly as in the proof of Proposition 6.12, we reduce Lemma 6.14 to the solvability of the equations

$$-V(U_1(x,y)) + E(y)U_1(x,y) - U_1(x,y)F(y) = W_1(x,y), \tag{6.14}$$

$$-V(U_2(x,y)) + F(y)U_2(x,y) - U_2(x,y)E(y) = W_2(x,y) \tag{6.15}$$

with respect to the matrices

$$U_1(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{s\times(p-s)},\ U_2(x,y) \in (QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{(p-s)\times s}.$$

Here $W_1(x,y)$ and $W_2(x,y)$ are arbitrary matrices in $(QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{s\times(p-s)}$ and $(QH^{(r)}_{S,\lambda}(\mathbb{R}^n))^{(p-s)\times s}$, respectively, and r is an arbitrary non-negative real number. Equations (6.14) and (6.15) are solvable by Proposition 6.7 because all eigenvalues of the matrix $F(0)$ are real and none of the eigenvalues of the matrix $E(0)$ is real. The proof of Theorem 5.8 is now complete. □

6.6. From formal series to the C^∞ category and the analytic category. In this section we prove results enabling the transition from formal series to the C^∞ category or the analytic category which were used in Sections 6.1–6.5.

Let $x = (x_1, \dots, x_k), y = (y_1, \dots, y_{n-k})$ be a local coordinate system on $\mathbb{R}^n$. Let V be a vector field on $\mathbb{R}^n$ satisfying the following conditions:

(a) V vanishes at any point of the manifold $x = 0$.
(b) The eigenvalues of V at 0 corresponding to directions transversal to the manifold $x = 0$ are positive.

It follows that the eigenvalues of V at a singular point $(0, y)$ close to $(0, 0)$ are also positive; in the statements below we do not require that they do not depend on y.

Let now $z = (z_1, \dots, z_p)$. Let $G(x, y, z)$ be a vector function on $\mathbb{R}^{n+p}$ with p components,

$$G(x,y,z) = (G_1(x,y,z), \dots, G_p(x,y,z))^t.$$

Consider the equation

$$V(H(x,y)) = G(x,y,H(x,y)) \tag{6.16}$$

with respect to the vector function $H(x,y) = (H_1(x,y), \dots, H_p(x,y))^t$.

PROPOSITION 6.15. *Equation* (6.16) *has a solution $H(x,y)$ provided that this equation is solvable on the level of formal series with respect to x.*

We emphasize that this statement holds in either the C^∞ or the analytic category. By formal series with respect to x we mean power series in x whose coefficients are smooth (analytic) functions of y.

In the analytic category Proposition 6.15 holds due to the absence of "small denominators" (see [1]), which is a corollary of assumption (b) on the vector field V. In the C^∞ category an analogue of Proposition 6.15 for functional equations (with $H(F)$ instead of $V(H)$, where F is a local diffeomorphism of $\mathbb{R}^n$) was proved in [2]. Proposition 6.15 can be proved by the same method, using techniques developed in [2] and [3]. In fact, in the C^∞ category the techniques developed in [2] allow us to prove Proposition 6.15 under the assumption that the eigenvalues of the vector field V corresponding to directions transversal to the manifold $x = 0$ have non-zero real part, i.e., V is hyperbolic with respect to the manifold $x = 0$. This assumption is much weaker than assumption (b), but the proof will be much more involved.

In the C^∞ category one more result was used in Sections 6.1–6.4.

PROPOSITION 6.16. *Assume that $G(x, y, 0) \equiv 0$ and assume that the function $H(x, y)$ has zero Taylor series with respect to x. Then* (6.16) *implies $H(x, y) \equiv 0$.*

This result is also similar to results in [2] and can be proved using the techniques in [2]. Note that Proposition 6.16 is not true if assumption (b) is replaced by the hyperbolicity of V with respect to S. For example, the equation $x_1 \frac{\partial H}{\partial x_1} - x_2 \frac{\partial H}{\partial x_2} = 0$ is an equation of form (6.16) ($n = k = 2, p = 1, G \equiv 0$). This equation has a flat solution (i.e., a solution with zero Taylor series) $\tau(x_1 x_2)$, where τ is any flat function of one variable.

Appendix. Another version of Poincare lemma property

Another version of Poincare lemma property for local analytic subsets $N \subset \mathbb{C}^n$ was studied in [9] and [10]. Let ω be a closed holomorphic $(p+1)$-form with vanishing pullback to the regular part N^{reg} of N. This means that $\omega|_{T_p N} = 0$ for any point p near which N has the structure of a smooth submanifold of $\mathbb{C}^n$. Is it true that ω is a differential of a p-form α on $\mathbb{C}^n$ with the same property? If this is the case, then we say that N has the Poincare lemma property for closed forms with vanishing pullback to N^{reg}. The Poincare lemma property considered in Sections 2 and 3 concerns closed forms vanishing at any point of N.

If a form ω vanishes at any point of N, i.e., for any $p \in N$ the coefficients of ω in some (and then any) local coordinate system vanish at p, then of course ω has zero pullback to the regular part of N. The inverse is not true. For example, the non-vanishing 1-form dx has zero pullback to the line $N : x = 0$. Therefore the assumption that ω vanishes at any point of N is stronger than the assumption that ω has zero pullback to N^{reg}. Nevertheless, in any version of the Poincare lemma property the p-form α must have the same property as ω. Therefore it is not clear a priori if one of the Poincare lemma properties implies the other. This remains an open question. We do not know an example of an analytic set N which has one of the Poincare lemma properties and does not have the other.

On the other hand, the known sufficient conditions for the two Poincare lemma properties of an analytic set N are the same. In [9] an analogue of Theorem 2.3 for closed forms with vanishing pullback to N^{reg}, was proved. In [10] it was noted that for analytic sets the quasi-homogeneity should be considered as a type of contractability, and it was shown that the quasi-homogeneity implies the Poincare lemma property for forms with vanishing pullback to N^{reg}. In fact, if N is an analytic set or, moreover, any variety with Whitney stratification, then all other results in Section 2 also hold if we replace the Poincare property for closed forms vanishing at any point of N by the Poincare lemma property for closed forms with vanishing pullback to N^{reg}. For such N the following holds: If $F : \mathbb{R}^k \to \mathbb{R}^n$ is a smooth map whose image is contained in N and ω is a form on $\mathbb{R}^n$ with zero pullback to N^{reg} then $F^*\omega = 0$. It is easy to see that this property of N allows us to repeat all proofs in Sections 2.

As in Section 3, one can construct de Rham cohomology groups for N such that their triviality is equivalent to the Poincare lemma property for closed forms with vanishing pullback to N^{reg}; see [9], [10], [4]. A priori these groups are different from the cohomology groups defined in Section 3. The cohomology groups $H^p_N(\mathbb{R}^n)$ in Section 3 are based on the factorization of the space $\Omega^p(\mathbb{R}^n)$ by the space $K^p_N(\mathbb{R}^n)$ consisting of p-forms ω of the form $\alpha + d\beta$, where α and β vanish at any point of N. To construct the cohomology groups $\tilde{H}^p_N(\mathbb{R}^n)$ corresponding to the Poincare lemma property for closed forms with vanishing pullback to N^{reg} one has to replace $K^p_N(\mathbb{R}^n)$ by the space of p-forms with vanishing pullback to N^{reg}. The reduction theorem (Theorem 3.1) remains true for these cohomology groups.

So, a singular set N defines cohomology groups $H^p_N(\mathbb{R}^n)$ and $\tilde{H}^p_N(\mathbb{R}^n)$, $p \geq 0$. We do not know if for analytic N these cohomology groups are always isomorphic. For example, if N is a stratified 1-dimensional submanifold of $\mathbb{R}^3$ then $\tilde{H}^2_N(\mathbb{R}^3) = \{0\}$ because the pullback to N^{reg} of any 3-form and any 2-form is equal to 0. On the other hand, we do not know if in this case $H^2_N(\mathbb{R}^3) = \{0\}$, i.e., whether for any analytic 1-dimensional stratified submanifold of $N \subset \mathbb{R}^3$ (say the union of curves) any 3-form vanishing at any point of N is a differential of a 2-form vanishing at any point of N.

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W. Domitrz, Warsaw University of Technology, Faculty of Mathematics and Information Science, Plac Politechniki 1, 00-661 Warsaw, Poland
E-mail address: domitrz@mini.pw.edu.pl

S. Janeczko, Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland
E-mail address: janeczko@ise.pw.edu.pl

M. Zhitomirskii, Department of Mathematics, Technion, 32000 Haifa, Israel
E-mail address: mzhi@techunix.technion.ac.il