Tw. Dla dowolnej liczby naturalnej n i dowolnej liczby rzeczywistej φ zachodzi

$$\cos(n\varphi) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} (-1)^l (\sin\varphi)^{2l} (\cos\varphi)^{n-2l}$$

$$\sin(n\varphi) = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (-1)^l (\sin\varphi)^{2l+1} (\cos\varphi)^{n-2l-1}$$

Dowód.: $(\cos\varphi + i\sin\varphi)^n = (\cos(n\varphi) + i\sin(n\varphi))$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \binom{n}{0} b^n + \binom{n}{1} a b^{n-1} + \ldots + \binom{n}{n-1} a^{n-1} b + \binom{n}{n} a^n$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad n! = 1 \cdot 2 \cdot \ldots \cdot (n-1)n$$

$$(\cos\varphi + i\sin\varphi)^n = (\overbrace{i\sin\varphi}^{a} + \overbrace{\cos\varphi}^{b})^n = \sum_{k=0}^{n} \binom{n}{k} (i\sin\varphi)^k (\cos\varphi)^{n-k}$$

$k = 2l$ (parzyste) $\quad 0 \leq 2l \leq n \quad 0 \leq l \leq \frac{n}{2} \quad 0 \leq l \leq \lfloor \frac{n}{2} \rfloor$, gdzie dla $a \in \mathbb{R}$, $a \geq 0$

$k = 2l+1$ (nieparzyste) $\quad 0 \leq 2l+1 \leq n \quad 0 \leq l \leq \frac{n-1}{2} \quad 0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$

$\lfloor a \rfloor$ to największa liczba całkowita niewiększa niż a.

Stąd mamy

$$\sum_{k=0}^{n} \binom{n}{k} (i \sin\varphi)^k (\cos\varphi)^{n-k} = \underset{(k \text{ parzyste})}{\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} i^{2l} (\sin\varphi)^{2l} (\cos\varphi)^{n-2}} + \underset{(k \text{ nieparzyste})}{\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} i^{2l+1} (\sin\varphi)^{2l+1} (\cos\varphi)^{n-2l-1}} =$$

$$(i)^{2l} = ((i)^2)^l = (-1)^l \qquad (i)^{2l+1} = (i)^{2l} \cdot i = (-1)^l \cdot i$$

$$= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} (-1)^l (\cos\varphi)^{2l} (\cos\varphi)^{n-2l} + i \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (-1)^l (\sin\varphi)^{2l+1} (\cos\varphi)^{n-2l-1} =$$

$$= (\cos\varphi + i \sin\varphi)^n = \cos(n\varphi) + i \sin(n\varphi)$$

Stąd

$$\cos(n\varphi) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} (-1)^l (\sin\varphi)^{2l} (\cos\varphi)^{n-2l}$$

$$\sin(n\varphi) = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (-1)^l (\sin\varphi)^{2l+1} (\cos\varphi)^{n-2l-1} \qquad \square$$