ON LOCAL INVARIANTS OF SINGULAR SYMPLECTIC FORMS.

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Abstract. We find a complete set of local invariants of singular symplectic forms with the structurally smooth Martinet hypersurface on a $2n$-dimensional manifold. In the $\mathbb{C}$-analytic category this set consists of the Martinet hypersurface Σ_2, the restriction of the singular symplectic form ω to $T\Sigma_2$ and the kernel of ω^{n-1} at the point $p \in \Sigma_2$. In the $\mathbb{R}$-analytic and smooth categories this set contains one more invariant: the canonical orientation of Σ_2. We find the conditions to determine the kernel of ω^{n-1} at p by the other invariants. In dimension 4 we find sufficient conditions to determine the equivalence class of a singular symplectic form-germ with the structurally smooth Martinet hypersurface by the Martinet hypersurface and the restriction of the singular symplectic form to it. We also study the singular symplectic forms with singular Martinet hypersurfaces. We prove that the equivalence class of such singular symplectic form-germ is determined by the Martinet hypersurface, the canonical orientation of its regular part and the restriction of the singular symplectic form to its regular part if the Martinet hypersurface is a quasi-homogeneous hypersurface with an isolated singularity.

1. Introduction.

A closed differential 2-form ω on a $2n$-dimensional smooth manifold M is **symplectic** if ω is nondegenerate. This means that ω satisfies the following condition

$$\omega^n|_p = \omega \wedge \cdots \wedge \omega|_p \neq 0, \text{ for } p \in M. \tag{1.1}$$

A closed differential 2-form ω on a $2n$-dimensional smooth manifold M is called a **singular symplectic** form if the set of points where ω does not satisfy (1.1):

$$\{p \in M : \omega^n|_p = 0\} \tag{1.2}$$

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is nowhere dense. We denote the set (1.2) by $\Sigma_2(\omega)$ or Σ_2. It is called the **Martinet hypersurface**.

Singular symplectic forms appear naturally if one studies classification of germs of submanifolds of a symplectic manifold. By Darboux-Givental theorem ([1], see also [6]) germs of submanifolds of the symplectic manifold are symplectomorphic if and only if the restrictions of the symplectic form to them are diffeomorphic. This theorem reduces the problem of local classification of generic submanifolds of the symplectic manifold to the problem of local classification of singular symplectic forms.

Singular symplectic forms can be applied in thermodynamics: in the modeling the absolute zero temperature region (see [12]). The first occurring singularity of singular symplectic forms is the Martinet singularity of type Σ_{20}. It has the following local normal form in coordinates $(x_1, y_1, \cdots, x_n, y_n)$ on $\mathbb{R}^{2n}$ ([13])

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i. \tag{1.3}$$

The Martinet hypersurface of the Martinet singular symplectic form of type Σ_{20} is smooth and the restriction of this singular form to the Martinet hypersurface has the maximal rank. This singular symplectic form gives a fine link between the thermodynamical postulate of positivity of absolute temperature and the stability of an applicable structure of thermodynamics ([11]).

By the classical Darboux theorem all symplectic forms on M are locally diffeomorphic i.e. there exists a diffeomorphism-germ of M mapping the germ of one symplectic form to the germ of the other.

This is no longer true if we consider singular symplectic forms. It is obvious that if germs of singular symplectic forms ω_1 and ω_2 are diffeomorphic then the germs of corresponding Martinet hypersurfaces $\Sigma_2(\omega_1)$ and $\Sigma_2(\omega_2)$ must be diffeomorphic and the restrictions of germs of singular symplectic forms ω_1 and ω_2 to the regular parts of $\Sigma_2(\omega_1)$ and $\Sigma_2(\omega_2)$ respectively must be diffeomorphic too.

In this paper we study if the inverse theorem is valid:

Do the Martinet hypersurface Σ_2 and the restriction of ω to the regular part of Σ_2 form a complete set of invariants of ω?

Because our consideration is local, we may assume that ω is a $\mathbb{K}$-analytic or smooth closed 2-form-germ at 0 on $\mathbb{K}^{2n}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Then $\omega^n = f\Omega$, where f is a function-germ at 0 and Ω is the germ at 0 of a volume form on $\mathbb{K}^{2n}$. The Martinet hypersurface has the form

$\Sigma_2 = \{f = 0\}$ and it is a called **structurally smooth at** 0 if $f(0) = 0$ and $df_0 \neq 0$. Then Σ_2 is a smooth hypersurface-germ. In dimension 4 such situation is generic.

Local invariants of singular contact structures were studied in [9] and [10]. B. Jakubczyk and M. Zhitomirskii show that local $\mathbb{C}$-analytic singular contact structures on $\mathbb{C}^3$ with structurally smooth Martinet hypersurfaces S are diffeomorphic if their Martinet hypersurfaces and restrictions of singular structures to them are diffeomorphic. In the $\mathbb{R}$-analytic category a complete set of invariants contains, in general, one more independent invariant. It is a canonical orientation on the Martinet hypersurface. The same is true for smooth local singular contact structures $P = (\alpha)$ on $\mathbb{R}^3$ provided $\alpha|_S$ is either not flat at 0 or $\alpha|_S = 0$. The authors also study local singular contact structures in higher dimensions. They find more subtle invariants of a singular contact structure $P = (\alpha)$ on $\mathbb{K}^{2n+1}$: a line bundle L over the Martinet hypersurface S, a canonical partial connection Δ_0 on the line bundle L at $0 \in \mathbb{K}^{2n+1}$ and a 2-dimensional kernel $ker(\alpha \wedge (d\alpha)^n)|_0$. They also consider the more general case when S has singularities.

For the first occurring singularities of singular symplectic forms on a 4-dimensional manifold the answer for the above question follows from Martinet's normal forms (see [13], [17], [8]). In fact it is proved that the Martinet hypersurface Σ_2 and a characteristic line field on Σ_2 (i.e. $\{X$ is a smooth vector field : $X \rfloor (\omega|_{T\Sigma_2}) = 0\})$ form a complete set of invariants of generic singularities of singular symplectic forms on a 4-dimensional manifold.

In this paper we show that a complete set of invariants for $\mathbb{C}$-analytic singular symplectic form-germs on $\mathbb{C}^{2n}$ with structurally smooth Martinet hypersurfaces consists of the Martinet hypersurface, the pullback of the singular form-germ ω to it and the 2-dimensional kernel of $\omega^{n-1}|_0$ (Theorem 2.2). The same is true for local $\mathbb{R}$-analytic and smooth singular symplectic forms on $\mathbb{R}^{2n}$ with structurally smooth Martinet hypersurfaces if we include in the set of invariants the canonical orientation of the Martinet hypersurface (Theorem 2.3).

In section 4 we also prove that an equivalence class of a smooth or $\mathbb{K}$-analytic singular symplectic form-germ ω on $\mathbb{K}^{2n}$ with the structurally smooth Martinet hypersurface is determined only by the Martinet hypersurface, its canonical orientation (only if $\mathbb{K} = \mathbb{R}$) and the pullback of the singular form-germ to it if the dimension of a vector space generated by the coefficients of the 1-jet at 0 of $(\omega|_{T\Sigma_2})^{n-1}$ is equal to 2.

In section 5 we consider singular symplectic forms on $\mathbb{K}^4$ with structurally smooth Martinet hypersurfaces. We show that an equivalence class of a smooth or $\mathbb{K}$-analytic singular symplectic form ω on $\mathbb{K}^4$ with a structurally smooth Martinet hypersurface is determined only by the Martinet hypersurface and the pullback of the singular form to it if the two generators of the ideal generated by coefficients of $\omega|_{T\Sigma_2}$ form a regular sequence.

In $\mathbb{C}$-analytic category we prove the same result for a wider class of singular symplectic forms. The analogous result in $\mathbb{R}$-analytic category requires the assumption on the canonical orientation. The preliminary versions of results of section 5 were presented in [3] (Theorems 5.1, 5.2, Proposition 5.3).

We also consider singular symplectic forms with singular Martinet hypersurfaces. We prove that if the Martinet hypersurface of a singular symplectic form-germ is a quasi-homogeneous hypersurface-germ with an isolated singularity then the complete set of local invariants of this singular form consists of the canonical orientation of the regular part of the Martinet hypersurface (for $\mathbb{K} = \mathbb{R}$ only) and the restriction of the singular form to the regular part of the Martinet hypersurface.

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2. The complete set of invariants for singular symplectic forms with structurally smooth Martinet hypersurfaces.

2.1. **The kernel of $\omega^{n-1}|_0$.** The kernel of $\omega^{n-1}|_0$ is the following 2-dimensional subspace of $T_0\mathbb{K}^{2n}$

$$\ker\left(\omega^{n-1}|_0\right) = \{v \in T_0\mathbb{K}^{2n} : v\rfloor\left(\omega^{n-1}|_0\right) = 0\}.$$

The kernel of $\omega^{n-1}|_0$ can be also described as a kernel of a $(2n-3)$-form on Σ_2. Let Y be a vector field-germ on $\mathbb{K}^{2n}$ that is transversal to Σ_2 at 0. Let $\iota : \Sigma_2 \hookrightarrow \mathbb{K}^{2n}$ be the inclusion. Then the kernel of $\iota^*(Y\rfloor\omega^{n-1})|_0$ is equal to $\ker\omega^{n-1}|_0$.

2.2. **The canonical orientation of Σ_2.** In $\mathbb{R}$-analytic and smooth categories there is one more invariant in general. This is a *canonical orientation of* Σ_2. The orientation may be defined invariantly. Let ω be a singular symplectic form-germ on $\mathbb{R}^{2n}$ with a structurally smooth Martinet hypersurface Σ_2 at 0. Then $\Sigma_2 = \{f = 0\}$ and $df|_0 \neq 0$. We define the volume form Ω_{Σ_2} on Σ_2 which determines the orientation of

Σ_2 in the following way

$$df \wedge \Omega_{\Sigma_2} = \frac{\omega^n}{f}.$$

If f is singular at 0 (see Section 6) then we define the canonical orientation on the regular part of $\Sigma_2 = \{f = 0\}$

This definition is analogous to the definition in [9] proposed by V. I. Arnold. It is easy to see that this definition of the orientation does not depend on the choice of f such that $\Sigma_2 = \{f = 0\}$ and $df|_0 \neq 0$. We call this orientation of Σ_2 the *canonical orientation of* Σ_2.

Example 2.1. Let ω_0, ω_1 be germs of the following singular symplectic forms on $\mathbb{K}^4$

$$\omega_0 = d(p_1(dx - zdy)) + xdx \wedge dy, \ \omega_1 = d(p_1(dy + zdx)) + xdx \wedge dy$$

in the coordinate system (p_1, x, y, z) on $\mathbb{K}^4$.

It is easy to see that $\omega_0^2 = \omega_1^2 = 2p_1 dp_1 \wedge dx \wedge dy \wedge dz$. Thus $\Sigma_2 = \Sigma_2(\omega_0) = \Sigma_2(\omega_1) = \{p_1 = 0\}$, $\sigma = \iota^*\omega_0 = \iota^*\omega_1 = xdx \wedge dy$ and the canonical orientations of Σ_2 are the same for ω_0 and ω_1.

But the kernels of $\omega_0|_0$ and $\omega_1|_0$ are different. One can check that

$$\ker(\omega_0|_0) = \ker(dp_1 \wedge dx)|_0 = \operatorname{span}\{\frac{\partial}{\partial y}|_0, \frac{\partial}{\partial z}|_0\}$$

and

$$\ker(\omega_1|_0) = \ker(dp_1 \wedge dy)|_0 = \operatorname{span}\{\frac{\partial}{\partial x}|_0, \frac{\partial}{\partial z}|_0\}.$$

Let $\Sigma_{22} = \{(x, y, z) \in \Sigma_2 : \sigma_{(x,y,z)} = 0\}$. It is easy to see that $\Sigma_{22} = \{(x, y, z) \in \Sigma_2 : x = 0\}$.

Then $\ker(\omega_0|_0)$ is tangent to Σ_{22} and $\ker(\omega_1|_0)$ is transversal to Σ_{22}. Therefore ω_0 and ω_1 are not equivalent.

2.3. **Main theorems for structurally smooth Martinet hypersurfaces.** In the $\mathbb{C}$-analytic category ω is determined by the restriction to $T\Sigma_2$ and the 2-dimensional kernel of $\omega^{n-1}|_0$.

Theorem 2.2. *Let ω_0 and ω_1 be germs of $\mathbb{C}$-analytic singular symplectic forms on $\mathbb{C}^{2n}$ with a common structurally smooth Martinet hypersurface Σ_2 at 0 and $rank(\iota^*\omega_0|_0) = rank(\iota^*\omega_1|_0) \leq 2n - 4$.*

If $\iota^*\omega_0 = \iota^*\omega_1$ and $\ker \omega_0^{n-1}|_0 = \ker \omega_1^{n-1}|_0$ then there exists a $\mathbb{C}$-analytic diffeomorphism-germ $\Psi : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{2n}, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

In $\mathbb{R}$-analytic and smooth categories ω are determined by the restriction to $T\Sigma_2$, the 2-dimensional kernel of $\omega^{n-1}|_0$ and the canonical orientation of Σ_2.

Theorem 2.3. *Let ω_0 and ω_1 be germs of smooth ($\mathbb{R}$-analytic) singular symplectic forms on $\mathbb{R}^{2n}$ with a common structurally smooth Martinet hypersurface Σ_2 at 0 and $rank(\iota^*\omega_0|_0) = rank(\iota^*\omega_1|_0) \leq 2n-2$.*

If the canonical orientations defined by ω_0 and ω_1 are the same, $\iota^*\omega_0 = \iota^*\omega_1$ and $\ker \omega_0^{n-1}|_0 = \ker \omega_1^{n-1}|_0$ then there exists a smooth ($\mathbb{R}$-analytic) diffeomorphism-germ $\Psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

Theorems 2.2 and 2.3 are corollaries of Theorem 3.4. Proofs of Theorems 2.2 and 2.3 are presented in the next section.

3. A normal form and a realization theorem for singular symplectic forms with structurally smooth Martinet hypersurfaces.

The main result of this section is Theorem 3.4. In this theorem a 'normal' form of ω with the given pullback to the Martinet hypersurface is presented and sufficient conditions for equivalence of germs of singular symplectic forms with the same pullback to the common Martinet hypersurface are found. We also show which germs of closed 2-forms on $\mathbb{K}^{2n-1}$ may be obtained as a pullback to a structurally smooth Martinet hypersurface of a singular symplectic form-germ on $\mathbb{K}^{2n}$. All results of this section hold in $\mathbb{C}$-analytic, $\mathbb{R}$-analytic and (C^∞) smooth categories.

Let Ω be a volume form-germ on $\mathbb{K}^{2n}$. Let ω_0 and ω_1 be two germs of singular symplectic forms on $\mathbb{K}^{2n}$ with structurally smooth Martinet hypersurfaces at 0. It is obvious that if there exists a diffeomorphism-germ of $\mathbb{K}^{2n}$ at 0 such that $\Phi^*\omega_1 = \omega_0$ then $\Phi(\Sigma_2(\omega_0)) = \Sigma_2(\omega_1)$. Therefore we assume that these singular symplectic forms have the same Martinet hypersurface.

If the singular symplectic form-germs are equal on their common Martinet hypersurface than we obtain the following result (see [4]).

Proposition 3.1. *Let ω_0 and ω_1 be two germs at 0 of singular symplectic forms on $\mathbb{K}^{2n}$ with the common structurally smooth Martinet hypersurface Σ_2.*

If $\frac{\omega_1^n}{\omega_0^n}|_0 > 0$ and $\omega_0|_{T_{\Sigma_2}\mathbb{K}^{2n}} = \omega_1|_{T_{\Sigma_2}\mathbb{K}^{2n}} = \tilde{\omega}$ then there exists a diffeomorphism-germ $\Phi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that

$$\Phi^*\omega_1 = \omega_0$$

and $\Phi|_{\Sigma_2} = Id_{\Sigma_2}$.

Remark 3.2. The assumption $\frac{\omega_1^n}{\omega_0^n}|_0 > 0$ is needed only in $\mathbb{R}$-analytic and smooth categories. In the $\mathbb{C}$-analytic category we may assume that

$\Re e\left(\frac{\omega_1^n}{\omega_0^n}|_0\right) > 0$ or $\Im m\left(\frac{\omega_1^n}{\omega_0^n}|_0\right) \neq 0$. But this is a technical assumption (see Remark 3.5).

Proof. We present the proof in $\mathbb{R}$-analytic and smooth categories. The proof in the $\mathbb{C}$-analytic category is similar. Firstly we simplify the form-germs ω_0 and ω_1. We find the local coordinate system such that $\omega_0^n = p_1\Omega$, $\omega_1^n = p_1(A+g)\Omega$, where $\Omega = dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n$, g is a function-germ, $g(0) = 0$ and $A > 0$ (see [13]). By assumptions, we have $\omega_i = p_1\alpha_i + \tilde{\omega}$, where α_i and $\tilde{\omega}$ are germs of 2-forms and $\tilde{\omega}|_{T_{\{p_1=0\}}\mathbb{R}^{2n}} = \omega_i|_{T_{\{p_1=0\}}\mathbb{R}^{2n}}$ for $i = 0, 1$. Then further on we use the Moser homotopy method (see [14]). Let $\omega_t = t\omega_1 + (1-t)\omega_0$, for $t \in [0,1]$.

We want to find a family of diffeomorphisms Φ_t, $t \in [0,1]$ such that $\Phi_t^*\omega_t = \omega_0$, for $t \in [0,1]$, $\Phi_0 = Id$. Differentiating the above homotopy equation by t, we obtain

$$\Phi_t^*\left(d(V_t\rfloor\omega_t) + \frac{d}{dt}\omega_t\right) = 0$$

where $V_t = \frac{d}{dt}\Phi_t$. Hence we get

$$d(V_t\rfloor\omega_t) = -\frac{d}{dt}\omega_t = \omega_0 - \omega_1 = p_1(\alpha_0 - \alpha_1).$$

Now we prove the following lemma.

Lemma 3.3. *If $p_1\alpha$ is a closed 2-form-germ on $\mathbb{R}^{2n}$ then there exists a 1-form-germ β such that $p_1\alpha = d(p_1^2\beta)$.*

Proof of Lemma 3.3. By the Relative Poincare Lemma (see [1], [5]) there exists a 1-form-germ γ such that $p_1\alpha = d(p_1\gamma) = dp_1 \wedge \gamma + p_1 d\gamma$. Therefore $dp_1 \wedge \gamma|_{T_{\{p_1=0\}}\mathbb{R}^{2n}} = 0$. Hence there exist a 1-form-germ δ and a smooth function-germ f such that $\gamma = p_1\delta + f dp_1$. If we take $\beta = \delta - \frac{df}{2}$ then

$$p_1\alpha = d(p_1\gamma - d(\frac{p_1^2 f}{2})) = d(p_1^2\beta),$$

which finishes the proof of Lemma 3.3. □

The 2-form $\omega_0 - \omega_1 = p_1(\alpha_0 - \alpha_1)$ is closed. By the above lemma we have

$$V_t\rfloor\omega_t = p_1^2\beta. \tag{3.1}$$

Now we calculate $\Sigma_2(\omega_t)$. It is easy to see that

$$\omega_i^n = (p_1\alpha_i + \tilde{\omega})^n = \tilde{\omega}^n + p_1\sum_{k=1}^{n}\binom{n}{k} p_1^{k-1}\alpha_i^k \wedge \tilde{\omega}^{n-k}.$$

But $\omega_i^n|_{T_{\{p_1=0\}}R^{2n}} = 0$. This clearly forces $\tilde{\omega}^n = 0$. By the above formula we get

$$n\alpha_0 \wedge \tilde{\omega}^{n-1} = \Omega - p_1 \sum_{k=2}^{n} \binom{n}{k} p_1^{k-2} \alpha_0^k \wedge \tilde{\omega}^{n-k}$$

and

$$n\alpha_1 \wedge \tilde{\omega}^{n-1} = (A+g)\Omega - p_1 \sum_{k=2}^{n} \binom{n}{k} p_1^{k-2} \alpha_1^k \wedge \tilde{\omega}^{n-k}.$$

The above formulas imply the following formula

$$\begin{aligned} \omega_t^n &= (p_1(t\alpha_1 + (1-t)\alpha_0) + \tilde{\omega})^n \\ &= p_1(tn\alpha_1 \wedge \tilde{\omega}^{n-1} + (1-t)n\alpha_0 \wedge \tilde{\omega}^{n-1}) + \\ &\quad + \sum_{k=2}^{n} \binom{n}{k} p_1^k (t\alpha_1 + (1-t)\alpha_0)^k \wedge \tilde{\omega}^{n-k} \\ &= p_1(1 + t(A+g-1))\Omega + \\ &\quad + p_1^2 \sum_{k=2}^{n} \binom{n}{k} p_1^{k-2} \left((t\alpha_1 + (1-t)\alpha_0)^k - t\alpha_1^k - (1-t)\alpha_0^k\right) \wedge \tilde{\omega}^{n-k}. \end{aligned} \tag{3.2}$$

From (3.2) we obtain

$$\omega_t^n = p_1(1 + t(A+g-1) + p_1 h_t)\Omega, \tag{3.3}$$

where h_t is a function-germ. But $1 + t(A-1) > 0$ for $A > 0$ and $t \in [0,1]$.

$\Sigma_2(\omega_t) = \{p_1 = 0\}$ is nowhere dense, therefore by direct algebraic calculation, it is easy to see that equation (3.1) is equivalent to the following equation

$$V_t \rfloor \omega_t^n = np_1^2 \beta \wedge \omega_t^{n-1}. \tag{3.4}$$

Combining (3.4) with (3.3) we obtain

$$V_t \rfloor (1 + t(A+g-1) + p_1 h_t)\Omega = np_1 \beta \wedge \omega_t^{n-1}. \tag{3.5}$$

But if $A > 0$ then $1 + t(A-1) > 0$ for $t \in [0,1]$. Therefore we can find a smooth (or $\mathbb{R}$-analytic) vector field-germ V_t that satisfies (3.5). The restriction of V_t to Σ_2 vanishes, because the right hand side of (3.5) vanishes on Σ_2. Hence there exists a diffeomorphism Φ_t such that $\Phi_t^* \omega_t = \omega_0$ for $t \in [0,1]$ and $\Phi_t|_{\Sigma_2} = Id_{\Sigma_2}$. This completes the proof of Proposition 3.1. □

If $\operatorname{rank}(\iota^*\omega|_0)$ is $2n-2$ then ω is equivalent to the Martinet singular symplectic form of type Σ_{20} (see the local normal form (1.3) and [13]). Therefore we study singular symplectic forms such that $\operatorname{rank}(\iota^*\omega|_0) \leq$

$2n-4$. In fact we will prove that structural smoothness of $\Sigma_2(\omega)$ implies that $\mathrm{rank}(\iota^*\omega|_0) = 2n-4$

In the next theorem we describe all germs of singular symplectic forms ω on $\mathbb{K}^{2n}$ with structurally smooth Martinet hypersurfaces at 0 and $\mathrm{rank}(\iota^*\omega|_0) \leq 2n-4$. We also find the sufficient conditions for equivalence of singular symplectic forms of this type. This is a generalization of the analogous result for singular symplectic forms on 4-dimensional manifolds ([3]).

We use the following mappings in the subsequent results $\iota : \Sigma_2 = \{p_1 = 0\} \hookrightarrow \mathbb{K}^{2n}$

$$\iota(p_2, \cdots, p_n, q_1, \cdots, q_n) = (0, p_2, \cdots, p_n, q_1, \cdots, q_n)$$

and $\pi : \mathbb{K}^{2n} \to \Sigma_2 = \{p_1 = 0\}$

$$\pi(p_1, p_2, \cdots, p_n, q_1, \cdots, q_n) = (p_2, \cdots, p_n, q_1, \cdots, q_n).$$

Theorem 3.4. *Let ω be a singular symplectic form-germ on $\mathbb{K}^{2n}$ with a structurally smooth Martinet hypersurface at 0.*

(a) If $\mathrm{rank}(\iota^*\omega|_0) \leq 2n-4$ then there exists a diffeomorphism-germ $\Phi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that

$$\Phi^*\omega = d\left(p_1\pi^*\alpha\right) + \pi^*\sigma,$$

where $\sigma = \iota^*\Phi^*\omega$ is a closed 2-form-germ on $\{p_1 = 0\}$ and α is a 1-form-germ on $\{p_1 = 0\}$ such that $\alpha\wedge\sigma^{n-1} = 0$ and $\alpha\wedge d\alpha\wedge\sigma^{n-2}|_0 \neq 0$.

(b) Moreover if $\omega_0 = d\left(p_1\pi^*\alpha_0\right) + \pi^*\sigma$ and $\omega_1 = d\left(p_1\pi^*\alpha_1\right) + \pi^*\sigma$ are two germs of singular symplectic forms satisfying the above conditions and

(1) $\frac{\alpha_1\wedge d\alpha_1\wedge\sigma^{n-2}}{\alpha_0\wedge d\alpha_0\wedge\sigma^{n-2}}|_0 > 0$, *(only for $\mathbb{K} = \mathbb{R}$)*
(2) $\alpha_1|_0 \wedge \alpha_0|_0 \wedge \sigma^{n-2}|_0 = 0$,

then there exists a diffeomorphism-germ $\Psi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

Remark 3.5. Assumption (1) is only needed in $\mathbb{R}$-analytic and smooth categories. In the $\mathbb{C}$-analytic category we have

$$\Phi^*(d\left(p_1\pi^*\alpha\right) + \pi^*\sigma) = d\left(p_1\pi^* i\alpha\right) + \pi^*\sigma,$$

where Φ is the following diffeomorphism

$$\Phi(p_1, p_2, \cdots, p_n, q_1, \cdots, q_n) = (ip_1, p_2, \cdots, p_n, q_1, \cdots, q_n)$$

and $i^2 = -1$. It is obvious that $\Phi|_{\Sigma_2} = Id_{\Sigma_2}$, where $\Sigma_2 = \{p_1 = 0\}$ and $i\alpha\wedge d(i\alpha)\wedge\sigma^{n-2} = -\alpha\wedge d\alpha\wedge\sigma^{n-2}$.

Proof. The proof is similar to the proof of analogous theorem for singular symplectic forms on a 4-dimensional manifold (see [3]). We can find a coordinate system $(p_1, q_1, \cdots, p_n, q_n)$ such that $\Sigma_2(\omega) = \{p_1 = 0\}$. Then by the Relative Poincare Lemma (see [1], [5]) there exists 1-form-germ γ on $\mathbb{K}^{2n}$ such that $\omega = d(p_1\gamma) + \pi^*\sigma$. It is clear that we can write γ in the following form $\gamma = \pi^*\alpha + p_1\delta + gdp_1$, where α is a 1-form-germ on $\{p_1 = 0\}$, g is a function-germ and δ is a 1-form-germ on $\mathbb{K}^{2n}$. Then

$$d(p_1(p_1\delta + gdp_1)) = p_1(2dp_1 \wedge \delta + p_1 d\delta + dg \wedge dp_1).$$

By Lemma 3.3 we have $\omega = d(p_1\pi^*\alpha) + \pi^*\sigma + d(p_1^2\theta)$.

Hence

$$\begin{aligned} \omega^n &= ndp_1 \wedge \pi^*\alpha \wedge \pi^*(\sigma^{n-1}) + 2np_1 dp_1 \wedge \pi^*\beta \wedge \pi^*(\sigma^{n-1}) \\ &\quad + n(n-1)p_1 dp_1 \wedge \pi^*\alpha \wedge d\pi^*\alpha \wedge \pi^*(\sigma^{n-2})) + p_1^2 v\Omega, \end{aligned}$$

where v is a function-germ at 0 on $\mathbb{K}^{2n}$. We have $\alpha \wedge \sigma^{n-1} = 0$, because $\omega^n|_{T_{\{p_1=0\}}\mathbb{K}^{2n}} = 0$. From $\sigma^{n-1}|_0 = 0$, we have

$$\omega^n = n(n-1)p_1 dp_1 \wedge \pi^*\alpha \wedge d\pi^*\alpha \wedge \pi^*(\sigma^{n-2}) + p_1 g\Omega.$$

where g is a function-germ on $\mathbb{K}^{2n}$ vanishing at 0. From the above we obtain that

$$\alpha \wedge d\alpha \wedge \sigma^{n-2}|_0 \neq 0.$$

Therefore

$$\operatorname{rank}(\sigma|_0) = 2n - 4. \tag{3.6}$$

Let

$$\omega_0 = d\left(p_1\pi^*\alpha\right) + \pi^*\sigma.$$

Then

$$\omega_0^n = n(n-1)p_1 dp_1 \wedge \pi^*\alpha \wedge d\pi^*\alpha \wedge \pi^*(\sigma^{n-2}) + p_1 h\Omega,$$

where h is a function-germ on $\mathbb{K}^{2n}$ vanishing at 0. One can check that

$$\tilde{\omega} = \omega_0|_{T_{\{p_1=0\}}\mathbb{K}^{2n}} = dp_1 \wedge \pi^*\alpha + \pi^*\sigma = \omega|_{T_{\{p_1=0\}}\mathbb{K}^{2n}}.$$

Therefore by Proposition 3.1 there exists a germ of a diffeomorphism $\Theta : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that $\Theta^*\omega = \omega_0$ and $\Theta|_{\{p_1=0\}} = Id_{\{p_1=0\}}$.

This finish the proof of part (a)

Now we prove part (b). (3.6) and (2) implies that there exists $B \neq 0$ such that $\alpha_1|_0 \wedge \sigma^{n-2}|_0 = B\alpha_0|_0 \wedge \sigma^{n-2}|_0$. If $B \neq 1$ then $\Phi^*\omega_1 = d(p_1\pi^*(B\alpha)) + \pi^*\sigma$ where Φ is a diffeomorphism-germ of the form $\Phi(p, q) = (Bp_1, p_2, ..., p_n, q_1, ..., q_n))$. Thus we may assume that $B = 1$.

We use the Moser homotopy method. Let $\alpha_t = t\alpha_1 + (1-t)\alpha_0$ and $\omega_t = d\left(p_1\pi^*\alpha_t\right) + \pi^*\sigma$ for $t \in [0,1]$. It is easy to check that $\alpha_t \wedge \sigma^{n-1} = 0$.

Now we look for germs of diffeomorphisms Φ_t such that

$$\Phi_t^*\omega_t = \omega_0, \ \text{for } t \in [0,1], \ \Phi_0 = Id. \tag{3.7}$$

Differentiating the above homotopy equation by t, we obtain

$$d(V_t \rfloor \omega_t) = d(p_1\pi^*(\alpha_0 - \alpha_1)),$$

where $V_t = \frac{d}{dt}\Phi_t$. Therefore we have to solve the following equation

$$V_t \rfloor \omega_t = p_1\pi^*(\alpha_0 - \alpha_1). \tag{3.8}$$

We calculate the Martinet hypersurface of ω_t.

$$\omega_t^n = n(n-1)p_1 dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}) + p_1^2 g_t \Omega,$$

where g_t is a smooth function-germ at 0, because $\sigma^n = 0$, $(d\alpha_t) \wedge \sigma^{n-1} = 0$ and $\alpha_t \wedge \sigma^{n-1} = 0$.

Now we calculate

$$\begin{aligned} &\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}|_0 \\ &= t^2\alpha_1 \wedge d\alpha_1 \wedge \sigma^{n-2}|_0 + t(1-t)\alpha_1 \wedge d\alpha_0 \wedge \sigma^{n-2}|_0 + \\ &+ t(1-t)\alpha_0 \wedge d\alpha_1 \wedge \sigma^{n-2}|_0 + (1-t)^2\alpha_0 \wedge d\alpha_0 \wedge \sigma^{n-2}|_0 \quad . \end{aligned}$$

From $\alpha_0 \wedge \sigma^{n-2}|_0 = \alpha_1 \wedge \sigma^{n-2}|_0$ we have

$$\begin{aligned} &\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}|_0 \\ &= (t^2 + t(1-t))d\alpha_1 \wedge \alpha_1 \wedge \sigma^{n-2}|_0 \\ &+ (t(1-t) + (1-t)^2)d\alpha_0 \wedge \alpha_0 \wedge \sigma^{n-2}|_0 \\ &= t\alpha_1 \wedge d\alpha_1 \wedge \sigma^{n-2}|_0 + (1-t)\alpha_0 \wedge d\alpha_0 \wedge \sigma^{n-2}|_0 \quad . \end{aligned}$$

But there exists $A > 0$ such that $\alpha_1 \wedge d\alpha_1 \wedge \sigma^{n-2}|_0 = A\alpha_0 \wedge d\alpha_0 \wedge \sigma^{n-2}|_0$, so we obtain

$$\begin{aligned} &\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}|_0 \\ &= (At + (1-t))\alpha_0 \wedge d\alpha_0 \wedge \sigma^{n-2}|_0 \neq 0 \end{aligned}$$

for $t \in [0,1]$. Therefore

$$dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2})|_0 \neq 0$$

for $t \in [0,1]$. Thus $\Sigma_2(\omega_t) = \{p_1 = 0\}$.

Because Σ_2 is nowhere dense, equation (3.8) is equivalent to

$$V_t \rfloor \omega_t^n = np_1\pi^*(\alpha_0 - \alpha_1) \wedge \omega_t^{n-1}$$

and $\omega_t^n = n(n-1)p_1 dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}) + p_1^2 g_t \Omega$, where g_t is a smooth function-germ at 0. Hence we have to solve the following equation

(3.9)
$$V_t \rfloor \left(n(n-1)dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}) + p_1 g_t \Omega\right) = n\pi^*(\alpha_0 - \alpha_1) \wedge \omega_t^{n-1}.$$

From the above calculation we have $\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}|_0 \neq 0$. Therefore $n(n-1)dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t \wedge \sigma^{n-2}) + p_1 g_t \Omega$ is a nondegenerate $2n$-form-germ on $\mathbb{K}^{2n}$ and

$$n\pi^*(\alpha_0 - \alpha_1) \wedge \omega_t^{n-1}|_0 =$$
$$n(n-1)dp_1 \wedge \pi^*(\alpha_1 \wedge \alpha_0 \wedge \sigma^{n-2})|_0 = 0,$$

because $\alpha_1 \wedge \alpha_0 \wedge \sigma^{n-2}|_0 = 0$. Hence we can find a smooth solution V_t of (3.9) such that $V_t|_0 = 0$. Thus there exit germs of diffeomorphisms Φ_t, which satisfy (3.7). For $t = 1$ we have $\Phi_1^* \omega_1 = \omega_0$. □

Now we can prove main theorems from the previous section.

Proof of Theorems 2.2 and 2.3. It is easy to see that if $\omega = d(p_1 \pi^* \alpha) + \pi\sigma$, where α and σ satisfy conditions of Theorem 3.4 then $\ker \omega^{n-1}|_0 = \ker(\alpha \wedge \sigma^{n-2})|_0$ and the canonical orientation of Σ_2 is defined by the volume form $\alpha \wedge d\alpha \wedge \sigma^{n-2}$. By Theorem 3.4 we get the result. □

We call a closed 2-form-germ σ on $\mathbb{K}^{2n-1}$ *realizable with a structurally smooth Martinet hypersurface* if there exists a singular symplectic form-germ ω on $\mathbb{K}^{2n}$ such that $\Sigma_2(\omega) = \{0\} \times \mathbb{K}^{2n-1}$ is structurally smooth and $\omega|_{T\Sigma_2(\omega)} = \sigma$.

From Martinet's normal form of a singular symplectic form-germ on $\mathbb{K}^{2n}$ of the rank $2n-2$ we know that all germs of closed 2-forms on $\mathbb{K}^{2n-1}$ of the rank $2n-2$ are realizable with a structurally smooth Martinet hypersurface. From part (a) of Theorem 3.4 we obtain the following realization theorem of closed 2-forms on $\mathbb{K}^{2n-1}$ of the rank less than $2n-2$ at $0 \in \mathbb{K}^{2n-1}$.

Theorem 3.6. *Let σ be a closed 2-form-germ on $\mathbb{K}^{2n-1}$ and rank$(\sigma|_0) < 2n-2$. σ is realizable with a structurally smooth Martinet hypersurface if and only if rank$(\sigma|_0) = 2n-4$ and there exists a 1 form-germ α on $\mathbb{K}^{2n-1}$ such that $\alpha \wedge \sigma^{n-1} = 0$ and $\alpha \wedge d\alpha \wedge \sigma^{n-2}|_0 \neq 0$.*

4. Determination by the restriction of ω to $T\Sigma_2$ and the canonical orientation of Σ_2.

In this section we find sufficient conditions to determine the equivalence class of a singular symplectic form by its restriction to the structurally smooth Martinet hypersurface Σ_2 and the canonical orientation of Σ_2.

Let $j_0^1 f$ denote the 1-jet at 0 of a smooth ($\mathbb{K}$-analytic) function-germ $f : \mathbb{K}^{2n-1} \to \mathbb{K}$. The vector space of all 1-jets at 0 of smooth $\mathbb{K}$-analytic) function-germs on $\mathbb{K}^{2n-1}$ is denoted by $J_0^1(\mathbb{K}^{2n-1}, \mathbb{K})$.

Let σ be a closed 2-form-germ at 0 on $\mathbb{K}^{2n-1}$. Then the closed $(2n-2)$-form-germ σ^{n-1} at 0 on $\mathbb{K}^{2n-1}$ has the following form in a local coordinates set $q = (q_1, \cdots, q_{2n-1})$ on $\mathbb{K}^{2n-1}$

$$\sigma^{n-1} = \sum_{i=1}^{2n-1} g_i dq_1 \wedge \cdots \wedge dq_{i-1} \wedge dq_{i+1} \wedge \cdots \wedge dq_{2n-1},$$

where $g_i : \mathbb{K}^{2n-1} \to \mathbb{K}$ is a smooth ($\mathbb{K}$-analytic) function-germ at 0 for $i = 1, \cdots, 2n-1$.

Hence the 1-jet at 0 of $2n-2$-form-germ σ^{n-1} has the following form

$$j_0^1\sigma^{n-1} = \sum_{i=1}^{2n-1} j_0^1 g_i dq_1 \wedge \cdots \wedge dq_{i-1} \wedge dq_{i+1} \wedge \cdots \wedge dq_{2n-1}.$$

We denote by span $j_0^1\sigma^{n-1}$ the vector space spanned by coefficients of $j_0^1\sigma^{n-1}$

$$\text{span } j_0^1\sigma^{n-1} = \text{span}\left(j_0^1 g_1, \cdots, j_0^1 g_{2n-1}\right).$$

If $g_i(0) = 0$ then $j_0^1 g_i = \sum_{k=1}^{2n-1} \frac{\partial g_i}{\partial q_k}(0) q_k$. Thus it is easy to check that if $\operatorname{rank}(\sigma|_0) = 2n-4$ then the definition of span $j_0^1\sigma^{n-1}$ does not depend on the choice of a local coordinate system .

Theorem 4.1. *Let ω_0 and ω_1 be germs of smooth ($\mathbb{K}$-analytic) singular symplectic forms on $\mathbb{K}^{2n}$ with a common structurally smooth Martinet hypersurface Σ_2 at 0 and $rank(\iota^*\omega_0|_0) = rank(\iota^*\omega_1|_0) = 2n-4$.*

If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$, ω_0 and ω_1 define the same canonical orientation of Σ_2 and the dimension of the vector space span $j_0^1\sigma^{n-1}$ is 2 then there exists a smooth ($\mathbb{K}$-analytic) diffeomorphism-germ $\Psi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

The proof is based on the following lemma.

Lemma 4.2. *Let σ be a closed smooth ($\mathbb{K}$-analytic) 2-form-germ at 0 on $\mathbb{K}^{2n-1}$ such that rank$(\sigma|_0) = 2n-4$. Let α_0, α_1 be smooth ($\mathbb{K}$-analytic) 1-form-germs at 0 on $\mathbb{K}^{2n-1}$ such that for $i = 0, 1$*

$$\alpha_i \wedge d\alpha_i \wedge \sigma^{n-2}|_0 \neq 0, \tag{4.1}$$

$$\alpha_i \wedge \sigma^{n-1} = 0. \tag{4.2}$$

If the dimension of a vector space span $j_0^1\sigma^{n-1}$ is 2 then there exists a number $A \neq 0$ such that $\alpha_0 \wedge \sigma^{n-2}|_0 = A\alpha_1 \wedge \sigma^{n-2}|_0$.

Proof of Lemma 4.2. Since rank$(\sigma|_0) = 2n-4$, there exists a local coordinate system $(x_1, \cdots, x_{2n-4}, y_1, y_2, y_3)$ on $\mathbb{K}^{2n-1}$ and function-germs a_i, b_{ij}, c_{ij} on $\mathbb{K}^{2n-1}$ vanishing at 0 such that

$$\sigma = \sum_{k=1}^{n-2} dx_{2k-1} \wedge dx_{2k} + \sum_{1 \leq i < j \leq 2n-4} c_{ij} dx_i \wedge dx_j \tag{4.3}$$
$$+ \sum_{i=1}^{3} \sum_{j=1}^{2n-4} b_{ij} dy_i \wedge dx_j + \sum_{\{i,j,k\}=\{1,2,3\}\ j<k} a_i dy_j \wedge dy_k.$$

It implies that the 1-jet of σ^{n-1} at 0 has the following form

$$j_0^1\sigma^{n-1} = \sum_{\{i,j,k\}=\{1,2,3\}\ j<k} j_0^1 a_i dy_j \wedge dy_k \wedge dx_1 \wedge \cdots \wedge dx_{2n-4}, \tag{4.4}$$

where $j_0^1 a_i$ denotes the 1-jet of the function-germ a_i at 0 for $i = 1, 2, 3$.

The vector space span$j_0^1\sigma^{n-1}$ is spanned by $j_0^1 a_1, j_0^1 a_2, j_0^1 a_3$.

There exist function-germs f_{ij} and g_{ik} for $i = 0, 1,\ j = 1, 2, 3,\ k = 1, \cdots, 2n-4$ such that

$$\alpha_i = \sum_{j=1}^{3} f_{ij} dy_j + \sum_{k=1}^{2n-4} g_{ik} dx_k.$$

By (4.1) we get that $f_{01} \neq 0$ or $f_{02} \neq 0$ or $f_{03} \neq 0$. Without loss of generality we may assume that $f_{03} \neq 0$, since we can change a coordinate system replacing y_j with y_3 if $f_{03} = 0$ and $f_{0j} \neq 0$ for $j \neq 3$.

By (4.2) we get $j_0^1(\alpha_0 \wedge \sigma^{n-1}) = 0$. By (4.4) it implies that

$$f_{01}(0) j_0^1 a_1 + f_{02}(0) j_0^1 a_2 + f_{03}(0) j_0^1 a_3 = 0,$$

since $a_i(0) = 0$ for $i = 1, 2, 3$. Since $f_{03}(0) \neq 0$ we get that

$$j_0^1 a_3 = -\frac{f_{01}(0)}{f_{03}(0)} j_0^1 a_1 - \frac{f_{02}(0)}{f_{03}(0)} j_0^1 a_2. \tag{4.5}$$

Thus the space span $j_0^1\, \sigma^{n-1}$ is spanned by $j_0^1 a_1, j_0^1 a_2$. Since dim span $j_0^1 \sigma^{n-1} = 2$ the 1-jets $j_0^1 a_1, j_0^1 a_2$ are $\mathbb{K}$-linearly independent. On the other hand by (4.2) we get $j_0^1(\alpha_1 \wedge \sigma^{n-1}) = 0$. By (4.4) it implies that

$$f_{11}(0) j_0^1 a_1 + f_{12}(0) j_0^1 a_2 + f_{13}(0) j_0^1 a_3 = 0,$$

since $a_i(0) = 0$ for $i = 1, 2, 3$. By (4.5) it implies that

$$\left(f_{11}(0) - \frac{f_{13}(0)}{f_{03}(0)} f_{01}(0) \right) j_0^1 a_1 + \left(f_{12}(0) - \frac{f_{13}(0)}{f_{03}(0)} f_{02}(0) \right) j_0^1 a_2 = 0.$$

Since the 1-jets $j_0^1 a_1, j_0^1 a_2$ are $\mathbb{K}$-linearly independent we get that

$$f_{11}(0) - \frac{f_{13}(0)}{f_{03}(0)} f_{01}(0) = f_{12}(0) - \frac{f_{13}(0)}{f_{03}(0)} f_{02}(0) = 0. \tag{4.6}$$

By (4.3) we get that $\sigma^{n-2}|_0 = (n-2)! dx_1 \wedge \cdots \wedge dx_{2n-4}|_0$. Thus we have for $i = 0, 1$

$$\alpha_i \wedge \sigma^{n-2}|_0 = (n-2)! \sum_{j=1}^{3} f_{ij}(0) dy_i \wedge dx_1 \wedge \cdots \wedge dx_{2n-4}|_0.$$

By (4.6) it implies that $\alpha_1 \wedge \sigma^{n-2}|_0 = \frac{f_{13}(0)}{f_{03}(0)} \alpha_0 \wedge \sigma^{n-2}|_0$. □

Proof of Theorem 4.1. By Theorem 3.4 we can find a local coordinate system such that the germs ω_0 and ω_1 have the following form $\omega_0 = d\,(p_1 \pi^* \alpha_0) + \pi^* \sigma$ and $\omega_1 = d\,(p_1 \pi^* \alpha_1) + \pi^* \sigma$, where $\alpha_0, \alpha_1, \sigma$ are form-germs satisfying the assumptions of Lemma 4.2. Thus there exists a number $A \neq 0$ such that $\alpha_0 \wedge \sigma^{n-2}|_0 = A \alpha_1 \wedge \sigma^{n-2}|_0$. By Theorem 3.4 it implies that there exists a smooth ($\mathbb{K}$-analytic) diffeomorphism-germ $\Psi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that

$$\Psi^* \omega_1 = \omega_0.$$

□

Example 4.3. Let ω be the following closed 2-form-germ on $\mathbb{K}^{2n}$

$$\begin{aligned} \omega = {} & d(p_1(dy_3 + y_1 dy_2)) + \textstyle\sum_{k=1}^{n-2} dx_{2k-1} \wedge dx_{2k} + \\ & (dy_3 + y_1 dy_2) \wedge (b(y_1, y_2, y_3) dy_1 - a(y_1, y_2, y_3) dy_2) \end{aligned} \tag{4.7}$$

where $(p_1, y_1, y_2, y_3, x_1, \cdots, x_{2n-4})$ is a coordinate system on $\mathbb{K}^{2n}$, b is a smooth ($\mathbb{K}$-analytic) function-germ on $\mathbb{K}^3$ vanishing at 0, h is a smooth ($\mathbb{K}$-analytic) function-germ on $\mathbb{K}^2$ vanishing at 0 and

(4.8)

$$a(y_1, y_2, y_3) = \int_0^{y_1} \left(t \frac{\partial b}{\partial y_3}(t, y_2, y_3) - \frac{\partial b}{\partial y_2}(t, y_2, y_3) \right) dt + h(y_2, y_3).$$

It is easy to see that the Martinet hypersurface is $\Sigma_2 = \{p_1 = 0\}$ and the restriction of ω to $T\Sigma_2$ has the following form

$$\sigma = (dy_3+y_1dy_2)\wedge(b(y_1,y_2,y_3)dy_1-a(y_1,y_2,y_3)dy_2)+\sum_{k=1}^{n-2} dx_{2k-1}\wedge dx_{2k}.$$

Thus $j_0^1\sigma^{n-1}$ is equal to

$$(n-2)!\,\big((j_0^1 b)dy_3\wedge dy_1+(j_0^1 a)dy_2\wedge dy_3\big)\wedge dx_1\wedge\cdots\wedge dx_{2n-4}.$$

Then the space span $j_0^1\sigma^{n-1}$ is span $\{j_0^1 a, j_0^1 b\}$. From (4.8) we get

$$a(0)=0,\quad \frac{\partial a}{\partial y_1}(0)=-\frac{\partial b}{\partial y_2}(0),\quad \frac{\partial a}{\partial y_i}(0)=\frac{\partial h}{\partial y_i}(0)\text{ for } i=2,3.$$

Hence span $j_0^1\sigma^{n-1}$ is spanned by

$$-\frac{\partial b}{\partial y_2}(0)y_1+\frac{\partial h}{\partial y_2}(0)y_2+\frac{\partial h}{\partial y_3}(0)y_3,\quad \frac{\partial b}{\partial y_1}(0)y_1+\frac{\partial b}{\partial y_2}(0)y_2+\frac{\partial b}{\partial y_3}(0)y_3.$$

Thus $\dim\operatorname{span} j_0^1\sigma^{n-1}$ is 2 if and only if the rank of the following matrix is 2.

$$\begin{bmatrix} -\frac{\partial b}{\partial y_2}(0) & \frac{\partial h}{\partial y_2}(0) & \frac{\partial h}{\partial y_3}(0) \\ \frac{\partial b}{\partial y_1}(0) & \frac{\partial b}{\partial y_2}(0) & \frac{\partial b}{\partial y_3}(0) \end{bmatrix}$$

For $n=2$ any closed 2-form-germ satisfying the assumptions of Theorem 3.4 is equivalent to (4.7) in a coordinate-set (p_1,y_1,y_2,y_3) on $\mathbb{K}^4$, since any contact form on $\mathbb{K}^3=\{p_1=0\}$ is equivalent to $dy_3+y_1dy_2$.

The set-germ $\Sigma_{22}=\{y\in\Sigma_2 : \sigma|_y=0\}$ can be described as

$$\{y\in\Sigma_2 : a(y)=b(y)=0\}.$$

If $\dim\operatorname{span} j_0^1\sigma^{n-1}$ is 2 then Σ_{22} is a germ of a smooth curve on Σ_2.

For $\mathbb{K}=\mathbb{R}$ if $(\frac{\partial b}{\partial y_2}(0))^2+\frac{\partial b}{\partial y_1}(0)\frac{\partial h}{\partial y_2}(0)$ is positive then ω has a hyperbolic Σ_{220} singularity, if it is negative then ω has an elliptic Σ_{220} singularity and if it is zero then ω has a parabolic Σ_{221} singularity [13]. Roussarie has shown the stability of Σ_{220} singularities [17]. Golubitsky and Tischler have proved that Σ_{221} singularity is not stable [8].

The normal forms of Σ_{220} singularities are presented below

$$\text{hyperbolic } \Sigma_{220}:$$
$$d(p_1(dy_3+y_1dy_2))+(dy_3+y_1dy_2)\wedge(y_1dy_1-y_2dy_2),$$

$$\text{elliptic } \Sigma_{220}:$$
$$d(p_1(dy_3+y_1dy_2))+(dy_3+y_1dy_2)\wedge(y_1dy_1+y_2dy_2).$$

5. Determination by the restriction of ω to $T\Sigma_2$ in dimension 4.

In [3] we proved the following result on determination of the equivalence class of a $\mathbb{C}$-analytic singular symplectic form-germ ω by its restriction to the structurally smooth Martinet hypersurface.

Theorem 5.1. *Let ω_0 and ω_1 be germs of $\mathbb{C}$-analytic singular symplectic forms on $\mathbb{C}^4$ with a common structurally smooth Martinet hypersurface Σ_2 at 0 and rank$(\iota^*\omega_0|_0) =$ rank$(\iota^*\omega_1|_0) = 0$.*

If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$ and there does not exist a $\mathbb{C}$-analytic vector field-germ X on Σ_2 at 0 such that $X\rfloor\sigma = 0$ and $X|_0 \neq 0$ then there exists a $\mathbb{C}$-analytic diffeomorphism-germ $\Psi : (\mathbb{C}^4, 0) \to (\mathbb{C}^4, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

In the analogous result in $\mathbb{R}$-analytic category ([3]) the fixed canonical orientation of the Martinet hypersurface is needed (see Example 5.5)

Theorem 5.2. *Let ω_0 and ω_1 be germs of $\mathbb{R}$-analytic singular symplectic forms on $\mathbb{R}^4$ with a common structurally smooth Martinet hypersurface Σ_2 at 0 and rank$(\iota^*\omega_0|_0) =$ rank$(\iota^*\omega_1|_0) = 0$.*

If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$, ω_0 and ω_1 define the same canonical orientation of Σ_2 and there does not exist an $\mathbb{R}$-analytic vector field-germ X on Σ_2 at 0 such that $X\rfloor\sigma = 0$ and $X|_0 \neq 0$ then there exists an $\mathbb{R}$-analytic diffeomorphism-germ $\Psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

One can also find the normal form of a singular symplectic form-germ on $\mathbb{K}^4$ at 0 which does not satisfy the assumptions of Theorems 5.2, 5.1. The following result is also true in the smooth category ([3]).

Proposition 5.3. *Let ω be a $\mathbb{K}$-analytic (smooth) singular symplectic form-germ on $\mathbb{K}^4$ with a structurally smooth Martinet hypersurface at 0 and rank$(\iota^*\omega|_0) = 0$.*

If there exists a $\mathbb{K}$-analytic (smooth) vector field-germ X on Σ_2 at 0 such that $X\rfloor\sigma = 0$ and $X|_0 \neq 0$ then there exists a $\mathbb{K}$-analytic (smooth) diffeomorphism-germ $\Psi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0)$ such that

$$\Psi^*\omega = d(p_1(dx + Cdy + zdy)) + g(x, y)dx \wedge dy$$

or

$$\Psi^*\omega = d(p_1(dy + Cdx + zdx)) + g(x, y)dx \wedge dy,$$

where $C \in \mathbb{K}$ and g is a $\mathbb{K}$-analytic function-germ on $\mathbb{K}^4$ at 0 that does not depend on p_1 and z.

In this section we find conditions for the determination of the equivalence class of a smooth or $\mathbb{R}$-analytic singular symplectic form on $\mathbb{R}^4$ by its pullback to the Martinet hypersurface only.

We need some notions from commutative algebra (see Appendix 1 of [9], [2]) to formulate the result in the smooth category. We recall that a sequence of elements $a_1, \cdots, a_r$ of a proper ideal I of a ring R is called *regular* if a_1 is a non-zero-divisor of R and a_i is a non-zero-divisor of $R/ < a_1, \cdots, a_{i-1} >$ for $i = 2, \cdots, r$. Here $< a_1, \cdots, a_i >$ denotes the ideal generated by $a_1, \cdots, a_i$. The *length* of a regular sequence $a_1, \cdots, a_r$ is r.

The *depth* of the proper ideal I of the ring R is the supremum of lengths of regular sequences in I. We denote it by $\mathrm{depth}(I)$. If $I = R$ then we define $\mathrm{depth}(I) = \infty$.

Let σ be a smooth ($\mathbb{K}$-analytic) closed 2-form-germ on $\Sigma_2 = \mathbb{K}^3$ and $\mathrm{rank}(\sigma|_0) = 0$. In the local coordinate system (x, y, z) on Σ_2 we have $\sigma = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$, where a, b, c are smooth ($\mathbb{K}$-analytic) function-germs on Σ_2. By $I(\sigma)$ we denote the ideal of the ring of smooth ($\mathbb{K}$-analytic) function-germs on Σ_2 generated by a, b, c i.e. $I(\sigma) =< a, b, c >$. It is easy to see that $I(\sigma)$ does not depend on the local coordinate system on Σ_2. σ satisfies the condition $\alpha \wedge \sigma = 0$, where α is a contact form-germ on $\mathbb{K}^3$. It implies that $I(\sigma)$ is generated by two function-germs.

In the $\mathbb{K}$-analytic category if $\mathrm{depth} I(\sigma) \geq 2$ then the two generators of $I(\sigma)$ form a regular sequence of length 2 (see [2]). One can easily check that it implies that there does not exist a $\mathbb{K}$-analytic vector field-germ on Σ_2 such that $X \rfloor \sigma = 0$ and $X|_0 \neq 0$. The inverse implication is not true in general. Now we can prove the following theorem.

Theorem 5.4. *Let ω_0 and ω_1 be germs of smooth or $\mathbb{R}$-analytic singular symplectic forms on $\mathbb{R}^4$ with a common structurally smooth Martinet hypersurface Σ_2 at 0 and rank$(\iota^*\omega_0|_0) =$ rank$(\iota^*\omega_1|_0) = 0$.*

If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$ and the two generators of the ideal $I(\sigma)$ form a regular sequence of length 2 then there exists a smooth or $\mathbb{R}$-analytic diffeomorphism-germ $\Psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ such that

$$\Psi^*\omega_1 = \omega_0.$$

Proof. By Theorem 3.4 (a) we obtain $\omega_0 = d(p_1\pi^*\alpha_0) + \sigma$ and $\omega_1 = d(p_1\pi^*\alpha_1) + \sigma$, where α_0, α_1 are germs of smooth contact forms on $\Sigma_2 = \{p_1 = 0\}$ such that $\alpha_0 \wedge \sigma = \alpha_1 \wedge \sigma = 0$.

α_0 is a contact form therefore we can find a coordinate system (x, y, z) on Σ_2 such that $\alpha_0 = dz + xdy$. Let $\sigma = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$, where a, b, c are function-germs on Σ_2 vanishing at 0. From $\alpha_0 \wedge \sigma = 0$

we get $c = -xb$. Thus $I(\sigma) =< a, b, c >=< a, b >$. The 2-form germ σ is closed. It implies that $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - x\frac{\partial b}{\partial z} = 0$. Thus we have

$$\frac{\partial a}{\partial x}(0) + \frac{\partial b}{\partial y}(0) = 0 \tag{5.1}$$

Let $\alpha_1 = fdx + gdy + hdz$, where f, g, h are function-germs on Σ_2. From $\alpha_1 \wedge \sigma = 0$ we obtain the equation

$$af + b(g - xh) = 0 \tag{5.2}$$

and $a(0) = b(0) = 0$.

By assumptions a, b is a regular sequence.

Therefore $f = rb$ and $g - xh = -ra$, where r is a smooth function-germ on Σ_2 at 0.

Thus 1-form germ α_1 has the following form

$$\alpha_1 = rbdx + (xh - ra)dy + hdz. \tag{5.3}$$

Thus $\alpha_1|_0 = h(0)dz$ since $a(0) = b(0) = 0$ and $h(0) \neq 0$, because α_1 is a contact form-germ. It implies that

$$\ker \alpha_0|_0 = \ker \alpha_1|_0. \tag{5.4}$$

By (5.3) we get

$$\alpha_1 \wedge d\alpha_1|_0 = \left((h(0))^2 - h(0)r(0)\left(\frac{\partial a}{\partial x}(0) + \frac{\partial b}{\partial y}(0)\right)\right) dx \wedge dy \wedge dz.$$

By (5.1) we obtain that

$$\alpha_1 \wedge d\alpha_1|_0 = (h(0))^2 dx \wedge dy \wedge dz, \quad \alpha_0 \wedge d\alpha_0|_0 = dx \wedge dy \wedge dz.$$

Since $h(0) \neq 0$ both 3-forms define the same orientation of Σ_2. Therefore from (5.4) we finish the proof by Theorem 3.4 (b). □

Example 5.5. Let ω be a closed 2-form-germ on $\mathbb{R}^4$ in coordinates (p_1, x, y, z) of the following form $d(p_1\alpha) + \sigma$, where

$$\alpha = dz + xdy, \quad \sigma = x(dz + xdy) \wedge (a(x, y, z)dx - b(x)dy),$$

$a(x, y, z) = a_1x + a_2y + a_3z$ and $b(x) = \frac{a_3}{3}x^2 - \frac{a_2}{2}x$.

It is easy to check that, $d\omega = 0$, $\Sigma_2(\omega) = \{p_1 = 0\}$, α is contact form-germ on $\{p_1 = 0\}$, $\omega|_{T\Sigma_2} = \sigma$ and $\alpha \wedge \sigma = 0$.

Let ω_1 be a closed 2-form-germ on $\mathbb{R}^4$ of the following form

$$d(p_1(h(x, y, z)\alpha + r(x, y, z)(a(x, y, z)dx - b(x)dy))) + \sigma,$$

where h and r are $\mathbb{R}$-analytic function-germs on $\{p_1 = 0\}$ and $h(0)r(0) \neq 0$. It is easy to check that $d\omega_1 = 0$, $\Sigma_2(\omega_1) = \{p_1 = 0\}$, $\omega_1|_{T\Sigma_2} = \sigma$ and

$$(h(x, y, z)\alpha + r(x, y, z)(a(x, y, z)dx - b(x)dy)) \wedge \sigma = 0.$$

The 1-form-germ $h(x,y,z)\alpha + r(x,y,z)(a(x,y,z)dx - b(x)dy)$ is a contact form-germ on $\{p_1 = 0\}$ if and only if $h(0)(h(0) - 1/2a_2 r(0)) \neq 0$.

Thus ω and ω_1 are two singular symplectic form-germs with the same restriction σ to the common Martinet hypersurface $\{p_1 = 0\}$. But the canonical orientations of the Martinet hypersurface defined by ω and ω_1 are different if $h(0)(h(0) - 1/2a_2 r(0)) < 0$.

6. The complete set of invariants for singular symplectic forms with singular Martinet hypersurfaces.

In this section we consider singular symplectic forms with singular Martinet hypersurfaces. For any smooth ($\mathbb{K}$-analytic) function f on $\mathbb{K}^{2n}$ there exists closed 2-form ω such that $\Sigma_2(\omega)$ is $f^{-1}(0)$. Such singular symplectic form can be constructed in the following way (see [4])

$$\omega = d(\frac{1}{n!}\int_0^{x_1} f(t, x_2, \cdots, x_{2n})dtdx_2 + \sum_{i=2}^{n} x_{2i-1}dx_{2i}),$$

where $(x_1, \cdots, x_{2n})$ is the coordinate system on $\mathbb{K}^{2n}$. Then $\omega^n = f(x)dx_1 \cdots \wedge dx_{2n}$.

We assume that the Martinet hypersurface is a quasi-homogeneous hypersurface with an isolated singularity. Under these assumptions we can prove that the equivalence class of a singular symplectic form is determine by its restriction to the regular part of the singular Martinet hypersurface and its canonical orientation.

First we recall the notion of quasi-homogeneity and its properties.

Definition 6.1. The germ at 0 of a set $N \subset \mathbb{K}^m$ is called quasi-homogeneous if there exist a local coordinate system $(x_1, \ldots, x_m)$ and positive integers $\lambda_1, \ldots, \lambda_m$ such that the following holds: if a point with coordinates $(x_1, \cdots, x_m)$ belongs to N then for any $t \in [0,1]$ the point with coordinates $(t^{\lambda_1}x_1, \cdots, t^{\lambda_m}x_m)$ also belongs to N.

A function-germ f at 0 on $\mathbb{K}^m$ is quasi-homogeneous if there exist a local coordinate system $(x_1, \ldots, x_m)$ and positive integers $\lambda_1, \ldots, \lambda_m, \delta$ such that $f(t^{\lambda_1}x_1, \cdots, t^{\lambda_m}x_m) = t^{\delta} f(x_1, \ldots, x_m)$ for any $t \in [0,1]$ and any $(x_1, \ldots, x_m)$.

It is obvious that if a function-germ f on $\mathbb{K}^m$ is quasi-homogeneous then $f^{-1}(0)$ is a quasi-homogeneous subset-germ of $\mathbb{K}^m$. The following property of quasi-homogeneous subset-germs is crucial for our study.

Theorem 6.2 ([16] in $\mathbb{C}$-analytic category, [5] in $\mathbb{R}$-analytic and smooth categories)**.** *If N is a quasi-homogeneous subset-germ of $\mathbb{K}^m$ then any closed k-form-germ vanishing at every point of N is a differential of a $(k-1)$-form-germ vanishing at every point of N.*

To prove our result we also need the following division property.

Definition 6.3. A differential 1-form-germ α on $\mathbb{K}^m$ has k-division property if for any differential k-form-germ β such that $\alpha \wedge \beta = 0$ there exists a differential $(k-1)$-form-germ γ such that $\beta = \alpha \wedge \gamma$.

Let $\mathcal{O}$ denotes the ring of $\mathbb{K}$-analytic or smooth function-germs at 0 and let $f \in \mathcal{O}$. We recall the definition of an isolated singularity.

Definition 6.4. A singular hypersurface-germ $\{f = 0\}$ has an isolated singularity at 0 if

$$\dim_{\mathbb{K}} \frac{\mathcal{O}}{< \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_m} >} < \infty.$$

The differential of a function-germ with an isolated singularity has the division property.

Theorem 6.5 ([15])**.** *If $\{f = 0\}$ has an isolated singularity at 0 then df has k-division property for $k = 1, \cdots, m-1$.*

Now we are ready to prove the main result in this section.

Theorem 6.6. *Let ω_0 and ω_1 be germs of smooth ($\mathbb{K}$-analytic) singular symplectic forms on $\mathbb{K}^{2n}$ with a common singular Martinet hypersurface Σ_2 at 0. Let Σ_2 be a quasi-homogeneous hypersurface-germ with an isolated singularity at 0.*

If ω_0 and ω_1 have the same restriction to the regular part of Σ_2 and ω_0, ω_1 define the same canonical orientation of the regular part of Σ_2 then there exists a smooth ($\mathbb{K}$-analytic) diffeomorphism-germ $\Psi : (\mathbb{K}^{2n}, 0) \to (\mathbb{K}^{2n}, 0)$ such that

$$\Psi^* \omega_1 = \omega_0.$$

Proof. We may find a coordinate system such that $\omega_0^n = f\Omega$, where f is a quasi-homogeneous function-germ with an isolated singularity at 0 and Ω is a volume form-germ on $\mathbb{K}^{2n}$. Thus $\omega_1^n = gf\Omega$, where g is a function-germ, such that $g(0) > 0$, because $\Sigma_2 = \Sigma_2(\omega_0) = \Sigma_2(\omega_1)$, ω_0 and ω_1 define the same orientation of the regular part of Σ_2. The singular symplectic form-germs ω_0 and ω_1 have the same restriction to the regular part of Σ_2. Thus there exists a 3-form-germ β such that

$$df \wedge (\omega_1 - \omega_0) = f\beta. \tag{6.1}$$

Multiplying both sides of the above formula by $df \wedge$ we obtain $f df \wedge \beta = 0$. But Σ_2 is nowhere dense thus this implies that $df \wedge \beta = 0$. The hypersurface-germ $\{f = 0\}$ has an isolated singularity at 0, therefore by Theorem 6.5 df has k-division property for $k = 1, \cdots, 2n-1$. Thus we obtain $\beta = df \wedge \gamma$, where γ is a 2-form-germ. From the above formula and (6.1) we obtain $df \wedge (\omega_1 - \omega_0 - f\gamma) = 0$. By 2-division property of df we get that

$$\omega_1 - \omega_0 = f\gamma + df \wedge \delta, \tag{6.2}$$

where δ is a 1 form-germ.

The 2-form-germ $\omega_1 - \omega_0 = f(\gamma - d\delta) + d(f\delta)$ is closed. It implies that the 2-form $f(\gamma - d\delta)$ is closed too and it vanishes at every point of $\Sigma_2 = \{f = 0\}$. Since Σ_2 is quasi-homogeneous by Theorem 6.2 we obtain that there exists a 1 form-germ α such that

$$\omega_1 - \omega_0 = d(f\alpha) \tag{6.3}$$

Now we use Moser's homotopy method ([14]). Let

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + td(f\alpha),$$

for $t \in [0,1]$. We look for germs of diffeomorphisms Φ_t such that

$$\Phi_t^* \omega_t = \omega_0, \text{ for } t \in [0,1], \ \Phi_0 = Id. \tag{6.4}$$

Differentiating the above homotopy equation by t, we obtain

$$d(V_t \rfloor \omega_t) = d(f\alpha),$$

where $V_t = \frac{d}{dt}\Phi_t$. Therefore we have to solve the following equation

$$V_t \rfloor \omega_t = f\alpha. \tag{6.5}$$

First we calculate $\Sigma_2(\omega_t)$. It is easy to see that

$$\omega_1^n = (\omega_0 + d(f\alpha))^n = \omega_0^n + n(fd\alpha + df \wedge \alpha) \wedge \omega_0^{n-1} + f\kappa,$$

where κ is a $2n$-form-germ such that $\kappa|_0 = 0$ (because $df|_0 = 0$). But $\Sigma_2(\omega_0) = \Sigma_2(\omega_1) = \{f = 0\}$. Thus if we restrict both sides of the above formula to $\{f = 0\}$ we obtain that $df \wedge \alpha \wedge \omega_0^{n-1}|_{\{f=0\}} = 0$. Hence there exists a function-germ h such that

$$df \wedge \alpha \wedge \omega_0^{n-1} = hf\Omega. \tag{6.6}$$

But $\omega_1^n = gf\Omega$. Thus we obtain that

$$g(0) = 1 + n\left(\frac{d\alpha \wedge \omega_0^{n-1}}{\Omega}|_0 + h(0)\right). \tag{6.7}$$

We calculate

$$\omega_t^n = (\omega_0 + td(f\alpha))^n = \omega_0^n + n(fd\alpha + df \wedge \alpha) \wedge \omega_0^{n-1}t + f\kappa_t = fg_t\Omega,$$

where κ_t is a $2n$-form-germ such that $\kappa_t|_0 = 0$ for $t \in [0,1]$ and g_t is a function-germ. Thus

$$g_t(0) = 1 + tn\left(\frac{d\alpha \wedge \omega_0^{n-1}}{\Omega}|_0 + h(0)\right).$$

From (6.7) we obtain that $g_t(0) = 1 + t(g(0) - 1)$. But $g(0) > 0$, therefore $g_t(0) > 0$ for $t \in [0,1]$. Thus $\Sigma_2(\omega_t) = \{f = 0\}$ and ω_t define the same orientation of Σ_2 for any t.

Because $\{f = 0\}$ is nowhere dense, equation (6.5) is equivalent to

$$V_t \rfloor \omega_t^n = nf\alpha \wedge \omega_t^{n-1}$$

and $\omega_t^n = fg_t\Omega$. Therefore we have to solve the following equation

$$V_t \rfloor g_t \Omega = n\alpha \wedge \omega_t^{n-1}. \tag{6.8}$$

Now we prove that the right hand side of (6.8) vanishes at 0. It is easy to see that

$$\alpha \wedge \omega_t^{n-1}|_0 = \alpha \wedge \omega_0^{n-1}|_0. \tag{6.9}$$

The function-germ f is quasi-homogeneous. Let E be the Euler vector field for f i.e. $E \rfloor df = f$ and $E|_0 = 0$ (see [7]). From (6.6) we get that

$$df \wedge \alpha \wedge \omega_0^{n-1} = hf\Omega.$$

Thus

$$df \wedge \alpha \wedge \omega_0^{n-1} = h(E \rfloor df)\Omega = df \wedge (hE \rfloor \Omega),$$

because

$$(E \rfloor df)\Omega = df \wedge E \rfloor \Omega.$$

Hence

$$df \wedge (\alpha \wedge \omega_0^{n-1} - hE \rfloor \Omega) = 0.$$

By $(2n-1)$-division property of df we get that

$$\alpha \wedge \omega_0^{n-1} - hE \rfloor \Omega = df \wedge \theta,$$

where θ is a $(2n-2)$-form-germ. From (6.9) we get

$$\alpha \wedge \omega_t^{n-1}|_0 = 0,$$

because $E|_0 = 0$ and $df|_0 = 0$. Hence we can find a smooth solution V_t of (6.8) such that $V_t|_0 = 0$. Therefore there exit germs of diffeomorphisms Φ_t, which satisfy (6.4). For $t = 1$ we have $\Phi_1^* \omega_1 = \omega_0$. □

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