VOLUME-PRESERVING DIFFEOMORPHISMS ON VARIETIES AND $\mathcal{A}_\Omega$-EQUIVALENCE OF MAPS

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ABSTRACT. We describe a cohomological criterion for a pair of diffeomorphic variety-germs to be volume-preserving diffeomorphic. And we show that for the class of varieties that are quasi-homogeneous with respect to a smooth submanifold this cohomology vanishes, so that volume-preserving diffeomorphic is equivalent to diffeomorphic (over $\mathbb{C}$) or orientation-preserving diffeomorphic (over $\mathbb{R}$). Likewise, we give a criterion for a pair of left-right equivalent map-germs to be $\mathcal{A}_\Omega$-equivalent (i.e. left-right equivalent by volume-preserving diffeomorphisms on the left). And we show that for the class of weakly quasi-homogeneous map-germs each left-right orbit corresponds to one $\mathcal{A}_\Omega$-orbit (over $\mathbb{C}$) or to one or two $\mathcal{A}_\Omega$-orbits (over $\mathbb{R}$).

1. Introduction

We study the local classification of singular varieties and maps under the action of the group of volume-preserving diffeomorphisms, which is not a geometric subgroup in the sense of Damon [11] (the tangent space fails to have the structure of a finitely generated module). The definition of a geometric subgroup $\mathcal{G}$ of the Mather groups $\mathcal{A}$ and $\mathcal{K}$ collects the properties that are required in the proofs of the finite determinacy and versal unfolding theorems for any such $\mathcal{G}$, the central theorems for any classification. For non-geometric subgroups of $\mathcal{A}$ and $\mathcal{K}$ moduli (and even functional moduli) often appear rather quickly already in low codimension (even in codimension 0), see e.g. [14]. On the other hand, Martinet wrote 30 years ago in his book (see p. 50 of the English translation [29]) on the classification problem in unimodular geometry – i.e. the classification of map-germs under the subgroup $\mathcal{A}_\Omega$ of $\mathcal{A}$ in which the target diffeomorphisms are volume preserving – that the groups involved "are big enough that there is still some hope of finding a reasonable classification theorem". It turns out that Martinet was right – the results of the present paper imply, for example, that the classifications of stable maps for $\mathcal{A}_\Omega$ and $\mathcal{A}$ agree (at least over $\mathbb{C}$). Apart from $\mathcal{A}_\Omega$-equivalence of map-germs, we will study volume-preserving diffeomorphisms on germs of varieties $V \subset (\mathbb{K}^n, 0)$.

The following earlier work is related to our results on volume-preserving diffeomorphisms on germs of varieties. In [39], [10] the local classification problem for functions under the action of volume-preserving diffeomorphisms was first considered (leading to the so-called isochore Morse Lemma), and more recently isochore versal deformations were studied in [9] and [21]. The results of [39], [10] were generalized in another direction in [16], [17] and [18]. The starting point of the present paper are the articles [2], [4], [38] and [23]. In [2] powers of $\mathbb{C}$-analytic

1991 *Mathematics Subject Classification.* 58C27, 58A10, 14F40.

Key words and phrases. Singularities of map-germs, Unimodular geometry, Volume-preserving (isochore) diffeomorphisms, de Rham complex, Normal forms.

volume forms were studied. In particular V. I. Arnol'd proved that two germs of quasi-homogeneous $\mathbb{C}$-analytic hypersurfaces with isolated singularity are volume-preserving diffeomorphic if and only if they are diffeomorphic. The generalization of this result was obtained in [38] (see also [25],[26],[27],[28]). In this paper A. N. Varchenko proved that the dimension of the moduli space of germs of $\mathbb{C}$-analytic volume forms under the action of diffeomorphism-germs preserving a fixed hypersurface with an isolated singularity is equal to the difference of the Milnor number and the Tjurina number of this hypersurface. (By a result of K. Saito [37] this difference is zero if and only if the hypersurface is quasi-homogeneous.)

And the results on $\mathcal{A}_\Omega$-equivalence of map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ in the present paper are related to the following earlier works. In [4] V. I. Arnol'd introduced the local symplectic algebra and classified singular parameterized curves of the type A_{2k} on a symplectic manifold. G. Ishikawa and S. Janeczko studied the local classification problem for parameterized curves on a 2-dimensional symplectic manifold. (In dimension 2 a symplectic form is a volume form.) All simple germs in this classification are quasi-homogeneous and the orbit of an $\mathcal{A}$-simple $\mathbb{C}$-analytic quasi-homogeneous germ under the action of $\mathbb{C}$-analytic symplectomorphisms is the same as the $\mathcal{A}$-orbit of this germ. As a consequence of our results on $\mathcal{A}_\Omega$-equivalence of map-germs we obtain the following generalization of this result on parameterized curve-germs in symplectic 2-manifolds: an $\mathcal{A}$-simple map-germ from an n-manifold into a symplectic 2-manifold is quasi-homogeneous if and only if its $\mathcal{A}$-orbit and its $\mathcal{A}_\Omega$-orbit coincide (in target dimension 2 the volume preserving diffeomorphisms are symplectomorphisms).

Here is an outline of the contents of the present paper. In Section 2 we describe the main equivalence relations for germs of volume forms and map-germs and we summarize our main results.

In Section 3 we define the notion of a weak algebraic restriction of the germ of a k-form to a germ of a singular variety V, which is a modification of the notion of algebraic restriction defined by M. Zhitomirskii ([41], see also [12]). The k-forms with zero weak algebraic restriction to V form a subcomplex $\mathcal{W}_0^*(V)$ of the de Rham complex on $\mathbb{K}^n$, and we will see that $H^n(\mathcal{W}_0^*(V))$ vanishes for germs V that are quasi-homogeneous with respect to a smooth submanifold (as defined in [13]: quasi-homogeneity with respect to a smooth submanifold may be understood as a quasi-homogeneity with positive and zero weights).

In Section 4 we use the cohomology of the complex $\mathcal{W}_0^*(V)$ introduced in the previous section to present necessary and sufficient conditions for a pair of diffeomorphic variety-germs to be volume-preserving diffeomorphic ($\mathcal{R}_\Omega$-equivalent for some given volume form Ω) and for a pair of volume forms to be $\mathcal{R}_V$-equivalent (i.e. to be related by pull-back by a diffeomorphism preserving a given variety-germ V). Furthermore, we show that $\dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V))$ is the dimension of the $\mathcal{R}_V$-moduli space and that $H^n(\mathcal{W}_0^*(V))$ vanishes for those V that are quasi-homogeneous with respect to a smooth submanifold.

In Sections 5 and 6 we study $\mathcal{A}_\Omega$- and $\mathcal{A}_f$-equivalence of map-germs and volume forms (in the same way as before $\mathcal{R}_\Omega$- and $\mathcal{R}_V$-equivalence of varieties and volume forms): for $\mathcal{A}_\Omega$-equivalence we fix a volume form Ω on the target and require that the target (left) diffeomorphisms in a left-right equivalence of a pair of map-germs preserve Ω, whereas for $\mathcal{A}_f$ equivalence the map-germ f is given and a pair of volume forms on the target of f is related by pull-back by some Φ, where (Φ, Ψ)

lies in the isotropy subgroup of f in the group $\mathcal{A}$ of left-right equivalences. We present necessary and sufficient conditions for $\mathcal{A}_\Omega$-equivalence of map-germs and $\mathcal{A}_f$-equivalence of volume forms, and we give formulas for the dimensions of the $\mathcal{A}_f$- and $\mathcal{A}_\Omega$-moduli spaces. Furthermore, we show that for the class of weakly quasi-homogeneous map-germs each left-right orbit corresponds to one $\mathcal{A}_\Omega$-orbit (over $\mathbb{C}$) or to one or two $\mathcal{A}_\Omega$-orbits (over $\mathbb{R}$). The study of $\mathcal{A}_\Omega$- and $\mathcal{A}_f$-equivalence in Section 5 is in the style of that of $\mathcal{R}_\Omega$- and $\mathcal{R}_V$-equivalence in Section 4. In Section 6 we study the $\mathcal{A}_\Omega$ classification of map-germs using infinitesimal methods (in the style of Mather, looking at the "tangent spaces" of $\mathcal{A}_\Omega$-orbits). Our results give the classification of $\mathcal{A}_\Omega$-orbits of maps $\mathbb{C}^n \to \mathbb{C}^p$ from the corresponding lists of $\mathcal{A}$-simple germs f as follows: f is $\mathcal{A}_\Omega$-simple if and only if it is weakly quasi-homogeneous and not $\mathcal{A}$-adjacent to any non-weakly quasi-homogeneous germ. In dimensions (n,p) with $p < 2n$ it is sufficient to work with the smaller class of quasi-homogeneous map-germs, but for $p \geq 2n$ the wider class of weakly quasi-homogeneous germs is required for obtaining the $\mathcal{A}_\Omega$-simple orbits – all our examples of $\mathcal{A}$-simple weakly quasi-homogeneous map-germs, which are not quasi-homogeneous, occur for $p \geq 2n$.

Acknowledgements. The authors wish to express their thanks to Goo Ishikawa, Stanislaw Janeczko, Andrew du Plessis, Maria Ruas and Michail Zhitomirskii for many helpful conversations and remarks during the writing of this paper. These results were obtained while the authors were guests at the Institute of Mathematics, Polish Academy of Sciences (IMPAN) and the Banach Centre at IMPAN, respectively, where WD was supported by IMPAN and JHR by the EU programme Centre of Excellence - IMPAN-Banach Centre, ICA1-CT-2000-70024.

2. Summary of the main equivalence relations and results

Our results on the equivalence of varieties V_0 and V_1 under volume preserving diffeomorphisms (i.e. under $\mathcal{R}_\Omega$ equivalence, where Ω is a fixed volume form on the ambient space) will be deduced from corresponding statements about the equivalence of pairs of volume forms Ω_0 and Ω_1 under pull-back by diffeomorphisms $\mathcal{R}_V$ preserving a fixed variety V. Likewise, our results about $\mathcal{A}_\Omega$ equivalence of map-germs $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ (where Ω is a fixed volume form on $\mathbb{K}^p$) will follow from corresponding statements about what we call $\mathcal{A}_f$ equivalence (with $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ fixed) of volume forms Ω_0 and Ω_1 on $\mathbb{K}^p$ below.

Convention: throughout this paper we will be considering germs of maps, coordinate changes, differential forms, vector fields etc. at the origin in $\mathbb{K}^n$ or $\mathbb{K}^p$, and all objects are real-analytic or smooth (C^∞) when $\mathbb{K} = \mathbb{R}$ and complex-analytic when $\mathbb{K} = \mathbb{C}$.

(i) $\mathcal{R}_V$ **and** $\mathcal{IR}_V$ **equivalence**. Given a germ of a variety $V \subset (\mathbb{K}^n, 0)$ and a pair of volume forms Ω_i, $i = 0, 1$ we have:

$$\Omega_0 \sim_{\mathcal{R}_V} \Omega_1 \iff \exists \Phi \in \mathrm{Diff}(\mathbb{K}^n, 0) \text{ such that } \Phi(V) = V,\ \Phi^*\Omega_1 = \Omega_0.$$

And if, in the above situation, Ω_0 and Ω_1 can be joined by a smooth path contained in a single $\mathcal{R}_V$-orbit then we write $\Omega_0 \sim_{\mathcal{IR}_V} \Omega_1$

(ii) $\mathcal{R}_\Omega$ **equivalence**. Given a volume form Ω on $\mathbb{K}^n$ and a pair of germs of varieties $V_i \subset \mathbb{K}^n$, $i = 0, 1$ we have:

$$V_0 \sim_{\mathcal{R}_\Omega} V_1 \iff \exists \Phi \in \mathrm{Diff}(\mathbb{K}^n, 0) \text{ such that } \Phi(V_0) = V_1,\ \Phi^*\Omega = \Omega.$$

Notice that we obtain an equivalence $V_0 \sim_{\mathcal{R}_\Omega} V_1$ by first applying an arbitrary diffeomorphism Φ with $\Phi(V_0) = V_1$ that changes the volume form to $\Omega_1 = \Phi^*\Omega$, and then mapping Ω_1 back to Ω by pulling back by some $\Psi \in \mathcal{R}_{V_0}$.

For map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ we consider the following equivalences.

(iii) $\mathcal{A}_f$ **and** $\mathcal{IA}_f$ **equivalence**. For the isotropy subgroup $\mathrm{Iso}(f) := \{(\Phi,\Psi) \in \mathcal{A} : \Phi \circ f \circ \Psi^{-1} = f\}$ of a map-germ f and a pair of volume forms Ω_i, $i = 0, 1$, in $\mathbb{K}^p$ we have:

$$\Omega_0 \sim_{\mathcal{A}_f} \Omega_1 \iff \exists(\Phi,\Psi) \in \mathrm{Iso}(f) \text{ such that } \Phi^*\Omega_1 = \Omega_0.$$

And again if, in the above situation, Ω_0 and Ω_1 can be joined by a smooth path contained in a single $\mathcal{A}_f$-orbit then we write $\Omega_0 \sim_{\mathcal{IA}_f} \Omega_1$

(iv) $\mathcal{A}_\Omega$ **equivalence**. Given a volume form Ω on the target $\mathbb{K}^p$ and a pair of map-germs f, g then we have:

$$f \sim_{\mathcal{A}_\Omega} g \iff \exists(\Phi,\Psi) \in \mathcal{A} \text{ such that } \Phi^*\Omega = \Omega,\ f = \Phi \circ g \circ \Psi^{-1}.$$

We can obtain an $\mathcal{A}_\Omega$-equivalence of f and g by first applying an arbitrary $\mathcal{A}$-equivalence mapping g to f and Ω to some $\Omega_1 = (\Phi^{-1})^*\Omega$ and then pulling-back Ω to Ω_1 using $\mathcal{A}_f$-equivalence, as in the diagram below:

$$\begin{tikzcd}
(\mathbb{K}^n,0) \arrow[r, "g"] \arrow[d, "\Psi"] & (\mathbb{K}^p,\Omega,0) \arrow[d, "\Phi"] \\
(\mathbb{K}^n,0) \arrow[r, "f"] \arrow[d, "\tilde\Psi"] & (\mathbb{K}^p,\Omega_1,0) \arrow[d, "\tilde\Phi"] \\
(\mathbb{K}^n,0) \arrow[r, "f"] & (\mathbb{K}^p,\Omega,0)
\end{tikzcd}$$

In fact, given f, g and Ω as above the following statements are equivalent: (i) f and g are $\mathcal{A}_\Omega$-equivalent and (ii) there exist $(\Phi,\Psi) \in \mathcal{A}$ and $(\tilde\Phi,\tilde\Psi) \in \mathrm{Iso}(f)$ such that the above diagram commutes.

The following theorem summarizes the main results on $\mathcal{R}_V$ and $\mathcal{R}_\Omega$ equivalence (it combines the statements of Theorems 4.3 and **??** and Proposition 4.5 below). Given a germ of a variety $V \subset (\mathbb{K}^n, 0)$, we say that a k-form has zero weak algebraic restriction to V if it is equal to $d\alpha+\beta$, where α is a $(k-1)$-form and β a k-form such that $df \wedge \alpha|_V = 0$ and $df \wedge \beta|_V = 0$ for any $f \in I(V)$. And we denote by $[\Omega]_V$ the class of Ω in the cohomology group $H^n(\mathcal{W}_0^*(V))$, where $\mathcal{W}_0^*(V))$ is the subcomplex of the de Rham complex of differential forms with zero weak algebraic restriction to V (see Section 3 below for precise definitions). Finally let $\mathrm{div}(Derlog(V)) \subset C_n$ be the $\mathbb{K}$-subspace of C_n (C_n the local ring of smooth or analytic function-germs on $(\mathbb{K}^n, 0)$) consisting of the divergences of the logarithmic vector fields on V, then we have the following

Theorem 2.1. *For a germ of a variety $V \subset (\mathbb{K}^n, 0)$ and of volume forms $\Omega, \Omega_0, \Omega_1$ we have the following.*

(i) Given V: $\Omega_0 \sim_{\mathcal{IR}_V} \Omega_1 \iff [\Omega_0]_V = [\Omega_1]_V$.

(ii) Fixing V, the number of $\mathcal{R}_V$-moduli of germs of volume forms on $(\mathbb{K}^n, 0)$ is equal to

$$\dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V)) = \dim_{\mathbb{K}} C_n/\mathrm{div}(Derlog(V)).$$

The next result summarizes the statements in Theorems 4.4 and 4.7 (varieties V that are quasi-homogeneous with respect to a smooth submanifold were introduced in [13], in Section 3 we will see that for such V we have $H^n(\mathcal{W}_0^*(V)) = 0$).

Theorem 2.2. *Consider germs of varieties $V, V_1 \subset (\mathbb{K}^n, 0)$ and of volume forms $\Omega, \Omega_0, \Omega_1$. If V is quasi-homogeneous with respect to a smooth submanifold then the following statements hold.*

(i) $\Omega_0 \sim_{\mathcal{R}_V} \Omega_1$, for any pair Ω_0, Ω_1 (over $\mathbb{K} = \mathbb{C}$) or for any pair Ω_0, Ω_1 such that $\Omega_0|_0, \Omega_1|_0$ define the same orientation (over $\mathbb{K} = \mathbb{R}$).

(ii) Fixing Ω: $V_1 \sim_{\mathcal{R}_\Omega} V \iff \Phi(V) = V_1$ for some germ, Φ, of a diffeomorphism (over $\mathbb{K} = \mathbb{C}$) or an orientation-preserving diffeomorphism (over $\mathbb{K} = \mathbb{R}$).

For map-germs f the following theorem summarizes the main results on $\mathcal{A}_f$ and $\mathcal{A}_\Omega$ equivalence (it combines the results of Theorems 5.7 and **??**, Corollary 5.8 and Proposition 6.7 below). Here $Lift(f)$ denotes the C_p-module (C_p local ring of functions on the target of f) of vector fields Y liftable over f, $Lift_0(f)$ the submodule of $Lift(f)$ consisting of those Y that lift to source vector fields vanishing at 0 and let $\mathrm{div}(Lift_0(f)) := \{\mathrm{div}(Y) : Y \in Lift_0(f)\} \subset C_p$. And $[\Omega]_f$ denotes the class of the volume form Ω under the following equivalence ($divLift_0(f)$-equivalence see Def. 5.5 below): Ω_0 and Ω_1 are equivalent if $\Omega_1 - \Omega_0 = \mathrm{div}(Y)\Omega$ for some $Y \in Lift_0(f)$ and some (or, in fact, any) volume form Ω. Then we have the following

Theorem 2.3. *For a map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ and germs of volume forms $\Omega, \Omega_0, \Omega_1$ in $(\mathbb{K}^p, 0)$ we have the following.*

(i) Given f: $\Omega_0 \sim_{\mathcal{IA}_f} \Omega_1 \iff [\Omega_0]_f = [\Omega_1]_f$.

(ii) Fixing f, the number of $\mathcal{A}_f$-moduli of germs of volume forms on $(\mathbb{K}^p, 0)$ is equal to

$$\dim_{\mathbb{K}} C_p/\mathrm{div}(Lift_0(f)) = \dim_{\mathbb{K}} T\mathcal{A}\cdot f/T\mathcal{A}_\Omega \cdot f,$$

where in the second term Ω is some fixed volume form and $T\mathcal{A}_\Omega \cdot f$ denotes the tangent space to the $\mathcal{A}_\Omega$-orbit through f at f.

A map-germ f is weakly quasi-homogeneous if it is quasi-homogeneous for integer weights, non-negative weighted degrees and positive total degree (for some choice of coordinates in source and target). The following is a summary of Theorems 5.14 and 5.17.

Theorem 2.4. *Consider map-germs $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ and germs of volume forms $\Omega, \Omega_0, \Omega_1$ in $(\mathbb{K}^p, 0)$. If f is weakly quasi-homogeneous then the following statements hold.*

(i) $\Omega_0 \sim_{\mathcal{A}_f} \Omega_1$, for any pair Ω_0, Ω_1 (over $\mathbb{K} = \mathbb{C}$) or for any pair Ω_0, Ω_1 such that $\Omega_0|_0, \Omega_1|_0$ define the same orientation (over $\mathbb{K} = \mathbb{R}$).

(ii) Fixing Ω: $g \sim_{\mathcal{A}_\Omega} f \iff g \sim_{\mathcal{A}} f$, where the target diffeomorphisms (in the elements of $\mathcal{A}$) are arbitrary (over $\mathbb{K} = \mathbb{C}$) or are orientation-preserving (over $\mathbb{K} = \mathbb{R}$).

Remark 2.5. (1) One could deduce Theorems 2.3 and 2.4 on map-germs f from Theorems 2.1 and 2.2 by considering the germ of a variety $\Delta(f) \subset (\mathbb{K}^p, 0)$, which denotes the discriminant of f (for $n \geq p$) or the image (for $n < p$), and using the results in [32], [33], [8] on critical normalizations. However, we shall give direct, self-contained proofs of these theorems that apply to a larger class of map-germs f than the ones considered in [32], [33], [8]. Also notice that one could try to obtain Theorem 2.4 from its "infinitesimal version" in Proposition 6.1, which says that for weakly quasi-homogeneous maps f we have the equality $T\mathcal{A}\cdot f = T\mathcal{A}_\Omega \cdot f$ of tangent spaces, using arguments similar to the ones in [19].

(2) Our results on mono-germs f can be generalized to multi-germs, this just complicates the notation without really changing the proofs.

3. Cohomology of forms with zero weak algebraic restriction to a singular variety

Let V be a germ of a variety on $\mathbb{K}^n$ at 0. Let $I(V)$ be the ideal of smooth ($\mathbb{K}$-analytic) function-germs vanishing on V.

Definition 3.1. We say that germs of k-forms ω_0 and ω_1 have the same *weak algebraic restriction to* V if there exists a germ of a $(k-1)$-form α and a germ of a k-form β such that $\omega_1 - \omega_0 = d\alpha + \beta$ and $df \wedge \alpha|_V = 0$, $df \wedge \beta|_V = 0$ for any $f \in I(V)$.

We say that the germ of a k-form ω has *zero weak algebraic restriction to* V if ω and a germ of the zero k-form have the same weak algebraic restriction to V.

This definition is a modification of the definition of an algebraic restriction to a singular set V introduced by M. Zhitomirskii for 1-forms in [41] and generalized to arbitrary k-forms in [12]. Germs of k-forms ω_0 and ω_1 have the same *algebraic restriction to* V if there exists a germ of a $(k-1)$-form α and a germ of a k-form β such that $\omega_1 - \omega_0 = d\alpha + \beta$ and $\alpha|_V = 0$, $\beta|_V = 0$.

It is obvious that if ω_0 and ω_1 have the same algebraic restriction to V then they have the same weak algebraic restriction to V. But it is easy to see that the weak algebraic restriction of any n-form on $\mathbb{K}^n$ to a variety $V \subset \mathbb{K}^n$ is zero (see Proposition 3.5) and that this is not true for algebraic restrictions of volume forms to singular hypersurfaces.

If V is a germ of a smooth submanifold then ω_0 and ω_1 have the same algebraic restriction to V if and only if ω_0 and ω_1 have the same pullback to V i.e. they have the same geometric restriction to V (see [12]). The same is true for the weak algebraic restriction to a smooth submanifold. On smooth submanifolds the notion of algebraic restriction, weak algebraic restriction and geometric restriction coincide. This is not true on singular varieties (see Example 3.4)

Proposition 3.2. *If V is a smooth submanifold then ω_0 and ω_1 have the same weak algebraic restriction to V if and only if ω_0 and ω_1 have the same pullback to V.*

Proof. It is enough to show that a germ of a k-form ω has zero weak algebraic restriction if and only if the pullback of ω to V vanishes. We may assume that $k < n$. Let V be a a germ of a p-dimensional submanifold. Then there exists a local coordinate system $(x_1, \cdots, x_n)$ on $\mathbb{K}^n$ such that $V = \{x_1 = \cdots = x_{n-p} = 0\}$. The pullback of ω to V vanishes if and only if $\omega = \sum_{i=1}^{n-p} x_i\alpha_i + dx_i \wedge \beta_i$, where α_i is a germ of a k-form and β_i is a germ of a $(k-1)$-form for $i = 1, \cdots, n-p$. But then

$$\omega = \sum_{i=1}^{n-p} x_i(\alpha_i - d\beta_i) + d(\sum_{i=1}^{n-p} x_i\beta_i).$$

Thus ω has zero weak algebraic restriction to V.

Now assume that ω has zero weak algebraic restriction to V. Thus

$$\omega = \alpha + d\beta,$$

where α is a germ of a k-form and β is a germ of $(k-1)$-form such that $df\wedge\alpha|_V=0$, $df\wedge\beta|_V=0$ for any $f\in I(V)$.

We may write α in the following way

$$\alpha=\sum_{i=1}^{n-p}x_i\gamma_i+dx_i\wedge\delta_i+\pi_V^*\sigma,$$

where γ_i is a germ of a k-form on $\mathbb{K}^n$, δ_i is a germ of a $(k-1)$-form on $\mathbb{K}^n$ $(i=1,\cdots,n-p)$, σ a germ of a k-form on V and $\pi_V:\mathbb{K}^n\ni(x_1,\cdots,x_n)\mapsto(x_{n-p+1},\cdots,x_n)\in V$ is the germ of the standard projection on V. But $dx_i\wedge\alpha|_V=0$. Thus we obtain that $\sigma=0$ (for $k<n$). Therefore the pullback of α to V vanishes. In the same way we can show that the pullback of β to V vanishes. $\square$

If V is a hypersurface then we have the following proposition.

Proposition 3.3. *If $I(V)=<f>$, where f is a smooth ($\mathbb{K}$-analytic) function-germ, then a k-form ω has zero weak algebraic restriction to V if and only if the pullback of ω to the regular part of V vanishes.*

Proof. The pullback of ω to the regular part of V vanishes if and only if $df\wedge\omega|_V=0$. But this means that ω has zero weak algebraic restriction to V. $\square$

Let V be a germ of a variety on $\mathbb{K}^n$, if ω has weak zero algebraic restriction to V then the pullback of ω to the regular part of V vanishes. If V is the germ of a singular curve in $\mathbb{K}^3$ then the pullback of the germ of any 2-form to the regular part of V vanishes. But it is easy to see that there are 2-forms which do not have zero weak algebraic restriction to V.

Example 3.4. Let $V=\{x_1^2-x_2^2-x_3^2=x_2x_3=0\}$ be a germ of a variety on $\mathbb{K}^3$ at 0. Then pullback of any 2-form to the regular part of V vanishes, because the regular part of V is 1-dimensional, but the germ of a 2-form $dx_1\wedge dx_2$ does not have zero weak algebraic restriction to V.

The germ of the form $\omega=x_1dx_2\wedge dx_3+x_2dx_3\wedge dx_1+x_3dx_1\wedge dx_2$ has zero weak algebraic restriction to V, because $d(x_1^2-x_2^2-x_3^2)\wedge\omega=2(x_1^2-x_2^2-x_3^2)dx_1\wedge dx_2\wedge dx_3$ and $d(x_2x_3)\wedge\omega=2x_2x_3dx_1\wedge dx_2\wedge dx_3$. But ω does not have zero algebraic restriction to V (see [12]).

We denote by $\mathcal{W}_0^k(V)$ the set of k-forms with zero weak algebraic restriction to V. One can show that

Proposition 3.5. *Let ω be a germ of an n-form on $\mathbb{K}^n$. If $I(V)\neq\{0\}$ then $\omega\in\mathcal{W}_0^n(V)$*

Proof. If $f\neq 0$ then $df\wedge\omega$ is the germ of an $(n+1)$-form on $\mathbb{K}^n$. Thus $df\wedge\omega=0$ for any $f\in I(V)$. $\square$

It is easy to see that $\mathcal{W}_0^*(V)$ is a differential subcomplex of the complex of differential forms on $\mathbb{K}^n$. Thus we can define the cohomology groups of this subcomplex

$$H^k(\mathcal{W}_0^*(V))=\frac{\{\omega\in\mathcal{W}_0^k(V):\ d\omega=0\}}{\{d\alpha:\ \alpha\in\mathcal{W}_0^{k-1}(V)\}}$$

By Proposition 3.3 if H is a hypersurface then $H^*(\mathcal{W}_0^*(H))$ is the cohomology of the subcomplex of forms with vanishing pullback to the regular part of H, which we denote by $H^*_{Giv}(H)$. These were considered in [15] and [22] in the $\mathbb{C}$-analytic category (see also [13]). In [22] (see also [15]) it was proved that if V is the germ of a quasi-homogeneous $\mathbb{C}$-analytic variety then $H^k_{Giv}(V) = \{0\}$ for $k > 0$. This result was generalized in [13].

We are interested in $H^n(\mathcal{W}^*_0(V))$. We define a class of varieties on $\mathbb{K}^n$ for which this cohomology group vanishes. These are the varieties which are quasi-homogeneous with respect to a smooth submanifold, as defined in ([13]).

Definition 3.6. A germ of a variety $V \subset \mathbb{K}^n$ is called *quasi-homogeneous with respect to a smooth submanifold* S if there exists a local coordinate system

$$(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$$

and **positive** numbers

$$w_1, \ldots, w_k$$

(called weights) such that S is given by equations $x_1 = 0, \ldots, x_k = 0$ and, for all t, the map

$$F_t : \ (x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \mapsto (t^{w_1} x_1, \ldots, t^{w_k} x_k, y_1, \ldots, y_{n-k})$$

sends any point $p \in V$ to a point $F_t(p) \in V$, provided that p and $F_t(p)$ are sufficiently close to 0.

This is a generalization of the classical quasi-homogeneity. Classical quasi-homogeneity is quasi-homogeneity with respect to $S = \{0\}$. The quasi-homogeneity with respect to a smooth submanifold is a classical quasi-homogeneity with some of the weights allowed to be 0.

Example 3.7. Let

$$N = \{(x_1, x_2, y) \in \mathbb{R}^3 : (x_1^2 - x_2^2)^2 + y x_1^2 x_2^2 = 0\}.$$

The set N is quasi-homogeneous with respect to the curve $S : \{x_1 = x_2 = 0\}$ with weights $(1, 1)$: if $(x_1, x_2, y) \in N$ then $(tx_1, tx_2, y) \in N$. In ([13]) it was shown that N is not quasi-homogeneous (in any coordinate system) in the classical sense.

Example 3.8. Let V be a germ of a subset of a smooth hypersurface H. Then V is quasi-homogeneous with respect to H. Indeed, there exists a coordinate system $(x_1, x_2, \cdots, x_n)$ on $\mathbb{K}^n$ such that $H = \{x_1 = 0\}$. Thus $F_t(x_1, x_2, \cdots, x_n) = (tx_1, x_2, \cdots, x_n)$ maps V to V.

The vector field

$$E_w = \sum_{i=1}^{k} w_i x_i \frac{\partial}{\partial x_i}.$$

is called the Euler vector field for a coordinate system $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ with weights $w = (w_1, \ldots, w_k)$. If V is quasi-homogeneous with respect to a submanifold $\{x_1 = \cdots = x_k = 0\}$ with positive weights $(w_1, \cdots, w_k)$ then E_w is tangent to V. In fact it was proved in ([13]) that

Theorem 3.9. *Let S be a smooth submanifold of $\mathbb{R}^n$. Let $N = \{H_1 = \cdots = H_p = 0\}$, where $H_1, \ldots, H_p$ generate the ideal of smooth (or $\mathbb{K}$-analytic) function-germs vanishing on N, and suppose that the set of non-singular points of N (the points near which N has the structure of a smooth submanifold of $\mathbb{R}^n$) is dense in N.*

Then N is quasi-homogeneous with respect to S if and only if there exists a vector field which is tangent to N, vanishes at any point $x \in S$, and whose eigenvalues at

$x \in S$ corresponding to directions transversal to S do not depend on x and are real positive numbers.

Proposition 3.10. *Let V be the germ of a variety on $\mathbb{K}^n$. If V is quasi-homogeneous with respect to a smooth k-dimensional submanifold ($k < n$) then $H^n(\mathcal{W}_0^*(V)) = 0$.*

The proof the above proposition is based on the following lemma.

Lemma 3.11. *Let E_w be a germ of the Euler vector field for a coordinate system $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ with positive weights $w = (w_1, \ldots, w_k)$ and let Ω_0 be the germ of the volume-form $dx_1 \wedge \ldots \wedge dx_k \wedge dy_1 \wedge \ldots \wedge dy_{n-k}$. If ω is the germ of a smooth ($\mathbb{K}$-analytic) n-form on $\mathbb{K}^n$ then there exists a smooth ($\mathbb{K}$-analytic) function-germ g on $\mathbb{K}^n$ such that $\omega = d(g(E_w \rfloor \Omega_0))$.*

Proof. Let $G_t(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) = (e^{w_1 t}x_1, \ldots, e^{w_k t}x_k, y_1, \ldots, y_{n-k})$ for $t \leq 0$ and $x = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in \mathbb{K}^n$. It is easy to see that

$$(G_t)' := \frac{d}{dt}G_t = E_w \circ G_t,\ G_0 = Id_{\mathbb{K}^n},\ \lim_{t \to -\infty} G_t(x, y) = (0, y)$$

for any $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in \mathbb{K}^n$. Thus

$$\omega = G_0^*\omega - \lim_{t \to -\infty} G_t^*\omega = \int_{-\infty}^{0} (G_t^*\omega)' dt. \tag{3.1}$$

But $\omega = f\Omega_0$, where f is a smooth ($\mathbb{K}$-analytic) function-germ and

$$(G_t^*\omega)' = G_t^* L_{E_w}\omega = G_t^* d(E_w \rfloor \omega) = d(G_t^*(E_w \rfloor \omega)).$$

Thus

$$(G_t^*\omega)' = d(G_t^*(E_w \rfloor f\Omega_0)) = d((f \circ G_t)G_t^*(E_w \rfloor \Omega_0)).$$

It is easy to check by direct calculation that $G_t^*(E_w \rfloor \Omega_0) = e^{t\sum_{i=1}^k w_i}(E_w \rfloor \Omega_0)$. Therefore $(G_t^*\omega)' = d((f \circ G_t)e^{t\sum_{i=1}^k w_i}(E_w \rfloor \Omega_0))$. Combining this with (3.1) we obtain

$$\omega = d(\int_{-\infty}^{0} ((f \circ G_t)e^{t\sum_{i=1}^k w_i})dt(E_w \rfloor \Omega_0)) = d(g(E_w \rfloor \Omega_0)),$$

where the function-germ g on $\mathbb{K}^n$ is defined in the following way:

$$g(x, y) = \int_{-\infty}^{0} (e^{t\sum_{i=1}^k w_i}(f(G_t(x, y)))dt.$$

It is easy to see that g is smooth ($\mathbb{K}$-analytic), because

$$\int_{-\infty}^{0} (e^{t\sum_{i=1}^k w_i}(f(G_t(x, y)))dt = \int_{0}^{1} (s^{\alpha} f(F_s(x, y))ds$$

where $\alpha = (\sum_{i=1}^k w_i) - 1$ and

$$F_s(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) = (s^{w_1}x_1, \ldots, s^{w_k}x_k, y_1, \ldots, y_{n-k})$$

for any $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ and $s \in [0, 1]$. Multiplying weights by a sufficiently large constant we may assume that $\alpha > 1$. □

Proof of Proposition 3.10. Let ω be a a germ of a n-form on $\mathbb{K}^n$. It is obvious that $df \wedge \omega = 0$ for any $f \in I(V)$. By Lemma 3.11 there exists a n-form β such that $\omega = d(E_w \rfloor \beta)$ where E_w is the Euler vector field for V. But for any $f \in I(V)$ $df \wedge (E_w \rfloor \beta) = (E_w \rfloor df)\beta$ and $E_w \rfloor df \in I(V)$. Thus $df \wedge (E_w \rfloor \beta)|_V = 0$. □

Proposition 3.12. *Let V be a germ of a variety on $\mathbb{K}^n$. If there exists a germ of a vector field Y tangent to V such that $Y|_0 \neq 0$ then $H^n(\mathcal{W}_0^*(V)) = 0$.*

Proof. There exists a coordinate system $(x_1, x_2, \cdots, x_n)$ on $\mathbb{K}^n$ such that $Y = \frac{\partial}{\partial x_1}$. If ω is a germ of an n-form then

$$\omega = g dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = d\Big(\int_0^{x_1} g(t, x_2, \cdots, x_n) dt \frac{\partial}{\partial x_1} \rfloor dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\Big),$$

where g is a function-germ. It is easy to see that

$$df \wedge \Big(\int_0^{x_1} g(t, x_2, \cdots, x_n) dt \frac{\partial}{\partial x_1} \rfloor dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\Big)|_V = 0$$

for any $f \in I(V)$. Thus $H^n(\mathcal{W}_0^*(V)) = 0$. □

Remark 3.13. We can also prove Proposition 3.12 in a different way. If $\frac{\partial}{\partial x_1}$ is tangent to V then $x_1 \frac{\partial}{\partial x_1}$ is tangent to V too. By Lemma 3.11, using the same argument as in the proof of Proposition 3.10, we obtain $H^n(\mathcal{W}_0^*(V)) = 0$. If V satisfies the assumptions of Theorem 3.9 then V is quasi-homogeneous with respect to $\{x_1 = 0\}$.

Let C_n be the ring of smooth ($\mathbb{K}$-analytic) function-germs at 0 on $\mathbb{K}^n$. Let $I(V)$ be the ideal in C_n of function-germs vanishing on V.

Let $Derlog(V)$ denote the module of germs at 0 of vector fields tangent to V i.e. $Y \in Derlog(V)$ if $Y(I(V)) \subset I(V)$.

Let $\mathrm{div}(Y)$ denotes the divergence of a vector field $Y = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$ with respect to the volume form $\Omega_0 = dx_1 \wedge \ldots \wedge dx_n$ i.e.

$$\mathrm{div}(Y) = \frac{\mathcal{L}_Y \Omega_0}{\Omega_0} = \frac{d(Y \rfloor \Omega_0)}{\Omega_0} = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}.$$

Then $\mathrm{div}(Derlog(V)) := \{\mathrm{div}(Y) \in C_n : Y \in Derlog(V)\}$ is a $\mathbb{K}$-vector subspace of C_n.

Proposition 3.14. $H^n(\mathcal{W}_0^*(V)) \cong \frac{C_n}{\mathrm{div}(Derlog(V))}$.

Proof. Let ω_0 and ω_1 be germs n-forms on $\mathbb{K}^n$. Let Ω_0 be the germ of a volume form on $\mathbb{K}^n$. Then $\omega_0 = g_0 \Omega_0$ and $\omega_1 = g_1 \Omega_0$, where g_0 and g_1 are function-germs on $\mathbb{K}^n$. And $[\omega_0]_V = [\omega_1]_V$ in $H^k(\mathcal{W}_0^*(V))$ if and only if $\omega_0 - \omega_1 = d\alpha$, where α is the germ of a $(n-1)$-form on $\mathbb{K}^n$ such that $df \wedge \alpha|_V = 0$ for any $f \in I(V)$. Let X be a germ of a vector field such that $\alpha = X \rfloor \Omega_0$ then

$$df \wedge \alpha = df \wedge (X \rfloor \Omega_0) = (X \rfloor df) \Omega_0.$$

Thus $df \wedge \alpha|_V = 0$ if and only if $X \rfloor df \in I(V)$.

We obtain that $[\omega_0]_V = [\omega_1]_V$ in $H^n(\mathcal{W}_0^*(V))$ if and only if $g_0 - g_1 = \mathrm{div}(X)$, where $X \in Derlog(V)$. □

Remark 3.15. Proposition 3.10 can be proved using Proposition 3.14. It is easy to show that if V is quasi-homogeneous with respect to the smooth submanifold then $\mathrm{div}(Derlog(V))$ is C_n.

Definition 3.16. We say that a cohomology class $a \in H^n(\mathcal{W}_0^*(V))$ is realizable by a volume form if there exists a germ Ω of a volume form such that $[\Omega]_V = a$. We denote by $[Vol]_V$ the set of all cohomology classes realizable by volume forms.

Proposition 3.17. *If there exists a germ of a volume form with zero cohomology class in $H^n(\mathcal{W}_0^*(V))$ then $[Vol]_V = H^n(\mathcal{W}_0^*(V))$*

Proof. Let Ω be a germ of a volume form such that $[\Omega]_V = [0]_V \in H^n(\mathcal{W}_0^*(V))$. Let α be a germ of a n-form on $\mathbb{K}^n$. If $\alpha|_0 = 0$ then $\beta = \alpha + \Omega$ is a germ of a volume form and $[\beta]_V = [\alpha]_V \in H^n(\mathcal{W}_0^*(V))$. □

Proposition 3.18. *If $H^n(\mathcal{W}_0^*(V)) \setminus [Vol]_V \neq \emptyset$ then $H^n(\mathcal{W}_0^*(V)) \setminus [Vol]_V$ is a $\mathbb{K}$-linear subspace of $H^n(\mathcal{W}_0^*(V))$.*

If L is a finite dimensional linear subspace of $H^n(\mathcal{W}_0^*(V))$ such that $L \cap [Vol]_V \neq \emptyset$ then there exists a base $a_1, \cdots, a_{\dim_{\mathbb{K}} L}$ of L such that $a_1, \cdots, a_{\dim_{\mathbb{K}} L}$ are realizable by volume forms.

Proof. By Proposition 3.17, $[0]_V$ is not realizable by a volume form. If $a \in H^n(\mathcal{W}_0^*(V))$ is not realizable by a volume form then neither is λa for any $\lambda \in \mathbb{K}$, because if $\lambda \neq 0$ and there exists a germ Ω of a volume form such that $\lambda a = [\Omega]_V$ then $a = [\frac{1}{\lambda}\Omega]_V$. In the same way we show that if $a, b \in H^n(\mathcal{W}_0^*(V))$ are not realizable by a volume form then neither is $a + b$.

Let us assume that $H^n(\mathcal{W}_0^*(V)) \setminus [Vol]_V \neq \emptyset$. Then by Proposition 3.17 any germ of a volume form has non-zero cohomology class in $H^n(\mathcal{W}_0^*(V))$. Let Ω be a germ of a volume form such that $[\Omega]_V \in L$. Then $[\Omega]_V \neq [0]_V$ and let $[\Omega]_V, [\beta_2]_V, \cdots, [\beta_{\dim_{\mathbb{K}} L}]_V$ be a base of L. Let $a_1 = [\Omega]_V$ and $a_i = [\beta_i]_V$ if $\beta_i|_0 \neq 0$ and $a_i = [\beta_i + \Omega]_V$ if $\beta_i|_0 = 0$ for $i = 2, \cdots, \dim_{\mathbb{K}} L$. It is easy to see that $a_1, \cdots, a_{\dim_{\mathbb{K}} L}$ is a base of L and $a_1, \cdots, a_{\dim_{\mathbb{K}} L}$ are realizable by volume forms. □

By Propositions 3.17 and 3.18 we obtain the following corollary.

Corollary 3.19. $\dim_{\mathbb{K}}[Vol]_V = \dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V))$.

Proof. If there exists a germ of a volume form with zero cohomology class in $H^n(\mathcal{W}_0^*(V))$ then $[Vol]_V = H^n(\mathcal{W}_0^*(V))$ by Proposition 3.17. But if $H^n(\mathcal{W}_0^*(V)) \setminus [Vol]_V \neq \emptyset$ then $[Vol]_V = H^n(\mathcal{W}_0^*(V)) \setminus T$, where T is a subspace of codimension at least 1 by Proposition 3.18. □

4. Volume preserving diffeomorphisms on varieties

Let Ω_0, Ω_1 be germs at 0 of smooth ($\mathbb{K}$-analytic) volume-forms on $\mathbb{K}^n$. Let V be the germ at 0 of a subvariety of $\mathbb{K}^n$. We define two equivalence relations on the space of volume forms on $\mathbb{K}^n$.

Definition 4.1. We say that Ω_0 and Ω_1 are $\mathcal{R}_V$-equivalent if there exists a germ of a diffeomorphism $\Phi : (\mathbb{K}^n, V, 0) \to (\mathbb{K}^n, V, 0)$ (preserving V) such that $\Phi^*\Omega_1 = \Omega_0$.

Definition 4.2. We say that Ω_0 and Ω_1 are $\mathcal{I}R_V$-equivalent if there exists a smooth family of germs of diffeomorphisms $\Phi_t : (\mathbb{K}^n, V, 0) \to (\mathbb{K}^n, V, 0)$ (preserving V) for $t \in [0, 1]$ such that $\Phi_0 = Id$ and $\Phi_1^*\Omega_0 = \Omega_1$.

It is obvious that $\mathcal{I}R_V$-equivalence implies $\mathcal{R}_V$-equivalence. We now prove one of the main results of the paper.

Theorem 4.3. *Let V be a germ of a subset of $\mathbb{K}^n$. Let Ω_0 and Ω_1 be germs of smooth (or $\mathbb{K}$-analytic) volume forms on $\mathbb{K}^n$. If $\mathbb{K} = \mathbb{R}$ we assume that Ω_0 and Ω_1 define the same orientation of $\mathbb{K}^n$.*

Then $[\Omega_0]_V = [\Omega_1]_V$ in $H^n(\mathcal{W}_0^*(V))$ if and only if Ω_0 and Ω_1 are $\mathcal{I}\mathcal{R}_V$-equivalent.

Proof. First we assume that any germ of a vector field tangent to V vanishes at 0. If $[\Omega_0]_V = [\Omega_1]_V$ in $H^n(\mathcal{W}_0^*(V))$ then there exists a germ of a $(n-1)$-form α on $\mathbb{K}^n$ such that

$$\Omega_0 - \Omega_1 = d\alpha,\ df \wedge \alpha|_V = 0\ \forall f \in I(V). \tag{4.1}$$

We use Moser's homotopy method ([31]). Let $\Omega_t = \Omega_0 + t(\Omega_1 - \Omega_0)$ for $t \in [0,1]$. It is easy to see that if Ω_0 and Ω_1 define the same orientation of $\mathbb{R}^n$ then Ω_t is a germ of a volume-form at 0 for any $t \in [0,1]$. We are looking for diffeomorphisms $\Phi_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ for $t \in [0,1]$ such that

$$\Phi_t^\star \Omega_t = \Omega_0 \tag{4.2}$$

and $\Phi_0 = Id_{\mathbb{K}^n}$, $\Phi_t(V) \subset V$. Differentiating (4.2) we obtain

$$\Phi_t^\star(L_{U_t}\Omega_t + \Omega_1 - \Omega_0) = 0,$$

where $U_t \circ \Phi_t = \frac{d}{dt}\Phi_t$. $\Phi_t(V) \subset V$ if and only if U_t is tangent to V. Thus by (4.1) we have

$$d(U_t \rfloor \Omega_t) = \Omega_0 - \Omega_1 = d\alpha. \tag{4.3}$$

Thus we want to find U_t tangent to V such that

$$U_t \rfloor \Omega_t = \alpha. \tag{4.4}$$

Let X be a germ of a vector field such that $X \rfloor \Omega_0 = \alpha$. Then $df \wedge \alpha = df \wedge (X \rfloor \Omega_0) = (X \rfloor df)\Omega_0$. We have that $df \wedge \alpha|_V = 0$ for any $f \in I(V)$. Thus $X \rfloor df \in I(V)$ for any $f \in I(V)$, which means that X is tangent to V. By the above assumption $X|_0 = 0$. But

$$\Omega_t = g_t \Omega_0$$

where g_t is a non-vanishing function-germ at 0. Hence $U_t = \frac{1}{g_t}X$ is a solution of (4.4), which is tangent to V and vanishes at 0. By integration of U_t we obtain a smooth family of germs of diffeomorphism $\Phi_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ for $t \in [0,1]$ such that $\Phi_t(V) \subset V$, $\Phi_0 = Id$ and $\Phi_1^\star \Omega_1 = \Omega_0$.

To conclude the proof of the forward direction, assume that there exists a germ of vector field Y tangent to V such that $Y|_0 \neq 0$. Then there exists a coordinate system $(x_1, \cdots, x_n)$ on $\mathbb{K}^n$ such that $Y = \frac{\partial}{\partial x_1}$. If ω is a germ of an n-form then

$$\Omega_0 - \Omega_1 = g dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = d\Big(\int_0^{x_1} g(t, x_2, \cdots, x_n)dt \frac{\partial}{\partial x_1} \rfloor dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\Big),$$

where g is some function-germ and $X = \int_0^{x_1} g(t, x_2, \cdots, x_n)dt \frac{\partial}{\partial x_1}$ vanishes at 0.

But $Y|_0 \neq 0$ also implies that $H^n(\mathcal{W}_0^*) = 0$, by Proposition 3.12. Now we can continue in the same way as in the first part of the proof.

For the converse, assume that Ω_0 and Ω_1 are $\mathcal{IR}_V$-equivalent. Then there exists a smooth family of germs of diffeomorphisms $\Phi_t : (\mathbb{K}^n, V, 0) \to (\mathbb{K}^n, V, 0)$ (preserving V) for all $t \in [0,1]$ such that $\Phi_0 = Id$ and $\Phi_1^* \Omega_0 = \Omega_1$. Let $(\Phi_t)' = \frac{d}{dt}\Phi_t = X_t \circ \Phi_t$. Then

$$\Omega_1 - \Omega_0 = \Phi_1^* \Omega_0 - \Omega_0 = \int_0^1 (\Phi_t^* \Omega_0)' dt = \int_0^1 (\Phi_t^* \mathcal{L}_{X_t} \Omega_0) dt = \int_0^1 \Phi_t^* d(X_t \rfloor \Omega_0) dt.$$

Thus

$$\Omega_1 - \Omega_0 = d\left(\int_0^1 \Phi_t^*(X_t \rfloor \Omega_0) dt\right).$$

X_t is tangent to V. Thus for any $f \in I(V)$

$$df \wedge (X_t \rfloor \Omega_0)|_V = (X_t \rfloor df) \wedge \Omega_0|_V = 0.$$

But

$$df \wedge \Phi_t^*(X_t \rfloor \Omega_0) = \Phi_t^*(d((\Phi_t^{-1})^* f) \wedge (X_t \rfloor \Omega_0))$$

and $\Phi_t(V) \subset V$. Hence $df \wedge \Phi_t^*(X_t \rfloor \Omega_0)|_V = 0$, which implies that

$$df \wedge \int_0^1 \Phi_t^* d(X_t \rfloor \Omega_0) dt|_V = 0.$$

Thus $[\Omega_0]_V = [\Omega_1]_V$ in $H^n(\mathcal{W}_0^*(V))$. □

In [2] (see also [38], [28]) it was shown that if V is the germ of a $\mathbb{C}$-analytic quasi-homogeneous hypersurface with an isolated singularity then any two holomorphic volume forms on $\mathbb{C}^n$ are $\mathcal{R}_V$-equivalent. More generally we have (for variety-germs V of any codimension and with possibly non-isolated singularities) the following

Theorem 4.4. *Let V be a germ of a variety on $\mathbb{K}^n$.*

1. *Let $\mathbb{K} = \mathbb{R}$ and let Ω_0 and Ω_1 be germs of smooth or $\mathbb{R}$-analytic volume-forms at 0. If V is quasi-homogeneous with respect to a smooth submanifold of $\mathbb{R}^n$ and $\Omega_0|_0$, $\Omega_1|_0$ define the same orientation of $T_0\mathbb{R}^n$ then Ω_0 and Ω_1 are $\mathcal{R}_V$-equivalent.*

2. *Let $\mathbb{K} = \mathbb{C}$ and let Ω_0 and Ω_1 be germs of $\mathbb{C}$-analytic volume-forms at 0. If V is quasi-homogeneous with respect to a $\mathbb{C}$-analytic submanifold of $\mathbb{C}^n$ then Ω_0 and Ω_1 are $\mathcal{R}_V$-equivalent.*

In [38] it was shown that if $H = \{f = 0\}$ is a germ at 0 of a $\mathbb{C}$-analytic hypersurface in $\mathbb{C}^n$ with isolated singularity at 0 then the number of $\mathcal{IR}_H$-moduli of volume forms on $\mathbb{C}^n$ is equal to $\mu - \tau$, where $\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{<\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}>}$ is the Milnor number of H and $\tau = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{<f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}>}$ is the Tjurina number of H.

Let V be the germ of a variety on $\mathbb{K}^n$ and let Ω be the germ of a volume form on $\mathbb{K}^n$. The $\mathcal{IR}_V$-orbit of Ω is the identity component of the $\mathcal{R}_V$-orbit of Ω. Thus the number of $\mathcal{R}_V$-moduli of germs of volume forms is equal to the number of $\mathcal{IR}_V$-moduli of germs of volume forms. By Theorem 4.3 and Proposition 3.14 we obtain the following result

Proposition 4.5. *Let V be a germ of a variety of $\mathbb{K}^n$. Then the number of $\mathcal{R}_V$-moduli of germs of volume forms is equal to*

$$\dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V)) = \dim_{\mathbb{K}} \frac{C_n}{\operatorname{div}(Derlog(V))}.$$

Proof. Let Ω_0 and Ω_1 be germs of volume forms on $\mathbb{K}^n$. If $[\Omega_0]_V = [\Omega_1]_V$ in $H^n(\mathcal{W}_0^*(V))$ then Ω_0 and Ω_1 are R_V-equivalent by Theorem 4.3. By Corollary 3.19 and Proposition 3.14 we have $\dim_{\mathbb{K}}[Vol]_V = \dim_{\mathbb{K}} H^n(\mathcal{W}_0^*(V)) = \dim_{\mathbb{K}} \frac{C_n}{\operatorname{div}(Derlog(V))}$. Now assume that Ω_0 and Ω_1 are $\mathcal{R}_V$-equivalent. We also may assume that they are in the same component of an $\mathcal{R}_V$-orbit. But then they are $\mathcal{IR}_V$-equivalent. Hence $[\Omega_1]_V = [\Omega_0]_V$ in $H^n(\mathcal{W}_0^*(V))$ by Theorem 4.3. □

Let Ω be the germ of a fixed smooth ($\mathbb{K}$-analytic) volume-form on $\mathbb{K}^n$ at 0, and let V_1, V_2 be germs of varieties of $\mathbb{K}^n$ at $0 \in \mathbb{K}^n$.

Definition 4.6. V_1 and V_2 are $\mathcal{R}_\Omega$-equivalent if there exists a volume-preserving diffeomorphism-germ $\Phi : (\mathbb{K}^n, \Omega, 0) \to (\mathbb{K}^n, \Omega, 0)$ (i.e. $\Phi^*\Omega = \Omega$) such that $\Phi(V_1) = V_2$.

Theorem 4.4 can be formulated in the following way.

Theorem 4.7. *Let V, V_1 be germs of varieties on $\mathbb{K}^n$ at p.*

(1) *Let $\mathbb{K} = \mathbb{R}$ and fix a germ Ω of a smooth (or $\mathbb{R}$-analytic) volume-form at p. If V is quasi-homogeneous with respect to a smooth submanifold then V and V_1 are $\mathcal{R}_\Omega$-equivalent if and only if there exists a germ of smooth ($\mathbb{R}$-analytic) orientation-preserving diffeomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\Phi(V) = V_1$.*

(2) *Let $\mathbb{K} = \mathbb{C}$ and fix a germ Ω of a $\mathbb{C}$-analytic volume-form at p. If V is quasi-homogeneous with respect to a $\mathbb{C}$-analytic submanifold then V and V_1 are $\mathcal{R}_\Omega$-equivalent if and only if V and V_1 are $\mathbb{C}$-analytically diffeomorphic.*

Here is an example of a germ of a singular curve on $\mathbb{R}^2$ which determines an orientation on $\mathbb{R}^2$.

Example 4.8. Let $V_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 - x_1^5 = 0\}$ and $V_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 + x_1^5 = 0\}$. V_1 and V_2 are quasi-homogeneous with weights $(3, 5)$ and they are diffeomorphic. For example take $\Phi(x_1, x_2) = (-x_1, x_2)$. But they are not $\mathcal{R}_\Omega$-equivalent, because any diffeomorphism mapping V_1 to V_2 changes the orientation of $\mathbb{R}^2$.

5. $\mathcal{A}_\Omega$-equivalence of maps

We now study $\mathcal{A}_\Omega$-equivalence of map-germs f, where Ω is a fixed volume form on the target of f. The results about $\mathcal{A}_\Omega$-equivalence will follow from those about $\mathcal{A}_f$-equivalence of volume forms, where the map-germ f is fixed, in the same way as the results about $\mathcal{R}_\Omega$ were obtained in the previous section from those about $\mathcal{R}_V$.

Furthermore, as mentioned already in Remark 2.5, slightly weaker versions of the results about maps f can be deduced from those about germs of varieties V, by considering vector fields tangent to the discriminant or image $\Delta(f)$ of f. More precisely, using the results in [32], [33], [8] – and in particular the necessary and sufficient conditions in theorem 2 of [8] for a vector field tangent to $\Delta(f)$ to lift over f – we could deduce our results about $\mathcal{A}_\Omega$ and $\mathcal{A}_f$ from those about $\mathcal{R}_\Omega$ and $\mathcal{R}_V$ (taking $V = \Delta(f)$) for the class of maps f satisfying the conditions in theorem 2 of [8]. This class would, for example, over $\mathbb{C}$ include all $\mathcal{A}$-finite map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ for all pairs (n, p) different from $(n, 2)$, $n \geq 2$. Nevertheless, working directly with liftable vector fields, we can avoid these restrictions – see Example 5.13 below for a weakly quasi-homogeneous map-germ f (to which our Theorem 2.4 applies) which does not satisfy the conditions in [8].

First, we recall the definition of a liftable vector field ([1], [8]).

Definition 5.1. The germ of a vector field Y on $\mathbb{K}^p$ is *liftable* or *lifts* over a map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ to a germ of a vector field X at 0 on $\mathbb{K}^n$ if

$$df(X) = Y \circ f.$$

(Considering homomorphisms tf from the C_n-module of source vector fields to the C_n-module θ_f of smooth sections of $f^*T\mathbb{K}^p$ and ωf from the C_p-module of target vector fields to θ_f, given by $X \mapsto df(X)$ and $Y \mapsto Y \circ f$ respectively, the above equation becomes $tf(X) = \omega f(Y)$ – this notation will be used in the next section when we study the $\mathcal{A}_\Omega$-tangent space of f.) The C_p-module of germs of vector fields liftable over f we denote by $Lift(f)$. The C_p-submodule of $Lift(f)$ of germs of vector fields liftable over f to germs of vector fields **vanishing at** 0 we denote by $Lift_0(f)$.

Let Ω_0, Ω_1 be germs at 0 of smooth ($\mathbb{K}$-analytic) volume-forms on $\mathbb{K}^p$ and let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth (or $\mathbb{K}$-analytic) map-germ.

Definition 5.2. We say that Ω_0 and Ω_1 are $\mathcal{A}_f$-equivalent if there exist germs of diffeomorphisms $\Phi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ and $\Psi : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\Phi^*\Omega_1 = \Omega_0$ and $\Phi \circ f = f \circ \Psi$.

Remark 5.3. The condition $\Phi \circ f = f \circ \Psi$ is equivalent to $\Phi \circ f \circ \Psi^{-1} = f$. It means that f is preserved by $\mathcal{A}$-action of (Ψ^{-1}, Φ).

Definition 5.4. We say that Ω_0, Ω_1 are $\mathcal{IA}_f$-equivalent if there exist smooth families of germs of diffeomorphisms $\Phi_t : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ and $\Psi_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ for $t \in [0, 1]$ such that $\Phi_0 = Id_{\mathbb{K}^p}$, $\Psi_0 = Id_{\mathbb{K}^n}$, $\Phi_1^*\Omega_0 = \Omega_1$ and $\Phi_t \circ f = f \circ \Psi_t$ for $t \in [0, 1]$.

Obviously $\mathcal{IA}_f$-equivalence implies $\mathcal{A}_f$-equivalence.

Let $\operatorname{div}(Y)$ denotes the divergence of a germ of a vector field Y with respect to the germ of the volume form Ω, i.e. $\operatorname{div}(Y)\Omega = d(Y \rfloor \Omega) = \mathcal{L}_Y\Omega$.

Definition 5.5. We say that germs of volume forms Ω_0 and Ω_1 are $divLift_0(f)$-equivalent if there exists a germ of a vector field $Y \in Lift_0(f)$ such that $\Omega_1 - \Omega_0 = \operatorname{div}(Y)\Omega$.

Remark 5.6. $divLift_0(f)$ is an equivalence relation. We denote the $divLift_0(f)$-equivalence class of a germ of a volume form Ω_0 by $[\Omega_0]_f$. The definition of $divLift_0(f)$ does not depend of the choice of germ of a volume form Ω. If $\tilde{\Omega}$ is the germ of another volume form then $\Omega = g\tilde{\Omega}$. If $\Omega_1 - \Omega_0 = d(Y \rfloor \Omega)$ for $Y \in Lift_0(f)$ then $\Omega_1 - \Omega_0 = d(gY \rfloor \tilde{\Omega})$ and $gY \in Lift_0(f)$.

Now we state one of the main results of this section.

Theorem 5.7. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth ($\mathbb{K}$-analytic) map-germ. Let Ω_0 and Ω_1 be germs of smooth ($\mathbb{K}$-analytic) volume forms on $\mathbb{C}^p$. If $\mathbb{K} = \mathbb{R}$ we assume that Ω_0 and Ω_1 define the same orientation of $\mathbb{R}^p$.*

Then Ω_0 and Ω_1 are $\mathcal{IA}_f$-equivalent if and only if Ω_0 and Ω_1 are $divLift_0(f)$-equivalent.

Proof. First we assume that $\Omega_1 - \Omega_0 = \operatorname{div}(Y)\Omega_0$ and $df(X) = Y \circ f$, where X is the germ of a vector field at 0 on $\mathbb{K}^n$ such that $X|_0 = 0$. We use Moser's homotopy method ([31]). Let $\Omega_t = \Omega_0 + t(\Omega_1 - \Omega_0)$ for $t \in [0, 1]$. It is easy to see that Ω_t is the germ of a volume-form at 0 for any $t \in [0, 1]$. We are looking for diffeomorphisms $\Phi_t : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$, $\Psi_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ for $t \in [0, 1]$ such that

$$\Phi_t^\star \Omega_t = \Omega_0 \tag{5.1}$$

and $\Phi_0 = Id_{\mathbb{K}^p}$, $\Psi_0 = Id_{\mathbb{K}^n}$, $\Phi_t \circ f = f \circ \Psi_t$. Differentiating (5.1) we obtain

$$\Phi_t^\star(L_{Y_t}\Omega_t + \Omega_1 - \Omega_0) = 0,$$

where $Y_t \circ \Phi_t = \frac{d}{dt}\Phi_t$. And $\Phi_t \circ f = f \circ \Psi_t$ if and only if $Y_t \circ f = df(X_t)$, where $X_t \circ \Psi_t = \frac{d}{dt}\Psi_t$. By the assumptions we have

$$d(Y_t \rfloor \Omega_t) = \Omega_0 - \Omega_1 = \operatorname{div}(Y)\Omega_0 = d(Y \rfloor \Omega_0). \tag{5.2}$$

But

$$\Omega_t = h_t \Omega_0,$$

where h_t is a non-vanishing function-germ at 0. Hence $Y_t = \frac{1}{h_t}Y$ is a solution of (5.2). Multiplying $df(X) = Y \circ f$ by $1/h_t \in C_p$ (using the C_p-module structure of $Lift(f)$) we have

$$Y_t \circ f = (\frac{1}{h_t}Y) \circ f = df((\frac{1}{h_t} \circ f)X) = df(X_t),$$

where $X_t := (\frac{1}{h_t} \circ f)X$ vanishes at 0, because X vanishes at 0. And Y_t vanishes at 0 too, because $Y_t|_0 = Y_t|_{f(0)} = df|_0(X_t|_0) = df|_0(0) = 0$.

Integrating Y_t and X_t we obtain diffeomorphism-germs $\Phi_t : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ and $\Psi_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\Phi_1^\star \Omega_1 = \Omega_0$ and $\Phi_t \circ f = f \circ \Psi_t$.

Now assume that Ω_1 and Ω_0 are $\mathcal{IA}_f$-equivalent. Then there exist smooth families of germs of diffeomorphisms $\Phi_t : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ and $\Psi_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\Phi_0 = Id_{\mathbb{K}^p}$, $\Psi_0 = Id_{\mathbb{K}^n}$, $\Phi_1^*\Omega_0 = \Omega_1$ and $\Phi_t \circ f = f \circ \Psi_t$ for all $t \in [0, 1]$. Thus

$$\Omega_1 - \Omega_0 = \int_0^1 \frac{d}{dt}\Phi_t^*\Omega_0 dt = \int_0^1 \Phi_t^* d(Y_t \rfloor \Omega_0) dt = d\int_0^1 \Phi_t^*(Y_t \rfloor \Omega_0) dt,$$

where Y_t is a germ of a vector field on $\mathbb{K}^p$ such that $Y_t \circ \Phi_t = \frac{d}{dt}\Phi_t$.

But

$$\int_0^1 \Phi_t^*(Y_t \rfloor \Omega_0) dt = \int_0^1 \big(d(\Phi_t^{-1})(Y_t \circ \Phi_t)\big) \rfloor \Phi_t^* \Omega_0 dt = \int_0^1 Jac(\Phi_t)\tilde{Y}_t dt \rfloor \Omega_0,$$

where $\tilde{Y}_t = d(\Phi_t^{-1})(Y_t \circ \Phi_t)$ is a germ of a vector field on $\mathbb{K}^p$ and $Jac(\Phi_t) = \det d\Phi_t$.

Differentiating $\Phi_t \circ f = f \circ \Psi_t$ we obtain $Y_t \circ f = df(X_t)$, where X_t is a germ of a vector field at 0 such that $X_t \circ \Psi_t = \frac{d}{dt}\Psi_t$.

Then $\tilde{Y}_t \circ f = d(\Phi_t^{-1})(Y_t \circ \Phi_t \circ f) = d(\Phi_t^{-1})(Y_t \circ f \circ \Psi_t) = d(\Phi_t^{-1})(df(X_t \circ \Psi_t)) = d(\Phi_t^{-1} \circ f)(X_t \circ \Psi_t) = d(f \circ \Psi_t^{-1})(X_t \circ \Psi_t) = df(d(\Psi_t^{-1})(X_t \circ \Psi_t))$. Thus

$$\tilde{Y}_t \circ f = df(\tilde{X}_t), \ t \in [0, 1],$$

where $\tilde{X}_t = d(\Psi_t^{-1})(X_t \circ \Psi_t)$ is a germ of a vector field on $\mathbb{K}^n$.

Hence

$$\int_0^1 Jac(\Phi_t)\tilde{Y}_t dt \circ f = df\left(\int_0^1 (Jac(\Phi_t) \circ f)\tilde{X}_t dt\right).$$

Thus a germ of a vector field $Y = \int_0^1 Jac(\Phi_t)\tilde{Y}_t dt$ lifts over f to a germ of a vector field $X = \int_0^1 (Jac(\Phi_t) \circ f)\tilde{X}_t dt$ and $\Omega_1 - \Omega_0 = d(Y \rfloor \Omega_0) = \operatorname{div}(Y)\Omega_0$. It is easy to see that X vanishes at 0, because $X_t|_0 = 0$. □

From Theorem 5.7 we get the following corollary, using the fact the $\mathcal{IA}_f$-orbit of Ω is the identity component of the $\mathcal{A}_f$-orbit of Ω.

Corollary 5.8. *The number of $\mathcal{A}_f$-moduli of germs of volume forms on $\mathbb{K}^n$ is equal to $dim_{\mathbb{K}} \frac{C_p}{\operatorname{div}(Lift_0(f))}$.*

We now define a class of map-germs f for which $\operatorname{div}(Lift_0(f))$ is C_p.

Definition 5.9. We say that a smooth ($\mathbb{K}$-analytic) map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is *weakly quasi-homogeneous* if it is quasi-homogeneous with **non-negative** degrees $\delta_1, \ldots, \delta_p$ and a **positive** total degree $\delta_1 + \ldots + \delta_p$ for **integer** weights $w_1, \ldots, w_n$ (fixing coordinates $(x_1, \ldots, x_n) \in \mathbb{K}^n$ and $(X_1, \ldots, X_p) \in \mathbb{K}^p$).

For convenience, we assume that the degree of the zero function-germ is 1 (see Example 5.12 below).

Define Euler-like vector fields on $\mathbb{K}^n$ (with $w_i \in \mathbb{Z}$)

$$E_w = \sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}.$$

and on $\mathbb{K}^p$ (with $\delta_i \geq 0$, $\sum_i \delta_i > 0$)

$$E_\delta = \sum_{i=1}^{p} \delta_i X_i \frac{\partial}{\partial X_i}.$$

The following easy proposition gives one more equivalent definition of weak quasi-homogeneity of a map.

Proposition 5.10. *The following conditions on a smooth ($\mathbb{K}$-analytic) map-germ f are equivalent:*

(i) *f is weakly quasi-homogeneous with degrees $\delta = (\delta_1, \ldots, \delta_p)$ for weights $w = (w_1, \ldots, w_n)$ in coordinate systems $x = (x_1, \ldots, x_n)$ on $\mathbb{K}^n$ and $X = (X_1, \ldots, X_p)$ on $\mathbb{K}^p$;*

(ii) *$E_\delta \circ f = df(E_w)$.*

All quasi-homogeneous map-germs are weakly quasi-homogeneous. But there are weakly quasi-homogeneous map-germs which are not quasi-homogeneous.

Example 5.11. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a map-germ $\mathcal{A}$-equivalent to

$$\tilde{f}(u, x_1, ..., x_{n-1}) = (u, g(x_1, \cdots, x_{n-1}))$$

for some map-germ $g : (\mathbb{K}^{n-1}, 0) \to (\mathbb{K}^{p-1}, 0)$. Then f is weakly quasi-homogeneous. For $\tilde{f}$ we take weights $(1, 0, \cdots, 0)$.

Example 5.12. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a map-germ $\mathcal{A}$-equivalent to

$$\tilde{f}(x_1, ..., x_n) = (g(x_1, \cdots, x_n), 0)$$

for some map-germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^{p-1}, 0)$. Then f is weakly quasi-homogeneous. For $\tilde{f}$ we take weights $(0, \cdots, 0)$. The total degree of f is 1, because the degree of the zero component function-germ is 1.

Example 5.13. In [8]

$$f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0), \; f(u, x, y) = (u, x^4 + y^4 + ux^2y^2)$$

is presented as an example of a map-germ which is not generically a trivial unfolding of a quasi-homogeneous germ. This f is also an example of a weakly quasi-homogeneous map-germ (for the weights $(0, 1, 1)$).

Notice that the map-germ f in the third example fails to be $\mathcal{A}$-finite and that in the first example f is either stable (if g is stable) or it fails to be $\mathcal{A}$-finite. In the next section we describe a more subtle example of a weakly quasi-homogeneous (but not quasi-homogeneous) $\mathcal{A}$-simple map-germ without zero component functions (see Example 6.4).

Using Lemma 3.11 we prove the following theorem.

Theorem 5.14. *1. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth ($\mathbb{R}$-analytic) weakly quasi-homogeneous map-germ and let Ω_0 and Ω_1 be germs of smooth ($\mathbb{R}$-analytic) volume-forms at 0. If $\Omega_0|_0$ and $\Omega_1|_0$ define the same orientation of $T_0\mathbb{R}^p$ then Ω_0 and Ω_1 are $\mathcal{A}_f$-equivalent.*
2. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a $\mathbb{C}$-analytic weakly quasi-homogeneous map-germ and let Ω_0 and Ω_1 be germs $\mathbb{C}$-analytic volume-forms at 0. Then Ω_0 and Ω_1 are $\mathcal{A}_f$-equivalent.

Proof. Let E_w be the Euler-like vector field on $\mathbb{K}^n$ for a coordinate system $(x_1, \ldots, x_n)$ with weights $w = (w_1, \ldots, w_n)$ of weak quasi-homogeneity for f and let E_δ be the Euler vector field on $\mathbb{K}^p$ for a coordinate system $(X_1, \ldots, X_p)$ with nonnegative weights $\delta = (\delta_1, \ldots, \delta_p)$, which are degrees of weak quasi-homogeneity of f. We may assume that $\delta_1, \ldots, \delta_k$ are positive integers and $\delta_{k+1} = \ldots = \delta_p = 0$. Therefore E_δ satisfies the conditions of Lemma 3.11. Let Ω_1 be a germ of volume form on $\mathbb{K}^p$ and let $\Omega_0 = \pm dX_1 \wedge \ldots \wedge dX_p$ (we choose of the sign of Ω_0 in such way that $\Omega_0|_0$ and $\Omega_1|_0$ define the same orientation orientation of $T_0\mathbb{R}^p$ for $\mathbb{K} = \mathbb{R}$, and $\Omega_0 = dX_1 \wedge \ldots \wedge dX_p$ for $\mathbb{K} = \mathbb{C}$).

We show that Ω_1 and Ω_0 are $\mathcal{A}_f$-equivalent.

By Lemma 3.11 we have

$$\Omega_0 - \Omega_1 = d(h(E_\delta \rfloor \Omega_0)), \tag{5.3}$$

where h is a function-germ. From Proposition 5.10 we have $E_\delta \circ f = df(E_w)$ and multiplying with $h \circ f$ we get

$$(hE_\delta) \circ f = df((h \circ f)E_w),$$

and $(h \circ f)E_w|_0 = 0$, because $E_w|_0 = 0$.

Hence, by Theorem 5.7, Ω_1 and Ω_0 are $\mathcal{IA}_f$-equivalent. □

Remark 5.15. Theorem 5.14 can also be proved using Proposition 5.8. It is easy to show that if f is weakly quasi-homogeneous then $\mathrm{div}(Lift_0(f))$ is C_p.

Let Ω be a germ of a volume form $\mathbb{K}^p$ and let $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be map-germs.

Definition 5.16. f and g are $\mathcal{A}_\Omega$-equivalent if there exists a germ of a volume-preserving diffeomorphism $\Phi : (\mathbb{K}^p, \Omega, 0) \to (\mathbb{K}^p, \Omega, 0)$ (i.e. $\Phi^*\Omega = \Omega$) and a germ of a diffeomorphism $\Psi : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$, such that

$$f \circ \Psi = \Phi \circ g.$$

From Theorem 5.14 we obtain the following corollary.

Theorem 5.17. *1. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth ($\mathbb{R}$-analytic) weakly quasi-homogeneous map-germ and let Ω be the germ of smooth ($\mathbb{R}$-analytic) volume-form at 0. Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be another smooth ($\mathbb{R}$-analytic) map-germ, then f and g are $\mathcal{A}_\Omega$-equivalent if and only if there exist a germ of a smooth ($\mathbb{R}$-analytic) orientation-preserving diffeomorphism $\Phi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ and a germ of a smooth ($\mathbb{R}$-analytic) diffeomorphism $\Psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\Phi \circ g \circ \Psi = f$.*
2. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a $\mathbb{C}$-analytic weakly quasi-homogeneous map-germ

and let Ω be the germ of a $\mathbb{C}$-analytic volume-form at 0. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be another $\mathbb{C}$-analytic map-germ, then f and g are $\mathcal{A}_\Omega$-equivalent if and only if f and g are $\mathcal{A}$-equivalent.

The following example explains the role of the orientation in the real case.

Example 5.18. Let $f_1 : \mathbb{R} \ni t \mapsto (t^3, t^5) \in \mathbb{R}^2$ and $f_2 : \mathbb{R} \ni t \mapsto (t^3, -t^5) \in \mathbb{R}^2$ (see [23]). It is easy to see that f_1 and f_2 are quasi-homogeneous and $\mathcal{A}$-equivalent. But they are not $\mathcal{A}_\Omega$-equivalent, because their images $f_1(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 - x_1^5 = 0\}$ and $f_2(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^3 + x_1^5 = 0\}$ fail to be $\mathcal{R}_\Omega$-equivalent as germs of varieties at $0 \in \mathbb{R}^2$ (see Example 4.8).

6. $\mathcal{A}_\Omega$-classification of maps

From the work of Mather it is clear that the $\mathcal{A}$-stable map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ are all quasi-homogeneous. The classification of the $\mathcal{A}_\Omega$-stable germs therefore coincides with the corresponding $\mathcal{A}$-classification for $\mathbb{K} = \mathbb{C}$, for $\mathbb{K} = \mathbb{R}$ a given $\mathcal{A}$-orbit corresponds to one or two $\mathcal{A}_\Omega$-orbits. Also, Mather's nice pairs of dimensions (n, p) are the same for $\mathcal{A}$ and $\mathcal{A}_\Omega$. In the present section we describe the classification of the $\mathcal{A}_\Omega$-simple orbits in those dimensions (n, p) for which the $\mathcal{A}$-simple orbits are known and we study the foliation of $\mathcal{A}$-orbits of modality one and two by $\mathcal{A}_\Omega$-orbits for $(n, 2)$, $n \geq 2$. We also relate the number of $\mathcal{A}_f$-moduli to the dimension of $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$, which requires some notation for the tangent spaces to orbits.

Given a map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ we use coordinates $x = (x_1, \ldots, x_n)$ in the source and $(X_1, \ldots, X_p)$ in the target, and we also denote the vector fields $\partial/\partial x_i$ and $\partial/\partial X_i$ by e_i. Recall the usual definition of the $\mathcal{A}$-tangent space $T\mathcal{A} \cdot f = tf(\mathcal{M}_n \theta_n) + \omega f(\mathcal{M}_p \theta_p)$, where $\mathcal{M}_n$ and $\mathcal{M}_p$ denote the maximal ideals in the local rings C_n and C_p of function-germs in the source and target of f, respectively, and where $tf : \theta_n \to \theta_f$, $a \mapsto df(a)$ and $\omega f : \theta_p \to \theta_f$, $b \mapsto b \circ f$ are homomorphisms from the C_n-module of source vector fields θ_n and the C_p-module of target vector fields θ_p into the C_n-module θ_f of smooth sections of $f^*T\mathbb{K}^p$. For the subgroup $\mathcal{A}_\Omega = \mathcal{R} \times \mathcal{L}_\Omega$ of $\mathcal{A}$ we have to restrict θ_p to divergence free vector fields, the left tangent space $T\mathcal{L}_\Omega \cdot f$ is no longer a C_p-module. Let Λ_d denote the $\mathbb{K}$-vector space of homogeneous divergence free vector fields in $\mathbb{K}^p$ of degree d. Notice that Λ_d is the kernel of the epimorphism

$$\mathrm{div} : (\theta_p)_{(d)} := \frac{\mathcal{M}_p^d \cdot \theta_p}{\mathcal{M}_p^{d+1} \cdot \theta_p} \to H_{(d-1)} := \frac{\mathcal{M}_p^{d-1}}{\mathcal{M}_p^d},$$

which maps a vector field on $\mathbb{K}^p$ of degree d to its divergence. Hence

$$\dim \Lambda_d = \dim(\theta_p)_{(d)} - \dim H_{(d-1)} = (p-1)\binom{p+d-1}{d} + \binom{p+d-2}{d}.$$

The $\dim \Lambda_d$ vector fields

$$\prod_{l \neq i} X_l^{\alpha_l} e_i, \ \sum_l \alpha_l = d, \ i = 1, \ldots, p$$

and (setting $h_{X_i} := \partial h / \partial X_i$)

$$-h_{X_j} e_1 + h_{X_1} e_j, \ h = \prod_l X_l^{\alpha_l}, \ \alpha_1, \alpha_j \geq 1, \sum_l \alpha_l = d+1, \ j = 2, \ldots, p$$

are clearly linearly independent and hence form a basis for Λ_d. The tangent space to the $\mathcal{L}_\Omega$-orbit at f is then given by $T\mathcal{L}_\Omega \cdot f = f^* \oplus_{d\geq 1} \Lambda_d$. As usual, for any group of equivalences $\mathcal{G}$ we denote the extended pseudogroup of non-origin preserving diffeomorphisms by $\mathcal{G}_e$.

The following result is the infinitesimal version (and, of course, a direct consequence) of Theorem 5.17, but we give a short direct proof.

Proposition 6.1. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a weakly quasi-homogeneous map-germ, then we have an inclusion*

$$T\mathcal{L} \cdot f \subset T\mathcal{A}_\Omega \cdot f,$$

and hence $T\mathcal{A}_\Omega \cdot f = T\mathcal{A} \cdot f$. One also has $T\mathcal{A}_{\Omega,e} \cdot f = T\mathcal{A}_e \cdot f$.

Proof. Fixing coordinates, we suppose that $f = (f_1, \ldots, f_p)$ is weighted homogeneous with non-negative degrees and positive total degree $\delta_1 + \ldots + \delta_p$ for integer weights $w_1, \ldots, w_n$. Let $X^\alpha = \prod_l X_l^{\alpha_l}$ and $|\alpha| \geq 0$. The following elements of $T\mathcal{A}_\Omega \cdot f$ yield $\omega f(X^\alpha X_i \cdot \partial/\partial X_i) \in T\mathcal{L} \cdot f$, $i = 1, \ldots, p$:

$$\omega f(-(1+\alpha_j)X_1 X^\alpha \cdot \partial/\partial X_1 + (1+\alpha_1)X_j X^\alpha \cdot \partial/\partial X_j),\ j = 2, \ldots, p$$

together with

$$tf(f^*(X^\alpha)\sum_{i=1}^{n} w_i x_i \cdot \partial/\partial x_i) - \sum_{j=2}^{p} \delta_j \cdot \omega f(-\frac{1+\alpha_j}{1+\alpha_1} X^\alpha X_1 \cdot \partial/\partial X_1 + X^\alpha X_j \cdot \partial/\partial X_j)$$

$$= (1+\alpha_1)^{-1} \sum_{j=1}^{p} (1+\alpha_j)\delta_j \cdot \omega f(X^\alpha X_1 \cdot \partial/\partial X_1)$$

(notice that $\sum_j (1+\alpha_j)\delta_j \neq 0$, for any exponent vector α, is equivalent to the weak quasi-homogeneity of f). And the remaining elements of $T\mathcal{L} \cdot f$ are of the form

$$\omega f(\prod_{l\neq i} X_l^{\alpha_l} \cdot \partial/\partial X_i) \in T\mathcal{L}_\Omega \cdot f.$$

The desired inclusion now follows, and the last statement is trivial, because the constant target vector fields are divergence free. □

Remark 6.2. Notice that the above proof for $\mathcal{A}_\Omega = \mathcal{R} \times \mathcal{L}_\Omega$ cannot be adapted to $\mathcal{A}'_\Omega := \mathcal{R}_\Omega \times \mathcal{L}$, but we can obtain a corresponding result for the subgroup $\mathcal{K}'_\Omega$ of the contact group $\mathcal{K}$ for which the diffeomorphisms on the right are volume-preserving (simply exchange the roles of source and target vector fields in the above proof, and use multiplication by $x^\alpha \in C_n$). For f quasi-homogeneous for non-negative weights w_i and positive total weight $w_1 + \ldots + w_n$ we obtain that $\mathcal{K}'_\Omega \cdot f = T\mathcal{K} \cdot f$, and for general f one obtains a formula for $\dim_{\mathbb{K}} T\mathcal{K} \cdot f / T\mathcal{K}'_\Omega \cdot f$. Notice that considering volume-preserving diffeomorphisms on the right is more closely related to earlier work on the isochore Morse lemma [39, 10] (and can be considered as an extension of this work on functions to the case of mappings). But, in fact, the $\mathcal{K}'_\Omega$-classification of map-germs f corresponds to the classification of varieties $V = f^{-1}(0)$ up to volume-preserving diffeomorphisms in Sections 3 and 4 in the same way as ordinary $\mathcal{K}$-equivalence corresponds to the equivalence of varieties up to ambient diffeomorphisms (at least when $C_n/f^*\mathcal{M}_p$ is reduced).

The criterion in the next easy lemma is sufficient for detecting in the existing classifications of $\mathcal{A}$-simple orbits those which are foliated by an r-parameter family, $r \geq 1$, of $\mathcal{A}_\Omega$-orbits.

Lemma 6.3. *Consider a germ $f_u : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of the form $f_u = f + u \cdot M$, where f is a quasi-homogeneous germ, $u \in \mathbb{K}$ and $M = X^\alpha \cdot \partial/\partial X_j \notin T\mathcal{A} \cdot f = T\mathcal{A}_\Omega \cdot f$ is a monomial vector of positive weighted degree (with respect to the weights of f). Then we have the following:*

(i) The coefficient u is not a modulus for $\mathcal{A}$-equivalence.

(ii) For a set of weights for which f is weighted homogeneous, let $(\theta_n)_0$, $(\theta_p)_0$ and $(\theta_f)_0$ denote the filtration-0 parts of the modules of source-, target-vector fields and vector fields along f, respectively. If the kernel of the linear map

$$\gamma(f) : (\theta_n)_0 \oplus (\theta_p)_0 \to (\theta_f)_0, \ (a, b) \mapsto tf(a) - \omega f(b),$$

of $\mathbb{K}$-vector spaces is 1-dimensional then u is an $\mathcal{A}_\Omega$-modulus of f_u.

Proof. Let f be weighted-homogeneous for the weights $w_1, \dots, w_n$, and associate to the target variables the weights $\delta_1, \dots, \delta_p$. Then the weighted degree of $\partial/\partial X_i$ is $-\delta_i$ so that f has filtration 0 and M has filtration $r > 0$.

For $\mathcal{A}$-equivalence we consider the following element of $T\mathcal{A} \cdot f_u$:

$$tf_u\Big(\sum_{i=1}^{n} w_i x_i \cdot \partial/\partial x_i\Big) - \omega f_u\Big(\sum_{j=1}^{p} \delta_j X_j \cdot \partial/\partial X_j\Big) = ruM.$$

From Mather's lemma (Lemma 3.1 in [30]) we conclude that the connected components of $\mathbb{K} \setminus \{0\}$ of the parameter axis lie in a single $\mathcal{A}$-orbit, hence u is not a modulus for $\mathcal{A}$.

For the second statement we observe that $\dim \ker \gamma(f) = 1$ implies that this kernel is spanned by the pair of Euler fields (E_w, E_δ) (which is unique up to a multiplication by an element of $\mathbb{K}^*$). And $M \notin T\mathcal{A} \cdot f$ implies that the only generator of M in $T\mathcal{A} \cdot f_u$ must be of the form $tf_u(a) - \omega f_u(b)$ with (a, b) a non-zero multiple of (E_w, E_δ). But E_δ has non-zero divergence, hence this generator does not belong to $T\mathcal{A}_\Omega \cdot f_u$. Now Mather's lemma implies that u is a modulus for $\mathcal{A}_\Omega$. $\square$

We can distinguish three types of weakly quasi-homogeneous map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$: (a) those that are quasi-homogeneous, (b) those that are not quasi-homogeneous and have an $\mathcal{A}$-representative with a zero component function (their image lies in a smooth hypersurface in $\mathbb{K}^p$) and (c) the remaining ones. Looking at the existing classifications of $\mathcal{A}$-simple germs we see that all non-weakly quasi-homogeneous map-germs satisfy condition (ii) of the lemma above and hence have an $\mathcal{A}_\Omega$-modulus along the positive filtration term. Then for dimensions (n, p) with $p < 2n$ the weakly quasi-homogeneous $\mathcal{A}$-simple germs are all in fact quasi-homogeneous (i.e. of type (a)). For $p = 2n$ and $n \geq 2$ the weakly quasi-homogeneous germs are of type (c) or are quasi-homogeneous, see the classification in [24]. For $p > 2n$ and $n \geq 2$ we have all three types of weakly quasi-homogeneous germs. Finally, for curves in $\mathbb{K}^p$ the weakly quasi-homogeneous germs for $p = 2$ are all quasi-homogeneous, for $p \geq 3$ they are quasi-homogeneous or of type (b) (see [6, 3]).

Example 6.4. Consider the series 22_k of map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{2n}, 0)$, $n \geq 3$, from the classification of $\mathcal{A}$-simple germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^{2n}, 0)$, $n \geq 2$, in [24] given by:

$$g_k = (x_1, \dots, x_{n-1}, x_1 y + y^3, x_2 y, \dots, x_{n-1} y, x_1 y^2 + y^{2k+1}, x_2 y^2 + y^4), k \geq 2.$$

The germs 22_k are not semi-quasihomogeneous: if we write $g_k = f + y^{2k+1} \cdot e_{2n-1}$ then the weighted homogeneous initial part f is not $\mathcal{A}$-finite. For $n = 3$ all the germs 22_k are $\mathcal{A}$-simple, for $n \geq 4$ only 22_2 is $\mathcal{A}$-simple (the germs $22_{\geq 3}$ do not have an $\mathcal{A}$-modulus, but they lie in the closure of non-simple $\mathcal{A}$-orbits).

Now consider $\mathcal{A}_\Omega$-equivalence. Writing $f_u = f + u \cdot y^{2k+1} \cdot e_{2n-1}$ we see that $\dim\ker\gamma(f) = n - 2$. For $n = 3$ part (ii) of Lemma 6.3 implies that the coefficient u is an $\mathcal{A}_\Omega$-modulus. For $n \geq 4$ the germs f_u are weakly quasi-homogeneous of type (c) (take $w(x_1) = w(x_2) = w(y) = 0$ and $w(x_i) = 1$, $i \geq 3$) and $\mathcal{A}_\Omega$-equivalent to g_k (for $u \neq 0$).

Next, one obtains the classification of $\mathcal{A}_\Omega$-simple orbits from the existing $\mathcal{A}$-classification using the following

Remark 6.5. An $\mathcal{A}$-simple germ in the existing classifications is $\mathcal{A}_\Omega$-simple if and only if it does not lie in the closure of the orbit of any non-weakly quasihomogeneous germ.

As an example, let us consider map-germs with target dimension $p = 2$. The classification of $\mathcal{A}_\Omega$-simple map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^2, 0)$ in the curve-case $n = 1$ has been described in [23] (for target dimension 2 the symplectic forms are volume forms). For any $n \geq 2$ we have the following classification over $\mathbb{K} = \mathbb{C}$.

Proposition 6.6. *(i) The following map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ represent the $\mathcal{A}_\Omega$-orbits within the $\mathcal{A}$-simple orbits (here "type r" refers to the notation in* [34, 36] *for the $\mathcal{A}$-orbits).*

$\mathcal{A}_\Omega$-modality 0: (x, y) (type 1); $(x, y^2 + \sum_{i=1}^{n-2} z_i^2)$ (type 2); $(x, xy + y^3 + \sum_{i=1}^{n-2} z_i^2)$ (type 3); $(x, y^3 + x^k y + \sum_{i=1}^{n-2} z_i^2)$, $k > 1$ (type 4_k); $(x, xy + y^4 + \sum_{i=1}^{n-2} z_i^2)$ (type 5).

$\mathcal{A}_\Omega$-modality ≥ 1: $(x, xy + y^5 + ay^7 + \sum_{i=1}^{n-2} z_i^2)$ (types 6,7); $(*)$ $(x, xy^2 + y^4 + \sum_{k \geq 2} a_k y^{2k+1} + \sum_{i=1}^{n-2} z_i^2)$ (types 11_{2k+1}); $(x, xy^2 + y^5 + ay^6 + by^9 + \sum_{i=1}^{n-2} z_i^2)$ (types 12,13,14); $(x, x^2 y + y^4 + ay^5 + \sum_{i=1}^{n-2} z_i^2)$ (types 16,17); $(*)$ $(x^2 + ay^{2l+1}, y^2 + x^{2m+1})$, $l \geq m \geq 1$ (type $\mathrm{II}_{2,2}^{l,m}$).

(ii) The $\mathcal{A}$-unimodal orbits of map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ of lowest codimension in their $\mathcal{K}$-orbit, containing the remaining $\mathcal{A}$-unimodal orbits in their closures, have the following representatives (see [35]*):*

$$(x, y^4 + x^3 y + ax^2 y^2 + x^3 y^2 + \sum_{i=1}^{n-2} z_i^2),\ a \neq -3/2 \text{ (type 19)}$$

$$(x, xy + y^6 + y^8 + ay^9 + \sum_{i=1}^{n-2} z_i^2) \text{ (type 8)}$$

$$(x, xy + y^3 + ay^2 z + z^3 + z^5) \text{ (type } \mathbf{I}).$$

For $\mathcal{A}_\Omega$-equivalence the corresponding normal forms are:

$$(x, y^4 + x^3 y + ax^2 y^2 + bx^3 y^2 + \sum_{i=1}^{n-2} z_i^2)$$

$$(x, xy + y^6 + ay^8 + by^9 + cy^{14} + \sum_{i=1}^{n-2} z_i^2)$$

$$(x, xy + y^3 + ay^2z + z^3 + bz^5).$$

All $\mathcal{A}$-unimodal germs therefore have $\mathcal{A}_\Omega$-modality at least two.

Proof. The classification of the orbits of $\mathcal{A}_\Omega$-modality 0 follows from the remark preceding the statement of the proposition.

For the semi-quasihomogeneous map-germs $f = f_0 + f_+$ the $\mathcal{A}$-determinacy degree, say r, of the initial part f_0 is an upper bound for the $\mathcal{A}_\Omega$-determinacy degree of f. The normal forms of such f can then be obtained with the usual techniques (Mather's lemma) applied to the Lie group $\mathcal{A}_\Omega^r$ of r-jets of elements of $\mathcal{A}_\Omega$ (we omit the details of these calculations).

Finally, the map-germs $f = f_0 + f_+$ marked by an $(*)$ are not semi-quasihomogeneous. As a consequence we cannot use the $\mathcal{A}$-determinacy degree of the weighted-homogeneous initial parts f_0 as an upper bound for the $\mathcal{A}_\Omega$-determinacy degree of f. For the germs $\mathrm{II}_{2,2}^{l,m}$ we can show that these are formally $(2l+1)$-$\mathcal{A}_\Omega$-determined (for $a \neq 0$). For the germs 11_{2k+1} we only know that a_2 is an $\mathcal{A}_\Omega$-modulus (and we can set a_3 and a_4 to zero, provided a_2 is non-zero). See also remark 6.8 (below) on the problem of $\mathcal{A}_\Omega$-determinacy. □

Denote the number of $\mathcal{G}$-moduli ($\mathcal{G} = \mathcal{A}$ or $\mathcal{A}_\Omega$) of a map-germ f by $m_{\mathcal{G}}(f)$, and let $m_{\mathcal{A}/\mathcal{A}_\Omega}(f) := \dim_{\mathbb{K}}(T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f)$ be the dimension of the $\mathcal{A}_\Omega$ moduli space. Then $m_{\mathcal{A}_\Omega}(f) = m_{\mathcal{A}}(f) + m_{\mathcal{A}/\mathcal{A}_\Omega}(f)$, where $m_{\mathcal{A}/\mathcal{A}_\Omega}(f)$ is equal to the number of $\mathcal{A}_f$ moduli by Corollary 5.8 and the following result.

Proposition 6.7. *There is an isomorphism*

$$\frac{T\mathcal{A} \cdot f}{T\mathcal{A}_\Omega \cdot f} \cong \frac{C_p}{\mathrm{div}(Lift_0(f))}.$$

Proof. We use the notation $T_{id}\mathcal{A} = T_{id}\mathcal{R} + T_{id}\mathcal{L}$ for the tangent space to the $\mathcal{A}$-group at the identity, rather than the usual notation $T_{id}\mathcal{R} = \mathcal{M}_n \cdot \theta_n$ and $T_{id}\mathcal{L} = \mathcal{M}_p \cdot \theta_p$.

The map $\beta : T_{id}\mathcal{A} \to C_p,\ (a, b) \mapsto \mathrm{div} b$ is surjective with kernel $T_{id}\mathcal{A}_\Omega$, hence there is an isomorphism $\bar{\beta} : T_{id}\mathcal{A}/T_{id}\mathcal{A}_\Omega \to C_p$. And the map

$$\bar{\gamma} : \frac{T_{id}\mathcal{A}}{T_{id}\mathcal{A}_\Omega} \to \frac{\mathcal{M}_n \cdot \theta_f}{T\mathcal{A}_\Omega \cdot f},\ [(a, b)] \mapsto [tf(a) - wf(b)]$$

is independent of the choice of $(\tilde{a}, \tilde{b}) \in [(a, b)]$ and hence well-defined. Its image is $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$ and

$$\ker \bar{\gamma} = \{[(a, b)] : tf(a) - wf(b) \in T\mathcal{A}_\Omega \cdot f\}.$$

Hence $C_p/\bar{\beta}(\ker \bar{\gamma})$ is isomorphic to $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$. And it is easy to see that $\bar{\beta}(\ker \bar{\gamma}) = \mathrm{div}(Lift_0(f))$. □

Remark 6.8. For $\mathcal{A}$-finite map-germs f, that are not semi-quasihomogeneous, two questions arise: (i) is such an f always $\mathcal{A}_\Omega$-finite? (ii) Is there an estimate for the $\mathcal{A}_\Omega$-determinacy degree?

(i) Composing the above isomorphism $\sigma : C_p/\mathrm{div}(Lift_0(f)) \to T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$ with an epimorphism $g : \mathcal{M}_n \cdot \theta_f / T\mathcal{A}_\Omega \cdot f \to \mathcal{M}_n \cdot \theta_f / T\mathcal{A} \cdot f$ with kernel $T\mathcal{A} \cdot f / T\mathcal{A}_\Omega \cdot f$

(coming from the second isomorphism theorem for the vector spaces $T\mathcal{A}_\Omega \cdot f \subset T\mathcal{A} \cdot f \subset \mathcal{M}_n \cdot \theta_f$) we get an exact sequence

$$0 \longrightarrow \frac{C_p}{\operatorname{div}(Lift_0(f))} \xrightarrow{\sigma} \frac{\mathcal{M}_n \cdot \theta_f}{T\mathcal{A}_\Omega \cdot f} \xrightarrow{g} \frac{\mathcal{M}_n \cdot \theta_f}{T\mathcal{A} \cdot f} \longrightarrow 0.$$

Hence we have:

$$\operatorname{cod}(\mathcal{A}_\Omega, f) = \operatorname{cod}(\mathcal{A}, f) + m_{\mathcal{A}/\mathcal{A}_\Omega}(f),$$

and for $\mathcal{A}$-finite f the germ $\tilde{f}$ of f at any point of a punctured neighborhood of the origin in the source is $\mathcal{A}$-stable, hence weighted homogeneous, so that $m_{\mathcal{A}/\mathcal{A}_\Omega}(\tilde{f}) = 0$. Unfortunately, $\operatorname{div}(Lift_0(f))$ is not an ideal in C_p, so we cannot conclude (from the Nullstellensatz) that $m_{\mathcal{A}/\mathcal{A}_\Omega}(f)$ is finite. For plane curves, Ishikawa and Janeczko [23] have shown that $\operatorname{cod}(\mathcal{A}_{\Omega,e}, f)$ is equal to the delta invariant $\delta(f)$, and it is known that $\delta(f) < \infty$ is equivalent to the $\mathcal{A}$-finiteness of f. The analogue of the delta invariant for map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^{2n}, 0)$, $n \geq 2$, is the double-point number $d(f)$, whose finiteness is again equivalent to the $\mathcal{A}$-finiteness of f (see [24]), but the equation $\operatorname{cod}(\mathcal{A}_{\Omega,e}, f) = d(f)$ does not hold for $n \geq 2$.

(ii) Suppose that for an r-$\mathcal{A}$-determined map-germ f the following holds: $\mathcal{M}_n^l \theta_f \subset T\mathcal{K}_e \cdot f$ ($\mathcal{K}_e$ is the extended contact group) and $\mathcal{M}_p^k \subset \operatorname{div}(Lift_0(f))$ (so that f is $\mathcal{A}_\Omega$-finite). Then f is $(r + l(k+2) - 1)$-$\mathcal{A}_\Omega$-determined. To see this, let $\mathcal{G}_s$ denote the subgroup of $\mathcal{G}$ consisting of elements of $\mathcal{G}$ with s-jet the identity and apply the inverse $\bar{\beta}^{-1}$ of the isomorphism $\bar{\beta} : T_{id}\mathcal{A}/T_{id}\mathcal{A}_\Omega \to C_p$ in the proof of the above proposition to $\mathcal{M}_p^k$. It follows that $T\mathcal{A}_{\Omega,k+1} \cdot f = T\mathcal{A}_{k+1} \cdot f$, and 2.7.3 of [7] then implies that f is $(r + l(k+2) - 1)$-determined for $\mathcal{A}_{k+1}$ and hence for $\mathcal{A}_\Omega$.

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