MEIL 2020/2021, Calculus 2, Lecture 6
Implicit function, manifolds, conditional extrema

Implicit Functions

Let be given $\boldsymbol{f} : D \to \mathbb{R}^m$, where $D \subset \mathbb{R}^k \times \mathbb{R}^m$. Let:
$H = \{(\boldsymbol{x}, \boldsymbol{y}) \in D \ : \ \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = 0\}$, where $\boldsymbol{x} \in \mathbb{R}^k$, $\boldsymbol{y} \in \mathbb{R}^m$.
We want to treat the set H as a graph of a function $\boldsymbol{y}(\boldsymbol{x})$. Using this point of view we say that the function $\boldsymbol{y}(\boldsymbol{x})$ is in implicit form (implicit definition of a function).
The equation $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = 0$ we can treat as system of:
m - equations: $f_1 = 0$, $f_2 = 0$, $\ldots f_m = 0$ with
m unknowns:mi $y_1, y_2, \ldots y_m$ and
k parameters $x_1, x_2, \ldots x_k$.
This usually a complicated, nonlinear system of equations that cannot be solved (or we don't want to solve it). However, we want to get some information about the solution indirectly (without solving, just by investigating properties of the system).
The Implicit Functions Theorem gives the conditions for:
1. existence of $\boldsymbol{y}(\boldsymbol{x}) \iff$ unique solution of that system
2. properties of $\boldsymbol{y}(\boldsymbol{x})$ - differentiability with respect to parameters $\boldsymbol{x}$

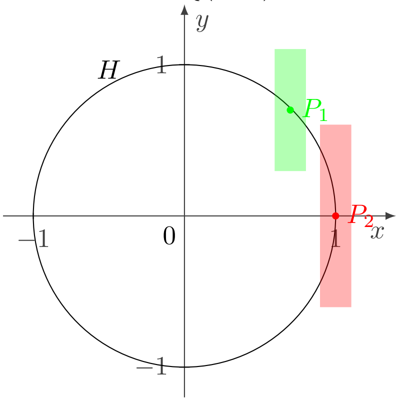
Theorem (Implicit Functions):

Let be given $(\boldsymbol{x_0}, \boldsymbol{y_0}) \in H$ and a ball $B_0 = B((\boldsymbol{x_0}, \boldsymbol{y_0}), r) \subset D$. Assume that f is C^1 class on B_0.
Assume that $\left|\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{x_0}, \boldsymbol{y_0})\right| \neq 0$ (determinant of $m \times m$ matrix). Then:
There exist balls $B_x = B(\boldsymbol{x_0}, r_1)$, $B_y = B(\boldsymbol{y_0}, r_2)$ such that $B_x \times B_y \subset B_0$ and
there exist a C^1 class function $\boldsymbol{g} : B_x \to B_y$ such that $H \cap (B_x \times B_y) = \{(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) : \boldsymbol{x} \in B_x\}$

Remark 1: When $\boldsymbol{f}$ is a C^n class function then so is $\boldsymbol{g}$.
Remark 2: The symbol $\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}$ means not a partial derivative but derivative of a function $\mathbb{R}^m \to \mathbb{R}^m$ thus a quadratic size m matrix.
Remark 3: The theorem conclusion states that locally (in small neighbourhood of $(\boldsymbol{x_0}, \boldsymbol{y_0})$) the system of equations for every $\boldsymbol{x}$ has a unique solution $\boldsymbol{y}$ and that arising function of parameters $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$ is regular (C^1 class).
Remark 4: The sketch of proof:
$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = 0$ and $\boldsymbol{f}(\boldsymbol{x_0}, \boldsymbol{y_0}) = 0 \implies$
$\Delta f = 0$
$\Delta f \approx \mathrm{d}f \implies$
$\mathrm{d}f = 0 \implies$
$\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\mathrm{d}\boldsymbol{x} + \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\mathrm{d}\boldsymbol{y} = 0$
$\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\mathrm{d}\boldsymbol{y} = -\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\mathrm{d}\boldsymbol{x}$
This is a linear system of equations (m-equations , m-unknowns , $k-$ parameters). This linear system has a unique solution when determinant of the matrix $\left[\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\right]$ is nonzero. Then nonlinear system of equations has nice properties (uniqueness and C^1 class).
Remark 5: Once we know that implicit function derivative exists, we can find the derivative using chain rule:
$\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) + \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{x}) \cdot \boldsymbol{g}'(\boldsymbol{x}) = 0 \implies$

$$[\boldsymbol{g}'(\boldsymbol{x})] = -\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x},\boldsymbol{y})\right]^{-1} \cdot \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x},\boldsymbol{y})\right]$$

Example 1: $k = 1$, $m = 1$, $f(x,y) = x^2 + y^2 - 1$
The set $H = \{(x,y) : x^2 + y^2 - 1 = 0\}$ is a circle.



Point $P_1(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \in H$

$\frac{\partial f}{\partial y} = 2y|_{P_1} = \sqrt{2} \neq 0$

Implicit function theorem says that there exists neighbourhood of P_1 (green box) such that set of solutions is a smooth curve (graph of C^1 class function).

Point $P_2(1,0) \in H$

$\frac{\partial f}{\partial y} = 2y|_{P_2} = 0$

Assumptions of implicit function theorem do not hold. We can observe that for any neighbourhood of P_2 (red box) set of solutions is not a graph of a function (on left hand side of 1 two solutions, on right hand side of 1 no solutions).

Example 1: $k = 1$, $m = 1$.
Find first and second derivative of implicit function $y(x)$ defined by equation: $xy + \ln(x-y) - 2 = 0$ at point $P(2,1)$.

Example 2: $k = 1$, $m = 2$, $\boldsymbol{y} = (y_1, y_2)$, $\boldsymbol{f}(x, y_1, y_2) = (f_1, f_2)$,
$f_1(x, y_1, y_2) =$, $f_1(x, y_1, y_2) = \ln(x - y_1) + xy_1y_2 - 2$, $f_2(x, y_1, y_2) = x + y_1 + y_2 + e^{y_1 - y_2} - 5$
Find derivative of implicit function $\boldsymbol{y}(x)$ at point $P(2,1,1)$.

Example 2 solution:
Set $H = \{\boldsymbol{f}(x, y_1, y_2) = 0\} \implies$

$$\begin{cases} \ln(x - y_1) + xy_1y_2 - 2 = 0 \\ x + y_1 + y_2 + e^{y_1 - y_2} - 5 = 0 \end{cases}$$

This is a system of equations with two unknowns y_1 and y_2 and one parameter x.
The set of solutions is a complicated curve in a 3-dimensional space. We treat this set locally as graph of $\boldsymbol{y}(x) = (y_1(x), y_2(x))$. We cannot solve this system of equations so we apply the implicit functions theorem. We check the assumptions. The only non trivial assumption is:

$$\left|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(P)\right| \neq 0$$

where

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

Partial derivatives:

$\dfrac{\partial f_1}{\partial y_1} = -\dfrac{1}{x-y_1} + xy_2\Big|_P = 1$

$\dfrac{\partial f_1}{\partial y_2} = xy_1|_P = 2$

$\dfrac{\partial f_2}{\partial y_1} = 1 + e^{y_1-y_2}\big|_P = 2$

$\dfrac{\partial f_2}{\partial y_2} = 1 - e^{y_1-y_2}\big|_P = 0$

Thus

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ \\ 2 & 0 \end{vmatrix} = -4 \neq 0$$

All assumptions hold so from the implicit functions theorem follows that there exists the function $\boldsymbol{y}(x) = (y_1(x), y_2(x))$ and the function is C^∞ class.

To find its derivative we use the chain rule.

$\boldsymbol{f}(x, \boldsymbol{y}(x)) = 0 \implies$

$x \to (x, y_1(x), y_2(x))$ - inside function (unknown)

$(x, y_1, y_2) \to (f_1, f_2)$ - outside function (known)

$x \to (0, 0)$ - composite function (known)

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We calculate partial derivatives:

$\dfrac{\partial f_1}{\partial x} = \dfrac{1}{x-y_1} + y_1y_2\Big|_P = 2$

$\dfrac{\partial f_2}{\partial x} = 1|_P = 1$

Hence

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies$$

$$\begin{cases} 2 + y_1' + 2y_2' = 0 \\ 2 + 2y_1' = 0 \end{cases} \implies$$

$y_1' = -1 \implies y_2' = -\frac{1}{2}$

$\boldsymbol{y}' = (-1, -\frac{1}{2})$

Remark 1: We cannot solve this system of equations but we can analyse properties of solutions using the implicit functions theorem and find derivative using chain rule.

Remark 2: System of equations for unknown y_1, y_2 is complicated (nonlinear) while the system of equations their derivatives y_1', y_2' is simple (nonlinear). We can observe the benefits of linearisation process at work.

End of example 2 solution

Example 3: Find local extrema of implicit function $y(x)$ defined by the equation: $x^3 + 2y^3 - 3xy = 0$.

Example 3 solution:

Let's define $f(x, y) = x^3 + 2y^3 - 3xy$

$f(x, y)$ is C^∞ class function, its domain is $\mathbb{R}^2$ an open set.

Th implicit function $y(x)$ exists and is C^∞ class in neighbourhood of such points that $\dfrac{\partial f}{\partial y} \neq 0$

$\dfrac{\partial f}{\partial y} = 6y^2 - 3x \neq 0$

Using chain rule we can find $y'(x)$:

$\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} y'(x) = 0 \implies y'(x) = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$\dfrac{\partial f}{\partial x} = 3x^2 - 3y$

$y'(x) = -\dfrac{3x^2 - 3y}{6y^2 - 3x}$

We are looking for critical points: $y' = 0 \implies \frac{\partial f}{\partial x} = 0$.

We solve the system of equations:

$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ f = 0 \end{cases} \implies$

$\begin{cases} 3x^2 - 3y = 0 \\ x^3 + 2y^3 - 3xy = 0 \end{cases} \implies$

$y = x^2 \implies x^3 + 2x^6 - 3x^3 = 0 \implies 2x^3(x^3 - 1) = 0 \implies x_1 = 0\,,\, x_2 = 1$

There are two solutions of the system: $P_1(0,0)$, $P_2(1,1)$. We have to check our assumption $\dfrac{\partial f}{\partial y} \neq 0$:

$\dfrac{\partial f}{\partial y}(P_1) = 0$ - we reject this point (this is probably a singular point of implicit function)

$\dfrac{\partial f}{\partial y}(P_2) = 3 \neq 0$ - this is regular, critical point of implicit function

We are going to find second derivative of implicit function: y'' . We different the equation

$\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} y'(x) = 0$ with respect to x . Using the chain rule:

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial x \partial y} y'(x) + \big(\dfrac{\partial^2 f}{\partial y \partial x} + \dfrac{\partial^2 f}{\partial y^2} y'(x)\big) y'(x) + \dfrac{\partial f}{\partial y} y''(x) = 0$

Because $y'(P_2) = 0$ then:

$y'' = -\frac{\frac{\partial^2 f}{\partial x^2}}{\frac{\partial f}{\partial y}}$

$\dfrac{\partial^2 f}{\partial x^2} = 6x \implies y'' = -\dfrac{6x}{6y^2 - 3x}\Bigg|_{P_2} = -2 < 0$

At the point $P_2(1,1)$ implicit function $y(x)$ has local maximum.

Remark 1: In the equation $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} y'(x) = 0$ argument are:

$\dfrac{\partial f}{\partial x}(x, y(x)) + \dfrac{\partial f}{\partial y}(x, y(x)) \cdot y'(x) = 0$

Remark 2: When $\dfrac{\partial f}{\partial y} = 0$ we don't even know if the function $y(x)$ exists. Thus the idea of extremum possibly doesn't make sense.

End of example 3 solution

Manifolds

Definition: A set $H \subset \mathbb{R}^{k+m}$ is said to be k-dimensional differential manifold (or k-dimensional smooth surface) when for each point $P(\boldsymbol{x_0}, \boldsymbol{y_0}) \in H$ there exist permutation of arguments, open sets $A \subset \mathbb{R}^k$, $B \subset \mathbb{R}^m$, $\boldsymbol{x_0} \in A$, $\boldsymbol{y_0} \in B$ and C^1-class function $\boldsymbol{h} : A \to B$ such, that

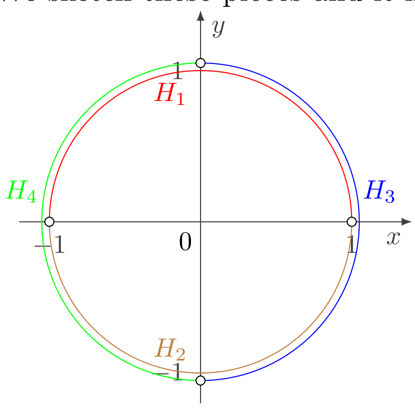
$H \cap (A \times B) = \{(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x})); \boldsymbol{x} \in A\}$.
Permutation of arguments means for example $P(x_1, y_1, y_2, x_2, y_3, \dots)$

Remark 1: Permutation of arguments means that $(\boldsymbol{x}, \boldsymbol{y})$ not necessary means $(x_1, x_2, \dots, x_k, y_1, y_2, \dots y_m)$ but for example $(x_1, y_1, y_2, x_2, y_3, \dots)$. This corresponds to rotation of the basis (coordinate system). An equivalent condition (to permutation): 'there exist basis'.
Remark 2: Sometimes the and C^1-class condition is replaced by C^n1-class or C^∞-class. It depends on applications of that idea (manifold idea).
Remark 3: Sometimes there is one more condition: 'H is connected' which means there is one piece only. According to the definition above there can be many (even infinitely many) pieces.
Permutation of arguments means that $(\boldsymbol{x}, \boldsymbol{y})$ not necessary means $(x_1, x_2, \dots, x_k, y_1, y_2, \dots y_m)$ but for example $(x_1, y_1, y_2, x_2, y_3, \dots)$. This corresponds to rotation of the basis (coordinate system). An equivalent condition (to permutation): 'there exist basis'.
Remark 4: 1-dimensional manifolds we call curves, 2-dimensional: surfaces.
Remark 5: When we say that a mechanical system has k degrees of freedom we mean that the set of all possible states of that system is a k-dimensional smooth manifold. Thus the definition above is at the same time the definition of dimension and the definition of objects for which the dimension idea makes sense.

Theorem: Let be given a C^1-class function $\boldsymbol{f} : D \to \mathbb{R}^m$, where $D \subset \mathbb{R}^{k+m}$ is open.
Let $H = \{\boldsymbol{x} \in D : \boldsymbol{f}(\boldsymbol{x}) = 0\}$. If
$(\forall \boldsymbol{x} \in H)$ rank $\boldsymbol{f}'(\boldsymbol{x}) = m$
then H is a smooth k-dimensional manifold.

Example 1: Investigate properties of the set $H = \{(x, y) : x^2 + y^2 - 1 = 0\}$

Method 1: As a Graph of a function
To investigate properties of this set using this point of view we have to split this set into 4 (not disjoint) pieces. This is the least number of pieces that we have to take into account.
$H = H_1 \cup H_2 \cup H_3 \cup H_4$
$H_1 = \{(x, h_1(x)) : x \in (-1, 1)\}$, $h_1(x) = \sqrt{1 - x^2}$
$H_2 = \{(x, h_2(x)) : x \in (-1, 1)\}$, $h_2(x) = -\sqrt{1 - x^2}$
$H_3 = \{(h_3(y), y) : y \in (-1, 1)\}$, $h_3(y) = \sqrt{1 - y^2}$
$H_4 = \{(h_4(y), y) : y \in (-1, 1)\}$, $h_4(y) = -\sqrt{1 - y^2}$
All the functions have on open domain in R^1 and are C^∞-class so H_1 , H_2 , H_3 , H_4 are smooth 1-dimensional manifolds. Because the pieces are not disjoint so generally it is not simple to conclude that also H is a smooth 1-dimensional manifold. We have to analyse how these pieces are connected. We sketch these pieces and it is clear that H is a smooth 1-dimensional manifold.



Method 2: Implicit form
$H = \{(x, y) : f(x, y) = 0\}$, where
$f(x, y) = x^2 + y^2 - 1$, domain of f : $D = \mathbb{R}^2$ - open, f is C^∞-class
We calculate:

$f' = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = [2x \quad 2y]$

rank $f' =$ rank $[2x \quad 2y] = \begin{cases} 0 & \text{for} \quad (x,y) = (0,0) \\ 1 & \text{otherwise} \end{cases}$

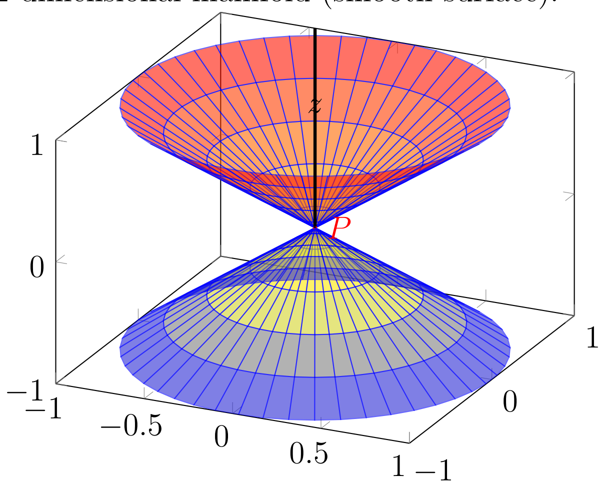
The point $(0,0) \notin H$ thus
for all $(x,y) \in H$ rank $f'(x,y) = 1$ so
H is a smooth 1-dimensional manifold (smooth curve).

Example 2: Investigate properties of the set $H = \{(x,y,z) : x^2 + y^2 - z^2 = 0\}$

We will use the implicit form method:
$H = \{(x,y,z) : f(x,y,z) = 0\}$, $f(x,y,z) = x^2 + y^2 - z^2$, domain $D = \mathbb{R}^3$ - an open set, f is C^∞-class
We calculate
$f' = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = [2x \quad 2y \quad -2z]$

rank $f' =$ rank $[2x \quad 2y \quad -2z] = \begin{cases} 0 & \text{for} \quad (x,y,z) = (0,0,0) \\ 1 & \text{otherwise} \end{cases}$

The point $P(0,0,0) \in H$ is then probably a singular point of H. Set H without this point is a 2-dimensional manifold (smooth surface).



Conditional Extrema

Definition: Let be given functions $f : D_f \to \mathbb{R}$ and $\boldsymbol{g} : D_g \to \mathbb{R}^m$ (constraints, conditions) where $D_g, D_f \subset \mathbb{R}^{k+m}$. Let $H = \{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) = 0\}$. Conditional extrema of the function f subject to constraints g are local extrema of f on the set $H \cap D_f$.

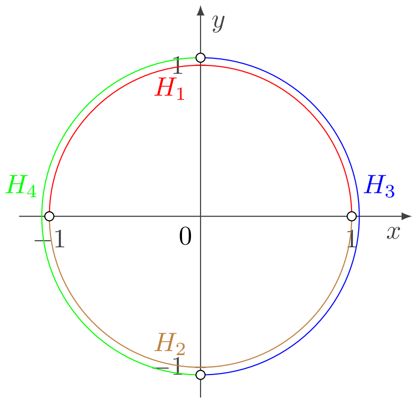
Remark: H is usually a manifold.

Theorem: Let H be the graph of a function: $H = \{(\boldsymbol{x}, \boldsymbol{w}(\boldsymbol{x})) : \boldsymbol{x} \in D_w\}$, $\boldsymbol{w} : D_g \to \mathbb{R}^m$, $D_w \subset \mathbb{R}^k$, $\boldsymbol{w}$ is continuous. Let $\boldsymbol{x_0} \in D_w$ and $(\boldsymbol{x_0}, \boldsymbol{w}(\boldsymbol{x_0})) \in D_f$. The function f has local minimum on H at $(\boldsymbol{x_0}, \boldsymbol{w}(\boldsymbol{x_0}))$ if and only if the function $h(\boldsymbol{x}) = f(\boldsymbol{x}, \boldsymbol{w}(\boldsymbol{x}))$ has local minimum at $\boldsymbol{x_0}$.

Remark 1: Similar theorem holds for local maximum.
Remark 2: To apply this theorem for conditional extrema we have to solve the constrains equations $\boldsymbol{g}(\boldsymbol{x}) = 0 \iff \boldsymbol{x} = (\boldsymbol{t}, \boldsymbol{w}(\boldsymbol{t}))$. Very often we have to split the set H into several pieces.

Example: Find local extrema of the function $f(x,y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$

Example solution:



We split the set $H = \{(x,y) : x^2 + y^2 = 1\}$ into four (not disjoint) pieces $H = H_1 \cup H_2 \cup H_3 \cup H_4$:
$H_1 = \{(x,y) : x \in (-1,1)\, ,\, y = w_1(x)\}$, $w_1(x) = \sqrt{1-x^2}$
$H_2 = \{(x,y) : x \in (-1,1)\, ,\, y = w_2(x)\}$, $w_2(x) = -\sqrt{1-x^2}$
$H_3 = \{(x,y) : y \in (-1,1)\, ,\, x = w_3(y)\}$, $w_3(y) = \sqrt{1-y^2}$
$H_4 = \{(x,y) : y \in (-1,1)\, ,\, x = w_4(y)\}$, $w_4(y) = -\sqrt{1-y^2}$
On H_1:
$h(x) = f(x, w_1(x)) = x^2 - (1-x^2) = 2x^2 - 1\ ,\ x \in (-1,1)\}$
We are looking for local extrema of $h(x)$.
Critical points: $h'(x) = 0 \implies$
$4x = 0$
$x = 0$
$h''(x) = 4 > 0 \implies$
the function h has local minimum at $x = 0$, thus the function f has conditional minimum at $P_1 = (0, w_1(0)) = (0,1)$.
Similarly we can find extrema on H_2 ,H_3 , and H_4 .
End of example solution: