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Singularities of Lagrangian Varieties and Related Topics

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Abstract. We study the classification problem for generic projections of Lagrangian submanifolds. A classification list for symmetric Lagrangian submanifolds is obtained and the generic evolutions of symmetric caustics are illustrated. We show how the singular Lagrangian varieties appear in the invariant theory of binary forms and we introduce the basic concepts of the desingularization procedure. Applications to differential geometry, geometrical optics, and mechanics are presented.

AMS subject classifications (1991). 57R45, 58F05, 58F14, 53B50, 78A05, 58C28.

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0. Introduction

Most of the basic concepts of symplectic geometry were introduced by Lagrange [32] in connection with the study of the motion of planets in the context of analytical mechanics (cf. [1]). During the long evolution of mechanics, from Huygens and Newton through to Jacobi, Hamilton, Poincaré, Birkhoff, and Weyl, the language of modern geometry was developed and symplectic geometry appeared to be a very universal and useful part of it (cf. [1], [23], [50]).

Working in the quasiclassical methods of quantum mechanics, Maslov [36], introduced the notion of a Lagrangian submanifold, which appeared to play an especially important role in symplectic geometry and its physical applications. Quite recent results of Guillemin–Sternberg show that the idea that Lagrangian submanifolds are the morphisms of the symplectic 'category' [23, 51] is very unifying for symplectic geometry. It is also very fruitful in applications, in view of [24, 50], which assign classes of generalized functions and differential systems (e.g. Gauss–Manin systems) to Lagrangian submanifolds of an appropriate cotangent bundle.

The focal sets of systems of rays, the wave-front evolution and caustics were investigated long ago by Huygens, Leibnitz, Bernoulli, Jacobi, and Morse (cf. [45, 30, 16]). However, R. Thom [46] was the first to emphasize the fundamental importance of the theory of stable singularities of smooth mappings. He initiated the application of singularity theory to these systems and suggested the extended use of stable Lagrangian submanifolds to model the internal peculiarities of physical systems in general. Following Thom's proposal, Arnol'd gave the classification of simple and stable singularities of the so-called Lagrange projections [3]. Then it became obvious that singularities of Lagrangian submanifolds, or the corresponding Lagrangian projections, and their stable or generic properties, are indispensable for understanding a large part of geometrical optics [14, 4], variational calculus [6], optimization and control theory [47, 39], Hamilton–Jacobi equations [22], holonomic systems of differential equations [41], classical mechanics [1], and field theory [23, 50]. All these directions indicate their own problems and methods of solution, as well suggesting new investigations in singularity theory itself [11, 12, 8, 26]. The recently found connection between oscillating integrals and mixed Hodge structures [48] on the one hand and the theory of transformations of differential systems [41] on the other hand, connects the theory of the singularities of Lagrangian varieties to the mainstream of contemporary mathematics. The standard theory of singularities of Lagrangian projections was extended and generalized by Arnol'd [5] (inspired by Melrose [37]) who showed that the singularities of Lagrangian varieties of systems of gliding rays along an obstacle in Euclidean space, substantially complete the amazing correspondence between the Coxeter groups generated by reflections A_k, B_k, C_k, D_k, E_k, F_4, G_2, H_2, H_3, H_4, and simple singularities [9]. In fact, the group of symmetries of the icosahedron governs the singularities of systems of rays on the plane in the presence of an obstacle with an inflection point [43, 4]. The standard singularities of Lagrangian projections also relate to phase transitions in the classical thermodynamical systems. The phenomena of the coexistence of phases and their equilibrium, even in the critical region, has been explained by singular models of Lagrangian submanifolds and their reduced local forms [28].

The present paper does not pretend to give a complete report on the achievements in the theory of singularities of Lagrangian varieties. We emphasize only some representative directions, which seem very important from the point of view of applications to mathematical physics. Section 1, is entirely basic and introductory. We give the classical classification results for stable Lagrange projections and their connection to elementary concepts in optics and mechanics. In Section 2, we introduce the notion of caustics, bicaustics, and quasicaustics, and show how they appear in symplectic bifurcation theory. Using symplectic canonical varieties, we reformulate the results of Scherbak ([43]) concerning the classification of wave fronts on smooth obstacles. We report on the new ideas and classification results in the theory of generic invariant Lagrangian submanifolds (i.e. Lagrangian submanifolds with symmetry). In Section 3, we present a complete list of generic Z_2-invariant bicaustics in R^2 and R^3. This is a new result. Behind Arnold's [7], [19] theory of singular Lagrangian varieties lies some interesting combinatorial and algebraic material concerning the theory of invariants of binary forms. In Section 4, we give the most elementary properties of the symplectic geometry of binary forms and construct the canonical generalization of open swallowtails.

Another natural question is how to desingularize singular objects. For singular

Lagrangian varieties, the most straightforward desingularization procedure is the local 'inverse reduction' procedure. This is presented in Section 5. We desingularize concrete classes of singular Lagrangian varieties and provide a list of local models of generic pullbacks of smooth Lagrangian submanifolds.

1. Singularities of Lagrange Projections

Let P be a manifold, and ω be a 2-form on P. The pair (P, ω) is called a symplectic manifold if ω is closed, i.e. $d\omega = 0$, and nondegenerate [1]. The last condition implies that the dimension of P is even. The representative model of a symplectic manifold is a cotangent bundle T^*X, endowed with the canonical 2-form $\omega_X = d\theta_X$, where the 1-form θ_X on T^*X is defined by

$$\langle u, \theta_X\rangle = \langle T^*\pi_X(u), \tau_{T^*X}(u)\rangle, \quad \text{for each } u \in TT^*X.$$

The mapping $T\pi_X: TT^*X \to TX$ is the tangent mapping of $\pi_X: T^*X \to X$, and $\tau_{T^*X}: TT^*X \to T^*X$ is the tangent bundle projection. If (x_i) are local coordinates introduced in X, and (x_i, ξ_i) are corresponding local coordinates in T^*X, then ω_X has the normal form $\omega_X = \sum_{i=1}^{n} d\xi_i \wedge dx_i$ (Darboux form [9]).

We say that an immersed submanifold $L \subset T^*X$ is Lagrangian if $\omega_X|_L = 0$ and $\dim L = \dim X$. Let $i: L \subset T^*X$ be a Lagrangian immersion of L, i.e. $i^*\omega = 0$. Then the mapping $\pi_X \circ i: L \to X$ is called a Lagrange projection (cf. [3]). In applications to physics, say static mechanical systems, the manifold X appears as the configuration space of a static system and T^*X is the generalized force bundle. The system under consideration is usually characterized by its configuration-force relation represented geometrically by a Lagrangian submanifold of T^*X (provided the system is reciprocal [50]). Let L be transversal, at a point $p \in L$, to the fibers of the canonical fibration π_X. Then, in a neighbourhood of the point $\pi_X(p) \in X$, there exists a smooth function $S: X \to R$, such that L is locally defined as the graph of the section $dS: X \to T^*X$. S is called the generating function or phase function (see, e.g., [24]). If the transversality condition is not fulfilled, which corresponds to the loss of static controllability of the system [1] (i.e. spontaneous flows, matter movements, phase transitions etc.), then L is represented by a family of functions on a manifold Λ, parametrized by X; $G: X \times \Lambda \to R$. It is called a Morse family [50] or generating family, and defines L by the canonical symplectic reduction procedure [1]:

$$L = \left\{(x, \xi) \in T^*X;\ \text{there exists } \lambda \in \Lambda, \text{ such that} \right.$$
$$\left. \xi = \frac{\partial G}{\partial x}(x, \lambda) \text{ and } 0 = \frac{\partial G}{\partial \lambda}(x, \lambda)\right\}$$

near $p \in T^*X$. The mapping $d_\lambda G: X \times \Lambda \to \Lambda$, $\Lambda \cong R^k$ is assumed to have maximal rank at $(\pi_X(p), 0) \in X \times \Lambda$.

The subset of X, say Σ_L, or, in another words, the set of critical values of the Lagrange projection $\pi_X \circ i$ is given by the following equations [16].

$$\Sigma_L = \left\{ x \in X;\ \text{there exists } \lambda \in \Lambda, \text{ such that } 0 = \frac{\partial G}{\partial \lambda}(x, \lambda) \text{ and } \det\left(\frac{\partial^2 G}{\partial \lambda_i \partial \lambda_j}\right)(x, \lambda) = 0 \right\}. \tag{1.1}$$

An analytic properties of Σ_L and its geometric structure are most important for applications. It represents the caustics in geometrical optics [33], lines of the phase diagram where phase transitions take place [17], and transition points from one equilibrium state to another inequivalent one in mechanical systems [23].

EXAMPLE 1.1. Let X be the Euclidean plane. The equations:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad f = -k(r-a)\cos\theta, \qquad g = -k(r-a)\sin\theta,$$

describe a Lagrangian submanifold L of T^*X with coordinates $(\theta, r) \in S^1 \times R$. This represents the position-force relation for a material point subject to a simple restoring force whose centre of attraction is allowed to move freely on the circle $x^2 + y^2 = a^2$. A Morse family for L defined on the manifold $X \times S^1$ with coordinates (x, y, θ), is given by the following function of potential energy,

$$G(x, y, \theta) = \frac{k}{2}\{(x - a\cos\theta)^2 + (y - a\sin\theta)^2\}.$$

It is easily seen that the loss of control appears at the critical values of the respective Lagrange projection, i.e. $\Sigma_L = \{(x, y) = 0\}$. This point is a very 'unusual' critical point. Its counterimage is $(\pi_X \circ i)^{-1}(\Sigma_L) = S^1$. Degeneration, or complexity, of this point is most interesting because it determines the singular sets for slightly deformed stabilized systems, which are more practical models of real open systems. For this sort of deformation, say L', we need more real elastic; namely with nonzero length of the unstretched elastics and some extra weak elastic joining the material point to the point $(0, -b)$ on the plane X.

$$\begin{aligned}(\varepsilon_1, \varepsilon_2, k_1) &\to G'(x, y, \theta)\\ &= \frac{k}{2}(((x_1 - a\cos\theta)^2 + (y_1 - a\sin\theta)^2)^{1/2} - \varepsilon_1)^2 +\\ &\quad + (k_1/2)(((a\cos\theta)^2 + (a\sin\theta + b)^2)^{1/2} - \varepsilon_2)^2, \quad b > 0.\end{aligned}$$

In effect, we obtain the three-parameter $-(\varepsilon_1, \varepsilon_2, k_1)-$family of Zeeman's machines (see [42], [53]) with critical sets illustrated in Figure 1.

These critical sets or, more precisely, their geometrical form, are not removable by small deformations of $\tilde{L}$ (illustrated in Figure 2) and they describe the interesting properties of the system.

To describe the structure of the position of the Lagrangian submanifold $L \subset T^*X$

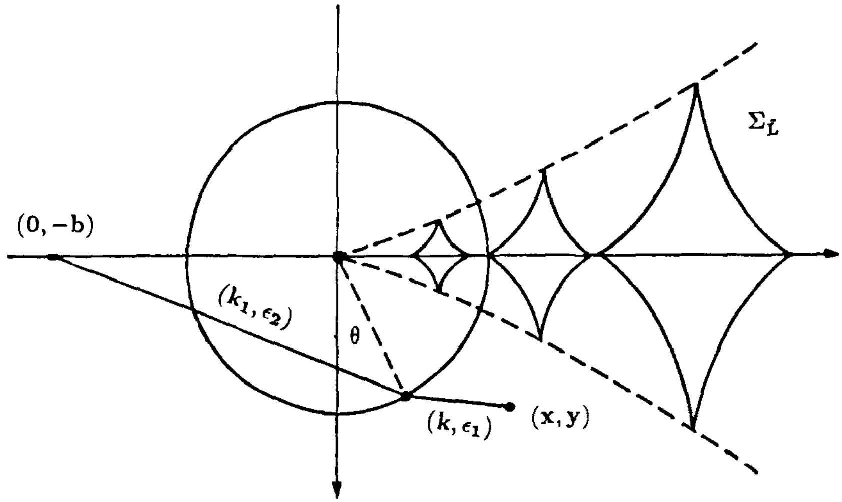


Fig. 1.

relative to the nontransversal fibre of T^*X, we must introduce the notion of fibered equivalence of germs of Lagrangian submanifolds [3]. To avoid inessential rigour, we frequently speak about submanifolds instead of their germs.

DEFINITION 1.2. Let (L_1, p_1), (L_2, p_2) be two germs of Lagrangian submanifolds of T^*X. They are called equivalent iff there exists a germ of a symplectomorphism

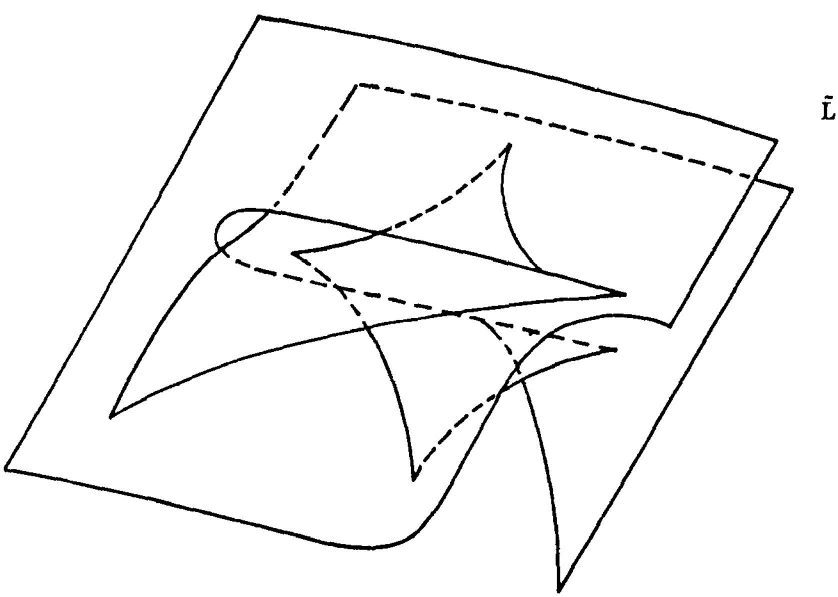


Fig. 2.

$\Phi:(T^*X, p_1) \to (T^*X, p_2)$, preserving the fibre structure $\pi_X: T^*X \to X$ and such that

$$\Phi(L_1) = L_2, \qquad \Phi(p_1) = p_2.$$

Let us introduce the Whitney C^∞-topology [35] on the space of Lagrange immersions $i: L \to T^*X$. Then we can say that a Lagrangian submanifold $L_1 \subset T^*X$ is close to L if the corresponding immersion $i_1: L_1 \to T^*X$ belongs to some Whitney-open neighbourhood of i.

DEFINITION 1.3. We say that $(L, p) \subset T^*X$ is Lagrange stable if, for any close Lagrangian submanifold $L_1 \subset T^*X$, one can find a point $p_1 \in T^*X$, close to p, such that (L_1, p_1), (L, p) are equivalent.

It is shown in [3] (cf. [8]) that Lagrange stability is a generic property for $n = \dim X \leqslant 5$. In this case, there are 16 stable models of Lagrange projections including the trivial one. They are given by the following Morse families:

$$A_1: F(x, \lambda) = 0, \qquad A_2: F(x, \lambda) = \lambda^3 + \lambda x_1,$$

$$A_3^\pm: F(x, \lambda) = \pm\lambda^4 + x_2\lambda^2 + x_1\lambda,$$

$$A_4: F(x, \lambda) = \lambda^5 + x_3\lambda^3 + x_2\lambda^2 + x_1\lambda,$$

$$A_5^\pm: F(x, \lambda) = \pm\lambda^6 + x_4\lambda^4 + x_3\lambda^3 + x_2\lambda^2 + x_1\lambda,$$

$$A_6: F(x, \lambda) = \lambda^7 + x_5\lambda^5 + x_4\lambda^4 + x_3\lambda^3 + x_2\lambda^2 + x_1\lambda,$$

$$D_4^\pm: F(x, \lambda) = \lambda_1^3 \pm \lambda_1\lambda_2^2 + x_3\lambda_1^2 + x_2\lambda_1 + x_1\lambda_2,$$

$$D_5^\pm: F(x, \lambda) = \pm\lambda_1^4 + \lambda_1\lambda_2^2 + x_4\lambda_1^3 + x_3\lambda_1^2 + x_2\lambda_1 + x_1\lambda_2,$$

$$D_6^\pm: F(x, \lambda) = \pm\lambda_1^5 + \lambda_1\lambda_2^2 + x_5\lambda_1^4 + x_4\lambda_1^3 + x_3\lambda_1^2 + x_2\lambda_1 + x_1\lambda_2,$$

$$E_6^\pm: F(x, \lambda) = \pm\lambda_2^4 + \lambda_1^3 + x_5\lambda_1^2\lambda_2 + x_4\lambda_1\lambda_2 + x_3\lambda_1^2 + x_2\lambda_1 + x_1\lambda_2.$$

Their appearance in the respective dimensions is the following:

$\dim X = 1$; A_1, A_2,

$\dim X = 2$; $A_1, A_2, A_3^\pm$,

$\dim X = 3$; $A_1, A_2, A_3^\pm, A_4, D_4^\pm$,

$\dim X = 4$; $A_1, A_2, A_3^\pm, A_4, D_4^\pm, A_5^\pm, D_5^\pm$,

$\dim X = 5$; $A_1, A_2, A_3^\pm, A_4, D_4^\pm, A_5^\pm, D_5^\pm, A_6, D_6^\pm, E_6^\pm$.

Illustrations of the corresponding critical sets (caustics) for the stable models in R^3 can be found in [8] (cf. also [9]). In general, the situation is much more complicated. There appear functional moduli and the normal forms for generic Lagrangian submanifolds have only obtained for $n \leqslant 10$ [8]. These results are valuable from the point of view of applications, so we briefly describe the procedure for obtaining the prenormal forms of generic Morse families.

Let $(L, p) \subset T^*X$ be a Lagrangian germ and $F: X \times R^k, 0 \to R$ be the germ of a corresponding Morse family. Let $\mathscr{E}_k$ denote the space of germs of smooth functions at

0 taking 0 to 0. A miniversal deformation of f is a function-term $G: R^{\mu-1} \times R^k, 0 \to R$,

$$G(u, \lambda) := f(\lambda) + \sum_{i=1}^{\mu-m-1} u_i \phi_i(\lambda) + \sum_{i=1}^{m} v_i \phi_{\mu-1-m+i}(\lambda), \tag{1.2}$$

where v_i are modal parameters [52].

By the versality property of G [13], we have the prenormal form of F.

THEOREM 1.4. *Generic Morse families on $X \times \Lambda$ are symplectically equivalent (i.e. the corresponding Lagrangian germs are equivalent) to families of the form*

$$F(x, \lambda) = f(\lambda) + \sum_{i=1}^{\mu-m-1} x_i \phi_i(\lambda) + \sum_{i=1}^{m} \psi_i(x) \phi_{\mu-1-m+i}(\lambda), \tag{1.3}$$

where $\mu - m - 1 \leqslant \dim X$ and $(\psi_1, \ldots, \psi_m): R^n, 0 \to R^m, 0$ is a generic map-germ when restricted to the subspace of modal directions $M := \{x \in R^n; x_1 = 0, \ldots, x_{\mu-m-1} = 0\}$.

Proof. Let $F: R^n \times \Lambda, 0 \to R, 0$ be a Morse family germ, $f(\lambda) := F(0, \lambda)$. By the theorem on miniversal deformations ([35]), F is induced from the miniversal deformation (1.2), i.e.

$$F(x, \lambda) = G \circ \Xi(x, \lambda),$$

where $\Xi: R^n \times \Lambda, 0 \to R^{\mu-1} \times \Lambda, 0$ is an appropriate pullback. Now we can make a generic assumption that Ξ is transversal to M. Thus, by right equivalence in the space of map-germs $R^n \times \Lambda, 0 \to R^{\mu-1} \times \Lambda, 0$, (cf. [35]), which induces a symplectic equivalence of the corresponding Lagrangian germs (cf. [16]), we can reduce F to the prenormal form (1.3) (see also [8], Section 21).

EXAMPLE 1.5. In case G is unimodal deformation of singularity we have

$$F(x, \lambda) = f(\lambda) + \sum_{i=1}^{\mu-2} x_i \phi_i(\lambda) + \psi_1(x) \phi_{\mu-1}(\lambda).$$

The generic function ψ_1 has the following form

(1) If $n = \mu - 2$, then ψ_1 is a smooth functional modulus.
(2) If $n > \mu - 2$, then ψ_1 can be reduced by the right equivalence on M to the following normal forms

$$\psi_1(x) = x_{\mu-1} \quad \text{or} \quad \psi_1(x) = \xi(x_1, \ldots, x_{\mu-2}) \pm x_{\mu-1}^2 \pm \cdots \pm x_n^2,$$

where $\xi: R^{\mu-2}, 0 \to R, 0$ is a smooth modulus.

As an example, we consider the D_4-symmetric X_9 singularity organizing the caustic structure in the catastrophe optics of liquid drop lenses [40]. The miniversal (nonsymmetric) deformation of X_9; $\lambda_1^4 + a\lambda_1^2\lambda_2^2 + \lambda_2^4$, is given by

$$G(u, \lambda) = \lambda_1^4 + v_1\lambda_1^2\lambda_2^2 + \lambda_2^4 + u_1\lambda_1 + u_2\lambda_2 + u_3\lambda_1^2 + u_4\lambda_2^2 + \\ + u_5\lambda_1\lambda_2 + u_6\lambda_1\lambda_2^2 + u_7\lambda_1^2\lambda_2.$$

Thus, the generic Lagrangian germs organized by X_9 do appear in at least of

$\dim X = 7$. We have the corresponding Morse family germs:

$$F(x, \lambda) = \lambda_1^4 + \psi(x)_1\lambda_1^2\lambda_2^2 + \lambda_2^4 + x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_1^2 + x_4\lambda_2^2 + \\ + x_5\lambda_1\lambda_2 + x_6\lambda_1\lambda_2^2 + x_7\lambda_1^2\lambda_2,$$

where ψ_1 is a smooth function if $\dim X = 7$, and $\psi_1(x) = x_8$, or $\psi_1(x) = \xi(x_1, \ldots, x_7) \pm x_8^2 \pm \cdots \pm x_n^n$, if $n > 7$.

Remark 1.6. Singularities of Lagrange projections are also universal in mathematics itself (cf. [6], [9], [24], [49]). Let us recall now only the simplest examples coming from differential geometry.

(A) Let M be a connected complete Riemannian manifold, let TM be its tangent bundle, and let $\tau_M: TM \to M$ be the projection map. The tangent bundle of TM, TTM, decomposes naturally under the Levi-Civita connection into the direct sum $H + V$ of a horizontal subbundle H and a vertical subbundle V. At each point $x \in TM$, V_x is the kernel of the tangent mapping $T\tau_M: TTM \to TM$. The horizontal part at x, H_x, consists of all vectors which are tangent at x to the curves obtained by parallel translation of x along smooth curves in H. Here, H is the kernel of the map which is called the connection map (cf. [1]),

$$K: TTM \to TM, \; Kw = K\left(\frac{\mathrm{d}X}{\mathrm{d}t}(0)\right) = \frac{D}{\mathrm{d}t}X(0),$$

where w is the initial tangent vector to a curve $X(t) \in TM$.

The Riemannian metric g on M gives rise to an isomorphism θ between TM and T^*M. Under this isomorphism, an element $x \in T_pM$ is mapped to the unique element $\theta(x) \in T_p^*M$, which satisfies $\theta(x)(v) = \langle x, v\rangle_g$ for all $v \in T_pM$. The canonical symplectic structure on T^*M pulls back via this isomorphism to a symplectic form ω on TM. It has the following form

$$(\theta^*\omega_M)(v, w) = \omega(v, w) = \langle T\tau_M v, Kw\rangle_g - \langle Kv, T\tau_M w\rangle_g,$$

and is invariant under the geodesic flow.

For a smooth n-dimensional manifold M, we consider the space Γ of all smooth complete Riemannian metrics on M endowed with the C^∞ Whitney topology. For each $g \in \Gamma$, $p \in M$, $\exp_p: T_pM \to M$ is the smooth map which assigns to $v \in T_pM$ the terminal point of the unique geodesic curve $\gamma: [0, 1] \to M$ with $\gamma(0) = p$, $\dot\gamma(0) = v$. We can write $\exp_p$ as a Lagrangian map,

$$\exp_p = \tau_M \circ \Phi_1$$

restricted to T_pM, where $\Phi_t: TM \to TM$ is the geodesic flow. It was proved (see [49]) that there exists a residual subset $\Gamma' \subset \Gamma$ with the property that for each $g \in \Gamma'$, the family $\{\exp_p: T_pM \to M, p \in M\}$ is a generic n-parameter family of Lagrangian maps with the above classified singularities.

(B) The Gauss map $f: M \to S^n$, $\dim M = n$, of a submanifold $M \subset R^{n+1}$, is defined by the conormal bundle to M in T^*R^{n+1} and forms the constrained Lagrangian submanifold (cf. [26]). As the standard result, one obtains (cf. [49]) that the Gauss map and the exponential map have generically the same types of singularities as the classified singularities of Lagrange projections.

2. Caustics, Bicaustics, and Quasicaustics. Singular Systems of Rays

2.1. BICAUSTICS

We define the caustic of a Lagrangian map $\pi_X|_L: L \to X$, $\pi_X: T^*X \to X$, to be the set, ΣL, of its critical values. In applications of singularity theory to the time evolution of potential systems, e.g. the evolution of noninteracting particles in a potential field, we need the classification of possible evolutions of caustics. The problem was substantially solved in [7]. We show here its use in symplectic bifurcation theory.

Let $H: T^*X \to R$ be a time-function on the phase space, i.e. $H(p, q) = t$, for which 0 is not a critical value. For each co-isotropic hypersurface $H^{-1}(t) \subset T^*X$ we have a natural projection along characteristics,

$$\pi_t: H^{-1}(t) \to H^{-1}(t)/\sim_t = M \cong T^*Y \quad \text{(locally)},$$

onto the reduced symplectic manifold which has the local form T^*Y. Let L be a Lagrangian submanifold in T^*X. Its caustic, called the 'big caustic' is useful for characterizing generic one-parameter families of caustics. We define the corresponding one-parameter family of Lagrangian submanifolds in T^*Y

$$L_t := \pi_t(L \cap H^{-1}(t)).$$

$\Sigma' L_t$ will denote the singular set of the caustic ΣL_t – called also the virtual caustic at time t. We consider the set $\Sigma' L = \bigcup_{t \in I} (\Sigma' L_t, t) \subset Y \times I$

DEFINITION 2.1.1. The image of the set $\Sigma' L$, by the natural projection on the first factor $\pi_1: Y \times I \to Y$ is called the bicaustic corresponding to L.

The bicaustic in Y is the set of all singularities of virtual caustics moving in Y. We see that if the caustic of an initial ('big') L has a singularity of type A_{k+1} in R^k then the generic bicaustic has a singularity of type A_k in R^{k-1}. The normal forms of bicaustics (and evolving virtual caustics) are classified in [7] for $Y \cong R^2$, and $Y \cong R^3$

PROPOSITION 2.1.2. *In the general position, there is the following correspondence between the caustics of 'big' Lagrangian submanifolds L and their bicaustics*:

The plane:

$\Sigma L = A_3$, *bicaustic*: $\{u = 0\}$,
$\Sigma L = A_4$, *bicaustic*: $\{u^3 - v^2 = 0\}$,
$\Sigma L = D_4^-$, *bicaustic*: $\{u^3 + av^2u^2 + v^6 = 0\}$,
$\Sigma L = D_4^+$, *bicaustic*: $\{u = 0\}$.

The three-space:

$\Sigma L = A_3$, *bicaustics*: $\{u^2 w - v^2 = 0\}$ (Whitney's cross cap, Fig. 3a), $\{u = 0\}$,

$\Sigma L = A_4$, *bicaustics*: $\{u^2 - v^3 = 0\}$, $\{u^2 - v^3 w^2 = 0\}$ (Fig. 3b),

$\Sigma L = A_5$, *bicaustic*: A_4,

$\Sigma L = D_4^-$, *bicaustics*: $\{w(w - v^2)(w - (a + u)v^2) = 0\}$,
$\{w(w - v^2)(w - (a \pm u^2)v^2) = 0\}$, $a \in R$,

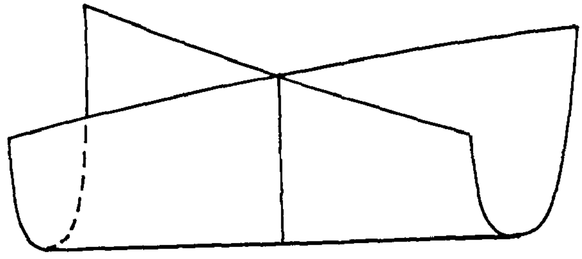
$\Sigma L = D_4^+$, *bicaustic*: $\{w = 0\}$

Particular cases of caustics are the caustics of geometrical optics. These are the envelopes of families of optical rays, i.e. geodesics, in X. Let X be endowed with a Riemannian metric g. Optical rays on X are defined by the geodesic flow of g on (T^*X, ω_X) [1]. X_g is a Hamiltonian vector field with energy function H_g: $T^*X \to R$, $H_g(p) = \frac{1}{2}\langle p, p\rangle_g$, where $\langle\cdot,\cdot\rangle_g$ is an inner product induced by g, so that $\omega_X(X_g, \cdot) = -\mathrm{d}H_g(\cdot)$. The optical caustics are singled out by the fact that the corresponding Lagrangian manifold, built by the rays, lies in the hypersurface of the phase space defined by the eikonal equation $\langle p, p\rangle_g = 1$. The classification lists of stable caustics are the same as the lists of optical caustics. The only differences start when considering the evolutions of caustics (see [40]). The global invariants and obstacles in appearing of some generic caustic evolutions as the optical ones was established by Chekanov [14].

2.2. QUASICAUSTICS

Structurally new caustics appear in optical diffraction through apertures and around

a)



b)

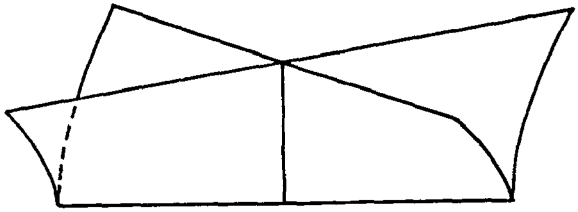


Fig. 3.

smooth obstacles [30]. These are caustics generated by the singular Lagrangian varieties first introduced by Pham [41]. Let g be the Euclidean metric on $X = R^3$. The associated phase space of rays (M, ω) is given by the symplectic reduction $\pi_M: H^{-1}(0) \to M \cong T^*S^2$, where $H^{-1}(0)$ is the zero level set of the Hamiltonian $H: T^*X \to R$, $H(p, q) = \frac{1}{2}(\|p\|^2 - 1)$. If we have a system of rays in X, we can look at it as a Lagrangian subvariety L of M. By taking its image under the symplectic relation $\operatorname{graph} \pi_M \subset (M \times T^*X; \pi_2^*\omega_X - \pi_1^*\omega)$ [1], we obtain the Lagrangian subvariety $\operatorname{graph} \pi_M(L)$ of T^*X (cf. [26]).

Consider now geometric diffraction due to a half-plane aperture [28]. Let $\{(x, y, z): z = \phi(x, y)\}$ (with ϕ normalized so that $\phi(0) = 0$, $D\phi(0) = 0$) be the initial wavefront in the presence of the aperture $\{(a, b, c);\ a \geqslant 0,\ c = mb - 1\}$, where $m \geqslant 0$ and $(a, b) \in R^2$ parametrize the aperture. Let $(a, b, x, y, u, v, w) \to F(a, b, x, y, u, v, w)$ be the optical distance function from the wavefront.

It is a function on a manifold with boundary $\{(a, b, x, y);\ a = 0\}$ parametrized by (u, v, w). One can write it explicitly here:

$$\begin{aligned} &F(a, b, x, y, u, v, w) \\ &\quad = ((x-a)^2 + (y-b)^2 + (\phi(x, y) - mb + 1)^2)^{1/2} + \\ &\qquad + ((u - a)^2 + (v - b)^2 + (w - mb + 1)^2)^{1/2}. \end{aligned}$$

The stationary condition leads to $m^2u^2 + v^2(m^2 - 1) - 2mv(1 + w) = 0$ and $v + m(1 + w) \leqslant 0$. These conditions define a half-cone of diffracted rays (Figure 4) [30]. The Lagrangian variety L of rays diffracted on this aperture is a union of two Lagrangian submanifolds; $L = L_1 \cup L_2$. L_1 is generated by F where a, b, x, y are the Morse parameters (cf. [28]) restricted to $a > 0$. L_2 is generated by

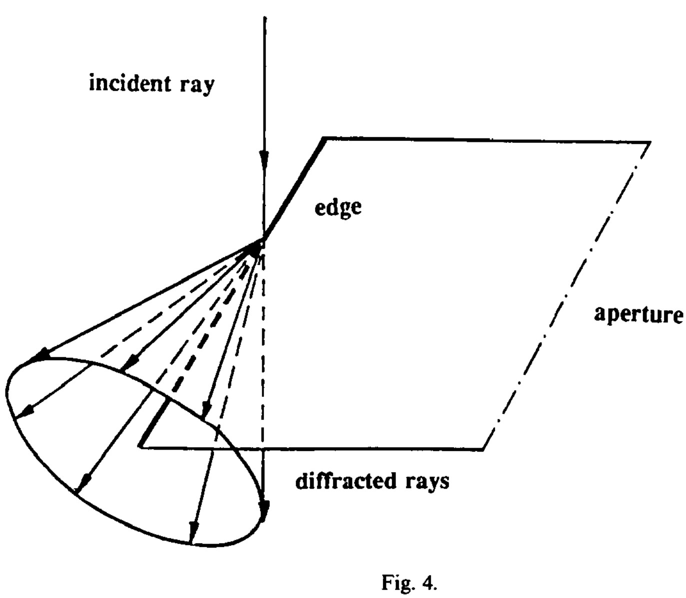


Fig. 4.

$\tilde{F}(b, x, y, u, v, w) := F(0, b, x, y, u, v, w)$ where b, x, y are the Morse parameters. Besides the ordinary caustic provide by the projections of L_1 and L_2, there is a caustic defined by the projection of intersection set $\pi_X(\bar{L}_1 \cap L_2)$.

DEFINITION 2.2.1. Let $F: R \times R^n \times X, 0 \to R$, $(y, x, a) \to F(y, x, a)$ be the family of function germs on hypersurface boundary $S = \{y = 0\}$. Then the set $Q_F = \pi_X(\bar{L}_1 \cap L_2)$ defined by

$$Q_F = \{a \in X;\ F(\cdot, a) \text{ has a critical point on } S\}$$

is called the quasicaustic of F.

Thus, caustics formed from the diffraction from simple apertures are determined by the singularities of functions on smooth boundaries classified in [8]. Their miniversal unfoldings are

$$\tilde{A}_\mu:\ \pm y \pm x^{\mu+1} + \sum_{i=1}^{\mu-1} a_i x^i, \quad \mu \geqslant 1,$$

$$B_\mu:\ \pm y^\mu \pm x^2 + \sum_{i=1}^{\mu-1} a_i y^{\mu-i}, \quad \mu \geqslant 2,$$

$$C_\mu:\ yx \pm x^\mu + \sum_{i=1}^{\mu-1} a_i x^{\mu-i}, \quad \mu \geqslant 2,$$

$$\tilde{D}_\mu:\ \pm y + x_1^2 x_2 \pm x_2^{\mu-1} + \sum_{i=1}^{\mu-2} a_i x_2^i + a_{\mu-1} x_1, \quad \mu \geqslant 4,$$

$$\tilde{E}_6:\ \pm y + x_1^3 \pm x_2^4 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_1 x_2^2,$$

$$\tilde{E}_7:\ \pm y + x_1^3 + x_1 x_2^3 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + {}$$
$$+ a_4 x_1 x_2 + a_5 x_2^3 + a_6 x_2^4,$$

$$\tilde{E}_8:\ \pm y + x_1^3 + x_2^5 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + {}$$
$$+ a_5 x_2^3 + a_6 x_1 x_2^2 + a_7 x_1 x_2^3,$$

$$F_4:\ \pm y^2 + x^3 + a_2 y + a_3 x + a_1 xy.$$

By the straightforward computations we obtain the following result,

PROPOSITION 2.2.2. *The quasicaustics for simple boundary singularities are*

$$\tilde{A}_\mu, \tilde{D}_\mu, \tilde{E}_k: Q_F = \varnothing,$$

$$B_\mu: Q_F = \{a \in C^{\mu-1};\ a_{\mu-1} = 0\},$$

$$C_\mu: Q_F = \{a \in C^{\mu-1};\ a_{\mu-1} = 0\},$$

$$F_4: Q_F = \{a \in C^3;\ a_2^2 + \tfrac{1}{3} a_1^2 a_3 = 0\} \quad \text{(Whitney's cross-cap).}$$

We see that the only stable singular quasicaustic for a half-plane aperture in R^3 is Whitney's cross-cap. It appears generically when the curve of rays passing through the edge of an aperture on the incident wave front is tangent to a constant curvature line on the wave front (cf. [43]).

Investigation of quasicaustics in the presence of system of apertures reduces the problem to the classification of singularities of functions on manifolds with corners and on singular varieties [11], [44]. As an example, we can consider the quasicaustic generated by the versal unfolding

$$F(y, x, a) = \pm y_1^2 \pm y_2^2 \pm x^2 + a_1 y_1 y_2 + a_2 y_1 + a_3 y_2$$

on the corner $\{y_1 = y_2 = 0\}$, which is illustrated in Figure 5.

2.3. SINGULAR SYSTEMS OF RAYS

A new area of investigation in the formation of caustics by diffraction around smooth obstacles was invented by Arnold [4] (cf. [30]). Consider an open subset S of an obstacle surface in R^3. Denote by l_1 the initial tangent line to a geodesic segment γ on S (incident ray belonging to M). Let l_2 be another tangent line to S. We say that l_2 is subordinate to l_1 with respect to an obstacle S if l_2 (or a piece in (R^3, S)) belongs to the geodesic segment with initial point and tangent vector γ (see [2]). The set

$$A = \{(l, \tilde{l}) \in \mathscr{P};\ \tilde{l} \text{ is subordinate to } l \text{ with respect to } S\}$$

where

$$A \subset \mathscr{P} = (M \times \tilde{M},\ \tilde{\omega} \ominus \omega),$$

and where $M = \tilde{M}$, M, $\tilde{M}$ are the symplectic spaces of lines in R^3. A is a Lagrangian subvariety of $\mathscr{P}$ defining the diffraction process around the obstacle S. If L is a Lagrangian subvariety of the space of incident rays M, then the outgoing diffracted rays form the Lagrangian subvariety of $\tilde{M}$. They are given by the image $A(L) = \{\tilde{l} \in \tilde{M}$: there exists $l \in M$ such that $(l, \tilde{l}) \in A\}$.

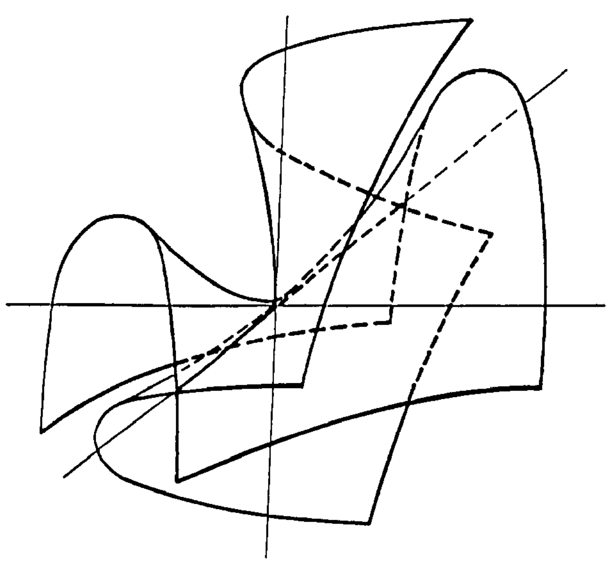


Fig. 5.

For the generic pairs (A, L) we have the following result (see Theorem 10 in [43]) concerning the classification of stable images $A(L)$ and their corresponding generating families.

PROPOSITION 2.3.1. (A). *For a generic obstacle curve in the plane, the only possible generic images by diffraction $A(L)$ are given by the following normal forms of generating families*

$$A_2: F_1(\lambda, q_1, q_2) = -\tfrac{1}{12}\lambda^3 + q_2\lambda - \tfrac{1}{2}q_1\lambda^2 \quad \textit{(obstacle curve-parabole).}$$

$$H_3: F_2(\lambda_1, \lambda_2, q_1, q_2) = \tfrac{9}{10}\lambda_1^5 - \lambda_2\lambda_1^3 + \tfrac{1}{2}\lambda_2^2\lambda_1 + q_2\lambda_2 - \tfrac{1}{2}q_1\lambda_2^2$$

(obstacle curve with an inflection point).

$$A_{2,2}: F_3(\lambda, q_1, q_2) = \tfrac{1}{2}\lambda|\lambda| + q_2\lambda - \tfrac{1}{2}q_1\lambda^2$$

(obstacle curve with a double tangent).

(B) *Let S be a generic obstacle-surface in R^3. Let the pair (A, L) be defined in a sufficiently small neighbourhood of a point on S. Then generically, the symplectic images $A(L)$ have the following generating families:*

$$\Xi_1: F = q_1,$$

$$\Xi_2: F = \tfrac{1}{5}\lambda^5 + \tfrac{2}{3}q_1\lambda^3 + q_1^2\lambda,$$

$$\Xi_3: F = \int_0^\lambda (t^3 + q_1 t + q_2)^2\, dt,$$

$$\Delta_2: F = \lambda^3 + q_1\lambda,$$

$$\Delta_3: F = \tfrac{1}{5}\lambda_1^5 + \tfrac{2}{3}\lambda_2\lambda_1^3 + \lambda_1\lambda_2^2 + q_1\lambda_2^2 + q_2\lambda_2,$$

$$\Delta_4: F = \int_0^{\lambda_1} (\lambda_2 + t^3 + q_1 t + q_2)^2\, dt + q_3\lambda_2,$$

$$\tilde{A}_3: F = \lambda^4 + q_1\lambda^2 + q_2\lambda,$$

$$\tilde{A}_4: F = \lambda^5 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda,$$

$$H_4: F = \tfrac{1}{5}\lambda_1^5 + \tfrac{2}{3}\lambda_1^3(q_1\lambda_2 + q_2) + \lambda_1(q_1\lambda_2 + q_2)^2 + \lambda_2^3 + q_3\lambda_2.$$

Remark 2.3.2. These families describe the strata of ordinary universal unfoldings with odd multiple singularities (see [43]).

3. Invariant Lagrangian Submanifolds and Classification of Symmetric Caustics

In the study of the wave pattern of high-frequency waves coming from a source and moving through the medium, the corresponding intensity of radiation is described by the asymptotics of the rapidly oscillating integrals [16], [43]. Asymptotically, as the frequency becomes infinite this intensity becomes infinite around the singularities of the Lagrange projections generated by the phase functions. If the source, as well as the boundary conditions (mirrors), exhibit symmetry properties, then the corresponding Lagrangian submanifolds describing the optical geometry of the system [28] possess these same symmetry properties. In fact, they are invariant with respect to the prescribed symplectic action of the compact Lie group G, defining the symmetry.

Let X be a smooth manifold with a smooth action of the compact Lie group G. This action extends to an action on the cotangent bundle T^*X which leaves the natural symplectic form invariant. If L is a G-invariant Lagrangian submanifold of T^*X, then the Lagrange projection $\pi_{X|L}: L \to X$ is G-equivariant and its discriminant, the caustic C_L of L, is a G-invariant subvariety of X. These symmetric discriminants determine the local bifurcation diagrams in the breaking of symmetry and structural phase transitions [21], [25]. In this section, we give a brief description of the theory of classification of G-invariant Lagrangian submanifolds. We use the theory to obtain *new* symmetric caustics.

We are interested in local properties of L so we may identify the manifold X with $V = R^n$, and assume that the action of G on $(V, 0)$ is linear and orthogonal. We identify T^*V with $V \oplus V^*$. The orthogonality of the action of G on V implies that V^* is isomorphic to V, as a G-space. If $(L, 0) \subset (T^*V, 0)$ is a G-invariant Lagrangian submanifold germ and $\pi_L: (L, 0) \to (V, 0)$ its associated G-invariant Lagrange projection then $\operatorname{Ker} D\pi_L(0) = T_0 L \cap V^*$ is a G-invariant subspace of V^* which we denote by W^*. We see that we can identify V with $W \oplus W^\perp$. Let $q_1, \ldots, q_k$ denote the coordinates for W, $q_{k+1}, \ldots, q_n$ for $W^\perp$, $p_1, \ldots, p_k$ for W^* and $p_{k+1}, \ldots, p_n$ for $(W^*)^\perp$. Thus, we have (see [25]) that there exists a smooth G-invariant function germ $F: \Lambda \oplus V \to R$, $F(\lambda_1, \ldots, \lambda_k, q_1, \ldots, q_n)$, the Morse family generating L. The Morse parameter space Λ is W.

DEFINITION 3.1. We say that two G-invariant Lagrangian submanifold germs $(L_j, 0) \subset (T^*V, 0)$, $(j = 1, 2)$ are symplectically equivalent if there exist germs of a G-equivariant symplectomorphism $\Phi: (T^*V, 0) \to (T^*V, 0)$ and a G-equivariant diffeomorphism $\phi: (V, 0) \to (V, 0)$ such that

$$\pi_V \circ \Phi = \phi \circ \pi_V \quad \text{and} \quad \Phi(L_1) \subseteq L_2.$$

For classification of local models of $(L, 0)$, we need to reformulate symplectic equivalence in terms of generating Morse families; two G-invariant Morse families $F_j: (\Lambda \oplus V, 0) \to R$, $(j = 1, 2)$ generate symplectically equivalent Lagrangian submanifolds if and only if there is a G-equivariant diffeomorphism germ $\Psi: (\Lambda \oplus V, 0) \to (\Lambda \oplus V, 0)$, a G-equivariant diffeomorphism germ $\psi: (V, 0) \to (V, 0)$ and a G-invariant function germ $\alpha: (V, 0) \to R$ such that

$$\pi_2 \circ \Psi = \psi \circ \pi_2,$$

and

$$F_1(\lambda, q) = F_2(\Psi(\lambda, q)) + \alpha(q),$$

where $\pi_2: \Lambda \oplus V \to V$ is the natural projection.

Infinitesimal Lagrange stability for invariant Morse families is defined in the usual way. Let $\mathscr{E}^G_{\lambda,q}$ (respectively $\mathscr{E}^G_q$) denote the ring of germs of G-invariant smooth functions on $\Lambda \oplus V$ (respectively, V). Let $\{\alpha_1, \ldots, \alpha_r\}$ denote a generating set for the $\mathscr{E}^G_{\lambda,q}$-module Θ^G_π consisting of germs of G-equivariant vector fields along the projection $\pi: \Lambda \oplus V \to \Lambda$. Let $\{\beta_1, \ldots, \beta_s\}$ denote a generating set for the $\mathscr{E}^G_q$-module, Θ^G_q, of germs of G-equivariant vector fields on $(V, 0)$. For any G-invariant function germ $F: (\Lambda \oplus V, 0) \to R$ we define the tangent space (cf. [35]),

$$T_G(F) = \mathscr{E}^G_{\lambda,q}\{\alpha_1 F, \ldots, \alpha_r F\} + \mathscr{E}^G_q\{\beta_1 F, \ldots, \beta_s F, 1\}$$

We denote the group of equivariant equivalences by $\mathscr{R}^+_G$. Then $T_G(F)$ is the tangent space to the $\mathscr{R}^+_G$ orbit of F. As in the standard singularity theory, we say that a G-invariant function germ $F: (\Lambda \oplus V, 0) \to R$ is infinitesimally stable ($\mathscr{R}^+_G$-stable) if $T_G(F) = \mathscr{E}^G_{\lambda,q}$.

Our approach to symplectic equivalence in the symmetric case is a generalization of the nonequivariant theory of Arnold and Zakalyukin [3], [52]. To specify the classification problem, we need to have the specified action of the group G. We consider corank 1 Lagrange projections from $\mathbf{Z}_2$-invariant Lagrangian submanifolds of T^*R^n, where $\mathbf{Z}_2$ acts on $V = R^n$ by

$$(x_1, \ldots, x_r, y_1, \ldots, y_s) \to (-x_1, \ldots, -x_r, y_1, \ldots, y_s), \quad n = r + s.$$

In this case, we have the following result (see [29])

PROPOSITION 3.2.

(a) *If $r \geqslant s$ then generic $\mathbf{Z}_2$-invariant Morse families are infinitesimally stable and are equivalent to the following families:*

$$\lambda^{2(k+1)} + \sum_{j=1}^{k} y_j \lambda^{2j} + \sum_{j=1}^{k} x_j \lambda^{2j-1}, \quad k \leqslant s.$$

(b) *If $s \geqslant r = 1$ then the infinitesimally stable Morse families are equivalent to the following families:*

$$\lambda^{2(k+1)} + \sum_{j=1}^{k} y_j \lambda^{2j} + \sum_{j=1}^{k-1} (\alpha_j + y_{k+j}) x_1 \lambda^{2j+1} + x_1 \lambda,$$

$$k \leqslant \tfrac{1}{2}(s+1); \ \alpha_j \in R.$$

(c) *If $r = 2$, $s < 5$ then the infinitesimally stable Morse families are equivalent to the families in* (a) *with $k \leqslant 2$. If $r = 2$, $s = 5$ then in addition to these, there are families equivalent to one of*

$$\lambda^8 + \sum_{j=1}^{3} y_j \lambda^{2j} + \{(\alpha_1 + y_4)x_1 + (\alpha_2 + y_5)x_2\}\lambda^5 + x_2\lambda^3 + x_1\lambda,$$

$$\alpha_1, \alpha_2 \in R,$$

$$\lambda^8 + \sum_{j=1}^{3} y_j \lambda^{2j} + \{(\alpha_1 + y_4)x_1 + y_5 x_2\}\lambda^3 + x_2\lambda^5 + x_1\lambda, \quad \alpha_1 \in R.$$

(d) *If $n \leqslant 7$ then the infinitesimally stable Morse families are equivalent to the families listed in* (a), (b) *and* (c).

PROPOSITION 3.3. (A) *If $n = 3$, $r = 1$ we have an extra two corank 2 infinitesimally stable $\mathbf{Z}_2$-invariant Morse families*

$$D_4^{\pm}: F(\lambda_1, \lambda_2, x_1, y_1, y_2) = \pm\lambda_1^2\lambda_2 + \lambda_2^3 + y_1\lambda_2^2 + x_1\lambda_1 + y_2\lambda_2,$$

representing the symmetric umbilics, where $\mathbf{Z}_2$ acts on Λ in the following way:

$$(\lambda_1, \lambda_2) \to (-\lambda_1, \lambda_2).$$

(B) *If $n = 3$, and $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ acts in V by*

$$(x_1, x_2, y_1) \to (-x_1, x_2, y_1), \qquad (x_1, x_2, y_1) \to (x_1, -x_2, y_1)$$

then we obtain only two stable G-invariant Morse families:

$$A_2: \quad F(\lambda, x_1, x_2, y_1) = \lambda^3 + \lambda y_1, \quad \lambda \to \lambda,$$

$$A_3: \quad F(\lambda, x_1, x_2, y_1) = \pm\lambda^4 + \lambda^2 y_1 + x_1\lambda, \quad \lambda \to -\lambda,$$

One can easily see that any extension of the list of Propositions 3.2 and 3.3, provides the functional moduli in the normal forms of Morse families. As far as we are interested in the structure of symmetric caustics, it is natural to introduce weaker equivalence, namely caustic equivalence [15], [29].

DEFINITION 3.4. Two G-invariant Morse families $F_j: (\Lambda \oplus V, 0) \to R$, $(j = 1, 2)$ are caustic equivalent if the following conditions hold.

(i) There exists a representation of G in U, a G-invariant Morse family $\mathscr{F}: \Lambda \oplus U \to R$ and G-invariant map germs $\phi_j: (V, 0) \to (U, 0)$ such that $F_j(\lambda, q)$ is $\mathscr{R}_G^+$-equivalent to $\mathscr{F}(\lambda, \phi_j(q))$.
(ii) There exists a pair of G-equivariant diffeomorphism germs (H, h) with $H: (V \times U, 0) \to (V \times U, 0)$ and $h: (V, 0) \to (V, 0)$ satisfying the following conditions
   (a) $H(q, y) = (h(q), \tilde{H}(q, y))$ where $\tilde{H}: V \times U \to U$ satisfies $\tilde{H}(q, 0) = 0$,
   (b) $H(V \times C_{\mathscr{F}}) \subseteq V \times C_{\mathscr{F}}$,
   (c) $H(q, \phi_1(q)) = (h(q), \phi_2(h(q)))$,
   where $C_{\mathscr{F}}$ is a caustic of $\mathscr{F}$.

Using this equivalence relation it is possible to reduce the functional modulus in the generic Morse family

$$\lambda^6 + y_2\lambda^4 + y_1\lambda^2 + \phi(x_1^2, y_1, y_2)x_1\lambda^3 + x_1\lambda,$$

in the case $s = 2$, $r = 1$, and obtain a new symmetric caustic in R^3, which we call the 'symmetric butterfly'. For the caustic equivalence, the list of Morse families of generic corank 1, $\mathbf{Z}_2$-invariant caustics in R^n when $n \leqslant 6$ is given in [29] and [Janeczko, S., Roberts, M., Classification of symmetric caustics II; Caustic equivalence, Warwick Preprints: 43/1991]. Using that list we can prove the following result in R^4.

PROPOSITION 3.5. *If $n = 4$ then the generic corank* 1, *$\mathbf{Z}_2$-invariant caustics are* *infinitesimally stable and are caustic equivalent to ones given by the following Morse families*

$r = 1$, $s = 3$:

A_1: $\lambda^2 + x_1\lambda$,

A_3: $\lambda^4 + y_1\lambda^2 + x_1\lambda$,

A_5^s: $\lambda^6 + y_1\lambda^2 + y_2\lambda^4 + x_1\lambda$,

A_7^s: $\lambda^8 + y_1\lambda^2 + y_2\lambda^4 + y_3\lambda^6 + x_1\lambda$.

$r = 2$, $s = 2$:

A_5: $\lambda^6 + y_1\lambda^2 + y_2\lambda^4 + x_2\lambda^3 + x_1\lambda$.

COROLLARY 3.6. *All generic $\mathbf{Z}_2$-invariant caustics in R^3 are illustrated in Figure* 6.

The first step in classification of generic evolutions of caustics is to find out the corresponding bicaustics traced out by the singular edges of virtual caustics.

PROPOSITION 3.7. *All generic $\mathbf{Z}_2$-invariant bicaustics in R^2 and their virtual caustic evolutions are illustrated in Figure* 7.

PROPOSITION 3.8. *Generic $\mathbf{Z}_2$-invariant bicaustics in R^3 and their virtual caustic evolutions corresponding to*

(a) A_3, A_5, $D_4^{\pm}$ *are the symmetric versions of the caustic evolutions illustrated in* [7], (*Figs.* 5 *and* 6, *Table* 1.2.3), *excluding the Whitney's cross-cap bicaustic,*
(b) A_5^s *are illustrated in Figure* 8,
(c) *the family of intersections of 'big' caustic* A_7^s *are illustrated in Figure* 9.

4. Tensor Invariants and Invariant Symplectic Geometry of Binary Forms

To obtain the previous results on bicaustics and wave-front singularities, one must use the information on the space of polynomials [7], [18]. It was shown in [6] that the interesting singularity of wave-front evolution around an obstacle in 3-space is diffeomorphic to the Lagrangian subvariety of the discriminant variety of the 4-space of polynomials $\{x^5 + Ax^3 + Bx^2 + Cx + D\}$. The natural symplectic structure of this space is given by reducing the unique $\mathrm{SL}_2(R)$-invariant symplectic structure on the space of all 5th degree polynomials. This structure, in turn, is given by a unique tensor invariant of degree two on the space of binary forms of odd degree. In this section, we show how to prove these facts and some other results concerning symplectic geometry of binary forms.

Let M^{n+1} be the space of binary forms of degree n, i.e. in coordinates x, y we have

$$M^{n+1} \ni f(x, y) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k y^{n-k}. \tag{4.1}$$

Let us consider the standard action v of $\mathrm{GL}_2(K)$ on M^{n+1} (cf. [31]). A nonconstant

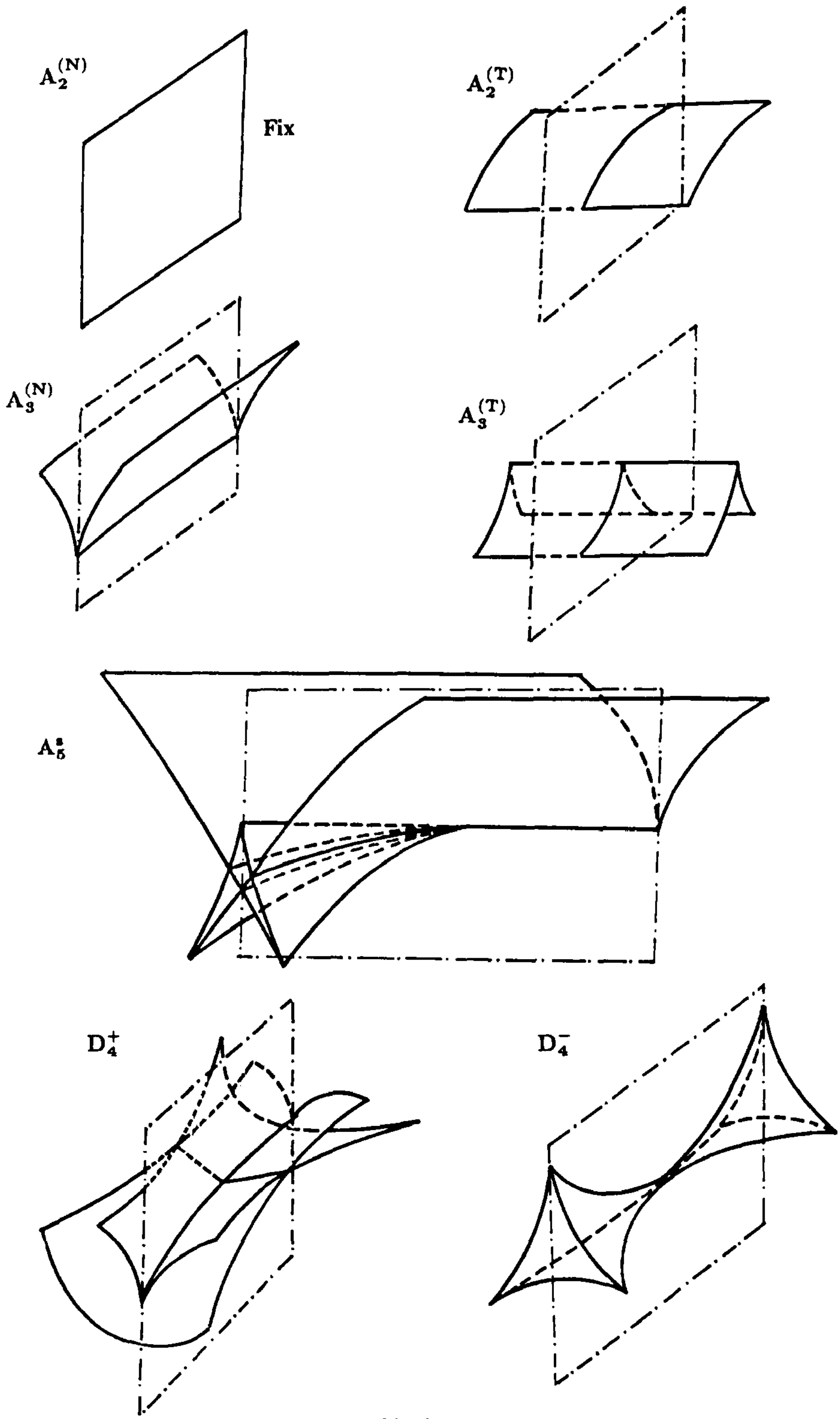


Fig. 6.

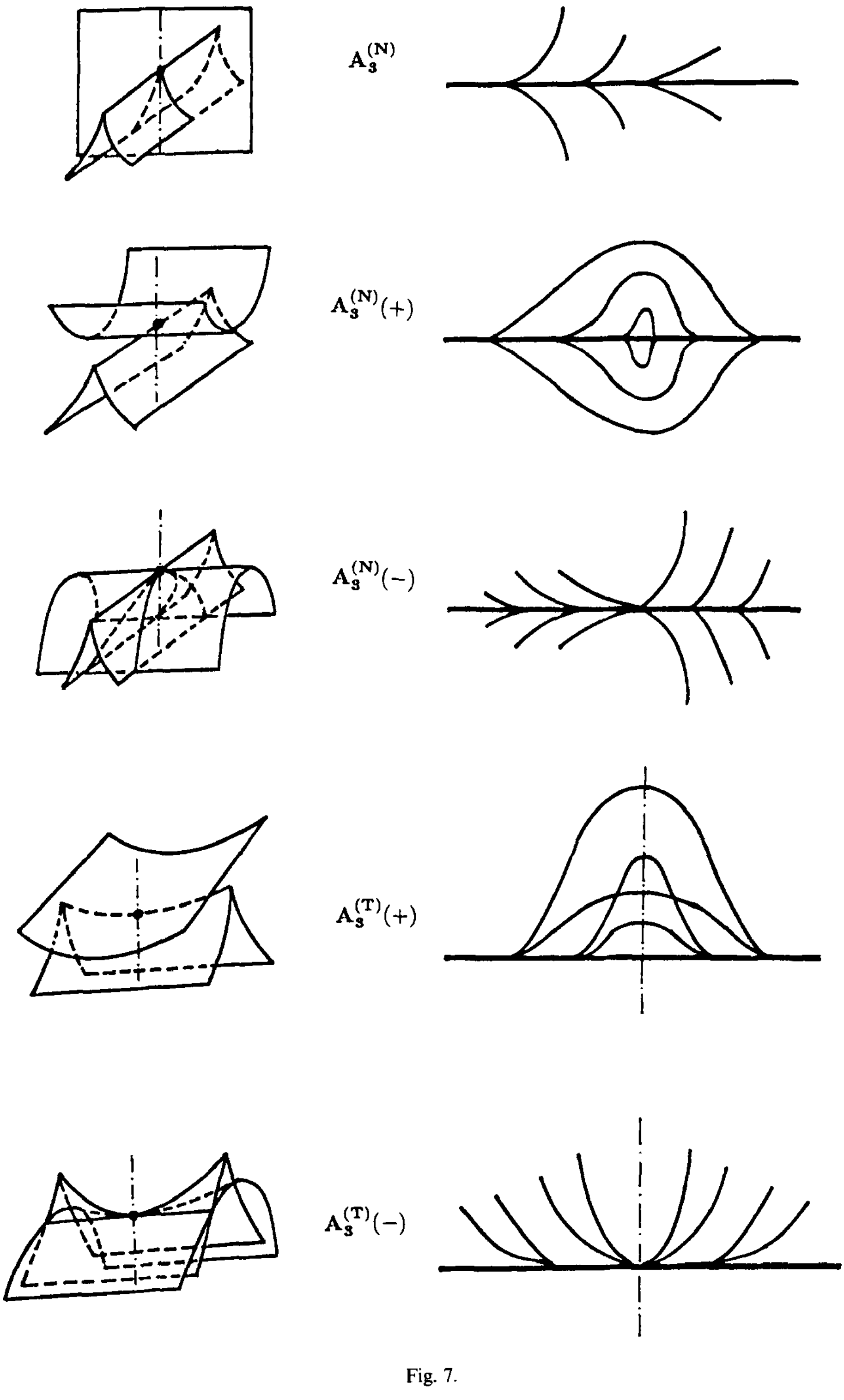


Fig. 7.

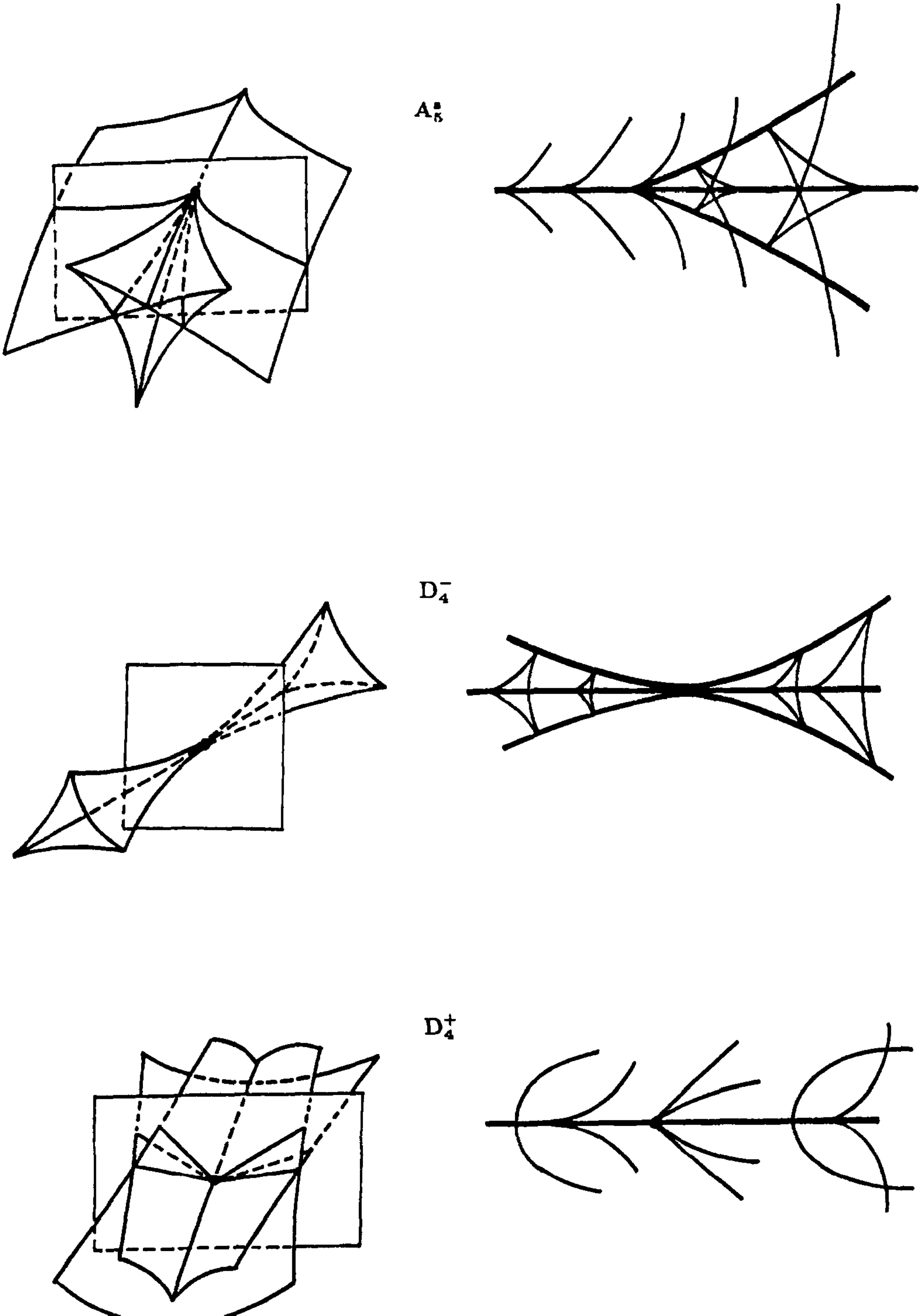


Fig. 7. (*Cont.*)

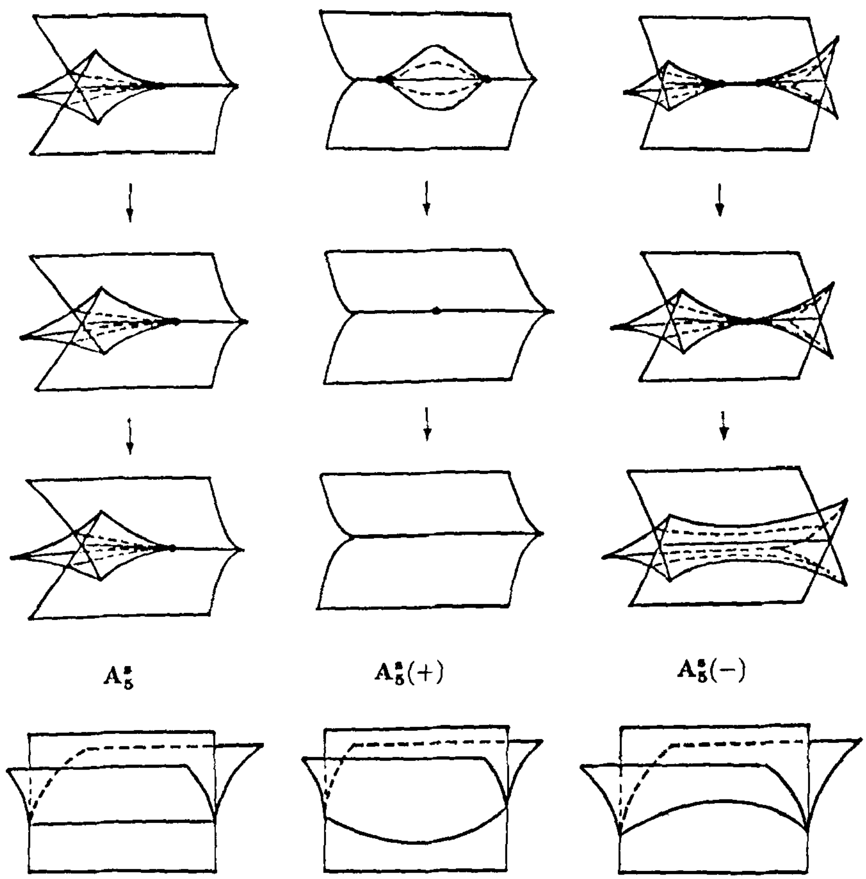


Fig. 8.

polynomial $I \in K[a_0, \dots, a_n, x, y]$ is said to be a covariant of index g of binary forms of degree n if for all $h \in GL_2(K)$, we have

$$\tilde{v}_h^* I = (\det h)^g I, \tag{4.2}$$

where $\tilde{v}$ is the canonical extension of v to $M^{n+1} \times K^2$. A polynomial function I defined only on M^{n+1} and invariant with respect to $v (g = 0)$, is said to be an invariant of binary forms. We assume that the coefficients of f belong to a field K of characteristic zero and the action v of $GL_2(K)$ is induced by the following transformations of variables x and y:

$$x = c_{11}\tilde{x} + c_{12}\tilde{y}, \qquad y = c_{21}\tilde{x} + c_{22}\tilde{y} \tag{4.3}$$

An effective method for indicating the polynomial covariants of binary forms is provided by the umbral calculus (see [31]) whose basic properties we now recall.

Let $P = \{\alpha, \beta, \dots, \omega, u\}$ be an alphabet consisting of an infinite (or finite) supply of Greek letters followed by the single Roman letter u. To each Greek letter, say α, and the Roman letter u, we associate two variables α_1, α_2, and u_1, u_2, respectively. The ring

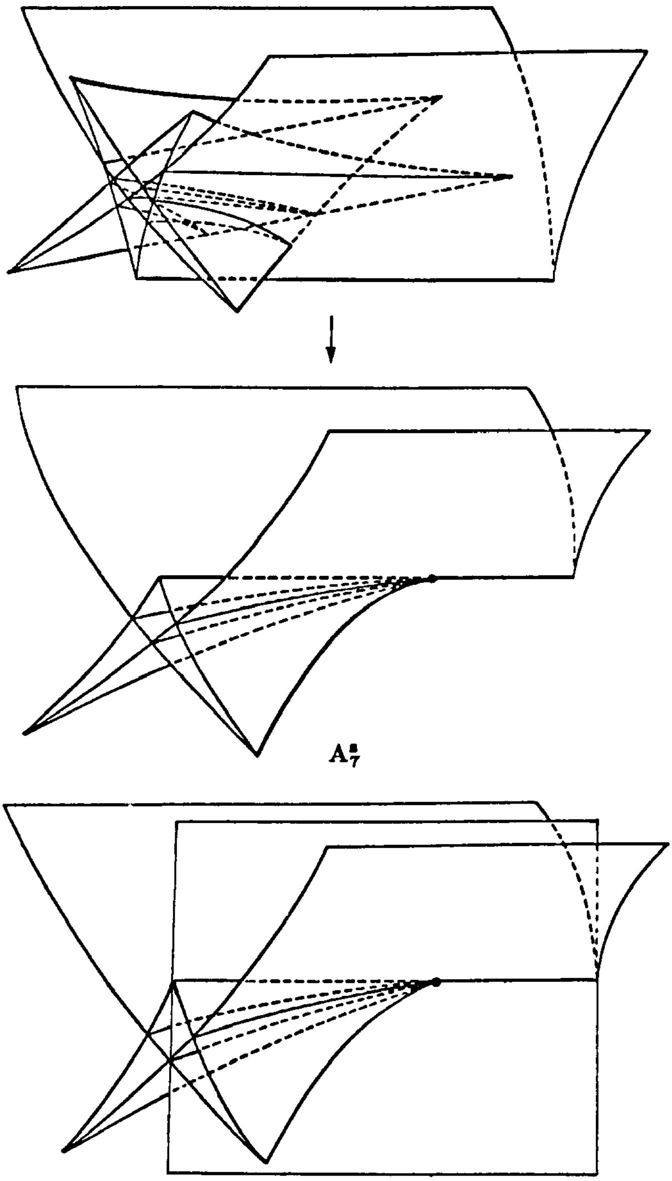


Fig. 9.

of polynomials in these variables is a vector space called the standard umbral space $\mathscr{U}$. With every space of binary forms we associate a linear operator, say $U(f)$ (also denoted by U), $U(f)\colon \mathscr{U} \to K[a_0, \ldots, a_n, x, y]$. $U(f)$ is defined by linearity on the corresponding monomials of $\mathscr{U} = K[\alpha_1, \alpha_2, \ldots, u_1, u_2]$;

$$\langle U(f) \mid \alpha_1^k \alpha_2^{n-k} \rangle = a_k, \qquad \langle U(f) \mid \alpha_1^j \alpha_2^k \rangle = 0, \quad \text{if } j + k \neq n,$$

$$\langle U(f) \mid u_1^k \rangle = (-y)^k, \qquad \langle U(f) \mid u_2^k \rangle = (x)^k,$$

and by the multiplicative rule

$$\langle U(f) \mid \alpha_1^i \alpha_2^j \cdots u_1^p u_2^q \rangle = \langle U(f) \mid \alpha_1^i \alpha_2^j \rangle \cdots \langle U(f) \mid u_1^p \rangle \langle U(f) \mid u_2^q \rangle.$$

Every polynomial $I(a_0, \ldots, a_n, x, y)$ can be written as $\langle U(f) \mid Q(\alpha_1, \alpha_2, \ldots, u_1, u_2)\rangle$ for some polynomial $Q \in \mathscr{U}$. The $\tilde{v}$-action of $\mathrm{GL}_2(K)$ implies the corresponding action of $\mathrm{GL}_2(K)$ on the umbral space $\mathscr{U}$. Let (c_{ij}) be defined as in (4.3). Then the corresponding change of umbral variables, say for some Greek letter α, is defined as follows

$$\alpha_1 = [\tilde{\alpha}\ c], \qquad \alpha_2 = [\tilde{\alpha}\ d],$$

where

$$\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2), \qquad c = (-c_{21}, c_{11}), \qquad d = (-c_{22}, c_{12})$$

and the bracket $[v\ w] = v_1 w_2 - v_2 w_1$, for two vectors $v = (v_1, v_2)$, $w = (w_1, w_2)$. So we easily have the following explicit expression for the representation v,

$$\tilde{a}_k = \langle U(f) \mid [\alpha\ c]^k [\alpha\ d]^{n-k} \rangle$$
$$= \sum_{m=0}^{n} \left(\sum_{i=m-n+k}^{\min(m,k)} \binom{k}{i} \binom{n-k}{m-i} c_{11}^i c_{12}^{m-i} c_{21}^{k-i} c_{22}^{n-k-m+i} \right) a_m.$$

We see that the bracket monomials, say $[\alpha, \beta]$, $[\alpha\ u]$, etc., represent covariants of index 1, thus we can consider, in the umbral space $\mathscr{U}$, the subspace of bracket polynomials defined as the linear combinations of bracket monomials, and obtain the following result:

(A) The umbral evaluation $\langle U \mid P \rangle$ of a bracket polynomial P, for which in every bracket monomial the number of brackets containing only Greek letters is constant and equal to $g \in \mathbf{N}$, is a covariant of index g.

(B) Let I be a covariant of index g of binary forms of degree n. Then there exists a bracket polynomial P of index g such that $I = \langle U \mid P \rangle$.

Now we apply the umbral approach to classify the tensor invariants of binary forms. By $\mathscr{U}_n(\alpha)$ we denote the subspace of $\mathscr{U}(\alpha)$ of all homogeneous polynomials of degree n. By $D_n(\alpha)$ we denote the vector space of all differential 1-forms: $a_1(\alpha)\,\mathrm{d}\alpha_1 + a_2(\alpha)\,\mathrm{d}\alpha_2$, $a_1, a_2 \in \mathscr{U}_{n-1}(\alpha)$. Let $E_n(\alpha) \subset D_n(\alpha)$ be the subspace of exact differential 1-forms. $E_n(\alpha)$ is generated by $\{d(\alpha_1^k \alpha_2^{n-k})\}_{k=0}^n$. Let $K_n(\alpha)$ denotes the subspace of $D_n(\alpha)$ generated by $\{\alpha_1^r \alpha_2^{n-r-2}[\alpha\ \mathrm{d}\alpha]\}_{r=0}^{n-2}$. We denote by μ the induced action of $\mathrm{GL}_2(K)$ on $D_n(\alpha)$. Thus, there exists a $\mathrm{GL}_2(K)$-invariant decomposition $D_n(\alpha) = E_n(\alpha) \oplus K_n(\alpha)$ with irreducible action of μ on each component. Now we can define the umbral operator U_α^* into the space of 1-forms on the space of binary forms M^{n+1}. Let $U' = U|_{\mathscr{U}_n(\alpha)}$. We define an operator $\bar{U}$ on $E_n(\alpha)$ satisfying condition $d \circ U' = \bar{U} \circ d$ and given on the basis elements;

$$\langle \bar{U} \mid d(\alpha_1^k \alpha_2^{n-k}) \rangle = \mathrm{d}a_k, \quad k = 0, \ldots, n$$

DEFINITION 4.1. The linear operator

$$U_\alpha^* := \bar{U} \circ \mathbf{P}: D_n(\alpha) \to (M^{n+1})^*,$$

where $\mathbf{P}$ is a linear projection $\mathbf{P}: E_n(\alpha) \oplus K_n(\alpha) \to E_n(\alpha)$, $\operatorname{Ker} \mathbf{P} = K_n(\alpha)$, is called the elementary umbral operator onto differential 1-forms on M^{n+1}. It is $GL_2(K)$-equivariant linear operator.

By the extension of U_α^* to the tensor product of p factors, say $W_{n,p} = D_n(\alpha) \otimes \cdots \otimes D_n(\beta)$, we obtain the umbral operator for representation of tensor invariants of binary forms of degree p

$$U^*_{(\alpha,\ldots,\beta)}: D_n(\alpha) \otimes \cdots \otimes D_n(\beta) \to \otimes^p(M^{n+1})$$

$$\langle U^*_{(\alpha,\ldots,\beta)} \mid w_1(\alpha) \otimes \cdots \otimes w_p(\beta)\rangle = \langle U^*_\alpha \mid w_1(\alpha)\rangle \otimes \cdots \otimes \langle U^*_\beta \mid w_p(\beta)\rangle.$$

We call $q \in W_{n,p}$ the bracket polynomial if it can be written as a linear combination of products (monomials) of brackets $[\alpha\ \beta] = \alpha_1\beta_2 - \alpha_2\beta_1$, $[\alpha\ \mathrm{d}\beta] = \alpha_1\,\mathrm{d}\beta_2 - \alpha_2\,\mathrm{d}\beta_1$, $[\mathrm{d}\alpha \otimes \mathrm{d}\beta] = \mathrm{d}\alpha_1 \otimes \mathrm{d}\beta_2 - \mathrm{d}\alpha_2 \otimes \mathrm{d}\beta_1$. The space of bracket polynomials is denoted by $B_{n,p}$. The index of $q \in B_{n,p}$ is the number of brackets in q. Now we immediately have the following proposition.

PROPOSITION 4.2. (A) *Let ϕ be the bracket polynomial of index g. Then the umbral evaluation of ϕ, $\langle U^*_{(\alpha,\ldots,\beta)} \mid \phi\rangle$ is an invariant of index g.*

(B) *Let Q be a tensor invariant of index g and degree p for binary forms of degree n. Then there exists a bracket polynomial ϕ of index g such that $Q = \langle U^*_{\alpha,\ldots,\beta)} \mid \phi\rangle$. Now we give the complete classification of the tensor invariants of degree two, $p = 2$. In fact, we have that $B_{n,2}$ is generated by two elements*

$$v_1 = [\alpha\ \beta]^{n-1}[\mathrm{d}\alpha \otimes \mathrm{d}\beta], \qquad v_2 = [\alpha\ \beta]^{n-2}[\alpha\ \mathrm{d}\alpha] \otimes [\beta\ \mathrm{d}\beta],$$

also we can easily check that $v_2 \in \operatorname{Ker} U^*_{\alpha,\beta}$, so $\dim \operatorname{Im}(U^*_{\alpha,\beta)}|_{B_{n,2}}) = 1$. By straightforward calculation we obtain Proposition 4.3.

PROPOSITION 4.3. *All tensor invariants of degree two on the space of binary forms of degree n are proportional to the following basic invariant:*

$$Q = \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j+1}((-1)^n \mathrm{d}a_{j+1} \otimes \mathrm{d}a_{n-j-1} + \mathrm{d}a_{n-j-1} \otimes \mathrm{d}a_{j+1}),$$

COROLLARY 4.4. (A) *If n is even, then there exists only one, up to constant multiples, $SL_2(K)$-invariant symplectic structure on M^{n+1}, which in coordinates*

$$q_r = \frac{n!}{r!}\, a_{n-r},\ p_r = (-1)^{k-r} \frac{n!}{(n-r)!}\, a_r, \quad r = 0, \ldots, k+1, (n = 2k+3)$$

on M^{n+1}, can be written in the Darboux form

$$\omega = \sum_{j=0}^{k+1} \mathrm{d}p_j \wedge \mathrm{d}q_j$$

and the elements of M^{n+1} in these coordinates have the following form:

$$f(x, y) = q_0 \frac{x^{2k+3}}{(2k+3)!} + \cdots + q_{k+1} \frac{x^{k+2}y^{k+1}}{(k+2)!} - p_{k+1} \frac{x^{k+1}y^{k+2}}{(k+1)!} + \cdots$$
$$+(-1)^{k+2}p_0 y^{2k+3}.$$

(B) *In a similar way, we obtain the* $SL_2(K)$*-invariant* 1*-form on* M^{n+1} (*cf.* [10]), *namely*

$$\theta = \sum_{j=0}^{k+1} (p_j \mathrm{d}q_j - q_j \mathrm{d}p_j)$$

and the canonical contact structure

$$\vartheta = \sum_{j=1}^{k+1} (p_j \mathrm{d}q_j - q_j \mathrm{d}p_j) - \mathrm{d}p_0$$

on the space of polynomials (*cf.* [4])

$$\left\{\frac{x^{2k+3}}{(2k+3)!} + q_1 \frac{x^{2k+2}}{(2k+2)!} + \cdots + q_{k+1} \frac{x^{k+2}}{(k+2)!} - p_{k+1} \frac{x^{k+1}}{(k+1)!} + \cdots + (-1)^{k+2}p_0\right\}.$$

$SL_2(K)$ acts symplectically on (M^{n+1}, ω). Thus, we find the momentum mapping of this action as the Ad*-equivariant quadratic momentum mapping (cf. [1]). It can be written as

$$J: M^{n+1} \to sl_2(K)^*;\ J(\bar{p}) = (H_+, H_-, H_d)(\bar{p}),$$

where

$$H_+(\bar{p}) = \sum_{r=1}^{k+1} p_r q_{r-1} + \tfrac{1}{2}q_{k+1}^2,$$

$$H_-(\bar{p}) = \sum_{r=0}^{k} (2k+3-r)(r+1)p_r q_{r+1} - \tfrac{1}{2}(k+1)^2 p_{k+1}^2,$$

$$H_d(\bar{p}) = \sum_{r=0}^{k+1} (2r-2k-3)p_r q_r,$$

and $\{H_+, H_-\} = H_d$.

Let (M, ω) be a symplectic manifold. The new symplectic structures associated to (M, ω) are provided by co-isotropic submanifolds in M (cf. [1]). We recall that a submanifold $C \subseteq M$ is co-isotropic if at each $x \in C$ we have

$$(T_xC)^{\diamond} = \{v \in T_xM;\ \langle v \wedge u, \omega\rangle = 0, \text{ for every } u \in T_xC\} \subseteq T_xC.$$

The distribution $\bigcup_{x\in C}(T_xC)^{\diamond}$ is the characteristic distribution of $\omega|_C$. Let B be the space of characteristics of it and $\rho\colon C \to B$ be its canonical projection. It is known (cf. [1]) that if B admits a differentiable structure and ρ is a submersion, then there is a unique symplectic structure β on B such that $\rho^*\beta = \omega|_C$. The symplectic manifold (B, β) associated in this way with the triplet (M, ω, C) is called the reduced symplectic manifold.

Let us consider two binary forms, $f(x, y)$, $g(x, y)$, where f is of degree n and g is of degree m, $m \leqslant n$. They can be written umbrally

$$f = \langle U \mid [\alpha\ u]^n \rangle,\ g = \langle U \mid [\beta\ u]^m \rangle.$$

Their apolar covariant $\langle f \mid g \rangle$ is the binary form of degree $n - m$ defined umbrally by

$$\langle f \mid g \rangle = \langle U \mid [\alpha\ \beta]^m [\alpha\ u]^{n-m} \rangle.$$

The apolar covariant $\langle f \mid f \rangle$ is an invariant for the binary forms of degree n. It can be expressed by

$$\langle f \mid f \rangle = \frac{1}{n!}(-1)^n \sum_{k=0}^{n} (-1)^k \frac{\partial^k f}{\partial x^k} \frac{\partial^{n-k} f}{\partial x^{n-k}} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k a_{n-k}.$$

Let (M^{n+1}, ω) be the symplectic space of binary forms. The canonical subspaces in M^{n+1}, say $C^{(l)}$, $0 \leqslant l \leqslant (n-1)/2$ of all binary forms apolar to its l-derivatives with respect to x are called the canonical apolar subspaces. It is easy to see that the apolar subspaces $C^{(l)}$ form the coisotropic varieties in (M^{n+1}, ω). They are described by the following systems of $l + 1$ equations:

$$P_0^{(l)} = \binom{n-l}{0} a_n a_0 - \binom{n-l}{1} a_{n-1} a_1 + \cdots \pm \binom{n-l}{n-l} a_l a_{n-l} = 0,$$

$$P_1^{(l)} = \binom{n-l}{0} a_n a_1 - \binom{n-l}{1} a_{n-1} a_2 + \cdots \pm \binom{n-l}{n-l} a_l a_{n-l+1} = 0,$$

......

$$P_l^{(l)} = \binom{n-l}{0} a_n a_l - \binom{n-l}{1} a_{n-1} a_{l+1} + \cdots \pm \binom{n-l}{n-l} a_l a_n = 0,$$

Now we pay more attention to the particular case; let $l = 1$, then the second apolar coisotropic variety $C^{(1)}$ can be expressed as follows:

$$C^{(1)} = \Big\{ f \in M^{n+1};\ \langle f \mid f'_x \rangle = n(P_0^{(1)} y + p_1^{(1)} x)$$

$$= (-1)^{k+1} \frac{1}{n!} (y H_d + 2x H_+) \equiv 0 \Big\},$$

where $\{H_+, H_d\} = H_+$. To the space of binary forms of degree n, one can easily associate the corresponding spaces of polynomials of one variable putting $y = 1$. In order to have the polynomial symplectic spaces adapted to the investigations of singularities in the variational obstacle problem (see [6], [43]), we associate to every symplectic space (M^{n+1}, ω) the canonically reduced symplectic space Q^{n-1} of polynomials of degree $n-1$. $Q^{n-1} = C_0/\sim$, where '$\sim$' is given by the characteristic fibration of the co-isotropic submanifold $C_0 = \{f \in M^{n+1}; n!a_n = 1\}$. Q^{n-1} is identified canonically with the space of derivatives $\mathrm{d}/\mathrm{d}x(f(x,1))$, $f \in M^{n+1}$ belonging to C_0, namely

$$Q^{n-1} \ni \phi(x) = \frac{x^{2k+2}}{(2k+2)!} + q_1 \frac{x^{2k+1}}{(2k+1)!} + \cdots + q_{k+1} \frac{x^{k+1}}{(k+1)!} - $$
$$- p_{k+1} \frac{x^k}{k!} + \cdots + (-1)^{k+1} p_1.$$

endowed with the reduced symplectic structure

$$\omega' = \sum_{j=1}^{k+1} \mathrm{d}p_j \wedge \mathrm{d}q_j.$$

One can easily find that the apolar subspaces $C^{(l)}$ induce the corresponding co-isotropic subspaces of (Q^{n-1}, ω'), say

$$\tilde{C}^{(l)}, \left(l = 1, \ldots, \frac{n-1}{2}\right),$$

described by

$$\tilde{C}^{(l)} = \{\phi \in Q^{n-1}; \tilde{P}_s^{(l)}(q, p) = 0, s = 1, \ldots, l\}, \quad l = 1, \ldots, k+1,$$

where

$$\tilde{P}_s^{(l)}(q, p) = \frac{(-1)^k}{n!} \sum_{i=1}^{k-s+1} \binom{n-l}{i} \binom{n}{i}^{-1} q_i p_i +$$
$$+ \frac{1}{(n!)^2} \sum_{i=k-s+2}^{k+1} (-1)^i \binom{n-l}{i} i!(n-s-i)! q_i q_{n-s-i} +$$
$$+ \frac{(-1)^k}{(n!)^2} \sum_{i=k+2}^{n-1} \binom{n-l}{i} i!(n-s-i)! p_{n-i} q_{n-s-i} + (-1)^{k-s} \frac{(n-s)!}{n!} p_s.$$

Now we investigate the properties of the symplectic space induced by the co-isotropic submanifold $\tilde{C}^{(1)}$ in (Q^{n-1}, ω'), $n = 2k+3$. The reduced symplectic space corresponding to the triplet $(Q^{n-1}, \omega', \tilde{C}^{(1)})$ (see, e.g., [4]) is identified with the following space of polynomials

$$Z = \left\{\frac{x^{2k+1}}{(2k+1)!} + q_1 \frac{x^{2k-1}}{(2k-1)!} + \cdots + q_k \frac{x^k}{k!} - p_k \frac{x^{k-1}}{(k-1)!} + \cdots + (-1)^k p_1\right\},$$

endowed with the reduced symplectic structure $\tilde{\omega} = \sum_{i=1}^k \mathrm{d}p_i \wedge \mathrm{d}q_i$. As a polynomial parametrisation of the characteristics of $\tilde{C}^{(1)}$, described by Z, we can write the following identification

$$\frac{(x-t)^{2k+1}}{(2k+1)!}+\bar{q}_1\frac{(x-t)^{2k}}{(2k)!}+\cdots+\bar{q}_{k-1}\frac{(x-t)^k}{k!}-$$
$$-\bar{p}_{k+1}\frac{(x-t)^{k-1}}{(k-1)!}+\cdots+(-1)^k\bar{p}^2$$
$$=\frac{x^{2k+1}}{(2k+1)!}+q_1\frac{x^{2k-1}}{(2k-1)!}+\cdots+q_k\frac{x^k}{k!}-p_k\frac{x^{k-1}}{(k-1)!}+\cdots+(-1)^k p_1,$$

i.e. the space of characteristics of the co-isotropic submanifold $\tilde{C}^{(1)}$ is identified with the derivatives of polynomials belonging to Q^{n-1}. Using that identity, we obtain the following result (cf. [5]).

PROPOSITION 4.5. (A) *If $m < k+1$, then the set of polynomials of Z having a root of multiplicity m, say $L^{(n-3)}_{2k-m+1}$, form the co-isotropic varieties in $(Z, \bar{\omega})$.*

(B) *If $m \geqslant k+1$, then the set of polynomials of Z having root of multiplicity m, $(L^{(n-3)}_{2k-m+1})$, form the isotropic varieties in $(Z, \bar{\omega})$. The maximal isotropic variety, i.e. $m = k+1$ is a Lagrangian variety symplectomorphic, in the case of $k = 2$, to the system of rays diffracted on a smooth obstacle, with the open swallowtail singularity (see [43]).*

5. Symplectic Images of Lagrangian Submanifolds and Resolution of Singularities

V. I. Arnold [5] first called attention to singular Lagrangian varieties. Originally they appeared in the forward scattering of light rays by a convex obstacle [38] and in the geometrical theory of diffraction [30]. The results by Arnold and coworkers provided an exhaustive geometrical classification of them [6], [43]. Quite independently, singular Lagrangian varieties appeared in describing the equilibrium structure of phase transitions and in the classical mechanics of constrained systems [27]. In addition, there is evidence [51], [26] that it is necessary to add singular Lagrangian varieties to the set of smooth Lagrangian relations, the morphisms of the symplectic 'category', in order to obtain an adequate category.

The first step toward the investigation of singular Lagrangian varieties is just to understand those singularities which are removable by some canonical procedure. That procedure is suggested by the methods used in the investigation of hidden-order parameters in composite systems or in searching the smooth general potentials for the reduced mechanical systems [47]. The idea of the resolution of singularities is also imposed by the theory of oscilating integrals with smooth phases corresponding to nonclassical symbols in the integral formula for the scattering amplitude [38].

Let (P, ω) be a symplectic manifold. Let (P, X, π) be a differential fibration and ϑ be a 1-form on P such that $\omega = \mathrm{d}\vartheta$. The quadruple (P, X, π, ϑ) is called a special symplectic structure on P if there is a diffeomorphism $\alpha\colon P \to T^*X$ such that

$\pi = \pi_X \circ \alpha$, $\vartheta = \alpha^* \vartheta_X$ (cf. [1]). Let $L \subset (P, \omega)$ be a germ of a singular Lagrangian variety, i.e. a stratified subset of P whose maximal strata are Lagrangian [27]. The question is:

Does there exist,

(i) a special symplectic structure (P, X, π, ϑ) on (P, ω),

(ii) a submersion $\rho: \Lambda \to X$ for some smooth manifold Λ,

(iii) a regular (i.e. *transversal to the fibers of* $T^*\Lambda$,) Lagrangian submanifold $N \subset (T^*\Lambda, d\vartheta_\Lambda)$,

such that L is an image of N with respect to the canonical cotangent bundle lifting of ρ? i.e. can we write $L = T^*\rho(N)$?

If the answer is positive, then N is called the symplectic resolution of L and its generating function F is called a generating family for L.

In this paper, we are interested only in the local version of the resolution problem. Let $(L, 0)$ be a germ of a Lagrangian subvariety in T^*V, $V \equiv R^n$ (we say Lagrangian germ for short). Assume that $(L, 0)$, given up to symplectomorphism of (T^*V, ω_V) is a solvable symplectically Lagrangian germ. If (x, λ) are coordinates of Λ adapted to the submersion

$$\rho: (\Lambda, 0) \to (V, 0),\ \Lambda \equiv V \times R^k$$

and $F: (V \times R^k, 0) \to R$ is a generating family for $(N, 0)$, then the image $(T^*\rho(N), 0) = (L, 0)$ is given implicitly by the system of equations

$$p_i = \frac{\partial F}{\partial x_i}(x, \lambda) \quad (i = 1, \ldots, n),$$

$$0 = \frac{\partial F}{\partial \lambda_j}(x, \lambda) \quad (j = 1, \ldots, k).$$

If the position of $(L, 0)$ and the cotangent fibration T^*V are fixed, and L is smooth, then the resolution problem is equivalent to the problem of finding the Morse families which define $(L, 0)$ in the form of these equations with an extra transversality condition [50]. For smooth L, $(L, 0)$ is always solvable in this way [24]. If $(L, 0)$ is not smooth, the existence of resolution is not obvious. Now we show the classes of Lagrangian varieties coming from related topics, which are solvable by the introduced procedure.

1. *Geometric interaction between holonomic components* [34].
Let V_1, V_2 be Lagrangian submanifolds of (P, ω). The Lagrangian subset $V_1 \cup V_2$ is called a regular geometric interaction if the following conditions are fulfilled: $V_1 \cap V_2$ is a submanifold of P, $\dim V_1 \cap V_2 = \dim V_1 - 1$ and for every point $p \in V_1 \cap V_2$ we have $T_p(V_1 \cap V_2) = T_pV_1 \cap T_pV_2$.

PROPOSITION 5.1. *Let* $(V_1 \cup V_2, p)$ *be a germ of a regular geometric interaction in* (P, ω). *Then there exist a symplectic manifold* $(\tilde{P}, \tilde{\omega})$ *and a symplectic reduction relation*

$R \subset (\tilde{P} \times P, \tilde{\omega} \ominus \omega)$, such that we have the canonical resolution formulae,

$$V_1 \cup V_2 = R(L),$$

for some regular Lagrangian submanifold $L \subset (\tilde{P}, \tilde{\omega})$.

By this result, one can classify the normal forms of Gauss–Manin systems on the boundary (see [41]).

EXAMPLE 5.2. Let $L = V_1 \cup V_2$, be given on the plane R^2 with coordinates p, q, $\omega = dp \wedge dq$, by the equation $p^2 - q^2 = 0$. Then the generating function for the symplectic resolution $N \subset T^*(R^2 \times R)$ is

$$F(q, \lambda) = -\tfrac{1}{2}q^2 + \sqrt{2}q\lambda^3 - \tfrac{1}{2}\lambda^6.$$

2. *The tangency classes of Lagrangian submanifolds*
Let $((V_1, V_2), p)$ be the tangency class of two Lagrangian submanifolds according to [20]. Locally it is represented by two components

$$V_1 = \{p_1 = \cdots = p_n = 0\} = X, \qquad V_2 = \left\{(q, p); p = \frac{\partial F}{\partial q}(q)\right\}, \tag{$*$}$$

for some smooth function F – the generating function for V_2.

PROPOSITION 5.3. *Let $((V_1, V_2), 0)$ be a pair of Lagrangian germs belonging to the same tangency class, i.e. represented locally like in $(*)$. Then there exists a symplectic resolution $N \subset (T^*(X \times R^k), \omega_{X \times R^k})$ of $(V_1 \cup V_2, 0)$, i.e.*

$$V_1 \cup V_2 = T^*\rho(N), \ \rho: X \times R^k \to X,$$

with the generating function:

$$\tilde{F}(q, v, \lambda, \mu) = -F(\lambda) + \sum_{i=1}^{n} v_i(\lambda_i + q_i) + \mu^3 \left(\sum_{i=1}^{n} \lambda_i^3 v_i\right),$$

where $k = 2n + 1$ and F is a generating function for V_2-component.

3. *The generalized open swallowtails* [6]:
The Lagrangian varieties $L_k^{(2k)}$ described in Proposition 4.5, are called the generalized open swallowtails (cf. [9]) or shortly open swallowtails. It is possible to solve them canonically (see [26]).

PROPOSITION 5.4. *An open, k-dimensional swallowtail is represented as a canonical pushforward of a regular Lagrangian submanifold, i.e.*

$$L_k^{(2k)} = T^*\pi(N_k),$$

where

(a) *N_k is a Lagrangian submanifold of (T^*Q, ω_Q),*
(b) *$\pi: Q \to Y$ is a projection given (in local coordinates of binary forms) in the*

following form $\pi:(q_1,\ldots,q_{k+1})\to(\bar{q}_1,\ldots,\bar{q}_k)$,

$$\bar{q}_i=\sum_{l=0}^{i-1}(-1)^l\frac{1}{l!}q_1^lq_i-(-1)^{i+1}\frac{i!}{(i+1)!}q_1^{i+1},\quad i=1,\ldots,k.$$

By choice of coordinates in which π is in the normal form, one can write the following formula for generating function of N_k;

$$P_k(\bar{q}_1,\ldots,\bar{q}_k,\lambda)=\frac{1}{2}\int_0^\lambda\left(\frac{k+2}{(k+1)!}x^{k+1}+\sum_{j=1}^k\frac{1}{(k-j)!}\bar{q}_jx^{k-j}\right)^2\mathrm{d}x.$$

This is a generating family (not Morse family) for the generalized open swallowtail $L_k^{(2k)}$.

The first step in the study of resolving Lagrangian varieties is the classification of symplectic pullbacks and pushforwards of smooth Lagrangian submanifolds. Let (T^*X,ω_X), (T^*Y,ω_Y) be two cotangent bundles. We form the product symplectic manifold $\Omega=(T^*X\times T^*Y,\omega_Y\ominus\omega_X)$. Let $f:X\to Y$ be a smooth mapping. We denote the graph of f by Γf, Γf is a submanifold of $X\times Y$. Any function on Γf can be pulled back onto X. The set

$$\{p\in T^*(X\times Y);\ \pi_{X\times Y}(p)\in\Gamma f \text{ and } \langle u,p\rangle=\langle u,\mathrm{d}\bar{g}\rangle$$
$$\text{for each } u\in T(\Gamma f)\subset T(X\times Y)\quad\text{such that } \tau_{X\times Y}(u)=\pi_{X\times Y}(p)\}\qquad(*)$$

is a Lagrangian submanifold of Ω. Here $\bar{g}$ is a smooth function on Γf. Let g denote the function $\bar{g}$ pulled back to X. The Lagrangian submanifold defined in (*), and denoted by (f,g) is called an f-constrained symplectic relation in Ω (cf. [26]). We denote by $\mathscr{F}$ the set of all f-constrained symplectic relations in Ω. We introduce in Ω (hence in $\mathscr{F}$) the canonical action of the group $\mathscr{G}=G_X\times G_Y$, where by G_X (resp. G_Y) we denote the group of symplectomorphisms preserving the fibre structure of T^*X (resp. T^*Y). A symplectomorphism $(\Phi,\Psi)\in\mathscr{G}$, locally has the following form:

$$\Phi(x,\xi)=(\phi(x),((D\phi(x))^{-1})^t(\xi+\mathrm{d}\alpha(x))):T^*X\to T^*X,$$
$$\Psi(y,\eta)=(\psi(y),((D\psi(y))^{-1})^t(\eta+\mathrm{d}\beta(y))):T^*Y\to T^*Y,$$

where ψ, ϕ are diffeomorphisms, $\phi:X\to X$, $\psi:Y\to Y$ and α, β are smooth functions on X and Y, respectively. This group acts on the pairs (f,g) in the following way

$$(f,g)\to(\Phi,\Psi)(f,g)$$
$$=(\psi\circ f\circ\phi^{-1},g\circ\phi^{-1}+\beta\circ f\circ\phi^{-1}-\alpha\circ\phi^{-1}).$$

Let $R=(f,g)$, the subset $R(N)=\{b\in T^*Y;$ there exists $a\in N$ that $(a,b)\in R\}$ ($R^t(L)=\{a\in T^*X;$ there exists $b\in L$ that $(a,b)\in R\}$) is called the pushforward of $N\subset T^*X$ (respectively, pullback of $L\subset T^*Y$) with respect to R.

One can easily see that if f is an immersion, then the pushforwards of Lagrangian submanifolds are always smooth Lagrangian submanifolds of T^*Y. Analogously, if f is an submersion, then the pullbacks of Lagrangian submanifolds are smooth

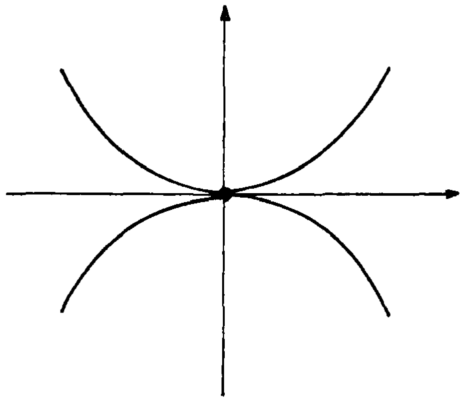


Fig. 10. Lagrangian variety of type (A_2, Σ^{10}).

Lagrangian submanifolds of T^*X. More generally, if $\pi_Y|_N: N \to Y$ and $f: X \to Y$ are transversal mappings (cf. [50]) then the pullback $R^t(N)$ is a Lagrangian submanifold of T^*X. An analogous result holds for pushforwards: if f has constant rank and L is transverse to $R^t(T^*Y)$ then $R(L)$ is a Lagrangian submanifold of T^*Y.

Let us denote the pushforward of Lagrangian submanifold $L \subset T^*X$ with respect to R by the pair (R, L); similarly for the pullback of $N \subset T^*Y$ with respect to R we use the notation (N, R). We say that the pushforwards (R_1, L_1), (R_2, L_2) (pullbacks (N_1, R_1), (N_2, R_2)) are equivalent if there exists $g \in \mathscr{G}$, $g = (\Phi, \Psi)$, such that

$$(R_2, L_2) = (g(R_1), \Phi(L_1))$$

$$((N_2, R_2) = (\Psi(N_1), g(R_1)), \text{ respectively}).$$

In small dimensions, we have the following classification theorem for pullbacks (see also Figure 10 and Figures 11a, b). There is an analogous classification result for generic pushforwards (see [26]).

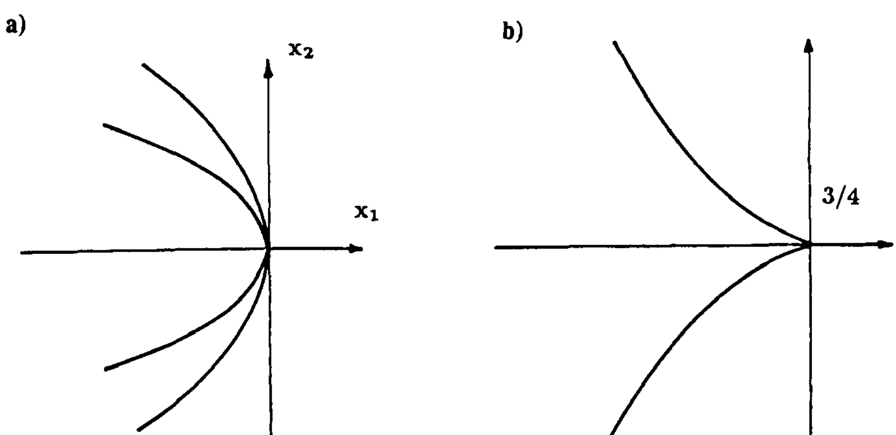


Fig. 11. Caustic of type (A_3, Σ^{20}), (a) $\phi'(0) \neq 0$, (b) $\phi'(0) = 0$.

Table I.

m, n	Type	$H: X \times R \to R$
1, 1	(A_1, Σ^0)	0
	(A_2, Σ^0)	$\lambda^3 + \lambda x$
	(A_1, Σ^{10})	0
	(A_2, Σ^{10})	$\lambda^3 \pm \lambda x^2$
1, 2	(A_1, Σ^0)	0
	(A_2, Σ^0)	$\lambda^3 + \lambda x$
	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x + \lambda\phi(x)$, $\phi(0) = 0$
1, 3	(A_1, Σ^0)	0
	(A_2, Σ^0)	$\lambda^3 + \lambda x$
	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x + \lambda\phi(x)$, $\phi(0) = 0$
2, 1	(A_1, Σ^1)	0
	(A_2, Σ^1)	$\lambda^3 + \lambda x_1$
	(A_1, Σ^{20})	0
	(A_2, Σ^{20})	$\lambda^3 + \lambda(\pm x_1^2 \pm x_2^2)$
2, 2	(A_1, Σ^0)	0
	(A_2, Σ^0)	$\lambda^3 + \lambda x_1$
	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x_1 + \lambda x_2$
	(A_1, Σ^{10})	0
	(A_2, Σ^{10})	$\lambda^3 + \lambda x_1$
	(A_3, Σ^{10})	$\lambda^4 + \lambda^2 x_1 + \lambda(x_2^2 + \phi(x_1))$, $\phi(0) = 0$
	(A_1, Σ^{110})	0
	(A_2, Σ^{110})	$\lambda^3 \pm \lambda x_1$
	(A_3, Σ^{110})	$\lambda^4 \pm \lambda^2 x_1 + \lambda(\phi_1(x_1)) + \phi_2(x_1)(x_2^3 + x_1 x_2))$, $\phi_2(0) > 0$
2, 3	(A_1, Σ^0)	0
	(A_2, Σ^0)	$\lambda^3 + \lambda x_1$
	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x_1 + \lambda x_2$
3, 1	(A_1, Σ^2)	0
	(A_2, Σ^2)	$\lambda^3 + \lambda x_1$
	(A_1, Σ^{30})	0
	(A_2, Σ^{30})	$\lambda^3 \pm \lambda(x_1^2 \pm x_2^2 \pm x_3^2)$
3, 2	(A_1, Σ^1)	0
	(A_2, Σ^1)	$\lambda^3 + \lambda x_1$
	(A_3, Σ^1)	$\lambda^4 + \lambda^2 x_1 + \lambda x_2$
	(A_1, Σ^{20})	0
	(A_2, Σ^{20})	$\lambda^3 \pm \lambda x_1$
	(A_3, Σ^{20})	$\lambda^4 + \lambda^2 x_1 + \lambda(x_2^2 \pm x_1^2 + \phi(x_1))$, $\phi(0) = 0$
	(A_1, Σ^{210})	0
	(A_2, Σ^{210})	$\lambda^3 \pm \lambda x_1$
	(A_3, Σ^{210})	$\lambda^4 \pm \lambda^2 x_1 + \lambda(\phi_1(x_1) + \phi_2(x_1)(x_2^2 + x_3^3 + x_1 x_3))$, $\phi_2(0) \neq 0$
3, 3	(A_1, Σ^0)	0
	(A_2, Σ^0)	$\lambda^3 + \lambda x_1$
	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x_1 + \lambda x_2$
	(A_1, Σ^{10})	0
	(A_2, Σ^{10})	$\lambda^3 + \lambda x_1$
	(A_3, Σ^{10})	$\lambda^4 + \lambda^2 x_1 + \lambda x_2$
	(A_1, Σ^{110})	0
	(A_2, Σ^{110})	$\lambda^3 + \lambda x_1$
	(A_3, Σ^{110})	$\lambda^4 + \lambda^2 x_1 + \lambda(\pm x_2 + \phi(x_1))$
	(A_1, Σ^{1110})	0
	(A_2, Σ^{1110})	$\lambda^3 \pm \lambda x_1$
	(A_3, Σ^{1110})	$\lambda^4 \pm \lambda^2 x_1 + \lambda(\alpha y_3 + \phi_1(x_1) + x_2\phi_2(x_1) + x_2^2\phi_3(x_1))$, $\alpha \neq 0$, $\phi_2(0) \neq 0$.

PROPOSITION 5.5. *Let $n = \dim X$, $m = \dim Y \leqslant 3$. Then the normal forms of the generating families for the generic pullbacks of the appropriate types are equivalent to these ones listed in Table I.*

Remark 5.6. The phase space of a simple thermodynamical system is the symplectic manifold $(R^{2(2+k)}, d\theta)$, where $\{T, -p, \mu_1, \dots, \mu_k, S, V, N_1, \dots, N_k\}$ are standard thermodynamical coordinates and $\theta = T\,dS - p\,dV + \sum_{i=1}^{k} \mu_i\,dN_i$ is a 1-form of internal energy (cf. [27]). The space of equilibrium states of the concrete thermodynamical system is identified with the Lagrangian subvariety of $(R^{2(2+k)}, d\theta)$. Singularities of the Lagrange projection $\pi: R^{2(2+k)} \to R^{2+k}$ onto $\{T-p, \mu_1, \dots, \mu_k\}$ of this variety correspond to the thermodynamical phase transitions and critical phenomena.

In an attempt to model the phase coexistence around the critical point as well as the phenomena near absolute zero temperature, it seems natural to admit some singularities of symplectic structure on the phase space.

HYPOTHESIS. *The natural structure in the absolute zero temperature region is the simplest stable degenerated symplectic structure* (cf. [27]), *defined by the following* 1-*form of internal energy*

$$\theta = \frac{1}{2} t^2\,dS - p\,dV + \sum_{i=1}^{k} \mu_i\,dN_i, \qquad (*)$$

where t is a parametric temperature.

The 2-form $d\theta$ has Martinet stable singularities of the form

$$x_1\,dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$$

along the hypersurface $\{t^2 = T = 0\}$ and is nonsingular elsewhere. The corresponding 'Lagrange projection' π is the projection of the thermodynamical phase space

$$\{t, -p, \mu_1, \dots, \mu_k, S, V, N_1, \dots, N_k\}$$

onto the space of 'thermodynamical forces' $\{t, -p, \mu_1, \dots, \mu_k\}$, which are natural control parameters in equilibrium.

Assuming $(*)$, we obtain a fine link between the thermodynamical postulate of positivity of absolute temperature and the stability of an applicable structure of thermodynamics. The normal states of equilibrium apart from $\{t = 0\}$ are described by Lagrangian varieties in agreement with classical approach. However, in our completed phase space, it is natural to ask for the classification of local forms of maximal isotropic submanifolds near the singular hypersurface $\{t = 0\}$. Extending the standard theory of singularities of Lagrange projections, one can obtain the following list of generating families for simple Lagrange projections of maximal isotropic submanifolds of codimension 1, near $\{x_1 = 0\}$, namely

$$\lambda^3 + x_2\lambda,$$
$$\lambda^3 + (\pm x_2^{k+1} \pm x_1 + q)\lambda, \quad k \geqslant 1,$$

$$\lambda^3 + (x_2x_3^2 \pm x_2^{k-1} \pm x_1 + q)\lambda, \quad k \geqslant 4,$$
$$\lambda^3 + (x_2^3 \pm x_3^4 \pm x_1 + q)\lambda,$$
$$\lambda^3 + (x_2^3 + x_2x_3^3 \pm x_1 + q)\lambda,$$
$$\lambda^3 + (x_2^3 + x_3^5 \pm x_1 + q)\lambda,$$
$$\lambda^3 + (\pm x_1^k + x_2^2 + q)\lambda, \quad k \geqslant 2,$$
$$\lambda^3 + (x_1x_2 \pm x_2^k + q)\lambda, \quad k \geqslant 2,$$
$$\lambda^3 + (\pm x_1^2 + x_2^3 + q)\lambda,$$

where q is a nondegenerate quadratic form of the remaining variables.

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