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DIFFERENTIAL GEOMETRY

On generic linear equations with skew-symmetric coefficient matrices

by

Takuo FUKUDA and Stanisław JANECZKO

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Summary. In the first part of the paper we study smooth solvability properties of linear equations. We prove an extension of Mather's theorem (1973) to skew-symmetric smooth function matrices. For the proof of the skew-symmetric case as well as of Mather's original results for the cases where the coefficient matrices are general matrices or symmetric matrices, the algebraic methods of Bochnak (1973) are applied. In the second part using criteria for solutions of linear equations we obtain sufficient conditions for smooth solvability of generalized Hamiltonian systems on smooth constraints.

1. Introduction and statement of results. In [10] Mather studied the existence of solutions to generic linear equations in the cases where the coefficient matrices are general matrices or symmetric matrices. For a $p \times q$ matrix $M(x)$ of smooth real valued functions on $\mathbb{R}^n$ he considered a linear equation

$$(*) \qquad M(x)u = f(x),$$

where $f(x)$ is a smooth column p-vector and u is a column q-vector. He proved that if $M(x)$ is generic, i.e. transversal to all Σ_r, $r=0,1,\dots,\min(p,q)$, at every $x \in \mathbb{R}^n$, where $\{\Sigma_r\}$ the canonical stratification by rank of the set of real $p \times q$ matrices, and $(*)$ has a solution $u(x)$ for every $x \in \mathbb{R}^n$, then $(*)$ has a local smooth solution. There are many examples showing that this conclusion is not true if the transversality assumption is dropped. For

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example, consider $n = p = q = 1$ and $M(x) = x^2$, $f(x) = x$; then there is not even a continuous solution of $(*)$ (see [10], [5, Example 5.1]).

In this paper we prove an extended version of Mather's theorem for C^∞ linear equations with skew-symmetric coefficient matrices using Bochnak's theorem [2]. In the same way we can prove Mather's original theorem with generic square matrices and generic symmetric matrices.

Let us fix the notation we will use. We denote by $\mathcal{E}_n$ the ring of germs of C^∞ functions at $0 \in \mathbb{R}^n$. The set of all real (resp. real symmetric, real skew-symmetric) $p\times p$ matrices will be denoted by $\mathcal{M}_p$ (resp. $\mathcal{S}_p$, $\mathcal{A}_p$). In this paper we do not distinguish between a germ at $0 \in \mathbb{R}^n$ and its representatives on a sufficiently small neighbourhood of $0 \in \mathbb{R}^n$.

Let $A(x) = (a_{ij}(x))$ be either an arbitrary $p \times p$ matrix, a symmetric $p \times p$ matrix, or a skew-symmetric $2k \times 2k$ matrix, with $a_{ij} \in \mathcal{E}_n$. Let $b_1, \ldots, b_p \in \mathcal{E}_n$, with $p = 2k$ in the skew-symmetric case, be function-germs such that

$$(1) \qquad b(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_p(x) \end{pmatrix} \quad \text{is in the image of } A(x) \text{ for all } x \in (\mathbb{R}^n, 0),$$

where "for all $x \in (\mathbb{R}^n, 0)$" means "for all x in some neighbourhood of $0 \in \mathbb{R}^n$".

We consider the linear equation

$$(2) \qquad A(x)\lambda = b(x), \quad p = 2k \text{ in the case } A(x) \in \mathcal{A}_{2k},$$

where $\lambda = (\lambda_1, \ldots \lambda_p)^t$ and we consider $A(x)$ as a map-germ from $(\mathbb{R}^n, 0)$ to $\mathcal{M}_p$, $\mathcal{S}_p$ or $\mathcal{A}_{2k}$ according to the case. We can stratify $\mathcal{M}_p$, $\mathcal{S}_p$ and $\mathcal{A}_{2k}$ by rank.

THEOREM 1.1. *Suppose that $b_1, \ldots, b_p \in \mathcal{E}_n$ satisfy condition* (1), *where $p = 2k$ in the skew-symmetric case. If the map-germ $A : (\mathbb{R}^n, 0) \to \mathcal{M}_p$ (resp. $\to \mathcal{S}_p, \mathcal{A}_{2k}$) is transversal to the stratification of $\mathcal{M}_p$ (resp. $\mathcal{S}_p, \mathcal{A}_{2k}$) by rank, then equation* (2) *has a smooth solution.*

We explain our basic ideas for the proof. For a $p \times p$ matrix A we know that

$$\hat{A}A = A\hat{A} = \det A \cdot E_p,$$

where $\hat{A}$ is the cofactor matrix of A and E_p is the unit $p \times p$ matrix. Then from equation (2) we have

$$(2') \qquad \det A(x) \cdot E_p\lambda = \hat{A}(x)b(x),$$

and if $b(x)$ satisfies condition (1) for $A(x)$, then $\hat{A}(x)b(x)$ also satisfies (1) for $\det A \cdot E_p$.

For a smooth function-germ $f \in \mathcal{E}_n$ with $f(0) = 0$ we say that f has *the property of zeros* if the following statement is true:

$$\text{if} \quad g \in \mathcal{E}_n \text{ vanishes on } f^{-1}(0) \quad \text{then} \quad g \in (f)_{\mathcal{E}_n},$$

where $(f)_{\mathcal{E}_n}$ is the ideal of $\mathcal{E}_n$ generated by f.

We will show that if $A : (\mathbb{R}^n, 0) \to \mathcal{M}_p$ (resp. $\to \mathcal{S}_p$) is transversal to the stratification of $\mathcal{M}_p$ (resp. $\mathcal{S}_p$) by rank, then

(i) $\det A(x)$ is irreducible in $\mathcal{E}_n$,
(ii) $\det A(x)$ has the property of zeros (see Lemma 2.13).

Property (ii) is essential to the proof of Theorem 1.1. If $\det A(x)$ has the property of zeros, then (2′) has a smooth solution and it is not difficult to show that so does (2). We prove (ii) using Bochnak's theorem (see Proposition 2.9)

When $A(x)$ is a skew-symmetric matrix, $\det A(x)$ is the square of a function called the pfaffian of $A(x)$ and $\det A(x)$ is not irreducible in $\mathcal{E}_n$. Therefore $\det A(x)$ does not have the property of zeros. However, if $A : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$ is transversal to the stratification of $\mathcal{A}_{2k}$ by rank, then we can show that the pfaffian of $A(x)$ is irreducible and has the property of zeros (Lemma 2.13).

There are many examples of smooth function-matrices which are not transversal to the stratification of $\mathcal{M}_p$ by rank, but for which equation (2) has a smooth solution. The following is a straightforward corollary of Theorem 1.1:

COROLLARY 1.2. *Let $A_1(x), \ldots, A_m(x)$ be matrices whose entries belong to $\mathcal{E}_n$ such that for each $i = 1, \ldots, m$, $A_i : (\mathbb{R}^n, 0) \to \mathcal{M}_p$ is transversal to the stratification of $\mathcal{M}_p$ by rank. Suppose that $(\det A_i)_{\mathcal{E}_n} \neq (\det A_j)_{\mathcal{E}_n}$ for $i \neq j$. Then the product of their determinants has the property of zeros and the equation*

$$A_1(x) \ldots A_m(x)\lambda = b(x)$$

has a smooth solution if $b(x) = (b_1(x), \ldots, b_p(x))^t$ satisfies condition (1) *for the product matrix $A_1(x) \ldots A_m(x)$.*

Also in the cases of symmetric matrices and of skew-symmetric matrices, a similar corollary holds, but very restrictive: In both cases, commutativity $A_i(x)A_j(x) = A_j(x)A_i(x)$ is necessary in order for the product matrix $A_1(x) \ldots A_m(x)$ to be symmetric or skew-symmetric. In the case of skew-symmetric matrices, m must moreover be an odd number.

2. Steps toward the proof of the theorem. In this section we prove Theorem 1.1 by using several propositions and lemmas.

2.1. Reduction of the problem

PROPOSITION 2.1.

(i) *Let* $A(x) = (a_{ij}(x))$ *be a* $p \times p$ *matrix with* $a_{ij} \in \mathcal{E}_n$. *Suppose that* $\operatorname{rank} A(0) = r$. *Then there exist invertible* $p \times p$ *matrices* $P(x) = (p_{ij}(x))$ *and* $Q(x) = (q_{ij}(x))$ *with* $p_{ij}, q_{ij} \in \mathcal{E}_n$ *and a* $(p-r) \times (p-r)$ *matrix* $\bar{A}(x) = (\bar{a}_{ij}(x))$ *with* $\bar{a}_{ij} \in \mathcal{E}_n$ *such that*

$$Q(x)A(x)P(x) = \begin{pmatrix} \bar{A}(x) & O \\ O & E_r \end{pmatrix} \quad \text{and} \quad \bar{A}(0) = O,$$

where E_r *is the unit* $r \times r$ *matrix and* O *is the zero matrix.*

(ii) *Let* $A(x) = (a_{ij}(x))$ *be a symmetric* $p \times p$ *matrix with* $a_{ij} \in \mathcal{E}_n$. *Suppose that* $\operatorname{rank} A(0) = r$. *Then there exist invertible* $p \times p$ *matrices* $P(x) = (p_{ij}(x))$ *with* $p_{ij} \in \mathcal{E}_n$ *and a* $(p-r) \times (p-r)$ *matrix* $\bar{A}(x) = (\bar{a}_{ij}(x))$ *with* $\bar{a}_{ij} \in \mathcal{E}_n$ *such that*

$$P(x)^t A(x)P(x) = \begin{pmatrix} \bar{A}(x) & O \\ O & E_r \end{pmatrix} \quad \text{and} \quad \bar{A}(0) = O.$$

(iii) *Let* $A(x) = (a_{ij}(x))$ *be a skew-symmetric* $2k \times 2k$ *matrix with* $a_{ij} \in \mathcal{E}_n$. *Suppose that* $\operatorname{rank} A(0) = 2r$. *Then there exist invertible* $2k \times 2k$ *matrices* $P(x) = (p_{ij}(x))$ *with* $p_{ij} \in \mathcal{E}_n$ *and a* $(2k-2r) \times (2k-2r)$ *matrix* $\bar{A}(x) = (\bar{a}_{ij}(x))$ *with* $\bar{a}_{ij} \in \mathcal{E}_n$ *such that*

$$P(x)^t A(x)P(x) = \begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix} \quad \text{and} \quad \bar{A}(0) = O,$$

where

$$J_{2r} = \begin{pmatrix} O & E_r \\ -E_r & O \end{pmatrix}.$$

Proof. We prove the skew-symmetric case (iii); items (i) and (ii) go along similar lines. Since $\operatorname{rank} A(0) = 2r$, there exists an invertible $2k \times 2k$ matrix Q such that

$$Q^t A(0)Q = \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix}.$$

Let

$$B(x) = Q^t A(x)Q$$

and write it in the form

$$B(x) = \begin{pmatrix} \bar{B}(x) & \bar{C}(x) \\ -\bar{C}(x)^t & \bar{D}(x) \end{pmatrix}, \quad \bar{B}(0) = O,\ \bar{C}(0) = O,\ \bar{D}(0) = J_{2r}.$$

Then there exists an invertible $2r \times 2r$ matrix $\bar{R}(x) = (r_{ij}(x))$ with $r_{ij} \in \mathcal{E}_n$ and $\bar{R}(0) = E_{2r}$ such that

$$\bar{R}(x)^t \bar{D}(x) \bar{R}(x) = J_{2r}.$$

Let

$$R(x) = \begin{pmatrix} E_{2k-2r} & O \\ O & \bar{R}(x) \end{pmatrix}.$$

Then

$$R(x)^t B(x) R(x) = \begin{pmatrix} \bar{B}(x) & \bar{C}(x)\bar{R}(x) \\ -\bar{R}(x)^t \bar{C}(x)^t & J_{2r} \end{pmatrix}.$$

For simplicity, we write $\bar{E}(x)$ for $\bar{C}(x)\bar{R}(x)$;

$$R(x)^t B(x) R(x) = \begin{pmatrix} \bar{B}(x) & \bar{E}(x) \\ -\bar{E}(x)^t & J_{2r} \end{pmatrix}.$$

We are going to show that there exist an invertible $2k \times 2k$ matrix $S(x) = (s_{ij}(x))$ with $s_{ij} \in \mathcal{E}_n$ and a skew-symmetric $(2k-2r) \times (2k-2r)$ matrix $\bar{A}(x) = (a_{ij}(x))$ with $a_{ij} \in \mathcal{E}_n$ such that

$$S(x)^t \begin{pmatrix} \bar{B}(x) & \bar{E}(x) \\ -\bar{E}(x)^t & J_{2r} \end{pmatrix} S(x) = \begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix}.$$

For this purpose, we look for an invertible $2k \times 2k$ matrix $T(x) = (t_{ij}(x))$ with $t_{ij} \in \mathcal{E}_n$ and a skew-symmetric $(2k-2r) \times (2k-2r)$ matrix $\bar{A}(x) = (a_{ij}(x))$ with $a_{ij} \in \mathcal{E}_n$ such that

$$(*) \qquad T(x)^t \begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix} T(x) = \begin{pmatrix} \bar{B}(x) & \bar{E}(x) \\ -\bar{E}(x)^t & J_{2r} \end{pmatrix}.$$

As a candidate for $T(x)$, we consider a matrix of the form

$$T(x) = \begin{pmatrix} E_{2k-2r} & O \\ \bar{F}(x) & E_{2r} \end{pmatrix}.$$

Then equation $(*)$ becomes

$$\begin{pmatrix} E_{2k-2r} & \bar{F}(x)^t \\ O & E_{2r} \end{pmatrix} \begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix} \begin{pmatrix} E_{2k-2r} & O \\ \bar{F}(x) & E_{2r} \end{pmatrix} = \begin{pmatrix} \bar{B}(x) & \bar{E}(x) \\ -\bar{E}(x)^t & J_{2r} \end{pmatrix},$$

that is,

$$\begin{pmatrix} \bar{A}(x) + \bar{F}(x)^t J_{2r} \bar{F}(x) & \bar{F}(x)^t J_{2r} \\ J_{2r}\bar{F}(x) & J_{2r} \end{pmatrix} = \begin{pmatrix} \bar{B}(x) & \bar{E}(x) \\ -\bar{E}(x)^t & J_{2r} \end{pmatrix}.$$

Thus we have two equations

$$\bar{F}(x)^t J_{2r} = \bar{E}(x), \qquad \bar{A}(x) + \bar{F}(x)^t J_{2r} \bar{F}(x) = \bar{B}.$$

From the first equation we obtain

$$\bar{F}(x) = J_{2r}\bar{E}(x)^t$$

and

$$T(x) = \begin{pmatrix} E_{2k-2r} & O \\ J_{2r}\bar{E}(x)^t & E_{2r} \end{pmatrix}.$$

From the second equation, we have

$$\begin{aligned} \bar{A}(x) &= \bar{B}(x) - \bar{F}(x)^t J_{2r}\bar{F}(x) = \bar{B}(x) - (-\bar{E}(x)J_{2r})(-J_{2r})J_{2r}\bar{E}(x)^t \\ &= \bar{B}(x) + \bar{E}(x)J_{2r}\bar{E}(x)^t. \end{aligned}$$

Setting $S(x) = T(x)^{-1}$ we have

$$S(x)^t \begin{pmatrix} \bar{B}(x) & \bar{E}(x) \\ -\bar{E}(x)^t & J_{2r} \end{pmatrix} S(x) = \begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix},$$

where

$$A(x) = \bar{B}(x) + \bar{E}(x)J_{2r}\bar{E}(x)^t.$$

This finishes the proof. ■

From Proposition 2.1, equation (2) is equivalent to either

$$\begin{pmatrix} \bar{A}(x) & O \\ O & E_r \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} = Q(x)A(x)P(x) \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} = Q(x) \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix}, \tag{3}$$

$$\begin{pmatrix} \bar{A}(x) & O \\ O & E_r \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} = P(x)^t A(x)P(x) \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} = P(x)^t \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix}, \tag{4}$$

or

$$\begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{2k} \end{pmatrix} = P(x)^t A(x)P(x) \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{2k} \end{pmatrix} = P(x)^t \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix}, \tag{5}$$

according to cases (i), (ii) or (iii) in Proposition 2.1.

We see that

(I) if $(\mu_1(x), \ldots, \mu_p(x))^t$ is a smooth solution of (3), (4) or (5), then $(\lambda_1(x), \ldots, \lambda_p(x))^t = P(x)(\mu_1(x), \ldots, \mu_p(x))^t$, with $p = 2k$ for (5), is a smooth solution of (2), and vice versa,

(II) $A : (\mathbb{R}^n, 0) \to \mathcal{M}_p$ (resp. $\to \mathcal{S}_p, \mathcal{A}_{2k}$) is transversal to the stratification of $\mathcal{M}_p$ (resp. $\mathcal{S}_p, \mathcal{A}_{2k}$) by rank if and only if so is $QAP : (\mathbb{R}^n, 0) \to \mathcal{M}_p$ (resp. $P^tAP : (\mathbb{R}^n, 0) \to \mathcal{S}_p$ or $P^tAP : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$).

So, instead of equation (2), we consider

$$\begin{pmatrix} \bar{A}(x) & O \\ O & E_r \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix} = \begin{pmatrix} b_1(x) \\ \vdots \\ b_p(x) \end{pmatrix}, \quad \bar{A}(0) = O, \tag{6}$$

$$\begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{2k} \end{pmatrix} = \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix}, \quad \bar{A}(0) = O. \tag{7}$$

Concerning (II), we have

PROPOSITION 2.2.

(i) *The map-germ*

$$\begin{pmatrix} \bar{A} & O \\ O & E_r \end{pmatrix} : (\mathbb{R}^n, 0) \to \mathcal{M}_p \quad (\textit{resp.} \to \mathcal{S}_p) \quad \textit{with} \quad \bar{A}(0) = O$$

is transversal to the stratification of $\mathcal{M}_p$ *(resp.* $\mathcal{S}_p$*) by rank if and only if* $\bar{A} : (\mathbb{R}^n, 0) \to \mathcal{M}_{p-r}$ *(resp.* $\to \mathcal{S}_{p-r}$*) is transversal to the stratification of* $\mathcal{M}_{p-r}$ *(resp.* $\mathcal{S}_{p-r}$*) by rank.*

(ii) *The map-germ*

$$\begin{pmatrix} \bar{A} & O \\ O & J_{2r} \end{pmatrix} : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k} \quad \textit{with} \quad \bar{A}(0) = O$$

is transversal to the stratification of $\mathcal{A}_{2k}$ *by rank if and only if* $\bar{A}$: $(\mathbb{R}^n, 0) \to \mathcal{A}_{2k-2r}$ *is transversal to the stratification of* $\mathcal{A}_{2k-2r}$ *by rank.*

Proof. We prove case (ii). For a non-negative even number $2m$, let

$$X_{2k,2m} = \{S \in \mathcal{A}_{2k} \mid \operatorname{rank} S = 2m\}. \tag{8}$$

Then the stratification of $\mathcal{A}_{2k}$ by rank is

$$\mathcal{W}(\mathcal{A}_{2k}) = \{X_{2k,2m} \mid m = 0, \dots, k\}. \tag{9}$$

In this proof, we set

$$A(x) = \begin{pmatrix} \bar{A}(x) & O \\ O & J_{2r} \end{pmatrix} \quad \text{and} \quad A(0) = \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix},$$

and for simplicity we write

$$X_{2m} := X_{2k,2m}.$$

Since the stratification of $\mathcal{A}_{2k}$ is a Whitney stratification [4] and $\bar{A}(0) = O$, the map-germ

$$A = \begin{pmatrix} \bar{A} & O \\ O & J_{2r} \end{pmatrix} : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k} \text{ is transversal to } \mathcal{W}(\mathcal{A}_{2k})$$

if and only if $A : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$ is transversal to X_{2r}. We know that

$$X_{2r} = \left\{ P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P \;\middle|\; P \in GL(2k, \mathbb{R}) \right\},$$

and it is enough to calculate the tangent space $T_{A(0)}X_{2r}$:

Lemma.

$$T_{A(0)}X_{2r} = \left\{ \begin{pmatrix} O & B \\ -B^t & D \end{pmatrix} \in \mathcal{A}_{2k} \;\middle|\; B \in M_{2k-2r,2r},\ D \in \mathcal{A}_{2r} \right\},$$

where $M_{p,q}$ is the set of all real $p \times q$ matrices.

By the Lemma, $A : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$ is transversal to X_{2r} if and only if $\bar{A} : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k-2}$ is transversal to the stratification of $\mathcal{A}_{2k-2r}$ by rank. This completes the proof of Proposition 2.2. ∎

Proof of Lemma. We have

$$T_{A(0)}X_{2r}$$
$$= \left\{ \lim_{\theta \to 0} \frac{1}{\theta} \left[(E_{2k} + \theta P)^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} (E_{2k} + \theta P) - \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} \right] \;\middle|\; P \in M_{2k,2k} \right\}.$$

Let $P \in M_{2k,2k}$. We decompose P as

$$P = \begin{pmatrix} Q & R \\ S & T \end{pmatrix},$$

where

$$Q \in M_{2k-2r,2k-2r}, \quad R \in M_{2k-2r,2r}, \quad S \in M_{2r,2k-2r}, \quad T \in M_{2r,2r}.$$

Then

$$P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} = \begin{pmatrix} O & S^t J_{2r} \\ O & T^t J_{2r} \end{pmatrix}, \qquad \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P = \begin{pmatrix} O & O \\ J_{2r}S & J_{2r}T \end{pmatrix}.$$

Now

$$(E_{2k} + \theta P)^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} (E_{2k} + \theta P)$$
$$= \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} + \theta \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P + \theta P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} + \theta^2\, P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P.$$

Thus

$$(E_{2k}+\theta P)^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} (E_{2k}+\theta P) - \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix}$$
$$= \theta \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P + \theta P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} + \theta^2 P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P$$
$$= \theta \begin{pmatrix} O & O \\ J_{2r}S & J_{2r}T \end{pmatrix} + \theta \begin{pmatrix} O & S^t J_{2r} \\ O & T^t J_{2r} \end{pmatrix} + \theta^2 P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P$$
$$= \theta \begin{pmatrix} O & S^t J_{2r} \\ J_{2r}S & J_{2r}T + T^t J_{2r} \end{pmatrix} + \theta^2 P^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} P.$$

Therefore

$$\lim_{\theta\to 0} \frac{1}{\theta} \left[(E_{2k}+\theta P)^t \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} (E_{2k}+\theta P) - \begin{pmatrix} O & O \\ O & J_{2r} \end{pmatrix} \right]$$
$$= \begin{pmatrix} O & S^t J_{2r} \\ J_{2r}S & J_{2r}T + T^t J_{2r} \end{pmatrix}.$$

Now, for any $B \in M_{2k-2r,2r}$, $D \in \mathcal{A}_{2r}$, there exist $S \in M_{2r,2k-2r}$, $T \in M_{2r,2r}$ such that

$$B = S^t J_{2r}, \qquad D = J_{2r}T + T^t J_{2r};$$

they are given by

$$S = J_{2r}B^t, \qquad T = -\frac{1}{2} J_{2r} D.$$

Thus we have

$$T_{A(0)} X_{2r} = \left\{ \begin{pmatrix} O & B \\ -B^t & D \end{pmatrix} \;\middle|\; B \in M_{2k-2r,2r},\ D \in \mathcal{A}_{2r} \right\},$$

as claimed. ■

We can prove Proposition 2.2(i) in the same way, calculating the tangent space to the orbit space of

$$\begin{pmatrix} O & O \\ O & E_r \end{pmatrix}$$

in $\mathcal{M}_p$ (resp. $\mathcal{S}_p$).

Remark 2.3. The stratifications of $\mathcal{M}_p$, $\mathcal{S}_p$ and $\mathcal{A}_{2k}$ by rank are Whitney stratifications. From Whitney's condition (a), the map-germ $\bar{A}$ is transversal to the stratification if and only if it is transversal to the stratum $\{O\}$ in $\mathcal{M}_{p-r}$ (resp. $\mathcal{S}_{p-r}, \mathcal{A}_{2k-2r}$).

Equation (6) can be decomposed into the following two equations:

$$\bar{A}(x)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_{p-r}\end{pmatrix}=\begin{pmatrix}b_1(x)\\ \vdots\\ b_{p-r}(x)\end{pmatrix},\qquad \bar{A}(0)=O, \tag{10}$$

and

$$E_r(x)\begin{pmatrix}\lambda_{p-r+1}\\ \vdots\\ \lambda_p\end{pmatrix}=\begin{pmatrix}b_{p-r+1}(x)\\ \vdots\\ b_p(x)\end{pmatrix}.$$

Since the latter has the obvious smooth solution

$$\lambda_{p-r+1}(x)=b_{p-r+1}(x),\ \ldots,\ \lambda_p(x)=b_p(x),$$

(6) has a smooth solution if and only if so does (10).

Similarly equation (7) has a smooth solution if and only if so does

$$\bar{A}(x)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_{2k-2r}\end{pmatrix}=\begin{pmatrix}b_1(x)\\ \vdots\\ b_{2k-2r}(x)\end{pmatrix},\quad \bar{A}(0)=O. \tag{11}$$

Thus in order to prove Theorem 1.1, in view of Remark 2.3, it is enough to prove it in the special case where $A(0)=O$:

THEOREM 2.4. *Suppose that $b_1,\ldots,b_p\in\mathcal{E}_n$ satisfy condition* (1), *where $p=2k$ in the skew-symmetric case. If the map-germ $A:(\mathbb{R}^n,0)\to\mathcal{M}_p$ (resp. $\to\mathcal{S}_p,\mathcal{A}_{2k}$) with $A(0)=O$ is transversal to $\{O\}$, then equation* (2) *has a smooth solution.*

2.2. Pfaffian of a skew-symmetric matrix. It is known that the determinant of a skew-symmetric $2k\times 2k$ matrix $A=(a_{ij})$ is the square of a polynomial in $a_{ij}, 1\ge i\ge j\ge 2k$, called the *pfaffian* of A, denoted by $\operatorname{Pf} A$:

$$\det A=(\operatorname{Pf} A)^2.$$

The pfaffian is given by

$$\operatorname{Pf} A:=\frac{1}{2^k k!}\sum_{\sigma\in S_{2k}}\operatorname{sgn}(\sigma)a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(2k-1)\sigma(2k)},$$

which is known to be equal to

$$\sum_{\sigma\in F_{2k}}\operatorname{sgn}(\sigma)a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(2k-1)\sigma(2k)},$$

where (with S_{2k} denoting the permutation group of $\{1,\ldots,2k\}$)

$$F_{2k}=\{\sigma\in S_{2k}\mid \sigma(2i-1)<\sigma(2i+1),\ \sigma(2i-1)<\sigma(2i),\ i=1,\ldots,k\}.$$

Let $k \geq 2$. For a skew-symmetric $2k \times 2k$ matrix $A = (a_{ij})$ and integers i, j with $i \neq j$, $1 \leq i, j \leq 2k$, let A_{ij}^{ij} denote the submatrix of A obtained by deleting the ith and jth rows and the ith and jth columns. Then A_{ij}^{ij} is again skew-symmetric.

For given A, let $\hat{A} = (\hat{a}_{ij})$ be the skew-symmetric $k \times k$ matrix defined by

$$\hat{a}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ (-1)^{i+j-1} \operatorname{Pf} A_{ij}^{ij} & \text{if } i < j, \\ (-1)^{i+j} \operatorname{Pf} A_{ij}^{ij} & \text{if } i > j, \end{cases}$$

For $k = 1$, the matrices A_{ij}^{ij} for (i, j) with $i \neq j$ make no sense, and

$$\text{for} \quad A = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}, \quad \text{we define} \quad \hat{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have the following well known formula for pfaffians.

PROPOSITION 2.5.

$$\hat{A}^t A = A \hat{A}^t = (\operatorname{Pf} A) E_{2k}. \tag{12}$$

2.3. Bochnak's theorem: The property of zeros for principal ideals of $\mathcal{E}_n$. First we recall Hilbert's Nullstellensatz. Let $\mathcal{O}_n^{\mathbb{C}}$ denote the ring of holomorphic function-germs at $0 \in \mathbb{C}^n$. Let $\mathcal{I}$ be an ideal of $\mathcal{O}_n^{\mathbb{C}}$ and let X be the germ of a complex analytic subset of $\mathbb{C}^n$ at $0 \in \mathbb{C}^n$ (see [12]). Let

$$\begin{aligned} V(\mathcal{I}) &:= \text{the germ of the set } \{z \in \mathbb{C}^n \mid f(z) = 0 \text{ for all } f \in \mathcal{I}\} \text{ at } 0 \in \mathbb{C}^n, \\ I(X) &:= \text{the ideal } \{f \in \mathcal{O}_n^{\mathbb{C}} \mid f \text{ vanishes on } X\}, \\ \operatorname{rad} \mathcal{I} &:= \text{the radical of } \mathcal{I} := \{f \in \mathcal{O}_n^{\mathbb{C}} \mid f^m \in \mathcal{I} \text{ for some } m \in \mathbb{N}\}. \end{aligned}$$

PROPOSITION 2.6 (Hilbert's Nullstellensatz).

$$I(V(\mathcal{I})) = \operatorname{rad} \mathcal{I}.$$

In the real analytic case and the smooth case, Proposition 2.6 does not hold in general. However, thanks to Bochnak's theorem [2] (in the smooth case) and Risler's theorem [13] (in the real analytic case), we know when it holds.

Using Risler's theorem, we can prove the real analytic version of Theorem 1.1 when the coefficient matrix $A(x)$ is skew-symmetric. With the same argument, we can establish the real analytic version of Theorem 1.1.

We will prove Theorem 2.4 (and hence Theorem 1.1) using Bochnak's theorem.

Let $\mathcal{I}$ be an ideal of $\mathcal{E}_n$ and let X be the germ of a subset of $\mathbb{R}^n$ at $0 \in \mathbb{R}^n$. Let us denote

$V(\mathcal{I})$:= the germ of the set $\{x \in \mathbb{R}^n \mid f(x) = 0$ for all $f \in \mathcal{I}\}$ at $0 \in \mathbb{R}^n$,

$I(X)$:= the ideal $\{f \in \mathcal{E}_n \mid f$ vanishes on $X\}$,

$(f)_{\mathcal{E}_n}$:= the ideal of $\mathcal{E}_n$ generated by $f \in \mathcal{E}_n$.

Definition 2.7. We say that $f \in \mathcal{E}_n$ *has the property of zeros* if the ideal $(f)_{\mathcal{E}_n}$ has the property of zeros, i.e. $I(V((f)_{\mathcal{E}_n})) = (f)_{\mathcal{E}_n}$.

Remark 2.8. In this terminology, we follow Bochnak [2] and Thom [14].

For a smooth function $\tilde{f}$ defined on an open subset U of $\mathbb{R}^n$ and a point $a \in U$, we introduce the following notation:

$$\begin{aligned}
G(\tilde{f}) &:= \{x \in \tilde{f}^{-1}(0) \mid d\tilde{f}(x) \neq 0\},\\
S(\tilde{f}) &:= \{x \in \tilde{f}^{-1}(0) \mid d\tilde{f}(x) = 0\},\\
\mathcal{O}_{n,a} &:= \text{the ring of analytic function-germs at } a \in U,\\
\mathcal{E}_{n,a} &:= \text{the ring of } C^\infty \text{ function-germs at } a \in U,\\
\tilde{f}_a &:= \text{the germ of } \tilde{f} \text{ at } a \in U.
\end{aligned}$$

Proposition 2.9 (Theorem of zeros for principal ideals of $\mathcal{E}_n$, [2, Corollary 2, p. 42]). *Let $f \in \mathcal{O}_n$ be an analytic function-germ. Then the following two conditions are equivalent:*

(i) *f has the property of zeros in $\mathcal{E}_n$: $I(V((f)_{\mathcal{E}_n})) = (f)_{\mathcal{E}_n}$,*

(ii) *if $\tilde{f}$ is a representative of f, then in a sufficiently small neighbourhood U of $0 \in \mathbb{R}^n$,*

$$\overline{G(\tilde{f})} \cap U = \tilde{f}^{-1}(0) \cap U,$$

where $\overline{G(\tilde{f})}$ is the topological closure of $G(\tilde{f})$, and for any point $a \in S(\tilde{f}) \cap U$, any $h \in \mathcal{O}_{n,a}$ and any

$$g \in \{\psi \in \mathcal{O}_{n,a} \mid \dim \psi^{-1}(0) \le n-2\},$$

if $\tilde{f}_a$ divides gh in $\mathcal{E}_{n,a}$, then $\tilde{f}_a$ divides h in $\mathcal{E}_{n,a}$.

Remark 2.10. Consider the function $f(x_1, x_2) = x_1^2 + x_2^2$. It is irreducible in $\mathcal{E}_2$ but it does not satisfy the condition $\overline{G(\tilde{f})} \cap U = \tilde{f}^{-1}(0) \cap U$: we have $\tilde{f}^{-1}(0) = \{0\}$ and $G(f) = \emptyset$. This phenomenon is characteristic of real functions. In the complex case, irreducible holomorphic functions have the property of zeros in the ring of holomorphic functions. And $f(z_1, z_2) = z_1^2 + z_2^2$ is not irreducible in the ring of holomorphic functions.

Remark 2.11. In Bochnak's original paper, another equivalent condition is given:

(iii) if f divides $\varphi_1^2 + \cdots + \varphi_p^2$ then f divides φ_i for $i = 1, \ldots, p$.

However, this condition is difficult to use, so we omitted it.

2.4. Some lemmas needed to apply Bochnak's theorem. We are going to apply Proposition 2.9 to $\det A(x)$ and $\operatorname{Pf} A(x)$. In general, $\det A(x)$ and $\operatorname{Pf} A(x)$ are not analytic function-germs. However, we have the following three lemmas.

LEMMA 2.12. *Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a germ of a diffeomorphism. Suppose that $f \in \mathcal{E}_n$ has the property of zeros in $\mathcal{E}_n$. Then $f \circ \varphi$ has the property of zeros.*

This lemma is obvious.

Now suppose that $A : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$ with $A(0) = O$ is a smooth map-germ transversal to $\{O\} \subset \mathcal{A}_{2k}$. Then the Jacobian matrix of the entries $a_{ij}(x)$, $i < j$, at $0 \in \mathbb{R}^n$ has rank $k(2k-1)$:

$$\operatorname{rank} \frac{\partial(a_{12}, a_{1,3}, \ldots, a_{2k-1,2k})}{\partial(x_1, \ldots, x_n)}(0) = k(2k-1).$$

Therefore there exists a local analytic coordinate system $(y_1, \ldots, y_n)$ in a neighbourhood of $0 \in \mathbb{R}^n$ such that

$$y_1 = a_{12}(x),\ y_2 = a_{13}(x),\ \ldots,\ y_{k(2k-1)} = a_{2k-1,2k}(x); \tag{13}$$

more precisely,

$$a_{ij}(x) = y_{k(2k-1)-(2k-i)(2k-i+1)/2+(j-1)}.$$

Thus, by the definition of pfaffians, $\operatorname{Pf} A(y)$ is a homogeneous polynomial of degree k in the variables $y_1, \ldots, y_{k(2k-1)}$. Moreover, we have

LEMMA 2.13. *Let $A : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$ be a smooth map-germ transversal to $\{O\} \subset \mathcal{A}_{2k}$ and let $(y_1, \ldots, y_n)$ be the local coordinate system satisfying* (13). *Let U be a sufficiently small open neighbourhood of $0 \in \mathbb{R}^n$ such that the mapping $A : U \to \mathcal{A}_{2k}$ is transversal to $\{O\} \subset \mathcal{A}_{2k}$.*

(i) $\operatorname{Pf} A(y)$ *is irreducible in $\mathcal{E}_n$ for all $y \in U$ and* $\dim(\operatorname{Pf} A)^{-1}(0) = n-1$.
(ii) *For any point a in U, the germ at a of the polynomial* $\operatorname{Pf} A(y)$ *is irreducible in $\mathcal{E}_{n,a}$ and* $\dim(\operatorname{Pf} A)^{-1}(0) = n-1$.

Property (ii) follows from (i) by tracing the arguments in §2.1, in particular Propositions 2.1 and 2.2. We will prove Lemma 2.13 in §4. In the same way we have

LEMMA 2.14. *Let $A : (\mathbb{R}^n, 0) \to (\mathcal{M}_p, O)$ (resp. $\to (\mathcal{S}_p, O), (\mathcal{A}_{2k}, O)$) be a smooth map-germ transversal to $\{O\} \subset \mathcal{M}_p$ (resp. $\subset \mathcal{S}_p, \mathcal{A}_{2k}$).*

(i) *There exists a local coordinate system $(y_1, \ldots, y_n)$ in a neighbourhood of $0 \in \mathbb{R}^n$ such that*

$$y_1 = a_{1,1}(x), \quad y_2 = a_{1,2}(x), \ \ldots, \ y_{p^2} = a_{p,p}(x),$$
when $A(x)$ is a general $p \times p$ matrix,

$$y_1 = a_{1,1}(x), \quad y_2 = a_{1,2}(x), \ \ldots, \ y_p = a_{1,p}(x),$$
$$y_{p+1} = a_{2,2}(x), \ \ldots, \ y_{p(p+1)/2} = a_{p,p}(x),$$
when $A(x)$ is a symmetric $p \times p$ matrix.

(ii) *If we express $A(x)$ as $A(y)$ in the above coordinate system $(y_1, \ldots, y_n)$, then* $\det A(y)$ *is a homogeneous polynomial in the variables $y_1, \ldots, y_{p^2}$ or $y_1, \ldots, y_{p(p+1)/2}$ according as $A(y)$ is a general $p \times p$ matrix or a symmetric $p \times p$ matrix.*

(iii) *In both cases,* $\det A(y)$ *is irreducible in $\mathcal{E}_{n,a}$ and* $\dim (\det A)^{-1}(0) = n - 1$.

Homogeneous polynomials are of course analytic functions. Therefore we can apply Proposition 2.9 to $\det A(y)$ and $\mathrm{Pf}\, A(y)$ if they satisfy the conditions in Proposition 2.9. In fact, we have

LEMMA 2.15 (Key Lemma). $\mathrm{Pf}\, A(y)$ *in Lemma* 2.13 *and* $\det A(y)$ *in Lemma* 2.14 *satisfy condition* (ii) *in Proposition* 2.9.

Lemma 2.15 will be proved for skew-symmetric matrices in §4; the other two cases can be treated similarly.

3. Proof of the theorem. We prove Theorem 2.4 under the assumption that Lemmas 2.13–2.15 hold.

We prove Theorem 2.4 for the skew-symmetric case. The theorem for the other two cases, i.e. the original Mather theorem, can be proved in a similar way.

Recall equation (2):

$$A(x)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_{2k}\end{pmatrix} = \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x)\end{pmatrix}.$$

We assume that $A(0) = O$ and $(b_1(x), \ldots, b_{2k}(x))^t \in A(x)(\mathbb{R}^{2k})$ at every x near the origin $0 \in \mathbb{R}^n$ and that $A : (\mathbb{R}^n, 0) \to \mathcal{A}_{2k}$ is transversal to $\{O\} \subset \mathcal{A}_{2k}$. Then from Lemmas 2.12 and 2.15, we may assume $A(x)$ is an analytic map-germ and $\mathrm{Pf}\, A(x)$ satisfies condition (ii) in Proposition 2.9.

From equation (2), we have

$$\hat{A}(x)^t A(x)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_{2k}\end{pmatrix} = \hat{A}(x)^t\begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x)\end{pmatrix}. \tag{14}$$

From Proposition 2.5 we know that $\hat{A}(x)^t A(x) = \operatorname{Pf} A(x) \cdot E_{2k}$. Then setting

$$\begin{pmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_{2k}(x) \end{pmatrix} = \hat{A}(x)^t \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix},$$

we obtain

$$\operatorname{Pf} A(x) \cdot E_{2k} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{2k} \end{pmatrix} = \begin{pmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_{2k}(x) \end{pmatrix}.$$

Since $A(0) = O$ and $(b_1(x), \dots, b_{2k}(x))^t \in A(x)(\mathbb{R}^{2k})$, we have $\operatorname{Pf} A(0) = 0$ and

$$(15) \qquad \begin{aligned} (\tilde{b}_1(x), \dots, \tilde{b}_{2k}(x))^t \in \hat{A}(x)^t A(x)(\mathbb{R}^{2k}) &= \operatorname{Pf} A(x) \cdot E_{2k}(\mathbb{R}^{2k}) \\ &= \operatorname{Pf} A(x)(\mathbb{R}^{2k}) \end{aligned}$$

for every x near $0 \in \mathbb{R}^n$. Since $\operatorname{Pf} A(x)(\mathbb{R}^{2k}) = \mathbb{R}^{2k}$ for those $x \in \mathbb{R}^{2k}$ where $\operatorname{Pf} A(x) \neq 0$, (15) is equivalent to

$$(16) \qquad \tilde{b}_1, \dots, \tilde{b}_{2k} \text{ vanish on } V((\operatorname{Pf} A)_{\mathcal{E}_n}) = \{x \in \mathbb{R}^n \mid \operatorname{Pf} A(x) = 0\},$$

which is equivalent to

$$\tilde{b}_1, \dots, \tilde{b}_{2k} \in I(V_{\mathbb{R}}((\operatorname{Pf} A)_{\mathcal{E}_n})).$$

Since, from Lemma 2.15, $\operatorname{Pf} A$ satisfies condition (ii) in Proposition 2.9, we have

$$(17) \qquad \tilde{b}_1, \dots, \tilde{b}_{2k} \in (\operatorname{Pf} A)_{\mathcal{E}_n}.$$

Thus, there exist $\lambda_1, \dots, \lambda_{2k} \in \mathcal{E}_n$ such that

$$\tilde{b}_1(x) = \lambda_1(x) \operatorname{Pf} A(x), \ \dots, \ \tilde{b}_{2k}(x) = \lambda_{2k}(x) \operatorname{Pf} A(x).$$

Therefore (14) has a smooth solution $(\lambda_1(x), \dots, \lambda_{2k}(x))^t$:

$$\hat{A}(x)^t A(x) \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_{2k}(x) \end{pmatrix} = \hat{A}(x)^t \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix}.$$

Then

$$A(x) \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_{2k}(x) \end{pmatrix} = \begin{pmatrix} b_1(x) \\ \vdots \\ b_{2k}(x) \end{pmatrix} \quad \text{where} \quad \det \hat{A}(x)^t \neq 0.$$

Since $\det A(x) = (\operatorname{Pf} A(x))^2$ and $\hat{A}(x)^t A(x) = \operatorname{Pf} A(x) \cdot E_{2k}$, we know that $\det \hat{A}(x)^t = (\operatorname{Pf} A(x))^{2k-2}$. Therefore

$$A(x)\begin{pmatrix}\lambda_1(x)\\ \vdots \\ \lambda_{2k}(x)\end{pmatrix} = \begin{pmatrix}b_1(x)\\ \vdots \\ b_{2k}(x)\end{pmatrix} \quad \text{where} \quad \operatorname{Pf} A(x) \neq 0.$$

Since $\dim (\operatorname{Pf} A)^{-1}(0) = n-1$ by Lemma 2.13, for an open neighbourhood U of 0 in $\mathbb{R}^n$, the set $\{x \in U \mid \operatorname{Pf} A(x) \neq 0\}$ is an open dense subset and we have

$$A(x)\begin{pmatrix}\lambda_1(x)\\ \vdots \\ \lambda_{2k}(x)\end{pmatrix} = \begin{pmatrix}b_1(x)\\ \vdots \\ b_{2k}(x)\end{pmatrix} \quad \text{for every } x \text{ near } 0 \in \mathbb{R}^n,$$

because $b_1(x), \ldots, b_{2k}(x)$ and $\lambda_1(x), \ldots, \lambda_{2k}(x)$ are smooth, hence continuous.

Thus $(\lambda_1(x), \ldots, \lambda_{2k}(x))^t$ is a smooth solution of (2). This completes the proof of Theorems 2.4 and 1.1 for $A(x)$ skew-symmetric. ∎

We can prove the theorem for the other two cases similarly using Proposition 2.9 and the properties of $\det A(y)$ shown in Lemmas 2.13–2.15.

4. Proof of Lemma 2.13. First we prove that $\operatorname{Pf} A(y)$ is irreducible in $\mathcal{E}_n$. Let $A(y) = (y_{ij})$, where

$$y_{1,2} = y_1, \quad y_{1,3} = y_2, \ \ldots, \ y_{2k-1,2k} = y_{k(2k-1)},$$
$$y_{i,i} = 0,\ i = 1, \ldots, 2k, \quad y_{ij} = -y_{j,i},\ i > j.$$

In this case

$$\operatorname{Pf} A(y) = \sum_{\sigma \in F_{2k}} \operatorname{sgn}(\sigma) y_{\sigma(1)\sigma(2)} y_{\sigma(3)\sigma(4)} \cdots y_{\sigma(2k-1)\sigma(2k)}.$$

Lemma. *Let S be the germ of an analytic submanifold of $\mathbb{R}^n$ at 0. Let $f \in \mathcal{O}_n$ be an analytic function-germ with $f(0) = 0$ whose restriction to S is not the zero function on S. If the restriction of f to S is irreducible in $\mathcal{E}_n$, then so is f.*

The proof is trivial.

We consider the two terms of $\operatorname{Pf} A(y)$ corresponding to

$$\sigma_1 = \begin{pmatrix}1 & 2 & 3 & 4 & \cdots & 2k\\ 1 & 2 & 3 & 4 & \cdots & 2k\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & \cdots & 2k\\ 1 & 2k & 2 & 3 & 4 & 5 & \cdots & 2k-1\end{pmatrix}.$$

Then

$$\operatorname{sgn}(\sigma_1) y_{\sigma_1(1)\sigma_1(2)} y_{\sigma_1(3)\sigma_1(4)} \cdots y_{\sigma_1(2k-1)\sigma_1(2k)} \\ + \operatorname{sgn}(\sigma_2) y_{\sigma_2(1)\sigma_2(2)} y_{\sigma_2(3)\sigma_2(4)} \cdots y_{\sigma_2(2k-1)\sigma_2(2k)} \\ = y_{1,2} y_{3,4} \cdots y_{2k-1,2k} + y_{1,2k} y_{23} \cdots y_{2k-2,2k-1}.$$

Renumber the variables $y_1, \ldots, y_n$ in such a way that the first $2k$ variables are

$$y_1 = y_{1,2}, \qquad y_2 = y_{3,4}, \ \ldots, \ y_k = y_{2k-1,2k},$$
$$y_{k+1} = y_{1,2k}, \quad y_{k+2} = y_{2,3}, \ \ldots, \ y_{2k} = y_{2k-2,2k-1}.$$

Let

$$S = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_{2k+1} = 0, \, y_{2k+2} = 0, \ldots, y_n = 0\}.$$

Then

$$(\operatorname{Pf} A)|_S = y_1 y_2 \ldots y_k + y_{k+1} y_{k+2} \cdots y_{2k},$$

which is obviously irreducible in $\mathcal{E}_n$. Therefore, from the above lemma, $\operatorname{Pf} A(y)$ is irreducible in $\mathcal{E}_n$.

The property $\dim (\operatorname{Pf} A)^{-1}(0) = n-1$ is almost obvious from the following facts:

(i) the set $\Sigma = \{B \in \mathcal{A}_{2k} \mid \operatorname{rank} B < 2k\}$ is a codimension 1 stratified subset of $\mathcal{A}_{2k}$,
(ii) $(\operatorname{Pf} A)^{-1}(0) = A^{-1}(\Sigma)$.

Thus, by Thom's transversality theorem, $\operatorname{codim} (\operatorname{Pf} A)^{-1}(0)$ in $\mathbb{R}^n$ is 1 and $\dim (\operatorname{Pf} A)^{-1}(0) = n - 1$. This completes the proof of Lemma 2.13. ∎

5. Proof of Lemma 2.15. We prove Lemma 2.15 for $\operatorname{Pf} A(y)$. The other case is handled similarly. We are going to show that $\operatorname{Pf} A(y)$ of Lemma 2.13 satisfies condition (ii) in Proposition 2.9. In this case $\tilde{f} = \operatorname{Pf} A(y)$.

First we show that $\operatorname{Pf} A$ satisfies the second condition in (ii) of Proposition 2.9.

Let $a \in S(\tilde{f}) \cap U$, $h \in \mathcal{O}_a$ and $g \in \{\psi \in \mathcal{O}_a \mid \dim \psi^{-1}(0) \le n-2\}$. Assume that $\operatorname{Pf} A_a$ divides gh in $\mathcal{E}_{n,a}$. Suppose that $\operatorname{Pf} A_a$ divides g in $\mathcal{E}_{n,a}$. Then $(\operatorname{Pf} A_a)^{-1}(0)$ must be a subset of $g^{-1}(0)$, but this contradicts the fact that $\dim (\operatorname{Pf} A_a)^{-1}(0) = n-1$ (by Lemma 2.13) and the assumption that $g \in \{\psi \in \mathcal{O}_a \mid \dim \psi^{-1}(0) \le n-2\}$. Hence $\operatorname{Pf} A_a$ does not divide g. On the other hand, from Lemma 2.13, $\operatorname{Pf} A_a$ is irreducible. Therefore $\operatorname{Pf} A_a$ divides h. Thus the second condition of (ii) of Proposition 2.9 is satisfied.

Now we prove that $\operatorname{Pf} A$ satisfies the first condition in (ii) of Proposition 2.9:

$$\overline{G(\operatorname{Pf} A)} \cap U = (\operatorname{Pf} A)^{-1}(0) \cap U.$$

LEMMA. *Let U be a small neighbourhood of* 0 *in* $\mathbb{R}^n$ *and let* $p \in U$. *If* $\operatorname{rank} A(p) = 2k - 2s$, *then the order of* $\operatorname{Pf} A(y)$ *at* p *is* s.

Proof. Since $\operatorname{rank} A(p) = 2k - 2s$, from Proposition 2.1, there exist invertible $2k \times 2k$ matrices $P(y) = (p_{ij}(y))$ with $p_{ij} \in \mathcal{E}_n$ and a skew-symmetric $2s \times 2s$ matrix $\bar{A}(y) = (\bar{a}_{ij}(x))$ with $\bar{a}_{ij} \in \mathcal{E}_n$ such that

$$P(y)^t A(y) P(y) = \begin{pmatrix} \bar{A}(y) & O \\ O & J_{2k-2s} \end{pmatrix} \quad \text{and} \quad \bar{A}(p) = O.$$

Since $\det P(y)^t A(y) P(y) = \det P(y)^2 \det A(y) = \det P(y)^2 (\operatorname{Pf} A(y))^2$, we have

$$\operatorname{Pf}(P(y)^t A(y) P(y)) = \pm \det P(y) \operatorname{Pf} A(y)$$

and

$$\operatorname{Pf}(P(y)^t A(y) P(y)) = \pm \operatorname{Pf} \bar{A}(y).$$

Thus $\det P(y) \operatorname{Pf} A(y) = \pm \operatorname{Pf} \bar{A}(y)$. Since $A : U \to \mathcal{A}_{2k}$ is transversal to the stratification of $\mathcal{A}_{2k}$ by rank, $\bar{A} : U \to \mathcal{A}_{2s}$ is transversal to the stratification of $\mathcal{A}_{2s}$ by rank and $\bar{A}(p) = O$, the order of the polynomial $\operatorname{Pf} \bar{A}(y)$ at p is s. Since $P(y)$ is an invertible matrix, $\det P(y) \neq 0$ and the order of $\operatorname{Pf} A(y)$ at p is equal to the order of $\operatorname{Pf} \bar{A}(y)$ at p, which is s. ∎

Therefore if $\operatorname{rank} A(p) = 2k - 2$, then the order of $\operatorname{Pf} A$ at p is 1 and $d(\operatorname{Pf} A)(p) \neq 0$. If $\operatorname{rank} A(p) < 2k - 2$, then $d(\operatorname{Pf} A)(p) = 0$. Thus

$$\begin{aligned} (\operatorname{Pf} A)^{-1}(0) &= \{p \in \mathbb{R}^n \mid \operatorname{rank} A(p) \leq 2k - 2\}, \\ G(\operatorname{Pf} A) &= \{p \in (\operatorname{Pf} A)^{-1}(0) \mid \operatorname{rank} A(p) = 2k - 2\}, \\ S(\operatorname{Pf} A) &= \{p \in (\operatorname{Pf} A)^{-1}(0) \mid \operatorname{rank} A(p) < 2k - 2\}. \end{aligned}$$

Recall that $X_{2k,2r} = \{B \in \mathcal{A}_{2k} \mid \operatorname{rank} B = 2r\}$, $0 \leq r \leq k$. Then

$$(\operatorname{Pf} A)^{-1}(0) \cap U = A^{-1}\Big(\bigcup_{s=0}^{k-1} X_{2k,2s}\Big) \cap U,$$

$$G(\operatorname{Pf} A) \cap U = A^{-1}(X_{2k,2k-2}) \cap U,$$

$$S(\operatorname{Pf} A) \cap U = A^{-1}\Big(\bigcup_{s=0}^{k-2} X_{2k,2s}\Big) \cap U,$$

where the A on the right hand sides is the mapping $A : \mathbb{R}^n \to \mathcal{A}_{2k}$.

It is known that $\{X_{2k,2k}, X_{2k,2k-2}, \ldots, X_{2k,0} = \{O\}\}$ is a Whitney stratification of $\mathcal{A}_{2k}$, and in particular

$$\overline{X_{2k,2r}} = \bigcup_{s=0}^{r} X_{2k,2s} \quad \text{for all } r \text{ with } 0 \leq r \leq k.$$

Therefore, $\overline{G(\operatorname{Pf} A)} \cap U = (\operatorname{Pf} A)^{-1}(0) \cap U$. This completes the proof of Lemma 2.15. ■

6. Hamiltonian vector fields on constraints. Let K be an even-dimensional submanifold of $(\mathbb{R}^{2n}, \omega)$ endowed with the symplectic structure in Darboux form $\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$ (see [1, 15]). A *generalized Hamiltonian system* on K (generalized Hamiltonian dynamics [3, 11]) is defined as a subbundle L of the tangent bundle $\tau : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ over K which is a Lagrangian submanifold of $(T\mathbb{R}^{2n}, \dot{\omega})$, that is, $\dot{\omega}|_L = 0$ for the associated symplectic form $\dot{\omega} = \sum_{i=1}^{n}(d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i)$. Then locally L is expressed as

(18)
$$L = \{v \in T\mathbb{R}^{2n} : \tau(v) \in K, \text{ and, for any } u \in T_{\tau(v)}K,\ \omega(u, v) = -dh(u)\},$$

for a function h which is locally defined on K. In the coordinates we use, the generalized Hamiltonian system (18) is generated by a Morse family $F : \mathbb{R}^{2n} \times \mathbb{R}^{2k} \to \mathbb{R}$, locally on K (see [15, 7]),

$$F(x, y, \lambda) = \sum_{i=1}^{2k} a_i(x, y)\lambda_i + b(x, y), \tag{19}$$

where K is defined by smooth functions $a_i : \mathbb{R}^{2n} \to \mathbb{R}$, $1 \le i \le 2k < 2n$, with the maximal rank condition $\operatorname{rank}\big(\frac{\partial a_i}{\partial x_l}(x, y), \frac{\partial a_i}{\partial y_l}(x, y)\big) = 2k$, so

$$K = \{(x, y) \in \mathbb{R}^{2n} : a_i(x, y) = 0,\ i = 1, \ldots, 2k\} \tag{20}$$

and $b : \mathbb{R}^{2n} \to \mathbb{R}$ is an arbitrary smooth extension of $h : K \to \mathbb{R}$. In what follows we consider mainly germs of functions and mappings at zero or representatives of such germs.

A fundamental property of such systems is their local smooth solvability, i.e. existence, for each $v \in L$, of a smooth family $\alpha : U \times (-\epsilon, \epsilon) \ni (\bar{v}, t) \mapsto \mathbb{R}^{2n}$ of smooth solutions of L in the neighbourhood U of v in L such that $\dot{\alpha}_{\bar{v}}(0) = \bar{v}$. A sufficient condition for smooth solvability of generalized Hamiltonian systems (see [5, 6, 8]) is the solvability of

$$\left\{\frac{\partial F}{\partial \lambda_i}, F\right\}(x, y, \lambda) = 0, \qquad i = 1, \ldots, 2k,\ (x, y, \lambda) \in K \times \mathbb{R}^{2k},$$

which is a linear equation for λ_j, $j = 1, \ldots, 2k$ (see [10, 5]) with matrix $A(x, y) = (\{a_i, a_j\}(x, y))$:

$$\big(\{a_i, a_j\}(x, y)\big)_{1 \le i,j \le 2k} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{2k} \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_{2k}\}(x, y) \end{pmatrix}, \qquad (x, y) \in K, \tag{21}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket induced by ω. The pointwise solvability of (21), that is,

$$\begin{pmatrix} \{b,a_1\}(x,y) \\ \vdots \\ \{b,a_{2k}\}(x,y) \end{pmatrix} \in \operatorname{Im} A(x,y), \qquad (x,y)\in K, \tag{22}$$

is a necessary condition for its smooth solvability.

Smooth solutions of (21), $\lambda(x,y)$ for $(x,y)\in K$, with any smooth extension to $\mathbb{R}^{2n}$ define sections of the bundle L as smoothly solvable (in the above sense) Hamiltonian vector fields with Hamiltonian

$$\begin{aligned} F_{a,b,\lambda}(x,y) &= F(x,y,\lambda(x,y)) \\ &= \sum_{i=1}^{2k} a_i(x,y)\lambda_i(x,y) + b(x,y); \end{aligned} \tag{23}$$

they are constrained Hamiltonian vector fields on K endowed with the restricted symplectic structure $\omega|_K$ (see [9]).

Let U be an open neighbourhood of 0 in $\mathbb{R}^{2n}$ and let $a_1(x,y),\ldots,a_{2k}(x,y)$ be C^∞ functions on U, defining the smooth submanifold K in (20) and such that

$$a_1(0) = \cdots = a_{2k}(0) = 0.$$

Using the skew-symmetric version of the Mather theorem 1.1 we get

PROPOSITION 6.1. *If A is transversal to the stratification $\mathcal{W}(\mathcal{A}_{2k})$ (see (9)) of $\mathcal{A}_{2k}$ by rank at $(0,0)$, and $b\in C^\infty(U)$ satisfies the pointwise solvability condition (22), then (21) has a smooth solution defined on a sufficiently small neighbourhood of $(0,0)\in K$.*

The solutions of (21) providing stationary Hamiltonian systems on K are given by solutions of the reduced equation

$$\left(\frac{\partial(a_1,\ldots,a_k)}{\partial(x,y)}(x,y)\right)^t \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{2k} \end{pmatrix} = -\begin{pmatrix} \frac{\partial b}{\partial x_1}(x,y) \\ \vdots \\ \frac{\partial b}{\partial y_n}(x,y) \end{pmatrix}. \tag{24}$$

It is easy to check that the solutions of (24), for which we can apply the usual Mather theorem, are also solutions of (21). The representation of the corresponding constrained Hamiltonian vector fields is given by the mapping

$$\Phi : K\times\mathbb{R}^{2k}\ (\subset \mathbb{R}^{2n}\times\mathbb{R}^{2k}) \to T\mathbb{R}^{2n}$$

defined by

$$
(25)\qquad
\begin{aligned}
x_i \circ \Phi(x,y,\lambda) &= x_i, \qquad y_i \circ \Phi(x,y,\lambda) = y_i,\\
\dot{x}_i \circ \Phi(x,y,\lambda) &= \sum_{\ell=1}^{2k} \frac{\partial a_\ell}{\partial y_i}(x,y)\lambda_\ell + \frac{\partial b}{\partial y_i}(x,y),\\
\dot{y}_i \circ \Phi(x,y,\lambda) &= \sum_{\ell=1}^{2k} -\frac{\partial a_\ell}{\partial x_i}(x,y)\lambda_\ell - \frac{\partial b}{\partial x_i}(x,y),
\end{aligned}
$$

for $i = 1, \ldots, n$.

Let Γ_λ denote the graph of the smooth solution $\lambda(x,y)$ of (21) with λ extended to an open neighbourhood V of zero in $\mathbb{R}^{2n}$,

$$
(26)\qquad \Gamma_\lambda(x,y) = (x,y,\lambda(x,y)), \quad (x,y) \in V.
$$

Now we regard $(\Phi \circ \Gamma_\lambda)(V)$ as a vector field tangent to K on V and we denote it by $X_{F_{a,b,\lambda}}$;

$$
(27)\qquad
\begin{aligned}
X_{F_{a,b,\lambda}}(x,y) = {} & \sum_{i=1}^{n}\Big(\sum_{\ell=1}^{2k} \frac{\partial a_\ell}{\partial y_i}(x,y)\lambda_\ell(x,y) + \frac{\partial b}{\partial y_i}(x,y)\Big)\frac{\partial}{\partial x_i}\\
& - \sum_{i=1}^{n}\Big(\sum_{\ell=1}^{2k} \frac{\partial a_\ell}{\partial x_i}(x,y)\lambda_\ell(x,y) + \frac{\partial b}{\partial x_i}(x,y)\Big)\frac{\partial}{\partial y_i}.
\end{aligned}
$$

$X_{F_{a,b,\lambda}}|_K$ is a section of L and $X_{F_{a,b,\lambda}}$ is a Lagrangian submanifold of $(T\mathbb{R}^{2n}, \dot{\omega})$, $X_{F_{a,b,\lambda}}(a_i) = 0$, $i = 1, \ldots, 2k$. For λ being a solution of (24) we get a stationary constrained Hamiltonian vector field

$$
X_{F_{a,b,\lambda}}(x,y) = \sum_{i=1}^{n} 0 \cdot \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} 0 \cdot \frac{\partial}{\partial y_i}\bigg|_K .
$$

The necessary conditions for b providing the pointwise solutions of (21) define the Poisson–Lie algebra of constrained Hamiltonian vector fields (see [5]). By applying Theorem 1.1 we know that (21) has a smooth solution on K iff the transversality condition for A is fulfilled and

$$
b \in \mathcal{B}_a = \langle a_1, \ldots, a_{2k}\rangle + \{a_1, \ldots, a_{2k}\}_K^{\perp},
$$

where $\{a_1, \ldots, a_{2k}\}_K^{\perp} = \{h \in \mathcal{E}_{x,y} \mid \{h, a_i\} = 0 \text{ on } K,\ 1 \le i \le 2k\}$. And the space of Hamiltonians

$$
H_{a,K} = \{F_{a,b,\lambda} \mid b \in \mathcal{B}_a,\ \lambda(x,y) \in \mathcal{S}_{a,b}\},
$$

where $\mathcal{S}_{a,b} = \{\lambda(x,y) \in \mathcal{E}_{x,y} \mid \lambda \text{ is a smooth solution of (21) on } K\}$, is a Poisson–Lie algebra with respect to ω.

For $b \in \mathcal{B}_a$ and for a smooth solution $\lambda(x,y)$ on K the Hamiltonian vector field $X_{F_{a,b,\lambda}}$ is tangent to K and its restriction to K is a constrained

Hamiltonian system on K. Since we are interested in constrained Hamiltonian systems on a submanifold K of $(\mathbb{R}^{2n},\omega)$, it is natural to identify two Hamiltonian functions $F, F' \in H_{a,K}$ such that $X_F|_K = X_{F'}|_K$. Let

$$\mathcal{N}_{a,K} = \{F \in H_{a,K} : X_F|_K = 0\}. \tag{28}$$

Then we have naturally

PROPOSITION 6.2. *$\mathcal{N}_{a,K}$ is an ideal of the Poisson–Lie algebra $H_{a,K}$.*

The Poisson–Lie algebra of constrained Hamiltonian systems is defined as the quotient algebra

$$\mathcal{H}_{a,K} := H_{a,K}/\mathcal{N}_{a,K}.$$

As an example we consider the case where K is a submanifold of $(\mathbb{R}^4,\omega)$ with $\omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$:

$$K = \{(x,y) \in \mathbb{R}^4 : x_1 = 0,\ y_2 - y_1^2 = 0\}.$$

We write

$$a_1(x,y) = x_1, \qquad a_2(x,y) = y_2 - y_1^2,$$

and $\omega|_K = 2y_1 dy_1 \wedge dx_2$ is singular on $\{y_1 = 0\}$.

For a function $b(x,y) \in \mathcal{E}_{x,y}$ we consider the linear equation

$$\begin{pmatrix} 0 & 2y_1 \\ -2y_1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \{b,a_1\} \\ \{b,a_2\} \end{pmatrix}, \quad \text{where} \quad 2y_1 = \{a_1,a_2\}. \tag{29}$$

A smooth solution of (29) on K is a pair $(\lambda_1(x,y), \lambda_2(x,y))$ of elements of $\mathcal{E}_{x,y}$ such that

$$\begin{pmatrix} 0 & 2y_1 \\ -2y_1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(x,y) \\ \lambda_2(x,y) \end{pmatrix} = \begin{pmatrix} \{b,a_1\}(x,y) \\ \{b,a_2\}(x,y) \end{pmatrix} \quad \text{for all } (x,y) \in K.$$

As $A(x,y)$ is transversal, we know that (29) has a smooth solution on K if and only if

$$b \in \mathcal{B}_a := \langle a_1,a_2\rangle_{\mathcal{E}_{x,y}} + \{a_1,a_2\}_K^{\perp}. \tag{30}$$

For $b \in \mathcal{B}_a$ and for a smooth solution $\lambda(x,y) = (\lambda_1(x,y), \lambda_2(x,y))$ of (29) on K, we consider the Hamiltonian function

$$F_{b,\lambda}(x,y) = x_1\lambda_1(x,y) + (y_2 - y_1^2)\lambda_2(x,y) + b(x,y).$$

The Hamiltonian vector field $X_{F_{b,\lambda}}$ is tangent to K, and its restriction to K is a constrained Hamiltonian system on K. We see that

$$\begin{aligned} X_{F_{b,\lambda}}|_K(x,y) = {} & \left(\frac{\partial b}{\partial y_1} - 2y_1\lambda_2\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial b}{\partial y_2} + \lambda_2\right)\frac{\partial}{\partial x_2} \\ & - \left(\frac{\partial b}{\partial x_1} + \lambda_1\right)\frac{\partial}{\partial y_1} - \frac{\partial b}{\partial x_2}\frac{\partial}{\partial y_2}, \quad (x,y) \in K. \end{aligned}$$

Let $\lambda'(x,y) = (\lambda_1'(x,y), \lambda_2'(x,y))$ be another smooth solution of (29) on K. Then we see that $\lambda'(x,y)|_K = \lambda(x,y)|_K$ and $X_{F_{b,\lambda'}}|_K = X_{F_{b,\lambda}}|_K$. Thus the restriction of $X_{F_{b,\lambda}}$ to K does not depend on the choice of a smooth solution of (29) on K. We denote

$$X_b := X_{F_{b,\lambda}}|_K. \tag{31}$$

Thus

$$\mathcal{H}_{a,K} = \{X_b(x,y) : b \in \mathcal{B}_a\}.$$

After some calculations we find that for any $b \in \mathcal{B}_a$, there exists a function $\bar{b}(x,y) \in \langle y_1^2 \rangle_{\mathcal{E}_{x,y}}$ such that $X_b = X_{\bar{b}}$. And if we consider $\bar{b}(x,y) = y_1^2 \bar{\alpha}(x,y) \in \langle y_1^2 \rangle_{\mathcal{E}_{x,y}}$, then the linear equation (29) has a smooth solution on K and we have

$$\begin{aligned} X_{\bar{b}} = {} & \left(\bar{\alpha} + \frac{1}{2} y_1 \frac{\partial \bar{\alpha}}{\partial y_1} + y_1^2 \frac{\partial \bar{\alpha}}{\partial y_2}\right)(x,y) \frac{\partial}{\partial x_2} \\ & - \frac{1}{2} y_1 \frac{\partial \bar{\alpha}}{\partial x_2}(x,y) \frac{\partial}{\partial y_1} - y_1^2 \frac{\partial \bar{\alpha}}{\partial x_2}(x,y) \frac{\partial}{\partial y_2}, \qquad (x,y) \in K. \end{aligned} \tag{32}$$

For $\bar{b} = y_1^2 x_2$ we have

$$\{\bar{b}, x_1\} = 2 y_1 x_2, \qquad \{\bar{b}, y_2 - y_1^2\} = -y_1^2$$

and the solution $(\lambda_1, \lambda_2) = (x_2, \frac{1}{2} y_1)$ of (29) gives the solvable Hamiltonian vector field on $(K, \omega|_K)$,

$$X_{\bar{b}} = x_2 \frac{\partial}{\partial x_2} - \frac{1}{2} y_1 \frac{\partial}{\partial y_1} - y_1^2 \frac{\partial}{\partial y_2}.$$

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Takuo Fukuda
Department of Mathematics
College of Humanities and Sciences
Nihon University
Tokyo, Japan
E-mail: fukuda@math.chs.nihon-u.ac.jp

Stanisław Janeczko
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland
E-mail: stanislaw.janeczko@pw.edu.pl