

Generating families for images of Lagrangian submanifolds and open swallowtails

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Summary. In this paper we study the symplectic relations appearing as the generalized cotangent bundle liftings of smooth stable mappings. Using this class of symplectic relations the classification theorem for generic (pre) images of lagrangian submanifolds is proved. The normal forms for the respective classified pullbacks and pushforwards are provided and the connections between the singularity types of symplectic relation, mapped lagrangian submanifold and singular image, are established. The notion of special symplectic triplet is introduced and the generic local models of such triplets are studied. We show that the open swallowtails are canonically represented as pushforwards of the appropriate regular lagrangian submanifolds. Using the $SL_2(\mathbb{R})$ invariant symplectic structure of the space of binary forms of an appropriate dimension we derive the generating families for the open swallowtails and the respective generating functions for its regular resolutions.

1. Introduction

The classification of singular lagrangian submanifolds as the sets of rays tangent to the geodesic flows on a hypersurface was carried out in the previous papers [15, 3]. This is connected to the theory of nested hypersurfaces in a symplectic manifold describing the geodesics on a Riemannian manifold with boundary [14]. In particular it is closely related to the problem of the shortest bypassing of the obstacle represented by a smooth hypersurface [3, 2], which we can briefly formulate as follows: Let $\mathbb{R}^{2n} = \{(x, p)\}$ be a phase space of a particle in classical mechanics [1], let $h(x, p) = \frac{1}{2}(p^2 - 1)$ be a hamiltonian function for this particle. Then the space of bicharacteristics in $H = \{h = 0\}$, say M , which forms a manifold of all oriented lines in \mathbb{R}^n , has a canonical symplectic structure. Let K be a hypersurface in \mathbb{R}^n (an obstacle) and γ a geodesic flow on K (e.g. the one defined on K by the variational problem of shortest bypassing of K). It is proved in [2] that the set of oriented lines tangent to γ on K forms a lagrangian submanifold in M which is not necessarily smooth. The appropriate local classification of these singular lagrangian submanifolds is carried out in the paper cited. It turned out that the generic singularities of this classification, so-called open swallowtails, can be conveniently described in the $SL_2(\mathbb{R})$ -invariant symplectic space of binary forms of an appropriate degree. We find that the open swallowtails can be obtained as images of the regular lagrangian submanifolds by means of a canonical symplectic procedure. This observation suggests the further generalization of the problem and classification of images of lagrangian submanifolds by means of (widely used in physical applications [5]) symplectic relations [16]. It appears that the beginning of the geometrical classifi-

cation presented in this paper provides new types of singular lagrangian submanifolds (cf. [12]).

Another motivation for the investigations presented here comes from thermodynamics of phase transitions [12] and independently from statics of controlled mechanical systems [18]. Let us consider a simple one-component thermodynamical system (cf. [12]) and admit the class of deformations onto two isolated subsystems of the same sample. The phase space for such deformations is as follows (cf. [12, 17]):

$$(T^*Y_1 \times T^*Y_2, -S_1 dT_1 - p_1 dV_1 + \mu_1 dN_1 - S_2 dT_2 - p_2 dV_2 + \mu_2 dN_2),$$

where $T^*Y_1, \{V_1, T_1, N_1, -p_1, -S_1, \mu_1\}$; $T^*Y_2, \{V_2, T_2, N_2, -p_2, -S_2, \mu_2\}$ are the phase spaces of the respective subsystems and $V_i, T_i, N_i, p_i, S_i, \mu_i$ are the standard thermodynamical coordinates. Let a lagrangian submanifold $L_1 \times L_2 \subset T^*Y_1 \times T^*Y_2$ be the space of equilibrium states of a composite isolated system. After removal of (chemical, thermal, mechanical) constraints, the virtual states of the system are defined by the coisotropic submanifold $C \subseteq T^*Y_1 \times T^*Y_2$ (cf. [5]),

$$C = \{T_1 = T_2, p_1 = p_2, \mu_1 = \mu_2, N_1 + N_2 = N = \text{const.} \quad (N_1 > 0, N_2 > 0)\}.$$

C provides the canonical characteristic submersion, say ρ , onto the phase space of the composite system $(T^*Y, -S dT - p dV)$,

$$\rho: C \rightarrow T^*Y, \rho(V_1, T_1, N_1, p_1, S_1, \mu_1, V_2, T_2, N_2, p_2, S_2, \mu_2) = (V_1 + V_2, T_1, p_1, S_1 + S_2).$$

Hence the space of equilibrium states of the composite system is an image $\rho(L_1 \times L_2)$, which for the Van der Waals gas forms a singular lagrangian submanifold in T^*Y well known in thermodynamics of coexistence states [12].

The aim of this paper is to set up a method of formalizing and generalizing these examples and derive the first results for further applications. We now outline the organization of the paper. In Section 2, in the beginning, we introduce some known but perhaps unfamiliar results of symplectic geometry, which we shall need later. Then we formulate the problem of classification of images of lagrangian submanifolds by means of special classes of symplectic relations, namely those generated by modified push-forwards and pullbacks of smooth mappings. This classification forces us to introduce a notion of singular lagrangian submanifold and to prove some results concerning the generating families (useful physical potentials) of these classified images. Restricting consideration to symplectic manifolds of dimension not greater than four, we prove the classification theorem for the normal forms of generic, generating families of the respective images of stable lagrangian submanifolds with respect to stable mappings. This classification substantially depends on the results of [13] but provides a more exact description of singular images and their maximally reduced generating families. Section 3 is devoted to the investigation of local properties of general symplectic triplets. We prove here the classification theorem for the so-called special symplectic triplets and derive the respective generating families for the respective lagrangian sets which it provides. In Section 4 we introduce the basis of Arnold's theory of open swallowtails represented in the symplectic space of binary forms. We show that the open swallowtails are provided by the appropriate special symplectic triplets. Using the methods of symplectic relations developed before, we prove that the open swallowtails are images of the regular lagrangian submanifolds by the canonical symplectic reduction relation. This fact allows us to perform the precise calculations for generating

families of the open swallowtails and compare them to those for the respective special symplectic triplets.

2. Symplectic relations and images of lagrangian submanifolds

Let $(P_1, \omega_1), (P_2, \omega_2)$ be two symplectic manifolds (see [1]). We define the product $(P_1, \omega_1) \times (P_2, \omega_2)$ as the symplectic manifold $(P_1 \times P_2, pr_1^* \omega_1 + pr_2^* \omega_2)$, where

$$pr_i: P_1 \times P_2 \rightarrow P_i \quad (i = 1, 2)$$

are the cartesian projections. We define a symplectic relation from (P_1, ω_1) to (P_2, ω_2) as an immersed lagrangian submanifold of $(P_1, -\omega_1) \times (P_2, \omega_2)$ and denote it by R (see [16, 5]).

We recall a notion of symplectic relation of a particular kind, namely the symplectic reduction relation. Such relations are morphisms in the category of symplectic manifolds and are very widely used in mathematical physics (cf. [19, 17, 18, 5, 1, 14]). A submanifold $C \subseteq (P, \omega)$ is called coisotropic if at each $x \in C$

$$(T_x C)^\S = \{v \in T_x P; \forall u \in T_x C \langle v \wedge u, \omega \rangle = 0\} \subset T_x C. \tag{1}$$

Let $D = \{v \in TC; v \lrcorner (\omega|_C) = 0\}$; we call D the characteristic distribution of C . Let B be the set of characteristics. We consider the following relation from P to B :

$$R = \{(x, b) \in P \times B; x \in C, b = \rho(x)\}, \tag{2}$$

where $\rho: C \rightarrow B$ is the canonical projection. If B admits a differentiable structure and the map ρ is a submersion (cf. [19]) then there is a unique symplectic structure β on B such that

$$\rho^* \beta = \omega|_C. \tag{3}$$

In this case (B, β) is called the reduced symplectic manifold, and R is a symplectic relation from (P, ω) to (B, β) . R is called the symplectic reduction relation of the symplectic manifold (P, ω) with respect to the coisotropic submanifold C (see [5, 16]).

Let $R \subset (P_1 \times P_2, pr_1^* \omega_2 - pr_2^* \omega_1)$ be a symplectic relation and $L \subset P_1$ a lagrangian submanifold of (P_1, ω_1) . The set

$$R(L) = \{p_2 \in P_2; \text{there exists } p_1 \in L \text{ such that } (p_1, p_2) \in R\} \tag{4}$$

is called the image of L under the symplectic relation R . Using the transpose relation tR (cf. [5]) we analogously define the counterimage of $N \subset (P_2, \omega_2)$, namely ${}^tR(N)$.

For the purpose of this paper we confine considerations to the typical example of symplectic manifolds, namely the cotangent bundle (i.e. the symplectic manifolds found in most applications are isomorphic to cotangent bundles [16, 12, 1]) (T^*X, ω_X) , where $\omega_X = d\vartheta_X$, and ϑ_X is the Liouville form in the cotangent bundle T^*X (over a smooth manifold X).

Let $(T^*X, \omega_X), (T^*Y, \omega_Y)$ be two cotangent bundles. the product

$$\Omega = (T^*X \times T^*Y, pr_2^* \omega_Y - pr_1^* \omega_X)$$

is a symplectic manifold which, for further purposes, will be identified with $T^*(X \times Y)$. Let $f: X \rightarrow Y$ be a smooth mapping. We denote the graph of f by Γf ; Γf is a submanifold of $X \times Y$. Any function on Γf can be pulled back onto X , so the smooth structure on

Γf is equivalent to the smooth structure on X . As we know (see [17], proposition 3·1) the set

$$\{p \in T^*(X \times Y); \pi_{X \times Y}(p) \in \Gamma f \text{ and } \langle u, p \rangle = \langle u, d\bar{g} \rangle \text{ for each } u \in T(\Gamma f) \subset T(X \times Y) \text{ such that } \tau_{X \times Y}(u) = \pi_{X \times Y}(p)\}. \quad (5)$$

is a symplectic relation in Ω . Here \bar{g} is a smooth function on Γf ; $\pi_{X \times Y}, \tau_{X \times Y}$ are the projections for cotangent and tangent bundles respectively. Let g denote the function \bar{g} pulled back to X .

Definition 2·1. Let $f: X \rightarrow Y, g: X \rightarrow \mathbb{R}$ be smooth functions. The symplectic relation defined in (5), and denoted by (f, g) , is called an f -constrained symplectic relation. We denote the set of all f -constrained symplectic relations in Ω by \mathcal{F} .

In the present paper we are interested only in local properties of symplectic relations as well as in local properties of images of lagrangian submanifolds. Hence X, Y will be open subsets of \mathbb{R}^n and \mathbb{R}^m respectively, and instead of lagrangian submanifolds or mappings we shall in fact consider their germs (see [6]). Later on, to avoid an inessential rigour, we speak about mappings, submanifolds etc. as representatives of germs.

Let us introduce in \mathcal{F} an action of a subgroup of the group of symplectomorphisms (an equivalence relation) such that for the images of lagrangian submanifolds this action reduces to the standard action (see [4, 22]) of the group of symplectomorphisms preserving the fibre structure of the cotangent bundle. Hence we introduce in Ω the canonical action of the group $\mathcal{G} = G_X \times G_Y$, where by G_X (resp. G_Y) we denote the group of symplectomorphisms preserving the fibre structure of T^*X (resp. T^*Y). It is evident that \mathcal{G} acts on \mathcal{F} , transforming a symplectic relation $R \subset \Omega$ into $(\Phi, \Psi)(R)$, where $(\Phi, \Psi) \in \mathcal{G}$. As we know (see [7, 22]) a symplectomorphism $(\Phi, \Psi) \in \mathcal{G}$, locally, has the following form:

$$\left. \begin{aligned} \Phi(x, \xi) &= (\phi(x), {}^tD\phi(x)^{-1}(\xi + d\alpha(x))) : T^*X \rightarrow T^*X, \\ \Psi(y, \eta) &= (\psi(y), {}^tD\psi(y)^{-1}(\eta + d\beta(y))) : T^*Y \rightarrow T^*Y, \end{aligned} \right\} \quad (6)$$

where ϕ, ψ are diffeomorphisms, $\phi: X \rightarrow X, \psi: Y \rightarrow Y$ and α, β are smooth functions on X and Y respectively. Thus the group \mathcal{G} is defined as a system of functions and diffeomorphisms: $(\phi, \alpha, \psi, \beta)$ with an appropriate composition formula.

By straightforward calculations, using [6] and the definition of a symplectic relation belonging to \mathcal{F} , we obtain

PROPOSITION 2·2. *For pairs (f, g) determining symplectic relations belonging to \mathcal{F} , we have the following transformation law (for the action of \mathcal{G} on Ω introduced above):*

$$(f, g) \rightarrow (\Phi, \Psi)(f, g) = (\psi \circ f \circ \phi^{-1}, g \circ \phi^{-1} + \beta \circ f \circ \phi^{-1} - \alpha \circ \phi^{-1}). \quad (7)$$

Taking $\beta = 0$ and $\alpha = g$, we see that the second component on the right hand side of (7) vanishes. Thus we have

COROLLARY 2·3. *For any orbit of the action (7) of \mathcal{G} in \mathcal{F} , there exists a representative of the form $(f, 0)$ i.e. a pure lifting of f to the cotangent bundle Ω , henceforth denoted by T^*f (cf. [19, 12]).*

If we consider the subgroup of \mathcal{G} , say \mathcal{G}' , whose elements are determined by triplets (ϕ, α, ψ) and act on a relation R by means of the symplectomorphism $(\phi, \alpha, \psi, \alpha \circ f)$, then immediately we obtain

COROLLARY 2.4. *The action (7) restricted to the subgroup $\mathcal{G}' \subset \mathcal{G}$ is a well-defined action on the space, say \mathcal{F} , of canonical liftings T^*f of smooth mappings $f: X \rightarrow Y$ to Ω . An element of \mathcal{F}' is represented by a pair $(f, 0)$.*

Let $R \in \mathcal{F}$ and L, N be the lagrangian submanifolds in (T^*X, ω_X) and (T^*Y, ω_Y) respectively.

Definition 2.5. Let $R = (f, g)$. The subset $R(L) \subset T^*Y$ (${}^tR(N) \subset T^*X$) is called the pushforward of L (respectively pullback of N) with respect to R .

It is well known (see [10]) that if f is an immersion then the pushforwards of lagrangian submanifolds are always smooth lagrangian submanifolds of T^*Y . Analogously, if f is a submersion then the pullbacks of lagrangian submanifolds are smooth lagrangian submanifolds of T^*X . More generally, if $\pi_Y|_N: N \rightarrow Y$ and $f: X \rightarrow Y$ are transversal mappings then the pullback ${}^tR(N)$ is a lagrangian submanifold of T^*X . An analogous result holds for pushforwards: if f has constant rank and L is transversal to ${}^tR(T^*Y)$ then $R(L)$ is a lagrangian submanifold of T^*Y .

In this paper we study the more general situation when the transversality conditions mentioned above are not assumed and f is a stable smooth mapping. We shall study a further possible approach to the classification of singular images (pullbacks and pushforwards), by specifying the various types of nontransversalities, in a forthcoming paper.

Let us denote the pushforward of L with respect to R by the pair (R, L) ; similarly, for the pullback of N with respect to R we use the notation (N, R) .

Definition 2.6. The pushforwards $(R_1, L_1), (R_2, L_2)$, (pullbacks: $(N_1, R_1), (N_2, R_2)$) are called equivalent if there exists $g \in \mathcal{G}, g = (\Phi, \Psi)$, such that

$$\left. \begin{aligned} (R_2, L_2) &= (g(R_1), \Phi(L_1)) \\ \text{(respectively } (N_2, R_2) &= (\Psi(N_1), g(R_1))) \end{aligned} \right\} \tag{8}$$

Before we proceed to classification of images we recall the very convenient notion of Morse family. It is well known (see [19, 5]) that any lagrangian submanifold L of the cotangent bundle, say T^*X , can be locally generated by a family of functions, the so-called Morse family, $F: X \times \mathbb{R}^k \rightarrow \mathbb{R}$ (for some $k \in \mathbb{N}, k \leq \dim X$) so that

$$L = \left\{ (x, \xi); \quad \xi = \frac{\partial F}{\partial x}(x, \lambda), \quad O = \frac{\partial F}{\partial \lambda}(x, \lambda) \right\}, \tag{9}$$

where $\text{rank} (\partial^2 F / \partial \lambda^2, \partial^2 F / \partial \lambda \partial x) = k$ in an appropriate source point of the germ F . Between Morse families with a minimal number of parameters (see [22]), there is the following notion of equivalence: two Morse families (or generating families as below) $F, F': X \times \mathbb{R}^k \rightarrow \mathbb{R}$ are equivalent if there exists a diffeomorphism

$$\Xi: X \times \mathbb{R}^k \rightarrow X \times \mathbb{R}^k, \quad \rho_X \circ \Xi = \rho_X,$$

such that $F = F' \circ \Xi$, where $\rho_X: X \times \mathbb{R}^k \rightarrow X$ is the projection. Let us note that equivalent Morse families represent the same lagrangian submanifold of T^*X (change of parametrization). For the proof of the converse statement see e.g. [22].

In this paper, most frequently, we use rather the following notion.

Definition 2.7. A family of functions on X , which describes a lagrangian subset in T^*X (possibly nondifferentiable but endowed with a Whitney stratification [11], the maximal strata of which are lagrangian) by the formula (9), not necessarily with the rank assumption, is called a generating family for the lagrangian subset in question.

PROPOSITION 2.8. *Let $L \subset T^*X$, $N \subset T^*Y$ be lagrangian submanifolds generated by the Morse families, say $G: X \times \mathbb{R}^k \rightarrow \mathbb{R}$ and $F: Y \times \mathbb{R}^l \rightarrow \mathbb{R}$ respectively. Let $R = (f, g) \in \mathcal{F}$, then the images of L and N with respect to R have the following generating families:*

(i) *for the pushforward (R, L) ; $P: Y \times \mathbb{R}^M \rightarrow \mathbb{R}$,*

$$P(y; \lambda, \mu, \nu) = \sum_{i=1}^m \lambda_i(y_i - f_i(\mu)) + g(\mu) + G(\mu, \nu),$$

where $\mu = (\mu_1, \dots, \mu_n)$, $\nu = (\nu_1, \dots, \nu_k)$, $M \leq m + n + k$,

(ii) *for the pullback (N, R) ; $H: X \times \mathbb{R}^l \rightarrow \mathbb{R}$,*

$$H(x; \lambda) = F(f(x), \lambda) - g(x),$$

in respective local Darboux coordinates on T^*X and T^*Y .

Proof. On the basis of (5) and Lecture 6 in [19] (see also [17]) a Morse family for the relation R is

$$A(x, y; \lambda) = \sum_{i=1}^m \lambda_i(y_i - f_i(x)) + g(x),$$

i.e. R , locally, can be expressed by the following equations:

$$\begin{aligned} -\xi_j &= -\sum_{i=1}^m \lambda_i \frac{\partial f}{\partial x_j}(x) + \frac{\partial g}{\partial x_j}(x) \quad (1 \leq j \leq n), \\ \eta_r &= \lambda_r \quad (1 \leq r \leq m). \end{aligned}$$

L is described by the equations

$$\xi_j = \frac{\partial G}{\partial x_j}(x, \nu), \quad 0 = \frac{\partial G}{\partial \nu_i}(x, \nu) \quad (1 \leq j \leq n, \quad 1 \leq i \leq k).$$

Hence using (4) for (R, L) we obtain (i). In the same way, reducing only an appropriate part of the parameters (as for the stable equivalence in [22]) we obtain (ii).

From (7), (8) and Proposition 2.8 we obtain immediately

COROLLARY 2.9. *Let $P(y; \lambda, \mu, \nu)$, $H(x; \lambda)$ be generating families for a pushforward (R, L) and pullback (N, R) respectively as in Proposition 2.8. Then the respective generating families, for the equivalent pushforward and pullback, are*

$$\begin{aligned} \tilde{P}(y, \lambda, \mu, \nu) &= \sum_{i=1}^m \lambda_i(y_i - (\psi \circ f \circ \phi^{-1})_i(\mu)) + g \circ \phi^{-1}(\mu) + \beta \circ f \circ \phi^{-1}(\mu) + G(\phi^{-1}(\mu), \nu), \\ \tilde{H}(x, \lambda) &= F(f \circ \phi^{-1}(x), \lambda) - g \circ \phi^{-1}(x) + \alpha \circ \phi^{-1}(x), \end{aligned}$$

where the equivalent symplectic relation \tilde{R} has the form (7).

Now we give the beginning of the classification of normal forms for the appropriate pushforwards and pullbacks. Let us denote by (Σ^{ijk}, A_r) for pushforward and (A_r, Σ^{ijk}) for pullback, the types of the respective equivalence classes, where Σ^{ijk} is a Boardman symbol of $f: (X, x_0) \rightarrow Y$ (cf. [11]) and A_r is the singularity type of L (or N) (cf. [4]) at a source or target point of the germ of the symplectic relation R .

PROPOSITION 2.10. *Let $\dim X, \dim Y < 3$. Then the normal forms for the generating families of generic pushforwards and pullbacks of the appropriate types are given in Table 1.*

Table 1

n, m	Type	$P: Y \times \mathbb{R}^N \rightarrow \mathbb{R}$	Type	$H: X \times \mathbb{R}^1 \rightarrow \mathbb{R}$
1, 1	(Σ^0, A_1)	0	(A_1, Σ^0)	0
	(Σ^0, A_2)	$\lambda y + \lambda^3$	(A_2, Σ^0)	$\lambda^3 + \lambda x$
	(Σ^{10}, A_1)	λy	(A_1, Σ^{10})	0
	(Σ^{10}, A_2)	λy	(A_2, Σ^{10})	$\lambda^3 \pm \lambda x^2$
1, 2	(Σ^0, A_1)	λy_2	(A_1, Σ^0)	0
	(Σ^0, A_2)	$\lambda_1^2 + \lambda_1 y_1 + \lambda_2 y_2$	(A_2, Σ^0)	$\lambda^3 + \lambda x$
			(A_3, Σ^0)	$\lambda^4 + \lambda^2 x + \lambda \phi(x), \phi(0) = 0$
2, 1	(Σ^1, A_1)	$0; \lambda^3 + y\lambda$	(A_1, Σ^1)	0
	(Σ^1, A_2)	$-y^3 + 0(y^4); \lambda^3 + y\phi(\lambda), \phi'(0) \neq 0$	(A_2, Σ^1)	$\lambda^3 + x_1 \lambda$
	(Σ^1, A_3)	$y^2 \phi(y)$	(A_1, Σ^{20})	0
	(Σ^{20}, A_1)	λy	(A_2, Σ^{20})	$\lambda^3 + \lambda(\pm x_1^2 \pm x_2^2)$
	(Σ^{20}, A_2)	λy		
	(Σ^{20}, A_3)	λy		
2, 2	(Σ^0, A_1)	0	(A_1, Σ^0)	0
	(Σ^0, A_2)	$\lambda^3 + \lambda y_1$	(A_2, Σ^0)	$\lambda^3 + \lambda x_1$
	(Σ^0, A_3)	$\lambda_4 + \lambda^2 y_1 + \lambda y_2$	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x_1 + \lambda x_2$
	(Σ^{10}, A_1)	$-\lambda_2 \lambda_1^2 \pm \lambda_1^2 + \lambda_2 y_2 + \lambda_1 y_1$	(A_1, Σ^{10})	0
	(Σ^{10}, A_2)	$\lambda_1 y_1 + \lambda_2 y_2 - \mu_1 \lambda_1 - \lambda_2 \mu_2^2 + \nu^3 + \nu(\mu_1 + \mu_2) + \alpha \mu_2^2 + \mu_2 \phi(\mu_1, \mu_2^2)$	(A_2, Σ^{10})	$\lambda^3 + \lambda x_1$
	(Σ^{10}, A_3)	$\lambda_1 y_1 + \lambda_2 y_2 - \lambda_1 \mu_1 - \lambda_2 \mu_2^2 + \nu^4 + \nu^2(\mu_1 + \mu_2) + \nu \phi_2(\mu_1, \mu_2) + \alpha \mu_2^2 + \mu_2 \phi_1(\mu_1, \mu_2^2)$	(A_1, Σ^{110})	0
			(A_2, Σ^{110})	$\lambda^3 \pm x_1 \lambda$
			(A_3, Σ^{110})	$\lambda^4 \pm \lambda^2 x_1 + \lambda(\phi_1(x_1) + \phi_2(x_1) \times (x_2^2 + x_1 x_2)), \phi_2(0) > 0$
	(Σ^{110}, A_1)	$\lambda_1(y_1 - \mu_1) + \lambda_2(y_2 - \mu_1 \mu_2 - \mu_2^3) \pm \mu_2^2 + \mu_2 \phi(\mu_1 \mu_2 + \mu_2^3)$		
	(Σ^{110}, A_2)	$\lambda_1(y_1 - \mu_1) + \lambda_2(y_2 - \mu_1 \mu_2 - \mu_2^3) + \nu^3 + \nu(\mu_2 + \phi_3(\mu_1, \mu_1 \mu_2 + \mu_2^3)) + \mu_2 \phi_1(\mu_1, \mu_2 \mu_1 + \mu_2^3) + \mu_2^2 \phi_2(\mu_1, \mu_1 \mu_2 + \mu_2^3) + \alpha \mu_2^3$		
	(Σ^{110}, A_3)	$\lambda_1(y_1 - \mu_1) + \lambda_2(y_2 - \mu_1 \mu_2 - \mu_2^3) + \nu^4 + \nu^2(\mu_2 + \phi_3(\mu_1 \mu_2 + \mu_2^3, \mu_1)) + \nu \phi_4(\mu_1, \mu_2) + \mu_2 \phi_1(\mu_1, \mu_1 \mu_2 + \mu_2^3) + \mu_2^2 \phi_2(\mu_1, \mu_1 \mu_2 + \mu_2^3) + \alpha \mu_2^3$		

Proof. By assumption $\dim X, \dim Y \leq 2$. The types (cf. [4, 11]) for f, L, N are those listed in the table. Hence for the respective germs, say for $L \subset T^*X$, we use the following Morse families (cf. [22]):

$$\left. \begin{aligned} A_1: G(x, \lambda) &= g_0(x), \\ A_2: G(x, \lambda) &= \lambda^3 + g_1(x) \lambda + g_0(x), \\ A_3: G(x, \lambda) &= \lambda^4 + g_1(x) \lambda^2 + g_2(x) \lambda + g_0(x). \end{aligned} \right\} \quad (10)$$

By [12], the classification of germs of pushforwards (R, L) , restricted to R belonging to \mathcal{F}' (see Corollary 2.3), reduces to the classification of germs of mapping diagrams

$$(\mathbb{R}^1 \times \mathbb{R}^s, 0) \xleftarrow{(\sigma, \theta)} (\mathbb{R}^n, 0) \xrightarrow{f} (\mathbb{R}^k, 0),$$

where $s = 0, 1, 2$, $g = g_1$ or $g = (g_1, g_2)$ for $s = 1$ or 2 respectively, with respect to the equivalence relation represented by the following commuting diagram:

$$\begin{array}{ccccc}
 (\mathbb{R}^1 \times \mathbb{R}^s, 0) & \xleftarrow{(g_0, g)} & (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}^k, 0) \\
 \text{id} \times (\text{id} + \alpha \circ f) \downarrow & & \downarrow h & & \downarrow \tilde{h} \\
 (\mathbb{R}^1 \times \mathbb{R}^s, 0) & \xleftarrow{(\tilde{g}_0, \tilde{g})} & (\mathbb{R}^n, 0) & \xrightarrow{\tilde{f}} & (\mathbb{R}^k, 0)
 \end{array}$$

The situation for pullbacks is analogous but simpler. Here any pullback of type (A_1, Σ^{ijk}) can be reduced to normal form with the trivial Morse family. Hence the problem of classification of pullbacks can be reduced to the problem of finding normal forms for mapping diagrams (cf. [4, 12])

$$(\mathbb{R}^n, 0) \xrightarrow{f} (\mathbb{R}^k, 0) \xrightarrow{g} (\mathbb{R}^1, 0) \quad (1 = 1, 2)$$

with a given type of singularity of the germ f and endowed with the following equivalence relation (cf. [4]):

$$\begin{array}{ccccc}
 (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}^k, 0) & & \\
 \downarrow h & & \downarrow \tilde{h} & \searrow g & \\
 (\mathbb{R}^n, 0) & \xrightarrow{f'} & (\mathbb{R}^k, 0) & \nearrow g' & (\mathbb{R}^1, 0)
 \end{array}$$

where $g_0 = 0$, $g = g_1$ or $g = (g_1, g_2)$.

Applying the Malgrange Preparation Theorem (cf. [6]), the generalized Morse Lemma and the method of liftable and lowerable vector fields (cf. [4]), we obtain the following list of normal forms for generic pairs (R, L) and (N, R) defining the push-forwards and pull-backs respectively (see also [13]).

Pushforwards:

$$(n, k) = (1, 1)$$

$$f(x) = x: \quad A_1: g_0(x) = 0; \quad A_2: (g_0, g)(x) = (0, x),$$

$$f(x) = x^2: \quad A_1: g_0(x) = \pm x^2 + ax^3; \quad A_2: (g_0, g)(x) = (ax^2 + x^3\phi_1(x^2), x + \phi_2(x^2))$$

$$(n, k) = (1, 2)$$

$$f(x) = (x, 0): \quad A_1: g_0(x) = 0; \quad A_2: (g_0, g)(x) = (0, x)$$

$$(n, k) = (2, 1)$$

$$f(x) = x_1: \quad A_1: g_0(x) = \pm x_2^2 \text{ or } g_0(x) = x_1x_2 + x_2^3;$$

$$A_2: (g_0, g)(x) = (x_1x_2 + x_2^2\phi(x), x_2) \text{ or } (g_0, g)(x) = (x_2\phi(x), x_1 \pm x_2^2),$$

$$A_3: (g_0, g)(x) = (x_2\phi_1(x), x_2, x_1 + x_2\phi_2(x)),$$

$$f(x) = x_1^2 \pm x_2^2: \quad A_1: g_0(x) = \pm x_1^2 + ax_2^2 + x_1\phi_1(f(x)) + x_2\phi_2(f(x)), \quad a \neq 1;$$

$$A_2: (g_0, g)(x) = (ax_1^2 + x_1\phi_1(f(x)) + x_2\phi_2(x), x_1 + \phi_3(f(x)));$$

$$A_3: (g_0, g)(x) = (ax_1^2 + x_1\phi_1(f(x)) + x_2\phi_2(x), x_1 + \phi_3(f(x)), \phi_4(x))$$

$$(n, k) = (2, 2)$$

$$f(x) = x: \quad A_1: g_0(x) = 0; \quad A_2: (g_0, g)(x) = (0, x_1); \quad A_3: (g_0, g)(x) = (0, x_1, x_2)$$

$$\begin{aligned}
 f(x) = (x_1, x_2^2): \quad & A_1: g_0(x) = x_1 x_2 \pm x_2^2; \\
 & A_2: (g_0, g)(x) = (ax_2^2 + x_2 \phi(f(x)), x_1 + x_2); \\
 & A_3: (g_0, g)(x) = (ax_2^2 + x_2 \phi_1(f(x)), x_1 + x_2, \phi_2(x)) \\
 f(x) = (x_1, x_1 x_2 + x_2^3): \quad & A_1: g_0(x) = \pm x_2^2 + x_2 \phi(x_1 x_2 + x_2^3) + ax_2^3; \\
 & A_2: (g_0, g)(x) = (x_2 \phi_1(f(x)) + x_2^2 \phi_2(f(x)) + ax_2^3, x_2 + \phi_3(f(x))); \\
 & A_3: (g_0, g)(x) = (x_2 \phi_1(f(x)) + x_2^2 \phi_2(f(x)) + ax_2^3, x_2 \\
 & \quad + \phi_3(f(x)), \phi_4(x)).
 \end{aligned}$$

Pullbacks:

$(n, k) = (1, 1)$

$$\begin{aligned}
 f(x) = x: \quad & A_2: g(x) = x \\
 f(x) = x^2: \quad & A_2: g(x) = \pm x^2,
 \end{aligned}$$

$(n, k) = (1, 2)$

$$f(x) = (x, 0): \quad A_2: g(x) = x; \quad A_3: g(x) = (x, \phi(x)),$$

$(n, k) = (2, 1)$

$$\begin{aligned}
 f(x) = x_1: \quad & A_2: g(x) = x_1, \\
 f(x) = x_1^2 \pm x_2^2: \quad & A_2: g(x) = x_1^2 \pm x_2^2.
 \end{aligned}$$

$(n, k) = (2, 2)$

$$\begin{aligned}
 f(x) = (x_1, x_2): \quad & A_2: g(x) = x_1; \quad A_3: g(x) = (x_1, x_2) \\
 f(x) = (x_1, x_2^2): \quad & A_2: g(x) = x_1; \quad A_3: g(x) = (x_1, \phi(x_1) + x_2^2) \\
 f(x) = (x_1, x_2^3 + x_1 x_2): \quad & A_2: g(x) = \pm x_1; \quad A_3: g(x) = (\pm x_1, \phi_1(x_1) + (x_2^3 + x_1 x_2) \phi_2(x_1)).
 \end{aligned}$$

Using these normal forms and Proposition 2·8, we can write down the generating families for the respective images of $L \subset T^*X$ and $N \subset T^*Y$. It is easy to check that the $R(L)$, for the types listed above, are germs of smooth lagrangian submanifolds in T^*Y . Hence on the basis of [22], theorem 4 and after some calculations, we can make a further reduction of the number of parameters for generating families of pushforwards. Thus the proof of Proposition 2·10 is completed.

Let us abbreviate the notation for pullbacks and pushforwards, writing $(A_r, \Sigma^{ijk})_{(n,m)}$ and $(\Sigma^{ijk}, A_r)_{(n,m)}$ respectively. So from Proposition 2·10 almost immediately we obtain

COROLLARY 2·11. *For the generic pullbacks and pushforwards listed in Proposition 2·10, we have the following relations:*

$$\begin{aligned}
 (\Sigma^0, A_i)_{(1,1)} &= A_i, \\
 (\Sigma^{10}, A_i)_{(1,1)} &= (\text{constrained lagrangian submanifold, so unstable in the standard} \\
 &\quad \text{sense (cf. [22, 12, 16])}, \\
 (\Sigma^0, A_i)_{(1,2)} &= (\text{constrained lagrangian submanifold}) \quad (i = 1, 2), \\
 (\Sigma^1, A_1)_{(2,1)} &= \{A_1, A_2\}, \\
 (\Sigma^1, A_2)_{(2,1)} &= \{A_1, A_2\}, \\
 (\Sigma^1, A_3)_{(2,1)} &= A_1, \\
 (\Sigma^{20}, A_i)_{(2,1)} &= (\text{constrained lagrangian submanifold}) \quad (i = 1, 2, 3), \\
 (\Sigma^0, A_i)_{(2,2)} &= A_i \quad (i = 1, 2, 3), \\
 (\Sigma^{10}, A_i)_{(2,2)} &= (\text{unstable}) \quad (i = 1, 2, 3), \\
 (\Sigma^{110}, A_i)_{(2,2)} &= (\text{unstable}) \quad (i = 1, 2, 3), \\
 (A_i, \Sigma^0)_{(1,1)} &= A_i \quad (i = 1, 2), \\
 (A_1, \Sigma^{10})_{(1,1)} &= A_1,
 \end{aligned}$$

- $(A_2, \Sigma^{10})_{(1,1)} = (\textit{singular}),$
- $(A_i, \Sigma^0)_{(1,2)} = A_i \quad (i = 1, 2),$
- $(A_3, \Sigma^0)_{(1,2)} = (\textit{unstable smooth if } \phi'(0) \neq 0),$
- $(A_i, \Sigma^1)_{(2,1)} = A_i \quad (i = 1, 2),$
- $(A_1, \Sigma^{20})_{(2,1)} = A_1,$
- $(A_2, \Sigma^{20})_{(2,1)} = (\textit{singular}),$
- $(A_i, \Sigma^0)_{(2,2)} = A_i \quad (i = 1, 2, 3),$
- $(A_i, \Sigma^{10})_{(2,2)} = A_i \quad (i = 1, 2),$
- $(A_3, \Sigma^{10})_{(2,2)} = (\textit{singular}),$
- $(A_i, \Sigma^{110})_{(2,2)} = A_i \quad (i = 1, 2),$
- $(A_3, \Sigma^{110})_{(2,2)} = (\textit{singular}).$

Example 2·12. An analogous phenomenon to that of the unstable pushforwards in Proposition 2·10 appears in many mechanical and thermodynamical systems (see e.g. [18, 12]). Let Y be the Euclidean plane. The equations

$$\begin{aligned} \eta_1 &= r \cos \theta, & \eta_2 &= r \sin \theta, \\ y_1 &= -k(r - a) \cos \theta, & y_2 &= -k(r - a) \sin \theta \end{aligned}$$

describe a lagrangian submanifold N of T^*Y with coordinates (θ, r) , $0 \leq \theta < 2\pi$, $-\infty < r < \infty$. (N can be obtained as a canonical pushforward, see [18]). N represents the position-force relation for a point subject to a simple restoring force whose centre of attraction is allowed to move freely on the circle $\eta_1^2 + \eta_2^2 = a^2$. We see that for $r = a$, $T^*_0 Y \cap N$ is the circle $\eta_1 = a \cos \theta$, $\eta_2 = a \sin \theta$ and for $(y_1, y_2) \neq 0$, N is transversal to the fibres $T^*_{(v_1, v_2)} Y$. Hence N is an unstable lagrangian submanifold like the ones listed in Proposition 2·10. The corresponding physically realizable reduction relation R and lagrangian submanifold L are constructed in [18].

Remark 2·13. Let us notice that every symplectic relation R , in general, is locally generated by the Morse family $(x, y; \lambda) \rightarrow G(x, y; \lambda)$ (λ -parameter). The classification of images (preimages) for more general symplectic relations R than those considered in this paper can be obtained using the following symplectic equivalence: let R_X, R_Y be two symplectic relations in $(T^*X \times T^*X, \pi_2^* \vartheta_X - \pi_1^* \vartheta_X)$ and in

$$(T^*Y \times T^*Y, \pi_2^* \vartheta_Y - \pi_1^* \vartheta_Y)$$

respectively, representing the appropriate elements of the group of symplectomorphisms of T^*X and T^*Y respectively (cf. [17]). We say that the symplectic relations $R, R' \subset (T^*X \times T^*Y, \pi_2^* \omega_Y - \pi_1^* \omega_X)$ are equivalent if there exist relations R_X, R_Y such that

$$R' = R_X \circ R \circ R_Y \quad (\text{cf. [10]}).$$

If $G_X: X \times X \rightarrow \mathbb{R}$, $G_Y: Y \times Y \rightarrow \mathbb{R}$ are Morse families for R_X and R_Y respectively then R' has the following Morse family:

$$G'(x, y; \mu, \nu, \lambda) = G_X(x, \mu) + G(\mu, \nu, \lambda) + G_Y(\nu, y). \tag{11}$$

It is easily seen that if $X = \mathbb{R}^1$, $Y = \mathbb{R}^1$ and R is transversal to the fibres of $T^*(X \times Y)$ then we can reduce G to the normal form

$$G(x, y; \mu, \nu) = xv + y\mu + \nu\mu f(\nu, \mu).$$

It seems to be interesting to consider the classification problem for images of lagrangian submanifolds with a more general class (than the one considered here) of symplectic relations. The more detailed analysis of this problem we leave to a forthcoming paper.

3. Special symplectic triplets

Now we pass to the images of lagrangian submanifolds provided by a symplectic reduction relation defined by a hypersurface H in a symplectic manifold (P, ω) . The first nontrivial step in the study of mutual intersection of a lagrangian submanifold $X \subseteq P$ and a hypersurface $H \subseteq P$ was carried out in [15] and [2]. It turned out that the nontransversal positions of X and H , i.e. a mutual tangency of the first order along the hypersurface $H \cap X$ of X , the so-called symplectic triplets $(H, X, H \cap X)$ provide the singular images $\rho(X)$ (see [2]), which are encountered in variational calculus of physical systems [2, 3] and in boundary value problems for differential operators [14, 15].

It is easy to establish that at any point, say $p \in H \cap X$, for a symplectic triplet $(H, X, H \cap X)$ one can choose a local special symplectic structure on P (see [17] and [20], theorem 4.1), the so-called Weinstein symplectic structure $T^*X \cong P$, such that

$$\left. \begin{aligned}
 H &= \left\{ (x, \xi) \in T^*X; h(x, \xi) = \sum_{i=1}^n a_i(x, \xi) \xi_i + \chi(x) = 0 \right\}, \\
 l &= H \cap X = \{x \in X; \chi(x) = 0\} \text{ is a submanifold of codimension } 1,
 \end{aligned} \right\} \tag{12}$$

where $a_i, \chi, 1 \leq i \leq n$ are smooth functions and in addition, graph $\chi \subseteq X \times \mathbb{R}$ has a first order tangency to X along l .

Definition 3.1. Let $(H, X, l = H \cap X)$ be a symplectic triplet in (P, ω) . We say that it is a special symplectic triplet if there exists a Weinstein symplectic structure, say T^*X , such that h generates a hamiltonian flow preserving this structure.

Locally, a special symplectic triplet is described by (12) with the additional assumption that $h(x, \xi) = \sum_{i=1}^n a_i(x) \xi_i + \chi(x)$. We see that the characteristics (provided by h) on X are defined by the vector field $V = \sum_i a_i(x) \partial/\partial x_i$. Using the symplectomorphisms preserving an affine form of h and zero section X (i.e. a class of special symplectic triplets, see [9] for contact equivalence) as well as the standard equivalence for Hamiltonians (i.e. $h \sim h'$ iff $h = ah'$ for some smooth function a such that $a(0) \neq 0$) we obtain the following result.

PROPOSITION 3.2. *Let (H, X, l) be a special symplectic triplet. Then, generically in a neighbourhood of any point of l , (H, X, l) can be reduced to one of the following normal forms:*

$$\left. \begin{aligned}
 H_k &= \{(x, \xi) \in T^*X; h = \xi_1 \pm a(x_1^{k+1} + x_2 x_1^{k-1} + \dots + x_{k+1})^2 = 0\}, \\
 l_k &= \{x \in X; x_1^{k+1} + x_2 x_1^{k-1} + \dots + x_{k+1} = 0\},
 \end{aligned} \right\} \tag{13}$$

where $k \leq \dim X - 1, a: (T^*X, 0) \rightarrow \mathbb{R}$ and $a(0) \neq 0$.

Proof. We show first that a germ $\chi: (X, 0) \rightarrow \mathbb{R}$, defining a symplectic triplet as in (12), can be brought to the form g^2 for some smooth function-germ $g: (X, 0) \rightarrow \mathbb{R}$ defining the hypersurface l . Let us take coordinates on X , and simultaneously symplectic coordinates in T^*X by cotangent bundle lifting (see [19]), such that

$$l = \{x \in X; x_1 = 0\}.$$

So $\chi(x) = x_1 g_1(x)$ and, since l is a hypersurface of nonisolated critical points for f , i.e.

$$\nabla\chi|_l = \left(g_1(x) + x_1 \frac{\partial g_1}{\partial x_1}(x), x_1 \frac{\partial g_1}{\partial x_2}(x), \dots, x_1 \frac{\partial g_1}{\partial x_n}(x) \right) = 0,$$

thus $g_1(x) = x_1 g_2(x)$. By the assumption of first order tangency of graph χ to X we have $g_2(0) \neq 0$. Hence we can write $\chi = \pm g^2$, where $g(x) = x_1 \sqrt{(\pm g_2(x))}$.

The vector field $V = \sum_i a_i(x) \partial/\partial x_i$ can be straightened in a neighbourhood of the considered point, so that $\phi_* V = \partial/\partial x_1$ for some diffeomorphism $\phi: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$. Taking the canonical lifting of ϕ to T^*X , we obtain the following normal form for h (for an equivalent special symplectic triplet):

$$h(x, \xi) = \xi_1 \pm g^2(x). \tag{14}$$

Now we have the natural group of equivalences for integral curves of $\partial/\partial x_1$, i.e. diffeomorphism germs preserving the fibre structure $(x_1, x_2, \dots, x_n) \xrightarrow{\pi} (x_2, \dots, x_n)$. Using these equivalences, we reduce the problem of description of mutual generic positions of characteristics of h and the submanifold $l (\{g(x) = 0\})$ to the classification problem for Whitney's projections (see [2, 11, 6])

$$\pi|_l: l \rightarrow \mathbb{R}^{n-1}.$$

Hence (14) can be brought into the normal form

$$h(x, \xi) = \xi_1 b(x) \pm (x_1^{k+1} + x_2 x_1^{k-1} + \dots + x_{k+1})^2,$$

where $b(0) \neq 0$. Taking an equivalent hamiltonian for H we obtain (13). Thus the proof of Proposition 3.2 is completed.

For any special symplectic triplet (H, X, l) there exists a canonical special symplectic structure on the space (B, β) of characteristics (bicharacteristics) on H (cf. (3)), say T^*Y , such that for the reduction relation

$$R = \{(p_1, p_2) \in T^*X \times T^*Y; p_2 = \rho(p_1), p_1 \in H\}, \tag{15}$$

where $\rho: H \rightarrow R^*Y$ is the canonical projection onto bicharacteristics, we have the following commuting diagram:

$$\begin{array}{ccc} T^*X \supseteq H & \xrightarrow{\rho} & T^*Y \\ \downarrow \pi_{X|H} & & \downarrow \pi_Y \\ X & \xrightarrow{\pi} & Y \end{array} ,$$

where π is a submersion (along characteristics).

COROLLARY 3.3. *Let (H, X, l) be a special symplectic triplet. Then a stationary lagrangian submanifold $R(X)$ (cf. [2]) is a canonical pushforward, i.e.*

$$R(X) = T^*\pi(L), \tag{17}$$

and for the respective types of triplets described in Proposition 3.2, we have:

$$\pi: X \rightarrow Y, \quad \pi: (x_1, x_2, \dots, x_n) \rightarrow (x_2, \dots, x_n).$$

Moreover L is generated (for the type of the triplet) by the following generating function:

$$F(x_1, \dots, x_n) = \pm \int_0^{x_1} a(s, x_2, \dots, x_n) (s^{k+1} + x_2 s^{k+1} + \dots + x_{k+1})^2 ds. \tag{18}$$

Proof. We see that the space of characteristics of H , described by the dynamical system

$$\begin{aligned} \dot{x}_1 &= 1, \dot{x}_2 = 0, \dots, \dot{x}_n = 0, \\ \dot{\xi}_1 &= -\frac{\partial h}{\partial x_1}, \dots, \dot{\xi}_n = -\frac{\partial h}{\partial x_n}, \end{aligned}$$

can be easily obtained (integrated). As the canonical variables

$$(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{\xi}_1, \dots, \bar{\xi}_{n-1}),$$

parametrizing the symplectic space of characteristics, we can take the initial values for $x_2, \dots, x_n, \xi_2, \dots, \xi_n$, where the initial value for x_1 is equal to zero. The corresponding symplectic form β , such that $\rho^*\beta = \sum_{i=1}^n d\xi_i \wedge dx_i|_H$, for this space may be chosen in the Darboux form. Now it is easy to check that (17) holds for each type of symplectic triplet of Proposition 3.2, with generating function (18) for L .

4. Generating families for the open swallowtails

Let us consider now the most representative example of the Arnold theory [3] for singular lagrangian submanifolds.

We are given the space of binary forms of degree $d = 2k + 3$, the dimension of which is equal to $2k + 4$ (cf. [2]). This space can be endowed with the unique $SL_2(\mathbb{R})$ -invariant symplectic form. In the appropriate Darboux coordinates $(q_0, \dots, q_{k+2}, p_0, \dots, p_{k+2})$ a binary form, say $\phi(x, y)$, can be written as follows:

$$\begin{aligned} \phi(x, y) &= q_0 \frac{x^{2k+3}}{(2k+3)!} + q_1 \frac{x^{2k+2}y}{(2k+2)!} + \dots + q_{k+1} \frac{x^{k+2}y^{k+1}}{(k+1)!} \\ &\quad + (-1)^1 p_{k+1} \frac{x^{k+1}y^{k+2}}{(k+1)!} + \dots + (-1)^{k+2} p_0 y^{2k+3}. \end{aligned}$$

We see that the space of characteristics of the coisotropic hypersurface $\{q_0 = 1\}$ is identified with the space of polynomials of degree $2k + 2$ (derivatives of the respective polynomials $\phi(x, 1)$), i.e.

$$T^*Q = \left\{ \frac{x^{2k+2}}{(2k+2)!} + q_1 \frac{x^{2k+1}}{(2k+1)!} + \dots + q_{k+1} \frac{x^{k+1}}{1(k+1)!} - p_{k+1} \frac{x^k}{k!} + \dots + (-1)^{k+1} p_1 \right\},$$

with the reduced symplectic form $\omega = \sum_{i=1}^{k+1} dp_i \wedge dq_i$, where (q_1, \dots, q_{k+1}) are coordinates on Q .

PROPOSITION 4.1 (cf. [2]). *The triplet (H, Q, l) , where*

$$H = \{h(q, p) = p_1 + q_1 p_2 + \dots + q_k p_{k+1} + q_{k+1}^2/2 = 0\},$$

*is a special symplectic triplet in (T^*Q, ω_Q) such that $\rho(l) \subseteq T^*Y$ is an open k -dimensional swallowtail.*

Proof. The space of characteristics of the hamiltonian system

$$\left. \begin{aligned} \dot{q}_1 &= 1, \dot{q}_2 = q_1, \dots, \dot{q}_{k+1} = q_k, \\ \dot{p}_1 &= -p_2, \dots, \dot{p}_k = -p_{k+1}, \dot{p}_{k+1} = -q_{k+1} \end{aligned} \right\} \quad (19)$$

can be identified with the space of polynomials of degree $2k + 2$ in $H \subset T^*Q$, such that $q_1 = 0$. We see that the zero section $Q \subseteq T^*Q$ intersected with H (i.e. l) forms the space of polynomials divisible by x^{k+2} , so the canonical projection $\rho(l)$ onto the space of characteristics T^*Y (endowed with the Darboux coordinates $(q_2, \dots, q_{k+1}, p_2, \dots, p_{k+1})$) can be identified with the polynomials of degree $2k + 1$ of the form

$$\phi(x, 1) = \frac{x^{2k+1}}{(2k+1)!} + q_2 \frac{x^{2k-1}}{(2k-1)!} + \dots + q_{k+1} \frac{x^k}{k!} - p_{k+1} \frac{x^{k-1}}{(k-1)!} + \dots + (-1)^k p_2$$

such that $\phi(x, 1) = (x - \xi)^{k+1} (x^k + \dots)$ for some $\xi \in \mathbb{R}$. But this is nothing else than the definition of open swallowtail introduced in [2], for example.

We can also use the initial values of $(q_1, \dots, q_{k+1}, p_1, \dots, p_{k+1})$ on characteristics to parametrize the space T^*Y . Remembering that h is a Hamiltonian of translations along the variable x , for the polynomial parametrization of characteristics we can make the following identification:

$$\begin{aligned} &\frac{(x-t)^{2k+2}}{(2k+2)!} + q_1 \frac{(x-t)^{2k+1}}{(2k+1)!} + \dots + q_{k+1} \frac{(x-t)^{k+1}}{(k+1)!} - p_{k+1} \frac{(x-t)^k}{k!} + \dots + (-1)^{k+1} p_1 \\ &= \frac{x^{2k+2}}{(2k+2)!} + \bar{q}_2 \frac{x^{2k}}{(2k)!} + \dots + \bar{q}_{k+1} \frac{x^{k+1}}{(k+1)!} - \bar{p}_{k+1} \frac{x^k}{k!} + \dots + (-1)^{k+1} \bar{p}_1, \end{aligned} \quad (20)$$

where $h(\bar{q}_1, \dots, \bar{q}_{k+1}, \bar{p}_1, \dots, \bar{p}_{k+1}) = 0$ and $\bar{q}_1 = 0$ implies $q_1 = t$. Hence we can take $(\bar{q}_2, \dots, \bar{q}_{k+1}, \bar{p}_2, \dots, \bar{p}_{k+1})$ as Darboux coordinates on T^*Y , where Y is parametrized by $(\bar{q}_2, \dots, \bar{q}_{k+1})$ and $\omega_Q|_H = \rho^* \omega_Y$. Likewise in (16) $\pi: (q_1, \dots, q_{k+1}) \rightarrow (\bar{q}_2, \dots, \bar{q}_{k+1})$ has the following form:

$$\bar{q}_j = \sum_{i=0}^{j-2} (-1)^i \frac{1}{i!} q_1^i q_{j-1} - (-1)^j \frac{(j-1)}{j!} q_1^j \quad (j = 2, \dots, k+1). \quad (21)$$

Let R be the canonical symplectic reduction relation connected with H , i.e. R is the graph of ρ in $(T^*Q \times T^*Y, \pi_2^* \omega_Y - \pi_1^* \omega_Q)$.

PROPOSITION 4.2. *An open, k -dimensional swallowtail can be represented as a canonical pushforward of a regular lagrangian submanifold, i.e.*

$$R(Q) = T^* \pi(L_k), \quad \dim Q = k + 1,$$

where L_k is a lagrangian submanifold of (T^*Q, ω_Q) with the following generating function:

$$\begin{aligned} F_k(q_1, \dots, q_{k+1}) &= \sum_{i=-1}^{k-2} \sum_{s=2}^{k-i-1} D_{k-i,s}^{(k)} q_1^{k+i-s+3} q_s q_{k-i} \\ &+ \frac{1}{2} \sum_{i=0}^{k-2} D_{k-i,k-i}^{(k)} q_1^{2i+3} q_{k-i}^2 + \sum_{i=0}^{k-2} E_{k-i}^{(k)} q_1^{k+i+3} q_{k-i} + \frac{1}{2} D_{k+1,k+1}^{(k)} q_1 q_{k+1}^2 \\ &+ E_{k+1}^{(k)} q_1^{k+2} q_{k+1} - \frac{E_2^{(k)}}{2k+3} q_1^{2k+3}, \end{aligned} \quad (22)$$

where

$$\left. \begin{aligned} D_{r,s}^{(k)} &= (-1)^{k-r} \sum_{j=s}^{k+1} \frac{(-1)^{j-s}}{(j-s)!(2k+3-j-r)!}, \\ E_r^{(k)} &= (-1)^{k-r} \left(\frac{1}{(2k+3-r)!} - \sum_{j=2}^{k+1} \frac{(-1)^j(j-1)}{j!(2k+3-j-r)!} \right) \quad (1 \leq r, s \leq k+1). \end{aligned} \right\} \quad (23)$$

Proof. On the basis of (21) $T^*\pi$ can be written as follows:

$$\left. \begin{aligned} p_1 &= \sum_{j=1}^k \bar{p}_{j+1} \sum_{l=1}^j (-1)^l \frac{1}{(l-1)!} q_1^{l-1} q_{j-1+1}, \\ p_r &= \bar{p}_r + \sum_{j=r}^k \bar{p}_{j+1} \frac{(-1)^{j-r+1}}{(j-r+1)!} q_1^{j-r+1} \quad (1 < r \leq k+1). \end{aligned} \right\} \quad (T^*\pi)$$

On the other hand, making further calculations, for R we obtain

$$\left. \begin{aligned} p_1 &= \sum_{j=1}^k \sum_{i=0}^{k-j} \frac{(-1)^{k-j-i+1}}{(k-i-j)!} q_j \bar{p}_{k-i+1} q_1^{k-j-i} - \sum_{j=1}^k \sum_{s=2}^{k+1} q_1^{2k+2-s-j} q_j \\ &\quad \times q_s D_{j+1,s}^{(k)} - \sum_{j=1}^k E_{j+1}^{(k)} q_j q_1^{2k+2-j} - \frac{1}{2} q_{k+1}^2, \\ p_r &= \sum_{j=0}^{k+1-r} (-1)^{k+1-r-j} \frac{1}{(k+1-r-j)!} \bar{p}_{k+1-j} q_1^{k+1-r-j} \\ &\quad + \sum_{s=2}^{k+1} D_{r,s}^{(k)} q_s q_1^{2k+3-s-r} + E_r^{(k)} q_1^{2k+3-r}, \end{aligned} \right\} \quad (R)$$

where $2 \leq r \leq k+1$ and $D_{r,s}^{(k)}, E_r^{(k)}$ are defined in (23).

Comparing equations for R and for $T^*\pi$, and remembering that Q is described by the equations $p_1 = p_2 \dots = p_{k+1} = 0$, after simple but long calculations we obtain (22).

Using Proposition 2.8 and the function (22) we obtain a generating family (not necessarily a Morse family) for the singular lagrangian submanifold in T^*Y called an open swallowtail (see [2]).

COROLLARY 4.3. *A generating family for an open, k -dimensional, swallowtail can be written in the form*

$$\begin{aligned} P_k(\bar{q}_2, \dots, \bar{q}_{k+1}; \mu_1, \dots, \mu_{k+1}, \lambda_1, \dots, \lambda_k) &= F_k(\mu_1, \dots, \mu_{k+1}) \\ &\quad + \sum_{i=1}^k \lambda_i \left(\bar{q}_{i+1} - \sum_{l=0}^{i-1} (-1)^l \frac{1}{l!} \mu_1^l \mu_{i-l+1} + (-1)^i \frac{i}{(i+1)!} \mu_1^{i+1} \right), \end{aligned}$$

where F_k is defined in (22) and $\mu_1, \dots, \mu_{k+1}, \lambda_1, \dots, \lambda_k$ are parameters of the family.

Example 4.4. Let $k = 1$ or 2 . Then the respective generating functions for smooth (resolvent) lagrangian submanifolds L_1, L_2 are

$$\begin{aligned} L_1: F_1(q_1, q_2) &= -\frac{1}{15} q_1^5 + \frac{1}{3} q_1^3 q_2 - \frac{1}{2} q_1 q_2^2, \\ L_2: F_2(q_1, q_2, q_3) &= -\frac{1}{840} q_1^7 + \frac{1}{120} q_1^5 q_2 - \frac{1}{8} q_1^4 q_3 - \frac{1}{8} q_1^3 q_2^2 + \frac{1}{2} q_1^2 q_2 q_3 - \frac{1}{2} q_1 q_3^2. \end{aligned}$$

Now by Corollary 4.3 and the standard method for reduction of parameters we obtain the generating one-parameter families for the cusp singularity (see [6]) and the two-dimensional open swallowtail singularity (see [3]) of the lagrangian submanifold:

cuspid:

$$P_1(\bar{q}_2, \lambda) = -\frac{1}{40}\lambda^5 - \frac{1}{8}\lambda^3\bar{q}_2 - \frac{1}{2}\lambda\bar{q}_2^2,$$

open swallowtail:

$$P_2(\bar{q}_2, \bar{q}_3; \lambda) = -\frac{1}{576}\lambda^7 - \frac{1}{30}\lambda^5\bar{q}_2 - \frac{1}{24}\lambda^4\bar{q}_3 - \frac{1}{8}\lambda^3\bar{q}_2^2 - \frac{1}{2}\lambda^2\bar{q}_2\bar{q}_3 - \frac{1}{2}\lambda\bar{q}_3^2.$$

Remark 4.5. Taking new coordinates on T^*Q , defined by (21) and $\bar{q}_1 = q_1$, we have $\pi: Q \rightarrow Y$, $\pi(q_1, \dots, q_{k+1}) = (q_2, \dots, q_{k+1})$. So after straightforward calculations (cf. [13]) we derive the following generating families for the respective open swallowtails

$$R(Q): P_k(\bar{q}_2, \dots, \bar{q}_{k+1}; \lambda) = \frac{1}{2} \int_0^\lambda \left(\frac{k+2}{(k+1)!} x^{k+1} + \sum_{i=2}^{k+1} \frac{1}{(k-i+1)!} \bar{q}_i x^{k-i+1} \right)^2 dx.$$

Comparing this formula with Corollary 3.3 (formula (18)) we see that the special symplectic triplets with $k = n - 1$ are diffeomorphic to the ones providing the open swallowtails.

Remark 4.6. One of the most interesting appearances of the open swallowtail ($k = 2$) is the one proposed by V. I. Arnold (and coworkers) [2], [3] in variational calculus, which is frequently called 'shortest bypassing of the obstacle'. It has some connection to geometrical optics (see [3]). Let us consider a piece D of a hypersurface (obstacle) in \mathbb{R}^3 , and define the geodesic flow on D by the time function $\tau: D \rightarrow \mathbb{R}$. Hence $(\nabla\tau)^2 = 1$. An appropriate symplectic triplet connected with this situation is defined by $\phi: T^*\mathbb{R}^3 \rightarrow \mathbb{R}$ (defining H), $\phi = p^2 - 1$ (all directions in the fibres) and the lagrangian submanifold L as all the extensions to $T_q\mathbb{R}^3$ of the 1-forms $p = d\tau|_q$ defined on the tangent space to D . It turns out the $(H, L, H \cap L)$ is a symplectic triplet diffeomorphic to the one considered in §§ 3, 4 of the present paper.

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