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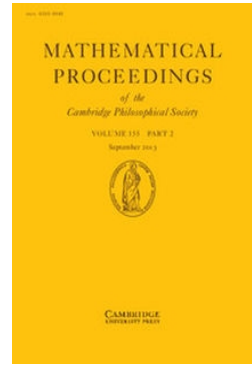
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Generic singularities of symplectic and quasi-symplectic immersions

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Abstract

For any $k < 2n$ we construct a complete system of invariants in the problem of classifying singularities of immersed k -dimensional submanifolds of a symplectic $2n$ -manifold at a generic double point.



1. Introduction

1.1. Symplectic and quasi-symplectic immersions

A smooth $2r$ -dimensional submanifold S of a $2n$ -dimensional symplectic manifold (M^{2n}, ω) is called symplectic if the restriction $\omega|_{TS}$ has the maximal possible rank $2r$. If $\dim S = 2r + 1$ then the maximal possible rank of this restriction is also $2r$ and in this case S is called quasi-symplectic.

The Darboux–Givental theorem (see [AG]) states that in the problem of local classification of pairs consisting of a symplectic form on M^{2n} and a smooth submanifold of M^{2n} , the pullback of the symplectic form to the submanifold is a complete invariant. This theorem implies that any two germs of smooth symplectic or quasi-symplectic submanifolds of the same dimension of a symplectic manifold can be brought one to the other by a local diffeomorphism preserving the symplectic form.

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This paper is devoted to the classification of first occurring singularities of immersed symplectic or quasi-symplectic submanifolds of a symplectic manifold, i.e. classification of the tuples

$$(\mathbb{R}^{2n}, \omega, S_1^k \cup S_2^k)_0 \quad (1.1)$$

where ω is a symplectic form on \mathbb{R}^{2n} and S_1^k, S_2^k are k -dimensional symplectic or quasi-symplectic submanifolds of $(\mathbb{R}^{2n}, \omega)$ whose intersection contains $0 \in \mathbb{R}^{2n}$. The notation $()_0$ means that objects in the parenthesis are germs at $0 \in \mathbb{R}^{2n}$. A tuple (1.1) is equivalent to a tuple of the same form with $\tilde{\omega}, \tilde{S}_1^k, \tilde{S}_2^k$ if there exists a local diffeomorphism of \mathbb{R}^{2n} which brings $\tilde{\omega}$ to ω and $\tilde{S}_1^k \cup \tilde{S}_2^k$ to $S_1^k \cup S_2^k$. We work in a fixed category which is smooth or real-analytic. We restrict ourselves to generic germs of (1.1) which means that our results concern a certain open and dense set in the space of such germs.

1.2. The cases of dimension 1 and codimension 1

Note that any hypersurface and any 1-dimensional submanifold of a symplectic manifold are quasi-symplectic. Within generic germs the cases $k = 1$ and $k = 2n - 1$ are much simpler than the case $2 \leq k \leq n - 2$ and in these two cases the classification of generic tuples (1.1) is contained in the work [Ar1] by V. Arnol'd ($k = 1$) and in the work [Me] by R. B. Melrose ($k = 2n - 1$). Theorems 1 and 2 below are the simplest particular results of these works.

THEOREM 1. *Let $k = 1$. All germs (1.1) with non-tangent strata S_1^1, S_2^1 where the restriction of ω to $T_0S_1^1 + T_0S_2^1$ has maximal rank 2 are equivalent.*

This theorem is the simplest case of the symplectic classification of singular curves which are diffeomorphic to $A_\mu = \{x \in \mathbb{R}^{2n} : x_1^{\mu+1} - x_2^2 = x_{\geq 3} = 0\}$, as obtained by V. Arnol'd in [Ar1], namely the case $\mu = 1$. All germs (1.1) with $k = 1$, non-tangent strata S_1^1, S_2^1 , and such that ω annihilates the space $T_0S_1^1 + T_0S_2^1$, are also equivalent. In the case of the tangent strata, with a finite order of tangency, the classification is more involved, but remains discrete. These results from the work [Ar1] are explained in [DJZ2] using the method of algebraic restrictions, developed in [Zh1] for classification of singular varieties in a contact space and in [DJZ2] for classification of singular varieties in a symplectic space. The work [DJZ2] contains the symplectic classification of singular curves with any fixed A or D or E singularity.

THEOREM 2. *Let $k = 2n - 1$. All germs (1.1) with transversal hypersurfaces S_1^{2n-1}, S_2^{2n-1} such that the restriction of ω to $T_0S_1^{2n-1} \cap T_0S_2^{2n-1}$ has maximal rank $2n - 2$ are equivalent.*

This theorem was proved by R. B. Melrose in [Me]. See [Me, proof of proposition 2.1] where Theorem 2 is formulated in a different, but equivalent, form. The main part of the work [Me] is devoted to a much more difficult case where S_1^{2n-1}, S_2^{2n-1} are transversal, but the restriction of ω to the manifold $S_1^{2n-1} \cap S_2^{2n-1}$ has the first occurring singularity within closed 2-forms on an even-dimensional manifold, the so called Σ_{20} singularity studied by J. Martinet (see [Ma] or Appendix G of [Zh2]). In particular, the restriction of ω to $T_0S_1^{2n-1} \cap T_0S_2^{2n-1}$ has rank $2n - 4$. In [Me] the hypersurfaces S_1^{2n-1}, S_2^{2n-1} are called glancing and this property is described in terms of the Poisson bracket of the functions f_1, f_2 defining hypersurfaces S_1^{2n-1}, S_2^{2n-1} . Melrose proved that in the C^∞ category all tuples (1.1) with glancing hypersurfaces are equivalent. In the analytic category not all tuples (1.1) with glancing hypersurfaces are equivalent as was showed in [Os].

1.3. Moduli in the case $2 \leq k \leq 2n - 2$

In this case the classification of generic tuples (1.1) is a much harder problem. The only related result we know of concerns the case $k = 2$ that is given in our work [DJZ2, section 7.4]. In the present work we classify generic tuples (1.1) for any k and n . Our theorems on a complete system of invariants in section 4 imply the following statement. The symbol $\lfloor x \rfloor$ denotes the largest integer not greater than x .

THEOREM 3. *In the problem of classification of generic tuples (1.1) with $2 \leq k \leq 2n - 2$, there are $\lfloor k/2 \rfloor$ moduli if $2 \leq k \leq n$, there is one modulus if $k = 2n - 2$ or $k = 2n - 3$, and in the remaining case $n < k \leq 2n - 4$ (which is possible for $2n \geq 10$ only) there are functional moduli which belong to the space of tuples of $(s - 1)$ functions of d variables, where $s = \lfloor (2n - k)/2 \rfloor$ and $d = 2(k - n)$.*

The precise meaning of the last statement, about the functional moduli, is explained in Corollary 28, Section 4.

1.4. Tools

Our starting point is the following proposition.

PROPOSITION 4 ([DJZ2]). *Let $N = S_1 \cup S_2 \cup \dots \cup S_r$ where S_i are germs at 0 of smooth submanifolds of \mathbb{R}^{2n} such that*

$$\dim(T_0S_1 + \dots + T_0S_r) = \dim S_1 + \dots + \dim S_r. \tag{1.2}$$

Let ω and $\tilde{\omega}$ be symplectic forms on \mathbb{R}^{2n} with the same restriction to the tangent bundles of S_i^k and the same restriction to the space $T_0S_1 + \dots + T_0S_r$. There exists a local diffeomorphism of \mathbb{R}^{2n} which sends $\tilde{\omega}$ to ω preserving N pointwise.

Strictly speaking, this proposition is not formulated in [DJZ2], but it is a logical corollary of two results from this work. The first one, [DJZ2, theorem A in section 2.7], states that given a germ of any quasi-homogeneous variety $N \subset \mathbb{R}^{2n}$ (in particular N in Proposition 4) any two symplectic forms ω and $\tilde{\omega}$ on \mathbb{R}^{2n} with the same algebraic restriction to N can be brought one to the other by a local diffeomorphism of \mathbb{R}^n which preserves N pointwise. We refer to [DJZ2] for the definition of algebraic restrictions and its role in the local classification of singular varieties in a symplectic manifold, and we refer to [DJZ1] for the definition of a quasi-homogeneous variety and its role in the local analysis of singular varieties. Proposition 4 is a logical corollary of the formulated theorem and another result from [DJZ2, theorem 7.1], stating that under the assumptions of Proposition 4, the symplectic forms ω and $\tilde{\omega}$ have the same algebraic restriction to N .

For the case $k > n$ we will also use the following result by Alan S. McRae.

PROPOSITION 5 ([MR]). *Let S_1 and S_2 be germs at 0 of submanifolds of \mathbb{R}^{2n} such that $T_0S_1 + T_0S_2 = T_0\mathbb{R}^{2n}$. Let ω and $\tilde{\omega}$ be symplectic forms on \mathbb{R}^{2n} coinciding at any point $z \in S_1 \cap S_2$ and having the same restrictions to $T S_1$ and $T S_2$. There exists a local diffeomorphism of \mathbb{R}^{2n} preserving pointwise S_1 and S_2 and bringing $\tilde{\omega}$ to ω .*

In fact, McRae proved a bit stronger result: Proposition 5 holds not only locally, but also in a neighbourhood of the union $S_1 \cup S_2$, provided S_1, S_2 are closed and ω can be deformed into $\tilde{\omega}$ inside the class of symplectic structures satisfying the hypotheses of Proposition 5. The latter certainly holds if S_1 and S_2 are germs at 0, given the deformation $\omega_t = \omega + t(\tilde{\omega} - \omega)$, $t \in [0, 1]$.

We also need the following proposition which is a slight generalization of the Darboux–Givental’ theorem.

PROPOSITION 6. *Let μ and $\tilde{\mu}$ be the germs at 0 of closed 2-forms on \mathbb{R}^k of maximal rank $2\lfloor k/2 \rfloor$ such that $\mu(z) = \tilde{\mu}(z)$ for any point z of a submanifold $Q \subset \mathbb{R}^k$. If k is odd we assume that the lines $\ker \mu(0)$ and $\ker \tilde{\mu}(0)$ do not belong to T_0Q . Then $\tilde{\mu}$ can be brought to μ by a local diffeomorphism of \mathbb{R}^k which preserves Q pointwise and has identity linear approximation at any point of Q .*

Proof. In the even-dimensional case Proposition 6 is exactly the Darboux–Givental’ theorem up to the assumption that μ and $\tilde{\mu}$ agree at points of Q and the requirement that the reducing diffeomorphism has identity linear approximation at points of Q . The proof is exactly the same as the proof of the Darboux–Givental’ theorem in [AG]. The odd-dimensional case reduces to the even-dimensional case as follows. Take a hypersurface H which contains Q and which is transversal to the kernels of μ and $\tilde{\mu}$. The restrictions of μ and $\tilde{\mu}$ to TH are symplectic. Take a local diffeomorphism $\hat{\Phi}$ of H which preserves Q pointwise, brings $\tilde{\mu}|_{TH}$ to $\mu|_{TH}$, and has identity linear approximation at any point of Q . Take vector fields X and \tilde{X} which generate the kernels of $\mu, \tilde{\mu}$ respectively and agree at any point of Q . Let Ψ^t and $\tilde{\Psi}^t$ be the flows of X and \tilde{X} . The required local diffeomorphism Φ of \mathbb{R}^k can be constructed as follows: for any point $p \in \mathbb{R}^k$, close to 0, we take $t = t(p)$ such that $\tilde{\Psi}^{t(p)} \in H$ and we set $\Phi(p) = (\Psi^{-t(p)} \circ \hat{\Phi} \circ \tilde{\Psi}^{t(p)})(p)$.

Finally, we need a simple part of the classification of couples of symplectic forms on the same vector space. This classification problem was solved in [GZ] by I. Gelfand and I. Zakharevich. We need the following statement formulated in terms of skew-symmetric matrices.

PROPOSITION 7 ([GZ, section 1]). *Let A and B be non-singular skew-symmetric $2s \times 2s$ matrices. The tuple of eigenvalues of the matrix $A^{-1}B$ is an invariant of the couple (A, B) with respect to the group of transformations $(A, B) \rightarrow (R^t A R, R^t B R)$, $\det R \neq 0$. The multiplicity of each of the eigenvalues of the matrix $A^{-1}B$ is greater than 1 and consequently this matrix has not more than s distinct eigenvalues. It has exactly s eigenvalues for a generic couple A and B . In this case the tuple of eigenvalues of $A^{-1}B$ is a complete invariant of (A, B) .*

1.5. Structure of the paper

In Section 2 we present linearization theorems which can be easily proved using Propositions 4–6. We believe that one of the main contributions of this work, maybe the main one, is the construction of invariants of tuples (1.1) which we call characteristic numbers. These characteristic numbers are constructed in Section 3. In the case $k > n$ a generic tuple (1.1) defines a manifold $Q = S_1^k \cap S_2^k$ endowed with a symplectic form $\omega|_{TQ}$ and our characteristic numbers can be extended to characteristic Hamiltonians on the symplectic manifold Q . The tuple of characteristic Hamiltonians, also constructed in Section 3, is an invariant of (1.1) with $k > n$ up to a symplectomorphism of Q . Our final theorems on *complete* systems of invariants are contained in Section 4, along with normal forms following from these theorems.

2. Linearization theorems

2.1. Regular intersection of S_1^k and S_2^k

Our final theorems in section 4 hold under a certain genericity condition, which we call the regularity of a tuple (1.1). It includes the regularity of the intersection of the strata S_1^k, S_2^k .

Definition 8. The strata S_1^k, S_2^k in (1.1) have regular intersection if $T_0S_1^k \cap T_0S_2^k = \{0\}$ for $k \leq n$ and $T_0S_1^k + T_0S_2^k = T_0\mathbb{R}^{2n}$ for $k > n$.

2.2. Linearization

The regularity of the intersection of the strata is a property of the linearization of (1.1) which is a tuple

$$(W^{2n}, \sigma, U_1^k \cup U_2^k) \tag{2.1}$$

consisting of a $2n$ -dimensional vector space W^{2n} , a symplectic (i.e. non-degenerate) 2-form σ on W^{2n} , and the union of the k -dimensional subspaces U_1^k, U_2^k .

Definition 9. The linearization of a tuple $(\mathbb{R}^{2n}, \omega, S_1^k \cup S_2^k)$ at a point $z \in S_1^k \cap S_2^k$ is the tuple (2.1) with $W^{2n} = T_z\mathbb{R}^{2n}$, $\sigma = \omega|_{W^{2n}}$, and $U_i^k = T_zS_i^k$, $i = 1, 2$.

2.3. Tuples (1.1) with the same linearization

The following two theorems can be easily proved using Propositions 4–6.

THEOREM 10. *Two tuples (1.1) with the same regularly intersecting symplectic or quasi-symplectic strata S_1^k, S_2^k of dimension $k \leq n$ and the same linearization at $0 \in \mathbb{R}^{2n}$ are equivalent.*

THEOREM 11. *Two tuples (1.1) with the same regularly intersecting symplectic or quasi-symplectic strata S_1^k, S_2^k of dimension $k > n$ and the same linearization at any point $z \in S_1^k \cap S_2^k$ close to $0 \in \mathbb{R}^{2n}$ are equivalent provided that the restrictions of ω and $\tilde{\omega}$ to $T_0S_1^k \cap T_0S_2^k$ have the maximal rank $2(n - k)$.*

Proof. Theorem 10 can be reduced to Proposition 4 with $r = 2$ as follows. Since the linearizations of the tuples are the same, we have $\omega(0) = \tilde{\omega}(0)$. By Proposition 6 with $Q = \{0\}$ there exist local diffeomorphisms ϕ_i of S_i^k , $i = 1, 2$ with identity linear approximations at 0 which bring the restriction of $\tilde{\omega}$ to $T S_i^k$ to the restriction of ω to $T S_i^k$, $i = 1, 2$. We can construct a local diffeomorphism Φ of \mathbb{R}^{2n} , also with identity linear approximation at 0, which preserves S_i^k and whose restriction to S_i^k coincides with ϕ_i , $i = 1, 2$. The diffeomorphism Φ brings $\tilde{\omega}$ to a symplectic form $\hat{\omega}$ such that ω and $\hat{\omega}$ have the same restriction to $T S_i^k$ and the same restriction to the space $T_0\mathbb{R}^{2n}$. Proving the equivalence of tuples (1.1) we may replace $\tilde{\omega}$ by $\hat{\omega}$. Now the equivalence follows from Proposition 4.

Theorem 11 can be reduced to Proposition 5 in exactly the same way using Proposition 6 with $Q = S_1^k \cap S_2^k$. If k is odd, we have a right to use Proposition 6 because the one-dimensional kernels of forms $\omega(0)$ and $\tilde{\omega}(0)$ are not tangent to Q . It follows from the assumptions in Theorem 11 that the restrictions of ω and $\tilde{\omega}$ to T_0Q have the maximal rank.

2.4. Isomorphic tuples (2.1). Linearization theorem.

Theorem 10 implies the following linearization theorem involving the natural definition of isomorphic tuples (2.1).

Definition 12. A tuple (2.1) is isomorphic to a tuple $(\tilde{W}^{2n}, \tilde{\sigma}, \tilde{U}_1^k \cup \tilde{U}_2^k)$ of the same form if there exists an isomorphism from W^{2n} to \tilde{W}^{2n} sending $\tilde{\sigma}$ to σ and sending $U_1^k \cup U_2^k$ to $\tilde{U}_1^k \cup \tilde{U}_2^k$.

THEOREM 13. *If two tuples (1.1) with regularly intersecting symplectic or quasi-symplectic strata are equivalent then their linearizations at $0 \in \mathbb{R}^{2n}$ are isomorphic. In*

the case $k \leq n$ the tuples are equivalent if and only if their linearizations at $0 \in \mathbb{R}^{2n}$ are isomorphic.

Proof. The first statement follows from the observation that if two tuples (1.1) are equivalent via a local diffeomorphism Φ then their linearizations at 0 are isomorphic via the isomorphism $d\Phi|_0$. The second statement is a direct corollary of Theorem 10 and the fact that for $k \leq n$ any pair of germs at 0 of smooth regularly intersecting k -dimensional submanifolds is diffeomorphic to its linearization by a diffeomorphism with identity linear approximation at 0.

Using Theorem 11 we could formulate a linearization theorem for the case $k > n$, with necessary and sufficient rather than only necessary conditions for the equivalence of tuples (1.1), but the formulation of such a theorem is rather involved, and we do not need it for the proof of our final theorem for the case $k > n$ (we use just Theorem 11).

3. Characteristic numbers and characteristic Hamiltonians

3.1. Regular tuples (1.1) and (2.1)

By Theorem 13 the problem of classifying tuples (1.1) reduces to the problem of classifying tuples (2.1) with respect to isomorphisms if $k \leq n$, and contains this problem if $k > n$. We solve this problem for generic tuples (2.1), namely for tuples (2.1) satisfying the following conditions.

Definition 14. A tuple (1.1) will be called *regular* if its linearization at $0 \in \mathbb{R}^{2n}$, a tuple of form (2.1), is regular. A tuple (2.1) is regular if its ingredients satisfy the following requirements:

- (1) The subspaces U_1^k and U_2^k are symplectic or quasi-symplectic, with regular intersection: $U_1^k \cap U_2^k = \{0\}$ for $k \leq n$; $U_1^k + U_2^k = W^{2n}$ for $k > n$;
- (2) If $k \leq n$ the restriction of σ to $U_1^k + U_2^k$ has maximal rank $2k$. If $k > n$ the restriction of σ to $U_1^k \cap U_2^k$ has maximal rank $2(k - n)$;
- (3) The skew-orthogonal complement to U_1^k in (W^{2n}, σ) is transversal to U_2^k ;
- (4) This condition is required only for odd k . In this case the previous conditions imply that $\ell_i = \ker \sigma|_{U_i^k}$, $i = 1, 2$ are different 1-dimensional subspaces of W^{2n} . We require that the 2-form σ does not annihilate the plane $\ell_1 + \ell_2$.

3.2. Reduction of dimensions

Our first step in classifying regular tuples (2.1) with $2 \leq k \leq 2n - 2$ is a reduction of dimensions $2n, k$ to $4s, 2s$. Namely we associate to a regular tuple (2.1) a tuple

$$(\widehat{W}^{4s}, \widehat{\sigma}, \widehat{U}_1^{2s} \cup \widehat{U}_2^{2s}) \tag{3.1}$$

$$s = s(k, n) = \min ([k/2], \lfloor (2n - k)/2 \rfloor) \tag{3.2}$$

constructed as follows, where $\ell_i = \ker \sigma|_{U_i^k}$ and the sign $^\perp$ denotes the skew-orthogonal complement in the symplectic space (W^{2n}, σ) :

$$k \text{ even : } \widehat{W}^{4s} = \begin{cases} U_1^k + U_2^k & \text{if } k \leq n \\ (U_1^k \cap U_2^k)^\perp & \text{if } k > n \end{cases} \tag{3.3}$$

$$k \text{ odd} : \widehat{W}^{4s} = \begin{cases} (U_1^k + U_2^k) \cap (\ell_1 + \ell_2)^\perp & \text{if } k \leq n \\ (U_1^k \cap U_2^k)^\perp \cap (\ell_1 + \ell_2)^\perp & \text{if } k > n \end{cases} \quad (3.4)$$

and for any parity of k we set

$$\widehat{\sigma} = \sigma|_{\widehat{W}^{4s}}, \quad \widehat{U}_i^{2s} = U_i^k \cap \widehat{W}^{4s}, \quad i = 1, 2.$$

PROPOSITION 15. For any regular tuple (2.1) the dimension of \widehat{W}^{4s} is $4s$, the dimension of \widehat{U}_i^{2s} is $2s$, and the form $\widehat{\sigma}$ on \widehat{W}^{4s} is symplectic, so that the tuple (3.1) has the same form as the tuple (2.1). The tuple (3.1) is also regular, i.e. it satisfies all the requirements in Definition 14. Two regular tuples (2.1) are isomorphic if and only if the corresponding reduced tuples (3.1) are as well.

Definition 16. The constructed tuple (3.1) will be called the reduced tuple, associated with a regular tuple (2.1).

Proposition 15, reducing classification of regular tuples (2.1) to the classification of regular tuples (3.1), is a simple statement and we leave its proof to the reader. The proof requires not more than the linear Darboux theorem stating that the rank of a 2-form on a vector space is its complete invariant with respect to isomorphisms.

3.3. Two linear operators defined by reduced tuples (3.1)

Our next step is the construction of two linear operators associated with such tuples. The regularity of (3.1) implies that we have the direct sums

$$\widehat{W}^{4s} = \widehat{U}_1^{2s} \oplus \widehat{U}_2^{2s} = \widehat{U}_1^{2s} \oplus (\widehat{U}_1^{2s})^\perp = \widehat{U}_2^{2s} \oplus (\widehat{U}_2^{2s})^\perp,$$

where as above the sign \perp means the skew-orthogonal complement in the space $(\widehat{W}^{4s}, \sigma)$. Consider the projections associated with the last two direct sums:

$$\begin{aligned} \widehat{W}^{4s} &= \widehat{U}_1^{2s} \oplus (\widehat{U}_1^{2s})^\perp, & \pi_1 : \widehat{W}^{4s} &\longrightarrow \widehat{U}_1^{2s}, \\ \widehat{W}^{4s} &= \widehat{U}_2^{2s} \oplus (\widehat{U}_2^{2s})^\perp, & \pi_2 : \widehat{W}^{4s} &\longrightarrow \widehat{U}_2^{2s}. \end{aligned}$$

Define linear operators $L_1 : \widehat{U}_1^{2s} \rightarrow \widehat{U}_1^{2s}$ and $L_2 : \widehat{U}_2^{2s} \rightarrow \widehat{U}_2^{2s}$ by the commutative diagram

$$\begin{array}{ccc} \widehat{U}_1^{2s} & \xrightarrow{L_1} & \widehat{U}_1^{2s} \\ & \searrow \pi_2 & \nearrow \pi_1 \\ & \widehat{U}_2^{2s} & \xrightarrow{L_2} \widehat{U}_2^{2s} \end{array} : \quad \begin{aligned} L_1 &= \pi_1 \circ (\pi_2|_{\widehat{U}_1^{2s}}) \\ L_2 &= \pi_2 \circ (\pi_1|_{\widehat{U}_2^{2s}}) \end{aligned}$$

LEMMA 17. For any regular tuple (3.1) the linear operators L_1 and L_2 are conjugate and consequently have the same eigenvalues.

Proof. Note that the given diagram implies that the diagram

$$\begin{array}{ccc} \widehat{U}_1^{2s} & \xrightarrow{L_1} & \widehat{U}_1^{2s} \\ \pi_2 \downarrow & & \downarrow \pi_2 \\ \widehat{U}_2^{2s} & \xrightarrow{L_2} & \widehat{U}_2^{2s} \end{array}$$

is commutative. Since (3.1) is a regular tuple, the three spaces \widehat{U}_1^{2s} , \widehat{U}_2^{2s} , and $(\widehat{U}_1^{2s})^\perp$ are transversal one to the other. It follows that π_2 restricted to \widehat{U}_1^{2s} is a bijection between \widehat{U}_1^{2s} and \widehat{U}_2^{2s} .

3.4. *Characteristic numbers*

Definition 18. Let (1.1) be a regular tuple, (2.1) its linearization at $0 \in \mathbb{R}^{2n}$, and (3.1) the reduced linearization. The eigenvalues (real and complex) of the constructed linear operators L_1 or L_2 will be called *characteristic numbers* of the tuples (1.1), (2.1), and (3.1).

The following statement is a direct corollary of Theorem 13 and Proposition 15.

THEOREM 19. *The characteristic numbers of a regular tuple (1.1) are invariants: if two regular tuples of form (1.1) are equivalent then their characteristic numbers are the same.*

The linear operators L_1 and L_2 are defined on vector spaces of dimension $2s$ and from the first glance it seems that a generic regular tuple (3.1) and consequently a generic regular tuple (1.1) has $2s$ distinct characteristic numbers. It is not so. The matrices of L_1, L_2 in some and then any bases of the vector spaces $\widehat{U}_1^{2s}, \widehat{U}_2^{2s}$ respectively is the product of two skew-symmetric $2s \times 2s$ matrices, and the eigenvalues of such matrices are not generic in the space of tuples of $2s$ complex numbers. To explain this claim, take any basis $B_1 = (u_{1,1}, \dots, u_{1,2s})$ of \widehat{U}_1^{2s} and any basis $B_2 = (u_{2,1}, \dots, u_{2,2s})$ of \widehat{U}_2^{2s} . The 2-form σ on \widehat{W}^{4s} is defined by a $4s \times 4s$ skew-symmetric matrix of the form

$$\sigma : \begin{pmatrix} A_1 & C \\ C^t & A_2 \end{pmatrix}, \quad A_1, A_2, C \in \text{Mat}(2s \times 2s), \quad A_1^t = -A_1, \quad A_2^t = -A_2. \quad (3.5)$$

The matrices A_1 and A_2 are non-singular. Since the tuple (3.1) is regular, the skew-orthogonal complement to \widehat{U}_1^{2s} is transversal to \widehat{U}_2^{2s} and it follows that the matrix C is also non-singular. The latter allows one to change the basis B_1 by the transition matrix C^{-1} to a new basis \widetilde{B}_1 of \widehat{U}_1^{2s} so that in the basis (\widetilde{B}_1, B_2) the 2-form σ is defined by matrix (3.5) with $C = I$ (certainly the matrices A_1 and A_2 will change). After this reduction of C to I , it is not hard to compute the matrix of the linear operator L_1 in the basis \widetilde{B}_1 , it is the skew-symmetric matrix $A_1^{-1}A_2$.

It is easy to see that when changing *both* bases B_1 and B_2 , the matrix C in (3.5) remains the identity matrix if and only if the transformations of B_1 and B_2 are defined by matrices R and $(R^t)^{-1}$, where R is any non-singular $2s \times 2s$ matrix. Such transformations of B_1 and B_2 bring the matrices A_1 and A_2 in (3.5) to the matrices $A_1 \rightarrow R^t A_1 R, A_2 \rightarrow R^t A_2 R$.

The outcome of this linear algebra computation (expressed without details which we leave to the reader) is as follows.

PROPOSITION 20. *One can associate to any regular tuple (3.1) two non-singular skew-symmetric $2s \times 2s$ matrices A_1, A_2 so that the characteristic numbers of (3.1) are the eigenvalues of the matrix $A_1^{-1}A_2$, and two tuples (3.1) are isomorphic if and only if the corresponding couples of skew-symmetric matrices can be brought one to the other by a transformation $(A_1, A_2) \rightarrow (R^t A_1 R, R^t A_2 R), \det R \neq 0$. Any couple (A_1, A_2) with two non-singular skew-symmetric $2s \times 2s$ is realizable, i.e. it is associated to some regular tuple (3.1).*

Consequently the classification of regular tuples (3.1) is exactly the same problem as the classification of couples of symplectic forms on a $2s$ -dimensional vector space. Now we can

use the classification of couples of symplectic forms given in [GZ]. We need a part of this classification given in Proposition 7 of this paper. The following theorem is a direct corollary of this proposition and Proposition 20.

PROPOSITION 21. *Each of the characteristic numbers of a regular tuple (3.1) is different from 0 and has multiplicity ≥ 2 . Consequently (3.1) has not more than s distinct characteristic numbers; if $s = 1$ then it has only one characteristic number. The multiplicity of each of the characteristic numbers of a generic regular tuple (3.1) is equal to 2 and consequently a generic regular tuple (3.1) has s distinct characteristic numbers. In this case (3.1) is isomorphic to another regular tuple of the same form if and only if the two tuples have the same characteristic numbers.*

The following statement is not more than a logical corollary of Proposition 21 and Definition 18.

COROLLARY 22. *Let $2 \leq k \leq 2n - 2$. The characteristic numbers of a regular tuple (1.1) have the same properties as in Proposition 21 with $s = s(k, n) = \min(\lfloor k/2 \rfloor, \lfloor (2n - k)/2 \rfloor)$.*

3.5. Characteristic Hamiltonians

In the case $k > n$ with the genericity assumption that each of the characteristic numbers has minimal possible multiplicity 2, we have $s = s(k, n) = \lfloor (2n - k)/2 \rfloor$ distinct characteristic numbers $\lambda_1, \dots, \lambda_s$, and these characteristic numbers can be extended to functions on the symplectic manifold

$$(Q, \omega_Q), \quad Q = S_1^k \cap S_2^k, \quad \omega_Q = \omega|_{TQ} \tag{3.6}$$

(the fact that Q is symplectic follows from the regularity of a tuple (1.1)) by associating to a point $z \in Q$, close to $0 \in \mathbb{R}^{2n}$, the characteristic numbers of the linearization of (1.1) at z . We obtain s smooth functions h_1, \dots, h_s on the symplectic manifold (3.6) taking the values $\lambda_1, \dots, \lambda_s$ at $z = 0$.

Definition 23. Let $k > n$. The constructed functions h_1, \dots, h_s , where $s = s(k, n) = \lfloor (2n - k)/2 \rfloor$, on the symplectic manifold (3.6) will be called the characteristic Hamiltonians of a regular tuple (1.1).

It is worthwhile to note that this definition works *only* under the assumption that each of the characteristic numbers of a regular tuple (1.1) has minimal possible multiplicity 2 so that the linearization of (1.1) at any point $z \in Q$ close to 0 has *the same* number $s = s(k, n) = \lfloor (2n - k)/2 \rfloor$ of distinct characteristic numbers.

4. Theorems on a complete system of invariants

4.1. The case $2 \leq k \leq n$

THEOREM 24. *Let $2 \leq k \leq n$. Assume that the characteristic numbers of two regular tuples (1.1) have minimal possible multiplicity 2 and consequently each of the tuples has $\lfloor k/2 \rfloor$ distinct characteristic numbers. The tuples are equivalent if and only if their characteristic numbers are the same.*

Proof. The “only if” part holds without the assumption on the multiplicities and it is a part of Theorem 13. The “if” part is a direct corollary of the same Theorem 13, Proposition 21, and Proposition 15. In fact, if the characteristic numbers of two tuples T and \tilde{T} of

form (1.1) are the same and have minimal multiplicity 2 then by Proposition 21 the reduced linearizations at 0 of T and \tilde{T} are isomorphic, by Proposition 15 their linearization at 0 are also isomorphic, and by Theorem 13 the tuples are isomorphic.

Note that the assumption on the multiplicities in Theorem 24 always holds for $k = 2$ and $k = 3$ where we have only one characteristic number. In the case $k = 2$ Theorem 24 was proved in [DJZ2, section 7.4], where the characteristic number was called the index of non-orthogonality between S_1^2 and S_2^2 .

4.2. *The case $n < k \leq 2n - 2$*

Consider two regular tuples of form (1.1):

$$T = (\mathbb{R}^{2n}, \omega, S_1^k \cup S_2^k)_0, \quad \tilde{T} = (\mathbb{R}^{2n}, \tilde{\omega}, \tilde{S}_1^k \cup \tilde{S}_2^k) \tag{4.1}$$

and the symplectic manifolds

$$(Q, \omega_Q), \quad Q = S_1^k \cap S_2^k, \quad \omega_Q = \omega|_{TQ} \tag{4.2}$$

$$(\tilde{Q}, \tilde{\omega}_{\tilde{Q}}), \quad \tilde{Q} = \tilde{S}_1^k \cap \tilde{S}_2^k, \quad \tilde{\omega}_{\tilde{Q}} = \tilde{\omega}|_{T\tilde{Q}}. \tag{4.3}$$

THEOREM 25. *Let $n < k \leq 2n - 2$. Assume that the characteristic numbers of two regular tuples (4.1) have minimal possible multiplicity 2 and consequently each of the tuples has $s = s(k, n) = \lfloor (2n - k)/2 \rfloor$ distinct characteristic numbers and the characteristic Hamiltonians h_1, \dots, h_s and $\tilde{h}_1, \dots, \tilde{h}_s$ are well-defined. The tuples T and \tilde{T} are equivalent if and only if there exists a local diffeomorphism $\phi : Q \rightarrow \tilde{Q}$ which sends $\tilde{\omega}_{\tilde{Q}}$ to ω_Q and the tuple of functions $(\tilde{h}_1, \dots, \tilde{h}_s)$ to (h_1, \dots, h_s) .*

Like in Theorem 24, the assumption on multiplicities always holds if $k = 2n - 3$ or $k = 2n - 2$ when we have only one characteristic number.

4.2.1. *Proof of the “only if” part*

Assume that the tuples (4.1) are equivalent via a local diffeomorphism Φ of \mathbb{R}^n . Since Φ sends S_1^k to \tilde{S}_1^k and S_2^k to \tilde{S}_2^k it sends Q to \tilde{Q} . It also sends $\tilde{\omega}$ to ω and consequently the restriction ϕ of Φ to Q sends the form $\tilde{\omega}_{\tilde{Q}}$ to the form ω_Q . The differential of the diffeomorphism Φ at a point $z \in Q$ sends the linearization of T at z to the linearization of \tilde{T} at the point $\phi(z)$. Therefore these two linearizations are isomorphic. By Proposition 15 the corresponding reduced linearizations are also isomorphic. By Theorem 19 these reduced linearizations have the same characteristic numbers. Therefore $\tilde{h}_i(\phi(z)) = h_i(z)$, up to numeration.

4.2.2. *Proof of the “if” part*

The proof of the “if” part is a reduction to Theorem 11. We will assume, without loss of generality, that $S_1^k = \tilde{S}_1^k$ and $S_2^k = \tilde{S}_2^k$. Let ϕ be a local diffeomorphism of Q which brings \tilde{h}_i to h_i , up to numeration. We can extend ϕ to a local diffeomorphism Ψ of \mathbb{R}^{2n} which preserves S_1^k and S_2^k . Applying Ψ to the tuple \tilde{T} we obtain a tuple with characteristic Hamiltonians coinciding with those of the tuple T , up to numeration. We can now assume that T and \tilde{T} satisfy the following conditions:

- (a) $S_i^k = \tilde{S}_i^k$ and consequently $Q = \tilde{Q}$;
- (b) the reduced linearizations of (4.1) at any point $z \in Q$ have the same characteristic numbers;
- (c) ω and $\tilde{\omega}$ have the same restriction to the tangent bundle of Q .

By Propositions 21 and 15 there is a family of isomorphisms $\tau_z : T_z\mathbb{R}^{2n} \rightarrow T_z\mathbb{R}^{2n}$, parameterized by a point $z \in Q$, which brings the linearization of T at $z \in Q$ to the linearization of \tilde{T} at the same point z . Condition (c) allows us to chose τ_z such that for any $z \in Q$ it preserves T_zQ and its restriction to T_zQ is the identity map. Having a family of isomorphisms τ_z with this property, we can construct a local diffeomorphism Φ of \mathbb{R}^{2n} which preserves S_1^k and S_2^k pointwise (and consequently preserves Q pointwise) and such that $d\Phi|_z = \tau_z$ for any $z \in Q$. Applying this diffeomorphism Φ to the tuple \tilde{T} we obtain a tuple \hat{T} such that T and \hat{T} have the same linearization at any point $z \in Q$. Now the equivalence of the tuples follows from Theorem 11.

4.3. The cases $k = 2n - 3, k = 2n - 2$

A short formulation of Theorem 25 is that, under the given condition on multiplicity of the characteristic numbers, the tuple of characteristic Hamiltonians, defined up to a symplectomorphism of the symplectic manifold (3.6), is a complete invariant of a regular tuple (1.1) with $n < k \leq 2n - 2$. Nevertheless, strictly speaking, Theorem 25 is a reduction theorem rather than a theorem on a complete system of invariants. It reduces the classification of generic tuples (1.1) with $n < k \leq 2n - 2$ to the classification of $\lfloor (2n - k)/2 \rfloor$ functions on a symplectic manifold of dimension $2(n - k)$ with respect to local symplectomorphisms of this manifold. It is well known that a single non-singular function h (such that $dh(0) \neq 0$) can be reduced to $h(0) + z_1$, where z_1 is one of local coordinates. Therefore Theorem 25 implies the following corollary.

COROLLARY 26. *Let $k = 2n - 2 \geq 4$ or $k = 2n - 3 \geq 5$ so that tuples (4.1) have only one characteristic number λ and $\tilde{\lambda}$. Assume that the characteristic Hamiltonians h and \tilde{h} are non-singular: $dh(0) \neq 0$ and $d\tilde{h}(0) \neq 0$. The tuples (4.1) are equivalent if and only if $\lambda = \tilde{\lambda}$.*

4.4. Normal forms

Using Theorems 24–25 and Corollary 26 it is easy to construct the following normal forms. If $2 \leq k \leq n$ then in suitable local coordinates $x, y \in \mathbb{R}^k, p, q \in \mathbb{R}^{n-k}$ a tuple (1.1) satisfying the assumptions of Theorem 24 has the form

$$S_1^k = \{y = p = q = 0\}, \quad S_2^k = \{x = p = q = 0\}, \tag{4.4}$$

$$\omega = \sum_{i=1}^k dx_i dy_i + \sum_{i=1}^{n-k} dp_i dq_i + \sum_{i=1}^s dx_{2i-1} dx_{2i} + \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{dy_{2i-1} dy_{2i}}{\lambda_i}. \tag{4.5}$$

If $n < k \leq 2n - 2$ then in suitable local coordinates $x, y \in \mathbb{R}^{2n-k}, p, q \in \mathbb{R}^{k-n}$ a tuple (1.1) satisfying the assumptions of Theorem 25 has the form

$$S_1^k = \{y = 0\}, \quad S_2^k = \{x = 0\}, \tag{4.6}$$

$$\omega = \sum_{i=1}^{2n-k} dx_i dy_i + \sum_{i=1}^{k-n} dp_i dq_i + \sum_{i=1}^{\lfloor (2n-k)/2 \rfloor} dx_{2i-1} dx_{2i} + \sum_{i=1}^{\lfloor (2n-k)/2 \rfloor} \frac{dy_{2i-1} dy_{2i}}{h_i(p, q)}. \tag{4.7}$$

The parameters λ_i in normal form (4.5) are moduli, and they are exactly the characteristic numbers. The functional parameters $h_i(p, q)$ in normal form (4.7) are exactly the characteristic Hamiltonians. In the case that some of the characteristic numbers $\lambda_i = h_i(0)$ are not real these normal forms hold in complex coordinates. Namely, if $\lambda_i = \bar{\lambda}_j \notin \mathbb{R}$ then

$x_{2i-1} = \bar{x}_{2j-1}$, $x_{2i} = \bar{x}_{2j}$, $y_{2i-1} = \bar{y}_{2j-1}$, $y_{2i} = \bar{y}_{2j}$ are complex valued conjugate coordinates and $h_i(p, q) = \bar{h}_j(p, q)$ are complex valued conjugate functions.

If $k = 2n - 2 \geq 4$ or $k = 2n - 3 \geq 5$ and a tuple (1.1) satisfies the assumptions of Corollary 26, then in suitable coordinates the tuple has the form (4.6)-(4.7) with $h_1(p, q) \equiv \lambda_1$, i.e. with only one parameter λ_1 .

4.5. The case $n < k \leq 2n - 4$. Functional moduli

Note that this case is possible only if the dimension of the symplectic space $(\mathbb{R}^{2n}, \omega)$ is at least 10. Theorem 25 implies that a generic tuple (1.1) has in suitable local coordinates the normal form (4.6)-(4.7) with $h_1(p, q) \equiv \lambda_1$. (The genericity conditions are the assumption of Theorem 25 and the requirement that at least one of the characteristic Hamiltonians is a non-singular function.) This normal form is parameterized by $s - 1$ functions $h_2(u, v), \dots, h_s(u, v)$, $s = \lfloor (2n - k)/2 \rfloor \geq 2$. Since the group of local symplectomorphisms can be parameterized by one function, it is almost clear that this normal form is asymptotically exact in the following sense.

Definition 27. Let m_ℓ be the number of moduli in the classification of generic germs (in any classification problem of local analysis). Assume that $m_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. A normal form, parameterized by functions, is called asymptotically exact if the number of parameters p_ℓ of its ℓ -jet satisfies $p_\ell = m_\ell(1 + o(1))$ as $\ell \rightarrow \infty$.

With this definition, we obtain one more corollary of Theorem 25.

COROLLARY 28. *Let $n < k \leq 2n - 4$ so that $s = s(k, n) = \lfloor (2n - k)/2 \rfloor \geq 2$. In this case the number of moduli in the classification of ℓ -jets of generic tuples (1.1) goes to ∞ as $\ell \rightarrow \infty$. A generic tuple (1.1) has in suitable coordinates normal form (4.6)-(4.7) with $h_1(u, v) \equiv \lambda_1$, parameterized by $(s - 1)$ functions of $2(k - n)$ variables. This normal form is asymptotically exact.*

At the beginning of this paper, in Theorem 3, we stated that in the case of dimensions k, n in Corollary 28 the functional moduli are $s - 1$ functions of $2(k - n)$ variables. Corollary 28 gives a precise meaning of what we mean by these words. A more detailed characterization of "functional codimension" of orbits in classification problems of local analysis requires Poincaré series of moduli numbers which was introduced by V. Arnol'd in [Ar2].

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