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HAMILTONIAN VECTOR FIELDS ON SINGULAR VARIETIES

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Abstract: We define Hamiltonian vector fields on singular subvarieties of the symplectic space. We describe Hamiltonian vector fields on smooth submanifolds, singular planar curves with ADE singularities and regular union singularities.
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1. INTRODUCTION

Let M be a smooth $2n$-dimensional manifold, endowed with a nondegenerate, closed 2-form ω. The 2-form ω is called symplectic and the pair (M,ω) is a symplectic manifold. We introduce the canonical symplectic structure $\dot{\omega}$ on TM using the vector bundle morphism $\beta : TM \ni u \mapsto \omega(u,\cdot) \in T^*M$, namely the pullback of the Liouville symplectic form $d\theta$ defined on the cotangent bundle T^*M, $\dot{\omega} = \beta^* d\theta$. A smooth vector field $X : M \to TM$ is said to be Hamiltonian if the form $\omega(X,\cdot)$ is exact. A function $H : M \to \mathbb{R}$ is called Hamiltonian for X if $\omega(X,\cdot) = -dH(\cdot)$. If X is Hamiltonian, then its image $X(M) \subset TM$ is a Lagrangian submanifold of $(TM,\dot{\omega})$ generated by H. In local Darboux coordinates, $M \cong \mathbb{R}^{2n}$, $\omega = \sum_{i=1}^n dy_i \wedge dx_i$, and $\dot{\omega} = \beta^* d\theta = \sum_{i=1}^n (d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i)$, where $(q,\dot{q}) = ((x,y),(\dot{x},\dot{y}))$ are coordinates on $T\mathbb{R}^{2n} \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

To generalize this notion, we introduce a concept of a Hamiltonian system as a general Lagrangian submanifold N of the symplectic tangent bundle $(TM,\dot{\omega})$. If $\tau |_N : N \to M$ is singular, where τ is tangent bundle projection, we also call N an implicit Hamiltonian system (cf. [12], [7]). Important property of such systems around singularities is their solvability, i.e. existence of smooth local curve $\gamma : (-\varepsilon,\varepsilon) \to M$ such that its tangent lifting $\dot{\gamma}(t)$ belongs to N around each point of N. An immediate necessary condition for solvability is tangential solvability condition, which is satisfied if $\dot{q} \in d(\tau\mid_N)_v(T_vN)$ for each point $v = (q, \dot{q}) \in N$. It is proved (cf. [7]) that, for certain naturally generic implicit Hamiltonian systems, they are solvable if they fulfill this tangential solvability condition. Another generalization following P.A.M. Dirac (cf. [3]) is provided by constrained Lagrangian submanifolds (cf. [11]) as Hamiltonian systems. The generalized Hamiltonian function for such system is a generating family (Morse family) for the corresponding Lagrangian submanifold L_h; $F(x,y,\lambda) = \sum_{i=1}^k a_i(x,y)\lambda_i + h(x,y)$ over the constraint K defined by smooth functions $a_i(x,y) = 0$. The condition of solvability $\{\frac{\partial F}{\partial \lambda_i}, F\} = 0$ for $(x,y,\lambda) \in S \times \mathbb{R}^{2n}$ defines the section of L_h which is tangent to K. The general sections of L_h give the vector fields which are Hamiltonian on the constrained submanifold.

In this work we concentrate on the vector fields of symplectic space (M, ω), which are Hamiltonian on a subvariety of M. As we do not exclude singularities, our approach is local and we consider mainly germs of subvarieties and germs of vector fields. We find the spaces of vector fields, which are Hamiltonian on symplectic, isotropic and coisotropic submanifolds of (M, ω) and we provide the classification of Hamiltonian vector fields on singular varieties: planar curves of type A_k, D_k, E_6, E_7, E_8, regular union of three 1-dimensional submanifolds, regular union of two 2-dimensional isotropic submanifolds, and regular union of two 2-dimensional symplectic submanifolds. We use the Mathematica package Exterior Differential Calculus for calculations.

2. HAMILTONIAN SYSTEMS ON SUBMANIFOLDS

Let K be a submanifold of $\mathbb{R}^{2n}$ and $h: K \to \mathbb{R}$ be a smooth function on K. The notion of generalized Hamiltonian system (generalized Hamiltonian dynamics) was introduced by P.A.M. Dirac in [3]. A generalized Hamiltonian system is the following sub-bundle L_h of $T\mathbb{R}^{2n}$ over K (cf. [13]):

$$L_h = \{v \in T\mathbb{R}^{2n} : \omega(v,u) = -dh(u) \quad \forall_{u \in TK}\}. \tag{1}$$

It is easy to see that L_h is a Lagrangian submanifold of $(T\mathbb{R}^{2n}, \dot{\omega})$.

In local coordinates, the generalized Hamiltonian system (1) can be written, using generating family $F : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$, in the following way:

$$F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + H(x,y), \tag{2}$$

where K is defined as a zero-level set of the mapping $a : (x,y) \mapsto (a_1(x,y), \ldots, a_k(x,y))$, $H(x,y)$ is an arbitrary smooth extension of the function $h : K \to \mathbb{R}$ and a is a maximal rank map-germ.

The generalized Hamiltonian system L is given by an immersion $\phi : C_F \to L \subset (T\mathbb{R}^{2n}, \dot{\omega})$ defined by

$$\phi(x,y,\lambda) = (x,y,\frac{\partial F}{\partial y}(x,y,\lambda), -\frac{\partial F}{\partial x}(x,y,\lambda)), \quad (x,y,\lambda) \in C_F.$$

Since $\frac{\partial F}{\partial \lambda_\ell}(x,y,\lambda) = a_\ell(x,y)$, we have $C_F = K \times \mathbb{R}^k$. Then L can be described as

$$L = \phi(C_F) = \{(x,y,\frac{\partial F}{\partial y}(x,y,\lambda), -\frac{\partial F}{\partial x}(x,y,\lambda)) \in T\mathbb{R}^{2n} : (x,y,\lambda) \in K \times \mathbb{R}^k\}.$$

L is a skew-conormal bundle to K and its smooth sections are called Hamiltonian systems on K with Hamiltonian H. This may be extended to Hamiltonian system on M taking Hamiltonian function

$$F(x,y) = \sum_{l=1}^{k} \lambda_l(x,y) a_l(x,y) + H(x,y)$$

for some smooth functions $\lambda_l(x,y)$.

Vector fields, which are Hamiltonian on K are given in the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{k} \lambda_j(x,y)(\frac{\partial a_j}{\partial y_i}(x,y)\frac{\partial}{\partial x_i} - \frac{\partial a_j}{\partial x_i}(x,y)\frac{\partial}{\partial y_i}) + \sum_{i=1}^{n}(\frac{\partial H}{\partial y_i}(x,y)\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}(x,y)\frac{\partial}{\partial y_i}). \tag{3}$$

If we consider the functions $\lambda_j(x,y)$ which are smooth solutions of the system of linear equations (cf. [8]),

$$\sum_{j=1}^{k} \{a_i, a_j\}(x,y)\lambda_j = \{H, a_i\}(x,y), \quad i = 1,\ldots,k, \tag{4}$$

then the vector fields (3) are the logarithmic Hamiltonian vector fields over K.

3. HAMILTONIAN VECTOR FIELDS ON SINGULAR VARIETIES

Let (M,ω) be a symplectic manifold. Let N be a subset of M.

Definition 1. *A smooth vector field X on M is called* **Hamiltonian on** *N if there exists a smooth function H on M such that*

$$(X \rfloor \omega)|_x = -dH|_x, \textit{ for every } x \in N. \tag{5}$$

Example 2. *Let* $(\mathbb{R}^{2n},\omega_0)$ *be the standard symplectic space. Let* $N\subset(\mathbb{R}^{2n},\omega_0)$ *be the germ of a hypersurface with isolated singularity at* 0. *Assume that the ideal of smooth function-germs vanishing on* N *is generated by a smooth function-germ* g *on* $\mathbb{R}^{2n}$. *Let* H *be a smooth function-germ on* $\mathbb{R}^{2n}$ *and let* X_H *be a Hamiltonian vector field-germ on* $(\mathbb{R}^{2n},\omega_0)$ *with a Hamiltonian* H *i.e.* $X_H\rfloor\omega=-dH$. *Let* Y *be a smooth vector field-germ on* $\mathbb{R}^{2n}$. *Then the vector field-germ* $gY+X_H$ *is Hamiltonian on* N.

A smooth k-form β on M vanishes on N if $\beta|_x=0$ for every $x\in N$.

Definition 3. *A smooth* k*-form* α *on* M *has zero algebraic restriction to* N *if there exist a smooth* k*-form* β *on* M *vanishing on* N *and a smooth* $(k-1)$*-form* γ *on* M *vanishing on* N *such that*

$$\alpha=\beta+d\gamma. \tag{6}$$

Let $\mathcal{A}_0^k(N,M)$ denote the space of smooth k-forms with zero algebraic restriction to N. Since $d(\mathcal{A}_0^k(N,M))\subset\mathcal{A}_0^{k+1}(N,M)$, the complex $(\mathcal{A}_0^*(N,M),d)$ is a subcomplex of the de Rham complex on M. We denote by $H^*(N,M)$ the cohomology groups of the complex $(\mathcal{A}_0^*(N,M),d)$.

Proposition 4. *A smooth vector field* X *on* M *is Hamiltonian on* N *if and only if there exists a smooth function* H *on* M *such that* $X\rfloor\omega+dH$ *has zero algebraic restriction to* N.

Proof. Definition 1 is equivalent to the following condition:

$$X\rfloor\omega+dH=\sum_{i=1}^{k}f_i\alpha_i, \tag{7}$$

where $\alpha_1,\cdots,\alpha_k$ are smooth 1-forms on M, $H,f_1,\cdots,f_k$ are smooth functions on M such that $f_1|_N=\cdots=f_k|_N=0$. But this implies that $X\rfloor\omega+dH$ has zero algebraic restriction to N.

On the other hand, if there exists a smooth function H on M such that $X\rfloor\omega+dH$ has zero algebraic restriction to N, then

$$X\rfloor\omega+dH=\sum_{i=1}^{k}f_i\alpha_i+dg, \tag{8}$$

where $\alpha_1,\cdots,\alpha_k$ are smooth 1-forms on M, $H,f_1,\cdots,f_k,g$ are smooth functions on M such that $f_1|_N=\cdots=f_k|_N=g|_N=0$. But this can be written in the following way:

$$X\rfloor\omega+d\,(H-g)=\sum_{i=1}^{k}f_i\alpha_i, \tag{9}$$

which implies that X is Hamiltonian on N. □

The above definition and proposition are the motivation for the following definition of the symplectic vector field on N:

Definition 5. *A smooth vector field X on M is called* **symplectic on** N *if $\mathcal{L}_X\omega$ has zero algebraic restriction to N.*

It is obvious that a vector field, which is Hamiltonian on N, is symplectic on N. The inverse implication is not always true. The necessary and sufficient conditions are given in the following proposition:

Proposition 6. *The vector field-germ X is Hamiltonian on N if and only if X is symplectic on N and $\mathcal{L}_X\omega$ define the zero cohomology class in $H^2(N,M)$.*

Corollary 7. *If $H^2(N,M) = \{0\}$, then any symplectic vector field-germ on N is Hamiltonian on N.*

Definition 8. *The germ at 0 of a set $N \subset \mathbb{R}^m$ is called quasi-homogeneous if there exist a local coordinate system $x_1,\ldots,x_m$ and positive numbers $\lambda_1,\ldots,\lambda_m$ such that the following holds: if a point with coordinates $x_i = a_i$ belongs to N, then for any $t \in [0,1]$ the point with coordinates $x_i = t^{\lambda_i}a_i$ also belongs to N.*

It was proved that if N is quasi-homogeneous, then $H^k(N,M) = \{0\}$ for $k > 0$. (e.g. see [4]). It implies the following proposition:

Proposition 9. *If N is quasi-homogeneous, then any symplectic vector field-germ on N is Hamiltonian on N.*

4. GERMS OF HAMILTONIAN VECTOR FIELDS ON SMOOTH SUBMANIFOLDS

If S is a smooth submanifold of M, then a smooth k-form α on M has zero algebraic restriction to M if and only if the pullback of α to M vanishes. Thus, we obtain the following result:

Corollary 10. *Let S be a smooth submanifold of M. Let $\iota : S \hookrightarrow M$ be an embedding of S. A smooth vector field X on M is Hamiltonian on S if and only if there exists a smooth function H on M such that*

$$\iota^*(X\rfloor\omega) = d(H\circ\iota). \tag{10}$$

Thus, by the above corollary we obtain the following:

$$\omega(X(x),v) = -dH(v), \text{ for every } x \in S, \text{ and for every } v \in T_xS. \tag{11}$$

It means that if the vector field X is Hamiltonian on a smooth submanifold S of M, then X is a section of the bundle L.

By Poincare Lemma and Corollary 10 we have

Proposition 11. *Let S be a smooth submanifold of M. Let $\iota : S \hookrightarrow M$ be an embedding of S. A smooth vector field X on M is Hamiltonian on S if and only if $d(\iota^*(X\rfloor\omega)) = 0$.*

4.1. SYMPLECTIC SUBMANIFOLDS

Let S be the germ of a symplectic submanifold of dimension $2k$ of the symplectic manifold $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$. Then, by the Darboux-Givental Theorem (see [1]), S is symplectomorphic to

$$S_0 = \{(x,y) \in \mathbb{R}^{2n} | x_i = y_i = 0 \text{ for } i = k+1, \cdots, n\}.$$

If $(\tilde{x}, \tilde{y}) = (x_1, \cdots, x_k, y_1, \cdots, y_k)$ and $\iota : S \ni (\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, 0, \tilde{y}, 0) \in \mathbb{R}^{2n}$, then a smooth vector field-germ

$$X = \sum_{i=1}^{n} f_i(x,y) \frac{\partial}{\partial x_i} + g_i(x,y) \frac{\partial}{\partial y_i}$$

at 0 on $\mathbb{R}^{2n}$ is Hamiltonian on S_0 if $d(\iota^*(X \rfloor \omega) = 0$.

It implies that $d(\sum_{i=1}^{k} f_i(\tilde{x}, 0, \tilde{y}, 0) dy_i - g_i(\tilde{x}, 0, \tilde{y}, 0) dx_i) = 0$. Thus, the vector field-germ

$$\tilde{X} = \sum_{i=1}^{k} f_i(\tilde{x}, 0, \tilde{y}, 0)) \frac{\partial}{\partial x_i} + g_i(\tilde{x}, 0, \tilde{y}, 0)) \frac{\partial}{\partial y_i}$$

on S_0 is Hamiltonian on a symplectic manifold $(S_0, \iota^* \omega = \sum_{i=1}^{k} dx_i \wedge dy_i)$. Let us notice that $\tilde{X}|_{\pi(x,y)} = \pi_*(X|_{(x,y)})$, where $\pi : \mathbb{R}^{2n} \ni (x,y) \mapsto (\tilde{x}, \tilde{y}) \in S_0$. Since $\pi \circ \iota = Id_{S_0}$, we obtain the following proposition:

Proposition 12. *A smooth vector field-germ X on $(\mathbb{R}^{2n}, \omega)$ is Hamiltonian on the symplectic submanifold-germ S_0, if the vector field-germ $\pi_*(X \circ \iota)$ on S_0 is Hamiltonian on the symplectic manifold $(S_0, \iota^* \omega)$.*

4.2. COISOTROPIC SUBMANIFOLDS

Let C be the germ of a coisotropic submanifold of codimension k of the symplectic manifold $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$. Then, by the Darboux-Givental Theorem (see [1]), C is symplectomorphic to

$$C_0 = \{(x,y) \in \mathbb{R}^{2n} | x_i = 0 \text{ for } i = 1, \cdots, k\}.$$

If $\tilde{x} = (x_{k+1}, \cdots, x_n)$ and $\iota : C_0 \ni (\tilde{x}, y) \mapsto (0, \tilde{x}, y) \in \mathbb{R}^{2n}$, then a smooth vector field-germ

$$X = \sum_{i=1}^{n} f_i(x,y) \frac{\partial}{\partial x_i} + g_i(x,y) \frac{\partial}{\partial y_i}$$

at 0 on $\mathbb{R}^{2n}$ is Hamiltonian on C_0 if $d(\iota^*(X \rfloor \omega) = 0$. It implies that the 1-form-germ

$$\iota^*(X \rfloor \omega) = \sum_{i=1}^{n} f_i(0, \tilde{x}, y) dy_i - \sum_{i=k+n}^{k} g_i(0, \tilde{x}, y) dx_i$$

on C_0 is exact. Hence, there exists a smooth function-germ on C_0 such that $g_i(0,\tilde{x},y) = -\frac{\partial h}{\partial x_i}(\tilde{x},y)$ for $i = k+1,\cdots,n$ and $f_i(0,\tilde{x},y) = \frac{\partial h}{\partial y_i}(\tilde{x},y)$ for $i = 1,\cdots,n$. Thus, we obtain the following proposition:

Proposition 13. *A smooth vector field-germ*

$$X = \sum_{i=1}^{n} f_i(x,y)\frac{\partial}{\partial x_i} + g_i(x,y)\frac{\partial}{\partial y_i}$$

on $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$ *is Hamiltonian on the coisotropic submanifold-germ*

$$C_0 = \{(x,y) \in \mathbb{R}^{2n} |\ x_i = 0\, for\ i = 1,\cdots,k\},$$

if there exists a smooth function germ h *on* C_0 *such that* $g_i(0,\tilde{x},y) = -\frac{\partial h}{\partial x_i}(\tilde{x},y)$ *for* $i = k+1,\cdots,n$ *and* $f_i(0,\tilde{x},y) = \frac{\partial h}{\partial y_i}(\tilde{x},y)$ *for* $i = 1,\cdots,n$.

4.3. ISOTROPIC SUBMANIFOLDS

Let I be the germ of an isotropic submanifold of dimension k of the symplectic manifold $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$. Then, by the Darboux-Givental Theorem (see [1]), I is symplectomorphic to

$$I_0 = \{(x,y) \in \mathbb{R}^{2n} | y = 0,\ x_i = 0 \text{ for } i = k+1,\cdots,n\}.$$

If $\tilde{x} = (x_1,\cdots,x_k)$ and $\iota : I_0 \ni \tilde{x} \mapsto (\tilde{x},0) \in \mathbb{R}^{2n}$, then a smooth vector field-germ

$$X = \sum_{i=1}^{n} f_i(x,y)\frac{\partial}{\partial x_i} + g_i(x,y)\frac{\partial}{\partial y_i}$$

at 0 on $\mathbb{R}^{2n}$ is Hamiltonian on I_0 if $d(\iota^*(X\rfloor\omega) = 0$. It implies that the 1-form-germ

$$\iota^*(X\rfloor\omega) = -\sum_{i=1}^{k} g_i(\tilde{x},0)dx_i$$

on I_0 is exact. Hence, there exists a smooth function-germ on I_0 such that $g_i(\tilde{x},0) = \frac{\partial h}{\partial x_i}(\tilde{x})$ for $i = 1,\cdots,k$. Thus, we obtain the following proposition:

Proposition 14. *A smooth vector field-germ*

$$X = \sum_{i=1}^{n} f_i(x,y)\frac{\partial}{\partial x_i} + g_i(x,y)\frac{\partial}{\partial y_i}$$

on $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$ *is Hamiltonian on the isotropic submanifold*

$$I_0 = \{(x,y) \in \mathbb{R}^{2n} | y = 0,\ x_i = 0\, for\ i = k+1,\cdots,n\},$$

if there exists a smooth function-germ h *on* I_0 *such that* $g_i(\tilde{x},0) = \frac{\partial h}{\partial x_i}(\tilde{x})$ *for* $i = 1,\cdots,k$.

In particular by Proposition 11 we obtain

Corollary 15. *If C is a regular curve (1-dimensional smooth submanifold), then any smooth vector field-germ on $\mathbb{R}^{2n}$ is Hamiltonian.*

Proof. Any smooth 1-form on C is closed. □

5. GERMS OF HAMILTONIAN VECTOR FIELDS ON SINGULAR CURVES

In this section we describe germs of Hamiltonian vector fields on singular curves at a singular point. By Corollary 15 any smooth vector field-germ is Hamiltonian on a regular curve.

PLANAR CURVES OF TYPES A_K, D_K, E_6, E_7, E_8

A planar curve in the symplectic space $(\mathbb{R}^{2n}, \omega)$ is a curve which is embedded in a smooth 2-dimensional submanifold S of $(\mathbb{R}^{2n}, \omega)$. Let $\iota : S \hookrightarrow \mathbb{R}^{2n}$ be an embedding of S.

We assume that the germ of the curve is locally diffeomorphic to $N = \{x \in \mathbb{R}^{2n} | G(x_1, x_2) = x_{\geq 3} = 0\}$, where G has the following properties:

1. $G(0,0) = 0$, $dG(0,0) = 0$,
2. the ideal of smooth function-germs on $\mathbb{R}^2$ vanishing on $\{(x_1,x_2) \in \mathbb{R}^2 | G(x_1,x_2) = 0\}$ is generated by G.
3. G is quasi-homogeneous polynomial.

Then, we can take locally $S = \{x \in \mathbb{R}^{2n} | x_{\geq 3} = 0\}$ and $\iota(x_3, \cdots, x_{2n}) = (0,0,x_3,\cdots,x_{2n})$. A smooth vector field-germ on $(\mathbb{R}^{2n}, \omega)$ is Hamiltonian on N if and only if $d(\iota^*(X\rfloor\omega)) = d(G(x_1,x_2)\alpha)$ for some smooth 1-form-germ α on $\mathbb{R}^2$.

By Theorem 4.11 in [5] any curve-germ in the symplectic space $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=0}^{n} dp_i \wedge dq_i)$, $n \geq 2$, which is diffeomorphic to the curve-germ at 0 $\{x \in \mathbb{R}^{2n} | G(x_1,x_2) = x_{\geq 3} = 0\}$ for smooth function-germs G in Tab. 1 is symplectomorphic to one and only one of the following curve-germs:

$$N^i = \{(p,q) \in \mathbb{R}^{2n} | G(p_1,p_2) = q_1 - \int_0^{p_2} F_i(p_1,t)dt = q_{\geq 2} = p_{\geq 3} = 0\} \subset (\mathbb{R}^{2n}, \omega_0), \quad (12)$$

for $i = 0, \cdots, \mu$, where smooth function-germs F_i are presented in Tab. 1.

Table 1

Classification of the algebraic restrictions to A_k, D_k, E_6, E_7, E_8

$G(x_1,x_2)$	$F_i(x_1,x_2),\ i=0,1,\ldots,\mu$
$A_k : x_1^{k+1}-x_2^2$ $k\geq 1$	$F_0=1$ $F_i=x_1^i,\ i=1,\ldots,k-1$ $F_k=0$
$D_k : x_1^2x_2-x_2^{k-1}$ $k\geq 4$	$F_0=1$ $F_i=bx_1+x_2^i,\ \ i=1,\ldots,k-4$ $F_{k-3}=(\pm 1)^k x_1+bx_2^{k-3}$, $F_{k-2}=x_2^{k-3},\ F_{k-1}=x_2^{k-2},\ F_k=0$
$E_6 : x_1^3-x_2^4$	$F_0=1,\ F_1=\pm x_2+bx_1,\ F_2=x_1+bx_2^2$, $F_3=x_2^2+bx_1x_2,\ F_4=\pm x_1x_2,\ F_5=x_1x_2^2,\ F_6=0$
$E_7 : x_1^3-x_1x_2^3$	$F_0=1,\ F_1=x_2+bx_1,\ F_2=\pm x_1+bx_2^2$, $F_3=x_2^2+bx_1x_2,\ F_4=\pm x_1x_2+bx_2^3$, $F_5=x_2^3,\ F_6=x_2^4,\ F_7=0$
$E_8 : x_1^3-x_2^5$	$F_0=\pm 1,\ F_1=x_2+bx_1,\ F_2=x_1+b_1x_2^2+b_2x_2^3$ $F_3=\pm x_2^2+bx_1x_2,\ F_4=\pm x_1x_2+bx_2^3$, $F_5=x_2^3+bx_1x_2^2,\ F_6=x_1x_2^2,\ F_7=\pm x_1x_2^3,\ F_8=0$

Let $\iota : \mathbb{R}^2 \to \mathbb{R}^{2n}$ be the following map-germ: $\iota(p_1,p_2)=(p_1,\int_0^{p_2}F_i(p_1,t)dt,p_2,0)$. A smooth vector field-germ

$$X=\sum_{i=1}^{n} f_i(p_1,q_1,\cdots,p_n,q_n)\frac{\partial}{\partial p_i}+g_i(p_1,q_1,\cdots,p_n,q_n)\frac{\partial}{\partial q_i}$$

on $(\mathbb{R}^{2n},\omega_0=\sum_{i=1}^n dp_i\wedge dq_i)$ is Hamiltonian on N^i if a smooth 2-form-germ at 0 on $\mathbb{R}^2$

$$\sigma=r(p_1,p_2)dp_1\wedge dp_2=d\left((f_1\circ\iota)d(\int_0^{p_2}F_i(p_1,t)dt)-(g_1\circ\iota)dp_1-(g_2\circ\iota)dp_2\right)$$

has zero algebraic restriction to $\{(p_1,p_2)\in\mathbb{R}^2 | G(p_1,p_2)=0\}$.

By the direct calculation we obtain that

$$\begin{aligned} r(p_1,p_2)=&\left(\tfrac{\partial g_1}{\partial p_2}-\tfrac{\partial g_2}{\partial p_1}\right)(p_1,\textstyle\int_0^{p_2}F_i(p_1,t)dt,p_2,0) \\ &+F_i(p_1,p_2)\left(\tfrac{\partial f_1}{\partial p_1}+\tfrac{\partial g_1}{\partial q_1}\right)(p_1,\textstyle\int_0^{p_2}F_i(p_1,t)dt,p_2,0) \\ &-\textstyle\int_0^{p_2}\tfrac{\partial F_i}{\partial p_1}(p_1,t)dt\left(\tfrac{\partial f_1}{\partial p_2}+\tfrac{\partial g_2}{\partial q_1}\right)(p_1,\textstyle\int_0^{p_2}F_i(p_1,t)dt,p_2,0). \end{aligned} \tag{13}$$

If G is quasi-homogeneous, then a smooth 2-form $r(p_1,p_2)dp_1\wedge dp_2$ has zero algebraic restriction to $\{(p_1,p_2)\in\mathbb{R}^2|G(p_1,p_2)=0\}$ if and only if r belongs to the ideal $<\nabla G>$ generated by $\frac{\partial G}{\partial p_1}(p_1,p_2)$, $\frac{\partial G}{\partial p_2}(p_1,p_2)$ (see [5]).

Thus, we obtain the following proposition:

Proposition 16. *A smooth vector field-germ*

$$X=\sum_{j=1}^{n} f_j(p,q)\frac{\partial}{\partial p_j}+g_j(p,q)\frac{\partial}{\partial q_j}$$

is Hamiltonian on

$$N^i=\{(p,q)\in\mathbb{R}^{2n}|G(p_1,p_2)=q_1-\int_0^{p_2}F_i(p_1,t)dt=q_{\geq 2}=p_{\geq 3}=0\}\subset(\mathbb{R}^{2n},\omega_0),$$

where G and F_i are presented in Tab. 1, if and only if the function-germ r given by (13) belongs to the ideal $<\nabla G>$.

5.1. PLANAR CURVES OF TYPES A_K^I

By Proposition 16 we obtain the following:

Proposition 17. *Let us fix $k\in\mathbb{N}$ and $i=0,1,\cdots,k$. A smooth vector field-germ*

$$X=\sum_{j=1}^{n} f_j(p,q)\frac{\partial}{\partial p_j}+g_j(p,q)\frac{\partial}{\partial q_j}$$

on $(\mathbb{R}^{2n},\omega_0=\sum_{j=1}^{n}dp_j\wedge dq_j)$ is Hamiltonian on

$$A_k^i=\{(p,q)\in\mathbb{R}^{2n}|p_1^{k+1}-p_2^2=q_1-p_1^ip_2=q_{\geq 2}=p_{\geq 3}=0\}\ (i=0,1,\cdots,k-1)$$

or on

$$A_k^k=\{(p,q)\in\mathbb{R}^{2n}|p_1^{k+1}-p_2^2=q_{\geq 1}=p_{\geq 3}=0\}$$

if and only if the following conditions are satisfied:

$$\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)=\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0)\ \mathit{for}\ j=0,\cdots,i-1,$$

$$\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)=\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0)-\frac{j!}{(j-i)!}\left(\frac{\partial^{j-i+1}f_1}{\partial p_1^{j-i+1}}(0)+\frac{\partial^{j-i+1}g_2}{\partial p_1^{j-i}\partial q_1}(0)\right)\ \mathit{for}\ j=i,\cdots,k-1.$$

Proof. For a planar curve of type A_k^i $(k\geq 1, i=0,\cdots,k)$, we have that $G(p_1,p_2)=p_1^{k+1}-p_2^2$ and $F_i(p_1,p_2)=p_1^i$ for $i=0,\cdots,k-1$ and $F_k(p_1,p_2)=0$ (see Tab. 1).

For A_k^0 singularity we have

$$r(p_1,p_2)=\left(\frac{\partial g_1}{\partial p_2}-\frac{\partial g_2}{\partial p_1}+\frac{\partial f_1}{\partial p_1}+\frac{\partial g_1}{\partial q_1}\right)(p_1,p_2,p_2,0).$$

For A_k^i singularity $i=1,\cdots,k-1$ the function-germ r has the following form:

$$\begin{aligned} r(p_1,p_2) = & \left(\tfrac{\partial g_1}{\partial p_2}-\tfrac{\partial g_2}{\partial p_1}\right)(p_1,p_1^i p_2,p_2,0) \\ & +p_1^i\left(\tfrac{\partial f_1}{\partial p_1}+\tfrac{\partial g_1}{\partial q_1}\right)(p_1,p_1^i p_2,p_2,0) \\ & -ip_1^{i-1}p_2\left(\tfrac{\partial f_1}{\partial p_2}+\tfrac{\partial g_2}{\partial q_1}\right)(p_1,p_1^i p_2,p_2,0). \end{aligned} \tag{14}$$

For A_k^k singularity we get $F_k(p_1,p_2)=0$ and

$$r(p_1,p_2)=\left(\frac{\partial g_1}{\partial p_2}-\frac{\partial g_2}{\partial p_1}\right)(p_1,0,p_2,0).$$

Since $<\nabla G>=<p_1^k,p_2>$, it is easy to see that $\mathcal{O}_2/<\nabla G>\cong\mathbb{R}\left\{1,p_1,\cdots,p_1^{k-1}\right\}$. The function-germ r belongs to $<\nabla G>$ if and only if $\frac{\partial^j r}{\partial p_1^j}(0,0)=0$ for $j=0,1,\cdots,k-1$. By a direct calculation we get that for $j=0,\cdots,i-1$

$$\frac{\partial^j r}{\partial p_1^j}(0,0)=\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)-\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0),$$

and for $j=i,\cdots,k-1$

$$\frac{\partial^j r}{\partial p_1^j}(0,0)=\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)-\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0)+\frac{j!}{(j-i)!}\left(\frac{\partial^{j-i+1}f_1}{\partial p_1^{j-i+1}}(0)+\frac{\partial^{j-i+1}g_2}{\partial p_1^{j-i}\partial q_1}(0)\right).$$

□

5.2. PLANAR CURVES OF TYPES D_K^I

For a planar curve of type D_k^i ($k\geq 4$, $i=0,\cdots,k$) we have that $G(p_1,p_2)=p_1^2p_2-p_2^{k-1}$. Then it is easy to see that $<\nabla G>=<p_1p_2,p_1^2-(k-1)p_2^{k-2}>$ and

$$\mathcal{O}_2/<\nabla G>\cong\mathbb{R}\left\{1,p_2,\cdots,p_2^{k-2},p_1\right\}.$$

For D_k^0 singularity we get $F_0(p_1,p_2)=1$ and

$$r(p_1,p_2)=\left(\frac{\partial g_1}{\partial p_2}-\frac{\partial g_2}{\partial p_1}+\frac{\partial f_1}{\partial p_1}+\frac{\partial g_1}{\partial q_1}\right)(p_1,p_2,p_2,0).$$

For D_k^i singularity $i=1,\cdots,k-4$ we get $F_i(p_1,p_2)=bp_1+p_2^i$ and

$$\begin{aligned} r(p_1,p_2) = & \left(\tfrac{\partial g_1}{\partial p_2}-\tfrac{\partial g_2}{\partial p_1}\right)(p_1,bp_1p_2+\tfrac{p_2^{i+1}}{i+1},p_2,0) \\ & +(bp_1+p_2^i)\left(\tfrac{\partial f_1}{\partial p_1}+\tfrac{\partial g_1}{\partial q_1}\right)(p_1,bp_1p_2+\tfrac{p_2^{i+1}}{i+1},p_2,0) \\ & -bp_2\left(\tfrac{\partial f_1}{\partial p_2}+\tfrac{\partial g_2}{\partial q_1}\right)(p_1,bp_1p_2+\tfrac{p_2^{i+1}}{i+1},p_2,0). \end{aligned} \tag{15}$$

For D_k^{k-3} singularity we have $F_{k-3}(p_1,p_2)=(\pm 1)^k p_1+bp_2^{k-3}$ and

$$\begin{aligned} r(p_1,p_2)=&\left(\tfrac{\partial g_1}{\partial p_2}-\tfrac{\partial g_2}{\partial p_1}\right)(p_1,(\pm 1)^k p_1p_2+b\tfrac{p_2^{k-2}}{k-2},p_2,0) \\ &+((\pm 1)^k p_1+bp_2^{k-3})\left(\tfrac{\partial f_1}{\partial p_1}+\tfrac{\partial g_1}{\partial q_1}\right)(p_1,(\pm 1)^k p_1p_2+b\tfrac{p_2^{k-2}}{k-2},p_2,0) \\ &-(\pm 1)^k p_2\left(\tfrac{\partial f_1}{\partial p_2}+\tfrac{\partial g_2}{\partial q_1}\right)(p_1,(\pm 1)^k p_1p_2+b\tfrac{p_2^{k-2}}{k-2},p_2,0). \end{aligned} \tag{16}$$

For D_k^{k-2} singularity we get $F_{k-2}(p_1,p_2)=p_2^{k-3}$ and

$$r(p_1,p_2)=\left(\frac{\partial g_1}{\partial p_2}-\frac{\partial g_2}{\partial p_1}\right)(p_1,\frac{p_2^{k-2}}{k-2},p_2,0)+p_2^{k-3}\left(\frac{\partial f_1}{\partial p_1}+\frac{\partial g_1}{\partial q_1}\right)(p_1,\frac{p_2^{k-2}}{k-2},p_2,0).$$

For D_k^{k-1} singularity we get $F_{k-2}(p_1,p_2)=p_2^{k-2}$ and

$$r(p_1,p_2)=\left(\frac{\partial g_1}{\partial p_2}-\frac{\partial g_2}{\partial p_1}\right)(p_1,\frac{p_2^{k-1}}{k-1},p_2,0)+p_2^{k-2}\left(\frac{\partial f_1}{\partial p_1}+\frac{\partial g_1}{\partial q_1}\right)(p_1,\frac{p_2^{k-1}}{k-1},p_2,0).$$

For D_k^k singularity we get $F_k(p_1,p_2)=0$ and

$$r(p_1,p_2)=\left(\frac{\partial g_1}{\partial p_2}-\frac{\partial g_2}{\partial p_1}\right)(p_1,0,p_2,0).$$

The function-germ r belongs to $<\nabla G>=<p_1p_2,p_1^2-(k-1)p_2^{k-2}>$ if and only if

$$\frac{\partial^j r}{\partial p_2^j}(0,0)=\frac{\partial r}{\partial p_1}(0,0)=0 \ \text{ for } j=0,1,\cdots,k-3 \tag{17}$$

and

$$\frac{\partial^2 r}{\partial p_1^2}(0,0)=\frac{2}{(k-1)!}\frac{\partial^{k-2} r}{\partial p_2^{k-2}}(0,0). \tag{18}$$

For general k conditions (17)-(18) are rather complicated in terms of partial derivatives of coefficient functions f_1, g_1, g_2 at 0. Therefore, we will present them only for D_4^i singularities for $i=0,1,\cdots,4$.

Let $X=\sum_{j=1}^n f_j(p,q)\frac{\partial}{\partial p_j}+g_j(p,q)\frac{\partial}{\partial q_j}$ be the smooth vector field-germ on $(\mathbb{R}^{2n},\omega_0=\sum_{j=1}^n dp_j\wedge dq_j)$. By a direct calculation Proposition 16 implies the following:

The vector field-germ X is Hamiltonian on

$$D_4^0=\{(p,q)\in\mathbb{R}^{2n}|p_1^2p_2-p_2^3=q_1-p_2=q_{\geq 2}=p_{\geq 3}=0\}$$

if and only if the following conditions are satisfied:

$$\frac{\partial^{j+1} g_2}{\partial p_1^{j+1}}(0) = \frac{\partial^{j+1} g_1}{\partial p_1^j \partial p_2}(0) + \frac{\partial^{j+1} g_1}{\partial p_1^j \partial q_1}(0) + \frac{\partial^{j+1} f_1}{\partial p_1^{j+1}}(0) \ \text{ for } j = 0, 1,$$

$$\frac{\partial^2 g_1}{\partial p_2^2}(0) + 2\frac{\partial^2 g_1}{\partial q_1 \partial p_2}(0) + \frac{\partial^2 g_1}{\partial q_1^2}(0) + \frac{\partial^2 f_1}{\partial p_1 \partial p_2}(0)$$
$$-\frac{\partial^2 g_2}{\partial p_1 \partial p_2}(0) + \frac{\partial^2 f_1}{\partial p_1 \partial q_1}(0) - \frac{\partial^2 g_2}{\partial p_1 \partial q_1}(0) = 0,$$

$$3\left(\frac{\partial^3 g_1}{\partial p_1^2 \partial p_2}(0) + \frac{\partial^3 g_1}{\partial p_1^2 \partial q_1}(0) + \frac{\partial^3 f_1}{\partial p_1^3}(0) - \frac{\partial^3 g_2}{\partial p_1^3}(0)\right)$$
$$= \frac{\partial^3 g_1}{\partial p_2^3}(0) + 3\frac{\partial^3 g_1}{\partial q_1 \partial p_2^2}(0) + 3\frac{\partial^3 g_1}{\partial q_1^2 \partial p_2}(0) + \frac{\partial^3 g_1}{\partial q_1^3}(0) + \frac{\partial^3 f_1}{\partial p_1 \partial p_2^2}(0) - \frac{\partial^3 g_2}{\partial p_1 \partial p_2^2}(0)$$
$$+2\frac{\partial^3 f_1}{\partial p_1 \partial q_1 \partial p_2}(0) - 2\frac{\partial^3 g_2}{\partial p_1 \partial q_1 \partial p_2}(0) + \frac{\partial^3 f_1}{\partial p_1 \partial q_1^2}(0) - \frac{\partial^3 g_2}{\partial p_1 \partial q_1^2}(0).$$

The vector field-germ X is Hamiltonian on

$$D_4^1 = \{(p,q) \in \mathbb{R}^{2n} | p_1^2 p_2 - p_2^3 = q_1 - p_1 p_2 - b\frac{p_2^2}{2} = q_{\geq 2} = p_{\geq 3} = 0\}$$

if and only if the following conditions are satisfied:

$$\frac{\partial g_1}{\partial p_2}(0) = \frac{\partial g_2}{\partial p_1}(0),$$

$$\frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) + \frac{\partial^2 g_1}{\partial p_1 \partial p_2}(0) - \frac{\partial^2 g_2}{\partial p_1^2}(0) = 0,$$

$$-\frac{\partial f_1}{\partial p_2}(0) + \frac{\partial^2 g_1}{\partial p_2^2}(0) - \frac{\partial g_2}{\partial q_1}(0) + b\left(\frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0)\right) - \frac{\partial^2 g_2}{\partial p_1 \partial p_2}(0) = 0,$$

$$6\left(\frac{\partial^2 g_1}{\partial p_1 \partial q_1}(0) + \frac{\partial^2 f_1}{\partial p_1^2}(0)\right) + 3\left(\frac{\partial^3 g_1}{\partial p_1^2 \partial p_2}(0) - \frac{\partial^3 g_2}{\partial p_1^3}(0)\right)$$
$$= \frac{\partial^3 g_1}{\partial p_2^3}(0) - \frac{\partial^3 g_2}{\partial p_1 \partial p_2^2}(0) - 2\left(\frac{\partial^2 f_1}{\partial p_2^2}(0) + \frac{\partial^2 g_2}{\partial q_1 \partial p_2}(0)\right)$$
$$+b\left(3\frac{\partial^2 g_1}{\partial p_1 \partial p_2}(0) + 2\frac{\partial^2 f_1}{\partial p_1 \partial p_2}(0) - \frac{\partial g_2}{\partial p_1}\partial q_1(0)\right).$$

The vector field-germ X is Hamiltonian on

$$D_4^2=\{(p,q)\in\mathbb{R}^{2n}|p_1^2p_2-p_2^3=q_1-\frac{p_2^2}{2}=q_{\geq 2}=p_{\geq 3}=0\}$$

if and only if the following conditions are satisfied:

$$\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)=\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0) \text{ for } j=0,1,$$

$$\frac{\partial^2 g_1}{\partial p_2^2}(0)+\frac{\partial g_1}{\partial q_1}(0)+\frac{\partial f_1}{\partial p_1}(0)-\frac{\partial^2 g_2}{\partial p_1\partial p_2}(0)=0,$$

$$3\left(\frac{\partial^3 g_1}{\partial p_1^2\partial p_2}(0)-\frac{\partial^3 g_2}{\partial p_1^3}(0)\right)=$$

$$\frac{\partial^3 g_1}{\partial p_2^3}(0)+\frac{\partial^2 g_1}{\partial q_1\partial p_2}(0)+2\left(\frac{\partial^2 g_1}{\partial q_1\partial p_2}(0)+\frac{\partial^2 f_1}{\partial p_1\partial p_2}(0)\right)-\frac{\partial^3 g_2}{\partial p_1\partial p_2^2}(0)-\frac{\partial^2 g_2}{\partial p_1\partial q_1}(0).$$

The vector field-germ X is Hamiltonian on

$$D_4^3=\{(p,q)\in\mathbb{R}^{2n}|p_1^2p_2-p_2^3=q_1-\frac{p_2^3}{3}=q_{\geq 2}=p_{\geq 3}=0\}$$

if and only if the following conditions are satisfied:

$$\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)=\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0) \text{ for } j=0,1,$$

$$\frac{\partial^2 g_1}{\partial p_2^2}(0)=\frac{\partial^2 g_2}{\partial p_1\partial p_2}(0),$$

$$3\left(\frac{\partial^3 g_1}{\partial p_1^2\partial p_2}(0)-\frac{\partial^3 g_2}{\partial p_1^3}(0)\right)=\frac{\partial^3 g_1}{\partial p_2^3}(0)+2\left(\frac{\partial g_1}{\partial q_1}(0)+\frac{\partial f_1}{\partial p_1}(0)\right)-\frac{\partial^3 g_2}{\partial p_1\partial p_2^2}(0).$$

The vector field-germ X is Hamiltonian on

$$D_4^4=\{(p,q)\in\mathbb{R}^{2n}|p_1^2p_2-p_2^3=q_{\geq 1}=p_{\geq 3}=0\}$$

if and only if the following conditions are satisfied:

$$\frac{\partial^{j+1}g_1}{\partial p_1^j\partial p_2}(0)=\frac{\partial^{j+1}g_2}{\partial p_1^{j+1}}(0) \text{ for } j=0,1,$$

$$\frac{\partial^2 g_1}{\partial p_2^2}(0) = \frac{\partial^2 g_2}{\partial p_1 \partial p_2}(0),$$

$$3\left(\frac{\partial^3 g_1}{\partial p_1^2 \partial p_2}(0) - \frac{\partial^3 g_2}{\partial p_1^3}(0)\right) = \frac{\partial^3 g_1}{\partial p_2^3}(0) - \frac{\partial^3 g_2}{\partial p_1 \partial p_2^2}(0).$$

In the same way by Proposition 16 one can obtain the necessary and sufficient conditions for the vector field-germ X to be Hamiltonian on planar curves with E_k^i singularities for $k = 6, 7, 8$ and $i = 0, 1, \cdots, k$ (see Tab. 1). Please notice that for E_k^i singularity there are k independent conditions, therefore we do not present them.

6. GERMS OF HAMILTONIAN VECTOR FIELDS ON REGULAR UNION SINGULARITIES.

A regular union singularity N at 0 in $\mathbb{R}^{2n}$ is the union

$$N = N_1 \cup \cdots \cup N_s,\ s \geq 2 \tag{19}$$

of germs at 0 of smooth submanifolds $N_1, \cdots, N_s$ of $\mathbb{R}^{2n}$ (in what follows - strata) such that the dimension of the space

$$W = T_0 N_1 + \cdots + T_0 N_s \tag{20}$$

is equal to the sum of the dimensions of the strata, i.e. the sum (20) is direct. If the number of strata and their dimensions are fixed, then all such N are diffeomorphic. By Theorem 7.1 in [5] the germ of a closed 2-form σ has zero algebraic restriction to N if and only if its pullback to each of the strata N_i ($i = 1, \cdots, s$) vanishes and the restriction of the germ σ to the space W vanishes. It implies the following:

Proposition 18. *A smooth vector field-germ X in the symplectic space* $(\mathbb{R}^{2n}, \omega)$ *is Hamiltonian on a regular union singularity N if and only if the pullback of the germ* $d(X\rfloor\omega)$ *to each of the strata* N_i *(*$i = 1, \cdots, s$*) vanishes and the restriction of the germ* $d(X\rfloor\omega)$ *to the space W vanishes.*

6.1. REGULAR UNION OF THREE 1-DIMENSIONAL SUBMANIFOLDS

Let us consider a regular union singularity of three germs at 0 of 1-dimensional submanifolds $N = N_1 \cup N_2 \cup N_3$ of the symplectic space $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^n dp_i \wedge dq_i)$. These symplectic singularities are classified in [5].

Proposition 19 (Theorem 7.4 in [5]). *Any regular union singularity N with three 1-dimensional strata in the symplectic space $(\mathbb{R}^{2n},\omega)$, $n \geq 3$ (resp. $n=2$) is symplectomorphic to one and only one of the varieties N^0, N^1, N^2, N^3 (resp. N^0, N^1, N^2) given in Tab. 2. It holds if and only if the pair (ω, N) satisfies the condition in the last column of the table.*

Table 2

Classification of symplectic regular union singularities with three 1-dimensional strata. W denotes the 3-space spanned by the tangent lines at 0 to the strata

	Symplectic normal forms	Geometric condition
N^0	$q_2 = p_1 + p_2$, $p_1q_1 = q_1p_2 = p_2q_2 = 0$, $p_{\geq 3} = q_{\geq 3} = 0$	$\omega\|_W \neq 0$, $\ker\omega\|_W \not\subset T_0N_i + T_0N_j$, for any $i,j \in \{1,2,3\}$;
N^1	$q_2 = p_1$, $p_1q_1 = q_1p_2 = p_2p_1 = 0$, $p_{\geq 3} = q_{\geq 3} = 0$	$\omega\|_W \neq 0$, $\ker\omega\|_W \subset T_0N_i + T_0N_j$, $\ker\omega\|_W \neq T_0N_i, T_0N_j$ for some $i,j \in \{1,2,3\}$;
N^2	$p_1q_1 = q_1p_2 = p_2p_1 = 0$, $p_{\geq 3} = q_{\geq 2} = 0$	$\omega\|_W \neq 0$, $\ker\omega\|_W = T_0N_i$ for some $i \in \{1,2,3\}$
N^3	$p_1p_2 = p_2p_3 = p_3p_1 = 0$, $p_{\geq 4} = q_{\geq 1} = 0$	$\omega\|_W = 0$.

Since the strata are 1-dimensional, by Proposition 18, a smooth vector-field germ X is Hamiltonian on N if and only if $d(X\rfloor\omega)|_W = 0$. Hence for singularities N^i for $i = 0,1,\cdots,3$ we obtain the following conditions:

Let $X = \sum_{i=1}^n f_i(p,q)\frac{\partial}{\partial p_i} + g_i(p,q)\frac{\partial}{\partial q_i}$ be a smooth vector field-germ on $\mathbb{R}^{2n}$.
The vector field-germ X is Hamiltonian on N^0 if and only if

$$\frac{\partial f_1}{\partial q_2}(0) + \frac{\partial f_1}{\partial p_2}(0) - \frac{\partial f_2}{\partial q_1}(0) + \frac{\partial g_2}{\partial q_1}(0) = 0,$$

$$\frac{\partial f_2}{\partial q_1}(0) + \frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) + \frac{\partial f_2}{\partial q_1}(0) = 0,$$

$$\frac{\partial g_1}{\partial q_2}(0) - \frac{\partial g_2}{\partial q_2}(0) - \frac{\partial f_2}{\partial p_2}(0) + \frac{\partial g_1}{\partial p_2}(0) + \frac{\partial f_2}{\partial p_1}(0) - \frac{\partial g_2}{\partial p_1}(0) = 0.$$

The vector field-germ X is Hamiltonian on N^1 if and only if

$$\frac{\partial f_1}{\partial p_2}(0) + \frac{\partial g_2}{\partial q_1}(0) = \frac{\partial f_1}{\partial q_2}(0) + \frac{\partial f_2}{\partial q_1}(0) + \frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) = 0,$$

$$\frac{\partial g_1}{\partial p_2}(0)-\frac{\partial g_2}{\partial q_2}(0)-\frac{\partial f_2}{\partial p_2}(0)-\frac{\partial g_2}{\partial p_1}(0)=0.$$

The vector field-germ X is Hamiltonian on N^2 if and only if

$$\frac{\partial f_1}{\partial p_2}(0)+\frac{\partial g_2}{\partial q_1}(0)=\frac{\partial g_1}{\partial q_1}(0)+\frac{\partial f_1}{\partial p_1}(0)=\frac{\partial g_1}{\partial p_2}(0)-\frac{\partial g_2}{\partial p_1}(0)=0.$$

The vector field-germ X is Hamiltonian on N^3 if and only if

$$\frac{\partial g_2}{\partial p_3}(0)-\frac{\partial g_3}{\partial p_2}(0)=\frac{\partial g_1}{\partial p_2}(0)-\frac{\partial g_2}{\partial p_1}(0)=\frac{\partial g_1}{\partial p_3}(0)-\frac{\partial g_3}{\partial p_1}(0)=0.$$

6.2. REGULAR UNION OF TWO 2-DIMENSIONAL ISOTROPIC SUBMANIFOLDS

Now we consider the regular union singularity of two 2-dimensional isotropic submanifold-germs of the symplectic space. The following classification proposition was proved in [5]:

Proposition 20. *Any regular union singularity N of two 2-dimensional isotropic submanifold-germs in a symplectic space $(\mathbb{R}^{2n},\omega=\sum_{i=1}^{n}dp_i\wedge dq_i)$ is symplectomorphic to one and only one of the varieties N^0, N^1, N^4 in Tab. 3. The orbit of N^i has codimension i in the class of all regular union singularities with two 2-dimensional isotropic strata. The normal form N^i holds if and only if the pair (ω,N) satisfies the condition given in the last column of Tab. 3.*

Table 3

Classification of symplectic regular union singularities of two 2-dimensional isotropic submanifold-germs. W denotes the 4-space spanned by the tangent planes at 0 to the strata

	Symplectic normal forms	Geometric condition	codim
N^0	$\{p_{\geq 3}=q_{\geq 1}=0\}\cup$ $\{p_{\geq 1}=q_{\geq 3}=0\}$	rank $\omega\vert_W=4$	0
N^1 ($n\geq 3$)	$\{p_{\geq 3}=q_{\geq 1}=0\}\cup$ $\{p_{\geq 1}=q_2=q_{\geq 4}=0\}$	rank $\omega\vert_W=2$	1
N^4 ($n\geq 4$)	$\{p_{\geq 3}=q_{\geq 1}=0\}\cup$ $\{p_1=p_2=p_{\geq 5}=q_{\geq 1}=0\}$	$\omega\vert_W=0$	4

By Proposition 18 a smooth vector field-germ X is Hamiltonian on N if and only if X is Hamiltonian on both of isotropic submanifold-germs N_1, N_2 and $d(X\rfloor\omega)|_W=0$.

Let $X=\sum_{i=1}^{n}f_i(p,q)\frac{\partial}{\partial p_i}+g_i(p,q)\frac{\partial}{\partial q_i}$ be a smooth vector field-germ on $\mathbb{R}^{2n}$. By Propositions 18 and 14 we obtain the following conditions:

The vector field-germ X is Hamiltonian on $N^0 = N_1^0 \cup N_2^0$ if and only if there exist a smooth function-germs h on $N_1^0 = \{p_{\geq 3} = q_{\geq 1} = 0\}$ and k on $N_2^0 = \{q_{\geq 3} = p_{\geq 1} = 0\}$ such that $g_i(p_1, p_2, 0) = \frac{\partial h}{\partial p_i}(p_1, p_2)$ and $f_i(0, q_1, q_2, 0) = \frac{\partial k}{\partial q_i}(q_1, q_2)$ for $i = 1, 2$, and

$$\frac{\partial g_2}{\partial q_2}(0) + \frac{\partial f_2}{\partial p_2}(0) = \frac{\partial f_1}{\partial p_2}(0) + \frac{\partial g_2}{\partial q_1}(0) = \frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) = \frac{\partial g_1}{\partial q_2}(0) + \frac{\partial f_2}{\partial p_1}(0) = 0.$$

The vector field-germ X is Hamiltonian on $N^1 = N_1^1 \cup N_2^1$ if and only if there exist a smooth function-germs h on $N_1^1 = \{p_{\geq 3} = q_{\geq 1} = 0\}$ and k on $N_2^1 = \{p_{\geq 1} = q_2 = q_{\geq 4} = 0\}$ such that $g_i(p_1, p_2, 0) = \frac{\partial h}{\partial p_i}(p_1, p_2)$ for $i = 1, 2$ and $f_j(0, q_1, 0, q_3, 0) = \frac{\partial k}{\partial q_j}(q_1, q_3)$ for $j = 1, 3$, and

$$\frac{\partial g_2}{\partial q_3}(0) + \frac{\partial f_3}{\partial p_2}(0) = \frac{\partial f_1}{\partial p_2}(0) + \frac{\partial g_2}{\partial q_1}(0) = \frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) = \frac{\partial g_1}{\partial q_3}(0) + \frac{\partial f_3}{\partial p_1}(0) = 0.$$

The vector field-germ X is Hamiltonian on $N^4 = N_1^4 \cup N_2^4$ if and only if there exist a smooth function-germs h on $N_1^4 = \{p_{\geq 3} = q_{\geq 1} = 0\}$ and k on $N_2^4 = \{p_1 = p_2 = p_{\geq 5} = q_{\geq 1} = 0\}$ such that $g_i(p_1, p_2, 0) = \frac{\partial h}{\partial p_i}(p_1, p_2)$ for $i = 1, 2$ and $g_j(0, p_3, p_4, 0) = \frac{\partial k}{\partial p_j}(p_3, p_4)$ for $j = 3, 4$, and

$$\frac{\partial g_2}{\partial p_3}(0) - \frac{\partial g_3}{\partial p_2}(0) = \frac{\partial g_2}{\partial p_4}(0) + \frac{\partial g_4}{\partial p_2}(0) = \frac{\partial g_1}{\partial p_3}(0) - \frac{\partial g_3}{\partial p_1}(0) = \frac{\partial g_1}{\partial p_4}(0) - \frac{\partial g_4}{\partial p_1}(0) = 0.$$

6.3. REGULAR UNION OF TWO 2-DIMENSIONAL SYMPLECTIC SUBMANIFOLDS

In this subsection we consider Hamiltonian vector field-germs on regular union singularities with two 2-dimensional *symplectic* strata in a symplectic space $(\mathbb{R}^{2n}, \omega)$. Recall that two germs of submanifolds N_1, N_2 of a symplectic space $(\mathbb{R}^{2n}, \omega)$ are called ω-orthogonal if $\omega(v, u) = 0$ for any vectors $v \in T_0N_1, u \in T_0N_2$. The symplectic classification of such N involves the following invariant:

Definition 21 (see Definition 7.6 in [5]). *The index of non-orthogonality between 2-dimensional symplectic submanifolds N_1 and N_2 of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is the number*

$$\alpha = \alpha(N_1, N_2) = 1 - \frac{(\omega \wedge \omega)(v_1, v_2, u_1, u_2)}{2 \cdot \omega(v_1, v_2) \cdot \omega(u_1, u_2)}$$

where v_1, v_2 is a basis of T_0N_1 and u_1, u_2 is a basis of T_0N_2.

It is easy to see that the index of non-orthogonality $\alpha(N_1, N_2)$ is well-defined, i.e. it does not depend on the choice of the bases of T_0N_1 and T_0N_2. It is equal to 0 if and only if there exists a non-zero vector $u \in T_0N_1$ such that $\omega(v, u) = 0$ for any $v \in T_0N_2$. It is equal to 1 if and only if the 4-form $\omega \wedge \omega$ has zero restriction to the space $W = T_0N_1 + T_0N_2$.

Theorem 22 (Theorem 7.9 in [5]). *Let $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. Let $N = N_1 \cup N_2$ be the regular union singularity with two 2-dimensional symplectic strata in the symplectic space $(\mathbb{R}^{2n}, \omega)$.*

If N_1 and N_2 are not ω-orthogonal, then N is symplectomorphic to the variety

$$N^{\alpha} = \{q_1 = p_2, p_1 = p_{\geq 3} = q_{\geq 3} = 0\} \cup \{p_2 = \alpha q_1, p_{\geq 3} = q_{\geq 2} = 0\},$$

where α is the index of non-orthogonality between N_1 and N_2.

If N_1 and N_2 are ω-orthogonal, then N has is symplectomorphic to

$$N^{\perp} = \{p_{\geq 2} = q_{\geq 2} = 0\} \cup \{p_1 = q_1 = p_{\geq 3} = q_{\geq 3} = 0\}.$$

If $n \geq 3$, then any of the normal forms is realizable and if $n = 2$, then any of the normal forms is realizable except the normal form N^1.

Theorem 22 was generalized in [6] to regular union singularities of two germs of symplectic or quasi-symplectic k-dimensional submanifolds of the symplectic space. For simplicity we present the case $k = 2$ only.

By Proposition 18 a smooth vector field-germ X is Hamiltonian on $N = N_1 \cup N_2$ if and only if X is Hamiltonian on both of symplectic submanifold-germs N_1, N_2 and $d(X \rfloor \omega)|_W = 0$.

Let $X = \sum_{i=1}^n f_i(p,q)\frac{\partial}{\partial p_i} + g_i(p,q)\frac{\partial}{\partial q_i}$ be a smooth vector field-germ on $\mathbb{R}^{2n}$. By Propositions 18 and direct calculations we obtain the following proposition:

Proposition 23. *The vector field-germ X is Hamiltonian on*

$$N^{\alpha} = \{q_1 = p_2, p_1 = p_{\geq 3} = q_{\geq 3} = 0\} \cup \{p_2 = \alpha q_1, p_{\geq 3} = q_{\geq 2} = 0\}$$

if and only if

$$\left(-\frac{\partial f_1}{\partial q_2} + \frac{\partial g_2}{\partial q_2} + \frac{\partial f_2}{\partial p_2} + \frac{\partial f_2}{\partial q_1}\right)|_{\{q_1=p_2, p_1=p_{\geq 3}=q_{\geq 3}=0\}} = 0,$$

$$\left(\alpha\frac{\partial g_1}{\partial p_2} + \frac{\partial g_1}{\partial q_1} + \frac{\partial f_1}{\partial p_1} - \alpha\frac{\partial g_2}{\partial p_1}\right)|_{\{p_2=\alpha q_1, p_{\geq 3}=q_{\geq 2}=0\}} = 0,$$

$$\frac{\partial g_2}{\partial q_2}(0) + \frac{\partial f_2}{\partial p_2}(0) = \frac{\partial f_1}{\partial p_2}(0) + \frac{\partial g_2}{\partial q_1}(0) = \frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) = \frac{\partial g_1}{\partial q_2}(0) + \frac{\partial f_2}{\partial p_1}(0) = 0,$$

$$\frac{\partial f_1}{\partial q_2}(0) - \frac{\partial f_2}{\partial q_1}(0) = \frac{\partial g_1}{\partial p_2}(0) - \frac{\partial g_2}{\partial p_1}(0) = 0.$$

Let us denote the stata of $N^{\perp}$ by

$$N_1^{\perp} = \{p_{\geq 2} = q_{\geq 2} = 0\}, \ \ N_2^{\perp} = \{p_1 = q_1 = p_{\geq 3} = q_{\geq 3} = 0\}.$$

In the same way we get the following result:

Proposition 24. *The vector field-germ X is Hamiltonian on $N^{\perp} = N_1^{\perp} \cup N_2^{\perp}$ if and only if*

$$\left(\frac{\partial g_1}{\partial q_1} + \frac{\partial f_1}{\partial p_1}\right)|_{N_1^{\perp}} = 0, \tag{21}$$

$$\left(\frac{\partial g_2}{\partial q_2} + \frac{\partial f_2}{\partial p_2}\right)|_{N_2^{\perp}} = 0, \tag{22}$$

$$\frac{\partial g_2}{\partial q_2}(0) + \frac{\partial f_2}{\partial p_2}(0) = \frac{\partial f_1}{\partial p_2}(0) + \frac{\partial g_2}{\partial q_1}(0) = \frac{\partial g_1}{\partial q_1}(0) + \frac{\partial f_1}{\partial p_1}(0) = \frac{\partial g_1}{\partial q_2}(0) + \frac{\partial f_2}{\partial p_1}(0) = 0,$$

$$\frac{\partial f_1}{\partial q_2}(0) - \frac{\partial f_2}{\partial q_1}(0) = \frac{\partial g_1}{\partial p_2}(0) - \frac{\partial g_2}{\partial p_1}(0) = 0.$$

The conditions (21)-(22) mean that the vector field-germ $f_i|_{N_i^{\perp}} \frac{\partial}{\partial p_i} + g_i|_{N_i^{\perp}} \frac{\partial}{\partial q_i}$ on the symplectic manifold-germ $(N_i^{\perp}, \omega|_{TN_i^{\perp}})$ is Hamiltonian (in the classical sense) for $i = 1, 2$ (see Proposition 12).

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