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Poisson–Lie Algebras and Singular Symplectic Forms Associated to Corank 1 Type Singularities

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Dedicated to Professor Armen Sergeev on his 70th birthday

Abstract—We show that there exists a natural Poisson–Lie algebra associated to a singular symplectic structure ω. We construct Poisson–Lie algebras for the Martinet and Roussarie types of singularities. In the special case when the singular symplectic structure is given by the pullback from the Darboux form, $\omega = F^*\omega_0$, this Poisson–Lie algebra is a basic symplectic invariant of the singularity of the smooth mapping F into the symplectic space $(\mathbb{R}^{2n}, \omega_0)$. The case of A_k singularities of pullbacks is considered, and Poisson–Lie algebras for $\Sigma_{2,0}$, $\Sigma^{\mathrm{e}}_{2,2,0}$ and $\Sigma^{\mathrm{h}}_{2,2,0}$ stable singularities of 2-forms are calculated.

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1. INTRODUCTION

Let ω be the germ of a closed 2-form at $0 \in \mathbb{R}^{2n}$. For a function-germ h at $0 \in \mathbb{R}^{2n}$ and nondegenerate ω, the Hamiltonian vector field of h with respect to ω is the vector field $X_{\omega,h}$ such that (see [11, 21])

$$\omega(X_{\omega,h}, \xi) = -\xi(h) \tag{1.1}$$

for any vector field ξ on $\mathbb{R}^{2n}$.

If ω is singular, then the smooth vector field $X_{\omega,h}$ defined by formula (1.1) may not exist (cf. [14, 19, 6]). Thus we define the space of Hamiltonians $\mathcal{H}_\omega$,

$$\mathcal{H}_\omega = \{h \in \mathcal{E}_{2n} \mid X_{\omega,h} \text{ is smooth}\}. \tag{1.2}$$

If $h, k \in \mathcal{H}_\omega$, we show that $\{h,k\}_\omega = \omega(X_{\omega,h}, X_{\omega,k})$ belongs to $\mathcal{H}_\omega$. And under a certain generic condition we prove that $\mathcal{H}_\omega$ equipped with the bracket $\{\cdot,\cdot\}_\omega$ is a Poisson–Lie algebra.

Let $(\mathbb{R}^{2n}, \omega_0)$ be a symplectic space with ω_0 in Darboux form. Let θ be the Liouville 1-form on the cotangent bundle $T^*\mathbb{R}^{2n}$. Then $d\theta$ is a standard symplectic structure on $T^*\mathbb{R}^{2n}$. Let $\beta\colon T\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ be the canonical bundle map defined by ω_0, $\beta\colon T\mathbb{R}^{2n} \ni v \mapsto \omega_0(v,\cdot) \in T^*\mathbb{R}^{2n}$. Then we can define the canonical symplectic structure $\dot\omega$ on $T\mathbb{R}^{2n}$, $\dot\omega = \beta^* d\theta = d(\beta^*\theta)$. Throughout the paper, unless otherwise stated, all objects are germs at 0 of smooth functions, mappings, forms, etc., or their representatives on an open neighborhood of 0 in $\mathbb{R}^{2n}$.

Let $\overline{F}\colon (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ be a smooth map-germ. We say that $\overline{F}$ is isotropic if $\overline{F}^*\dot\omega = 0$. If we assume that $\overline{F}\colon (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is an isotropic map-germ, then the germ of the differential of the

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1-form $(\beta \circ \overline{F})^*\theta$ vanishes, $d(\beta \circ \overline{F})^*\theta = \overline{F}^*\beta^* d\theta = \overline{F}^*\dot{\omega} = 0$. Thus $(\beta \circ \overline{F})^*\theta$ is a germ of a closed 1-form. And there exists a smooth function-germ $g\colon (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ such that

$$(\beta \circ \overline{F})^*\theta = -dg. \tag{1.3}$$

For each smooth isotropic map-germ $\overline{F}$ the function-germ g is uniquely defined up to an additive constant.

Let $F\colon \mathbb{R}^{2n} \to (\mathbb{R}^{2n}, \omega_0)$ be a smooth map, $\pi\colon T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $F = \pi \circ \overline{F}$. In general, $\overline{F}$ can be regarded as a vector field along F, i.e., a section of an induced fiber bundle $F^*T\mathbb{R}^{2n}$. By $\mathcal{E}_U$ ($\mathcal{E}_{\mathbb{R}^{2n}}$, respectively) we denote the $\mathbb{R}$-algebra of smooth function-germs at 0 on U (and on "the target space" $\mathbb{R}^{2n}$, respectively). For each isotropic map-germ $\overline{F}$ along F there exists a unique g belonging to the maximal ideal $\mathbf{m}_U$ of $\mathcal{E}_U$, $g \in \mathbf{m}_U$, which is a generating function-germ for $\overline{F}$. If $\overline{F}$ is an embedding, then its image $M = \overline{F}(\mathbb{R}^{2n}) \subset T\mathbb{R}^{2n}$ is an implicit differential system branching along singular values of F (cf. [7]). Singularities of such systems were studied by many authors (cf. [3, 4, 19]). In this paper we assume the smooth solvability of M and find their local classification and invariants.

To F we associate a symplectically invariant algebra $\mathcal{R}_F$ of all function-germs generating isotropic map-germs $\overline{F}$ along F. Let $F\colon \mathbb{R}^{2n} \to (\mathbb{R}^{2n}, \omega_0)$ be as above; then F induces a possibly degenerate 2-form $F^*\omega_0$. For a smooth function h defined on $U \subset \mathbb{R}^{2n}$, we formally define the Hamiltonian vector field X_h (which may not be smooth) on U by equality (1.1) with ω replaced by $F^*\omega_0$. To F we associate the Poisson–Lie algebra (1.2),

$$\mathcal{H}_F = \{h \in \mathcal{E}_{2n} \mid X_h \text{ is smooth}\}. \tag{1.4}$$

Then $\mathcal{H}_F \subset \mathcal{R}_F$ is a Poisson–Lie algebra endowed with the Poisson–Lie bracket

$$\{k, h\}_{F^*\omega_0} := F^*\omega_0(X_k, X_h). \tag{1.5}$$

Assume $\overline{F}\colon (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is a smooth isotropic map-germ along a smooth map-germ $F\colon (\mathbb{R}^{2n}, 0) \to \mathbb{R}^{2n}$ such that the regular point set of F is dense, and $h\colon (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ is a generating function-germ of $\overline{F}$. Then $\overline{F}$ is smoothly solvable (cf. [8, 9]) as an implicit differential system if and only if h belongs to the Poisson–Lie algebra $\mathcal{H}_F$. Thus the elements of $\mathcal{H}_F$ are considered to be Hamiltonians, which satisfy the equation

$$(\beta \circ dF(X_h))^*\theta = -dh.$$

In this paper we introduce the symplectic $\mathcal{A}$-equivalence to classify the smooth map-germs F into a symplectic space. We use this equivalence to classify the normal forms of such mappings in Section 2. Then, in Section 3 we use the classified normal forms to investigate the structure of the singular pullback $F^*\omega_0$. In Section 4 we find conditions for a smooth map-germ F under which $F^*\omega_0$ is a stable 2-form. Calculations are done for Martinet and Roussarie normal forms, but in Section 5 for the special case of A_k type singularities of mappings. The Poisson–Lie algebra of a singular symplectic form is introduced in Section 6 (cf. [8–10]). And the Poisson–Lie algebras for $\Sigma_{2,0}$, $\Sigma^{\mathrm{e}}_{2,2,0}$ and $\Sigma^{\mathrm{h}}_{2,2,0}$ stable singularities of 2-forms are calculated in Section 7.

2. NORMAL FORMS OF MAPPINGS INTO A SYMPLECTIC SPACE

Let $F\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ and $G\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be two C^∞ map-germs, where the target space $\mathbb{R}^{2n}$ is endowed with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dy_i \wedge dx_i$. We say that F and G are *symplectomorphic* if there exist a diffeomorphism-germ $\phi\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the source space and a symplectomorphism $\Phi\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the target space such that

$$G = \Phi \circ F \circ \phi. \tag{2.1}$$

In this paper, we use new (modified) pre-normal forms of A_k singularities of map-germs (cf. [1, 2, 5, 12, 13]). Before that, we give an introductory pre-normal form of not necessarily stable corank 1 map-germs $F\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$.

Proposition 2.1 (introductory pre-normal form). *Let $G\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ be a C^∞ map-germ of corank 1. Then G is symplectomorphic to a map-germ of the form*

$$\begin{gathered} F=(f_1,\dots,f_{2n}),\\ f_i(u)=u_i \quad (i\le 2n-1), \qquad f_{2n}(u) \text{ is a } C^\infty \text{ function.} \end{gathered}\tag{2.2}$$

Proof. Suppose $G\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ is a C^∞ map-germ of corank 1. Then there exist a C^∞ diffeomorphism $h\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ of the source space and a C^∞ diffeomorphism $\varphi=(\varphi_1,\dots,\varphi_{2n})\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ of the target space such that

$$\begin{gathered} \varphi_i\circ G\circ h(u_1,\dots,u_{2n})=u_i \qquad (i<2n),\\ \varphi_{2n}\circ G\circ h(u_1,\dots,u_{2n})=g(u_1,\dots,u_{2n}), \end{gathered}$$

where g is a C^∞ function with $\partial g/\partial u_{2n}(0)=0$.

Now we use this differential normal form to construct a symplectomorphic change of coordinates of the target space. There is a symplectic diffeomorphism on the target space

$$\psi=(\psi_1,\dots,\psi_{2n})\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0) \qquad \text{such that} \quad \psi_{2n}=\varphi_{2n}.$$

Next, let

$$v_i=\psi_i\circ G\circ h(u_1,\dots,u_{2n}) \quad (i<2n), \qquad v_{2n}=u_{2n}.$$

Then, $(v_1,\dots,v_{2n})$ are new coordinates on the source space and we have

$$\psi_i\circ G\circ h=v_i \quad (i<2n), \qquad \psi_{2n}\circ G\circ h=g(v_1,\dots,v_{2n}). \qquad \square$$

Now for A_k map-germs, we have

Proposition 2.2. *Let $G\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ be an A_k type singularity.*

1. *If G is a fold map-germ, i.e., A_1, then G is symplectomorphic to a map-germ of the form*

$$\begin{gathered} F=(f_1,\dots,f_{2n}),\\ f_i(u)=u_i \quad (i\le 2n-1), \qquad f_{2n}(u)=u_{2n}^2. \end{gathered}\tag{2.3}$$

2. *If G is an A_k type map-germ with $k\ge 2$, then G is symplectomorphic to a map-germ of the form*

$$\begin{aligned} f_i(u)&=u_i \qquad (i\le 2n-1),\\ f_{2n}(u)&=u_{2n}^{k+1}+\sum_{i=1}^{k-1}a_i(u_1,\dots,u_{2n-1})u_{2n}^i+b(u_1,\dots,u_{2n-1}), \end{aligned}\tag{2.4}$$

where $a_1(u_1,\dots,u_{2n-1}),\dots,a_{k-1}(u_1,\dots,u_{2n-1})$ and $b(u_1,\dots,u_{2n-1})$ are smooth functions and the differentials $da_1,da_2,\dots,da_{k-1}$ are linearly independent at the origin.

3 (*cusp for $n=1$*). *If $G\colon (\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ is an A_k map-germ with $k\ge 2$, then $k=2$ and it is symplectomorphic to the normal form of a cusp*:

$$F=(f_1,f_2), \qquad f_1(u)=u_1, \qquad f_2(u)=u_2^3+u_1u_2.\tag{2.5}$$

Proof. The proof of assertion 1 is almost the same as the proof of Proposition 2.1. Suppose that G is a fold map-germ, i.e., A_1 map-germ. Then there exist a C^∞ diffeomorphism $h\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ of the source space and a C^∞ diffeomorphism $\varphi=(\varphi_1,\dots,\varphi_{2n})\colon (\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ of the target space such that

$$\varphi_i \circ G \circ h(u_1, \dots, u_{2n}) = u_i \quad (i < 2n), \qquad \varphi_{2n} \circ G \circ h(u_1, \dots, u_{2n}) = u_{2n}^2.$$

Then, there is a symplectic diffeomorphism on the target space

$$\psi = (\psi_1, \dots, \psi_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \qquad \text{such that} \quad \psi_{2n} = \varphi_{2n}.$$

Let

$$v_i = \psi_i \circ G \circ h(u_1, \dots, u_{2n}) \quad (i < 2n), \qquad v_{2n} = u_{2n}.$$

Then, $(v_1, \dots, v_{2n})$ are coordinates on the source space and we have

$$\psi_i \circ G \circ h = v_i \quad (i < 2n), \qquad \psi_{2n} \circ G \circ h = u_{2n}^2 = v_{2n}^2.$$

Now suppose that G is an A_k map-germ. Then, by Morin's theorem (cf. [17]), there exist a C^∞ diffeomorphism $h\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the source space and a C^∞ diffeomorphism $\varphi = (\varphi_1, \dots, \varphi_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the target space such that

$$\begin{aligned} \varphi_i \circ G \circ h(u_1, \dots, u_{2n}) &= u_i \qquad (i < 2n), \\ \varphi_{2n} \circ G \circ h(u_1, \dots, u_{2n}) &= u_{2n}^{k+1} + \sum_{i=1}^{k-1} u_i u_{2n}^i. \end{aligned} \tag{2.6}$$

Then, there is a symplectic diffeomorphism on the target space

$$\psi = (\psi_1, \dots, \psi_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \qquad \text{such that} \quad \psi_{2n} = \varphi_{2n}. \tag{2.7}$$

Let

$$v_i = \psi_i \circ G \circ h(u_1, \dots, u_{2n}) \quad (i < 2n), \qquad v_{2n} = u_{2n}. \tag{2.8}$$

Then, $(v_1, \dots, v_{2n})$ are new coordinates on the source space, and from (2.6) and (2.8) we have

$$\begin{aligned} \psi_i \circ G \circ h(v_1, \dots, v_{2n}) &= v_i \qquad (i < 2n), \\ \psi_{2n} \circ G \circ h(v_1, \dots, v_{2n}) &= u_{2n} = v_{2n}^{k+1} + \sum_{i=1}^{k-1} u_i v_{2n}^i. \end{aligned}$$

Taking the inverse of the source coordinates (2.8), we get the final form

$$\begin{aligned} \psi_i \circ G \circ h(v_1, \dots, v_{2n}) &= v_i \qquad (i < 2n), \\ \psi_{2n} \circ G \circ h(v_1, \dots, v_{2n}) &= u_{2n} = v_{2n}^{k+1} + \sum_{i=1}^{k-1} u_i(v) v_{2n}^i. \end{aligned}$$

Note that the coefficients $u_i(v)$ are functions of the variables $v_1, v_2, \dots, v_{2n-1}, v_{2n}$. However, the coefficients $u_i(v)$ are desirable to be functions of the variables $v_1, v_2, \dots, v_{2n-1}$.

Since $u_i(v)$'s are functions of the variables $v_1, \dots, v_{2n}$, they can be expressed in the form

$$u_i(v_1, \dots, v_{2n}) = \sum_{j=1}^{2n-1} v_j \alpha_{i,j}(v_1, \dots, v_{2n}) + \beta_i(v_{2n}).$$

Since G is an A_k type map-germ, the order of $\beta_i(v_{2n})$ must be greater than $k - i$:

$$\operatorname{ord} \beta_i(v_{2n}) > k - i;$$

indeed, if $\operatorname{ord} \beta_i(v_{2n}) \le k - i$, then G must be an A_ℓ-singularity for some $\ell < k$.

Then with the coordinates

$$w_i = v_i \quad (i < 2n), \qquad w_{2n} = \sqrt[k+1]{u_{2n}^{k+1} + \sum_{i=1}^{k-1} \beta_i(v_{2n}) v_{2n}^i}$$

in the source space, $\psi_{2n} \circ G \circ h(w_1, \dots, w_{2n})$ becomes an unfolding of w_{2n}^{k+1} with parameters $w_1, \dots, w_{2n-1}$ in the sense of unfolding theory (see, e.g., [20]):

$$\psi_{2n} \circ G \circ h(0, \dots, 0, w_{2n}) = w_{2n}^{k+1}.$$

Then again under new coordinates of the form

$$\overline{w}_i = w_i = v_i \quad (i < 2n), \qquad \overline{w}_{2n} = \overline{w}_{2n}(v_1, \dots, v_{2n}),$$

$\psi_{2n} \circ G \circ h$ becomes of the form

$$\psi_{2n} \circ G \circ h = \overline{w}_{2n}^{k+1} + \sum_{i=1}^{k-1} \overline{a}_i(\overline{w}_1, \dots, \overline{w}_{2n-1}) \overline{w}_{2n}^i + b(\overline{w}_1, \dots, \overline{w}_{2n-1}). \tag{2.9}$$

Note that after (2.7) we have not changed coordinates in the target space. So the map-germ G and the map-germ $\psi \circ G \circ h$,

$$\psi_i \circ G \circ h(\overline{w}) = \overline{w}_i \qquad (i < 2n),$$
$$\psi_{2n} \circ G \circ h(\overline{w}) = \overline{w}_{2n}^{k+1} + \sum_{i=1}^{k-1} \overline{a}_i(\overline{w}_1, \dots, \overline{w}_{2n-1}) \overline{w}_{2n}^i + b(\overline{w}_1, \dots, \overline{w}_{2n-1}),$$

are symplectomorphic. This completes the proof of assertion 2.

The proof of assertion 3 is a straightforward application of assertion 2. $\square$

3. INDUCED CLOSED 2-FORMS FROM THE SYMPLECTIC STRUCTURE

Now we want to investigate the induced closed 2-forms $F^*\omega_0$. In order to avoid unnecessarily complicated calculations, we choose the following new coordinates in the target space ($\mathbb{R}^{2n}$, $\omega_0 = \sum_{i=1}^n dy_i \wedge dx_i$):

$$z_1 = -x_1, \quad z_2 = y_1, \quad \dots, \quad z_{2n-1} = -x_n, \quad z_{2n} = y_n.$$

Then

$$\omega_0 = dz_1 \wedge dz_2 + \dots + dz_{2n-1} \wedge dz_{2n}.$$

Following the above change, we also use the corresponding new coordinates in the source space:

$$v_1 = -u_1, \quad v_2 = u_{n+1}, \quad \dots, \quad v_{2n-1} = -u_n, \quad v_{2n} = u_{2n}.$$

In this section, we formulate our results on the induced closed 2-forms $F^*\omega_0$. This is stated for the corank 1 map-germ and expressed for the symplectic pre-normal form (2.2) of F.

Let $(z_1, \dots, z_{2n})$ be the standard coordinates in the target space $\mathbb{R}^{2n}$ and let $\omega_0 = dz_1 \wedge dz_2 + \dots + dz_{2n-1} \wedge dz_{2n}$ be the symplectic form on the target space $\mathbb{R}^{2n}$. With the assumptions of Proposition 2.1 we have the following result.

Proposition 3.1. *Let F be in the pre-normal form* (2.2). *Then*

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta(v)\, dv_{2n-1} \wedge dv_{2n} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i}\, dv_i \wedge dv_{2n-1}, \tag{3.1}$$

where $\Delta(v) = \partial f_{2n}/\partial v_{2n}(v)$ is the Jacobian of F.

From now on, we assume that

$$d\Delta(0) \neq 0. \tag{3.2}$$

Let us introduce the notation

$$\Sigma_2(F^*\omega_0) = \{v \in \mathbb{R}^{2n} \mid \Delta(v) = 0\}, \tag{3.3}$$

$$A_{F^*\omega_0}(v) = \{w \in T_v\mathbb{R}^{2n} \mid i(w)F^*\omega_0(v) = 0\}, \quad \text{the kernel of } F^*\omega_0(v), \tag{3.4}$$

where $i(w)F^*\omega_0(v)$ denotes the inner product of the vector w and the 2-form $F^*\omega_0(v)$.

Since $d\Delta(0) \neq 0$, $\Sigma_2(F^*\omega_0)$ is a $(2n-1)$-dimensional submanifold of $\mathbb{R}^{2n}$.

Proposition 3.2. *Suppose that* $d\Delta(0) \neq 0$. *If* $v \in \Sigma_2(F^*\omega_0)$, *then* $\dim A_{F^*\omega_0}(v) = 2$ *and it is spanned by the following two vectors*:

$$e_1 = -\sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i}} \frac{\partial}{\partial v_{2i-1}} + \sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i-1}} \frac{\partial}{\partial v_{2i}} + \frac{\partial}{\partial v_{2n-1}}, \qquad e_2 = \frac{\partial}{\partial v_{2n}}. \tag{3.5}$$

Proof. Let $v \in \Sigma_2(F^*\omega_0)$. Since $\dim A_{F^*\omega_0(v)} = 2$ and e_1 and e_2 are linearly independent, it is enough to show that $e_1, e_2 \in A_{F^*\omega_0(v)}$.

From Proposition 3.1, we have

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta\, dv_{2n-1} \wedge dv_{2n} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1},$$

where $\Delta = \partial f_{2n}/\partial v_{2n}$ is the Jacobian of F.

Since $v \in \Sigma_2(F^*\omega_0)$, $\Delta(v) = 0$. Thus

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1} \qquad \text{on} \quad \Sigma_2(F^*\omega_0).$$

Let

$$e = \sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i} \in T_v\mathbb{R}^{2n}.$$

Then

$$e \in A_{F^*\omega_0}(v) \qquad \text{if and only if} \qquad F^*\omega_0(v)\left(e, \frac{\partial}{\partial v_j}\right) = 0 \quad (j = 1, \dots, 2n).$$

Now we solve the following equation for the coefficients $w_1, \dots, w_{2n}$:

$$F^*\omega_0\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = 0 \qquad (j = 1, \dots, 2n).$$

We have

$$\begin{aligned} 0 &= F^*\omega_0\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2j-1}}\right) \\ &= \left(\sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1}\right)\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2j-1}}\right) \\ &= -w_{2j} + \frac{\partial f_{2n}}{\partial v_{2j-1}} w_{2n-1} \qquad (j < n) \end{aligned}$$

and

$$0 = F^*\omega_0\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2j}}\right) = w_{2j-1} + \frac{\partial f_{2n}}{\partial v_{2j}} w_{2n-1} \qquad (j < n).$$

Thus we obtain

$$w_{2j-1} = -\frac{\partial f_{2n}}{\partial v_{2j}} w_{2n-1}, \qquad w_{2j} = \frac{\partial f_{2n}}{\partial v_{2j-1}} w_{2n-1}. \tag{3.6}$$

Note that

$$F^*\omega_0\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2n}}\right) = 0 \qquad \text{for arbitrary } w_1, \dots, w_{2n-1},$$

since $F^*\omega_0$ does not contain the term $\partial/\partial v_{2n}$.

We also see that if we let

$$w_{2i-1} = -\frac{\partial f_{2n}}{\partial v_{2i}} w_{2n-1}, \qquad w_{2i} = \frac{\partial f_{2n}}{\partial v_{2i-1}} w_{2n-1},$$

then we immediately have

$$F^*\omega_0\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2n-1}}\right) = 0.$$

Thus we have no relations between $w_1, \dots, w_{2n}$ other than (3.6). Therefore, as a basis of $A_{F^*\omega_0}(v)$, we can choose

$$e_1 = -\sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i}} \frac{\partial}{\partial v_{2i-1}} + \sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i-1}} \frac{\partial}{\partial v_{2i}} + \frac{\partial}{\partial v_{2n-1}}, \qquad \text{letting} \quad w_{2n-1} = 1, \quad w_{2n} = 0,$$

$$e_2 = \frac{\partial}{\partial v_{2n}}, \qquad \text{letting} \quad w_{2n-1} = 0, \quad w_{2n} = 1.$$

This completes the proof of Proposition 3.2. □

4. CLASSIFICATION OF MAPPINGS BY INDUCED CLOSED 2-FORMS

In this section we find the classification of singularities of corank 1 maps induced by the classification of "stable" singularities of closed differential 2-forms (cf. [15, 18, 16]).

Let

$$\omega = \sum_{1 \le i < j \le 2n} \alpha_{i,j}\, dv_i \wedge dv_j$$

be the germ of a closed 2-form on $\mathbb{R}^{2n}$ at 0. As a volume form on $\mathbb{R}^{2n}$, we choose

$$\Omega = dv_1 \wedge dv_2 \wedge \ldots \wedge dv_{2n}.$$

Let

$$\omega^n = f\Omega.$$

If $f(0) \neq 0$, then by Darboux's theorem, ω is isomorphic to the Darboux form

$$dv_1 \wedge dv_2 + dv_3 \wedge dv_4 + \ldots + dv_{2n-1} \wedge dv_{2n}.$$

Now we assume that $f(0) = 0$ while $df(0) \neq 0$. Let

$$\Sigma_2(\omega) = \{v \in \mathbb{R}^{2n} \mid f(v) = 0\}.$$

By the condition $df(0) \neq 0$, $\Sigma_2(\omega)$ is a dimension $2n-1$ submanifold of $\mathbb{R}^{2n}$ and at a point $v \in \Sigma_2(\omega)$ the kernel

$$A_\omega(v) = \{w \in T_v\mathbb{R}^{2n} \mid i(w)\omega(v) = 0\}$$

of $\omega(v)$ is a two-dimensional vector subspace of $T_v\mathbb{R}^{2n}$, where $i(w)\omega(v)$ denotes the inner product of a tangent vector w and a 2-form ω.

Definition 4.1 (J. Martinet)**.** Suppose that $f(0)=0$ while $df(0) \neq 0$. If $A_\omega(0)$ is transversal to $T_0\Sigma_2(\omega)$, we say that ω has a $\Sigma_{2,0}$ *singularity at* 0.

Theorem 4.1 (J. Martinet)**.** *If a closed* 2*-form* ω *has a* $\Sigma_{2,0}$ *singularity at* 0, *then* ω *is isomorphic to the following closed* 2*-form*:

$$v_1 dv_1 \wedge dv_2 + dv_3 \wedge dv_4 + \ldots + dv_{2n-1} \wedge dv_{2n}.$$

Let us consider the set

$$\Sigma_{2,2}(\omega) = \{v \in \Sigma_2(\omega) \mid A_\omega(v) \subset T_v\Sigma_2(\omega)\}.$$

It is known that $\Sigma_{2,2}(\omega)$ is a dimension $2n-3$ submanifold of $\mathbb{R}^{2n}$.

Definition 4.2 (J. Martinet)**.** Suppose that $0 \in \Sigma_{2,2}(\omega)$. If $A_\omega(0)$ is transversal to $T_0\Sigma_{2,2}(\omega)$ in $T_0\Sigma_2(\omega)$, then we say that ω has a $\Sigma_{2,2,0}$ *singularity at* 0.

Since $\Sigma_{2,2,0}$ singularities of closed 2-forms are classified only for $n=2$, from now on we only consider closed 2-forms on $\mathbb{R}^4$.

Theorem 4.2 (R. Roussarie)**.** *If a closed* 2*-form* ω *on* $\mathbb{R}^4$ *has a* $\Sigma_{2,2,0}$ *singularity at* 0, *then* ω *is isomorphic to one of the following two closed* 2*-forms*:

$$dv_1 \wedge dv_2 + v_3 dv_2 \wedge dv_3 + d\left(v_1v_3 + v_2v_4 - \frac{v_3^3}{3}\right) \wedge dv_4,$$

$$dv_1 \wedge dv_2 + v_3 dv_2 \wedge dv_3 + d\left(v_1v_3 - v_2v_4 - \frac{v_3^3}{3}\right) \wedge dv_4.$$

Definition 4.3. If ω is isomorphic to the first of the above two forms, we say that ω has a $\Sigma^{\mathrm{e}}_{2,2,0}$ (elliptic $\Sigma_{2,2,0}$) singularity at 0, and if ω is isomorphic to the second of the above forms, we say that ω has a $\Sigma^{\mathrm{h}}_{2,2,0}$ (hyperbolic $\Sigma_{2,2,0}$) singularity at 0.

These two cases are distinguished as follows: Suppose that a closed 2-form ω on $\mathbb{R}^4$ has a $\Sigma_{2,2,0}$ singularity at 0. Let Ω be a positive volume form of $\mathbb{R}^4$ with coordinates $v_1, \ldots, v_4$, say, $\Omega = dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4$. Then ω^2 has the form

$$\omega^2 = f\Omega$$

for a function f such that $f(0)=0$ and $df(0) \neq 0$.

Let $\overline{\Omega}_{\Sigma_2(\omega)}$ be a volume form on $\Sigma_2(\omega)$ such that

$$\overline{\Omega}_{\Sigma_2(\omega)} \wedge df \quad \text{and} \quad \Omega \quad \text{define the same orientation on } \mathbb{R}^4.$$

Let $\Sigma_2(\omega)$ be oriented in such a way that $\overline{\Omega}_{\Sigma_2(\omega)}$ is a positive volume form on $\Sigma_2(\omega)$. It is known (see [18, p. 147]) that there exists a smooth vector filed X on $\Sigma_2(\omega)$ such that

$$(\omega_{|\Sigma_2(\omega)}) = i(X)(\overline{\Omega}_{\Sigma_2(\omega)}),$$

where $i(X)(\overline{\Omega}_{\Sigma_2(\omega)})$ is the inner product of the vector filed X with the 3-form $\overline{\Omega}_{\Sigma_2(\omega)}$.

Let w_1, w_2, w_3 be coordinates at 0 on $\Sigma_2(\omega)$ which define a positive orientation on $\Sigma_2(\omega)$. Then the vector field X has the form

$$X = \sum_{i=1}^{3} a_i(w)\frac{\partial}{\partial w_i}.$$

By the definition of $\Sigma_{2,2}(\omega)$, ω vanishes on $\Sigma_{2,2}(\omega)$. So, the Jacobian matrix of X at 0

$$\left(\frac{\partial a_i}{\partial w_j}(0)\right)$$

has rank 2 and it has two nonzero eigenvalues $\lambda_{\omega,1}$ and $\lambda_{\omega,2}$, which are known to be either both real or both imaginary.

Theorem 4.3 (R. Roussarie)**.** *Let ω have a $\Sigma_{2,2,0}$ singularity at* 0*.*

1. *If the two eigenvalues $\lambda_{\omega,1}$ and $\lambda_{\omega,2}$ are real, then ω has a $\Sigma^{\mathrm{h}}_{2,2,0}$ singularity at* 0*.*
2. *If the two eigenvalues $\lambda_{\omega,1}$ and $\lambda_{\omega,2}$ are imaginary, then ω has a $\Sigma^{\mathrm{e}}_{2,2,0}$ singularity at* 0*.*

Theorem 4.4. *Let F be a map-germ of the form* (2.2)*. Then $F^*\omega_0$ is isomorphic to the Martinet's normal form of $\Sigma_{2,0}$ singularities of closed* 2*-forms,*

$$\sum_{i=1}^{n-1} dv_{2i-1}\wedge dv_{2i} + v_{2n-1}\,dv_{2n-1}\wedge dv_{2n}, \tag{4.1}$$

if and only if

$$(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0,0). \tag{4.2}$$

Proof. By (2.2) we have

$$(F^*\omega_0)^n = n\Delta\, dv_1\wedge dv_2\wedge\ldots\wedge dv_{2n}.$$

Since by the assumption $da_1(0)\neq 0$, we have $d\Delta(0) = da_1(0)\neq 0$. So, by the definition of $\Sigma_{2,0}$, it is enough to seek the condition for $A_\omega(0)$ to be transversal to $T_0\Sigma_2(\omega)$ at 0.

Since

$$\Sigma_2(\omega) = \{v\in\mathbb{R}^{2n} \mid \Delta(v) = 0\}$$

and $A_\omega(0)$ is spanned by e_1 and e_2, we know that $A_\omega(0)$ is transversal to $T_0\Sigma_2(\omega)$ at 0 if and only if $(e_1(\Delta)(0), e_2(\Delta)(0))\neq(0,0)$. Thus, from Martinet's theorem, $F^*\omega_0$ is isomorphic to Martinet's normal form of $\Sigma_{2,0}$ if and only if $(e_1(\Delta)(0), e_2(\Delta)(0))\neq(0,0)$. $\square$

Theorem 4.5. *Suppose that $F^*\omega_0$ is not isomorphic to Martinet's normal form of $\Sigma_{2,0}$ type singularities*; *i.e., suppose that*

$$(e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0).$$

Then $F^*\omega_0$ is isomorphic to Roussarie's $\Sigma_{2,2,0}$ normal forms if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0)\\ e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0)\end{pmatrix} = 2.$$

Proof. Since

$$\Sigma_{2,2}(F^*\omega_0) = \bigl\{v\in\mathbb{R}^4 \mid \Delta(v) = 0,\ e_1(\Delta)(v) = 0,\ e_2(\Delta)(v) = 0\bigr\}$$

and $A_\omega(0)$ is spanned by e_1 and e_2, we know that $A_\omega(0)$ is transversal to $T_0\Sigma_{2,2}(\omega)$ in $T_0\Sigma_2(\omega)$ at 0 if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2.$$

Therefore, by the definition of $\Sigma_{2,2,0}$, $F^*\omega_0$ is isomorphic to Roussarie's $\Sigma_{2,2,0}$ if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2,$$

which holds if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2,$$

for $(e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0)$ by assumption. $\square$

Let $F = (f_1,\dots,f_4)\colon (\mathbb{R}^4,0) \to (\mathbb{R}^4,0)$ be the pre-normal form of corank 1 map-germ given in Proposition 2.1 such that

$$d\Delta(0) \neq 0, \qquad (e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0),$$
$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2,$$

where

$$\Delta = \frac{\partial f_4}{\partial v_4}, \qquad e_1 = -\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1} + \frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3}, \qquad e_2 = \frac{\partial}{\partial v_4}.$$

Then by Theorem 4.5, $F^*\omega_0$ is of type $\Sigma_{2,2,0}$.

Since $d\Delta(0) \neq 0$,

$$\Sigma_2(F^*\omega_0) = \{v = (v_1,\dots,v_4) \in \mathbb{R}^4 \mid \Delta = 0\}$$

is a three-dimensional submanifold of $\mathbb{R}^4$ and

$$\frac{\partial \Delta}{\partial v_i}(0) \neq 0 \qquad \text{for some} \quad i = 1,\dots,4.$$

Since $(e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0)$, we have

$$\frac{\partial \Delta}{\partial v_4}(0) = 0, \qquad \left(-\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1} + \frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3}\right)\Delta(0) = 0.$$

If $\partial\Delta/\partial v_1(0) = 0$ and $\partial\Delta/\partial v_2(0) = 0$, then by the above formula we have $\partial\Delta/\partial v_3(0) = 0$, which contradicts the fact that $d\Delta(0) \neq 0$. Thus we have

Lemma 4.1.

$$\frac{\partial \Delta}{\partial v_1}(0) \neq 0 \qquad \textit{or} \qquad \frac{\partial \Delta}{\partial v_2}(0) \neq 0.$$

So after the changes of coordinates

$$\overline{z}_1 = -z_2, \quad \overline{z}_2 = z_1, \quad \overline{z}_3 = -z_3, \quad \overline{z}_4 = z_4 \qquad \text{in the target space,}$$
$$\overline{v}_1 = -v_2, \quad \overline{v}_2 = v_1, \quad \overline{v}_3 = -v_3, \quad \overline{v}_4 = v_4 \qquad \text{in the source space,}$$

we may assume that

$$\frac{\partial \Delta}{\partial v_1}(0) \neq 0.$$

Then, by the implicit function theorem, there is a function $\varphi(v_2, v_3, v_4)$ such that

$$\Sigma_2(F^*\omega_0) = \{v \in \mathbb{R}^4 \mid \Delta(v) = 0\} = \big\{(v_1, \dots, v_4) \in \mathbb{R}^4 \mid v_1 = \varphi(v_2, v_3, v_4)\big\},$$

and we can choose v_2, v_3, v_4 as coordinates on $\Sigma_2(F^*\omega_0)$. Let us define

$$\alpha_2 = -\frac{\partial f_4}{\partial v_1}\frac{\partial \varphi}{\partial v_4}, \qquad \alpha_3 = -\frac{\partial \varphi}{\partial v_4}, \qquad \alpha_4 = \frac{\partial \varphi}{\partial v_3} - \frac{\partial f_4}{\partial v_1}\frac{\partial \varphi}{\partial v_2} - \frac{\partial f_4}{\partial v_2}.$$

Considering the Jacobian matrix of $\alpha_2, \alpha_3, \alpha_4$, we have

$$\operatorname{rank}\left(\frac{\partial \alpha_i}{\partial v_j}(0)\right)_{2\le i,j\le 4} = 2. \tag{4.3}$$

Theorem 4.6. *Let the assumptions of Theorem* 4.5 *be fulfilled. Then*

(1) $F^*\omega_0$ *is isomorphic to Roussarie's normal form* $\Sigma^{\mathrm{h}}_{2,2,0}$ *if and only if the two nonzero eigenvalues of*

$$\left(\frac{\partial \alpha_i}{\partial v_j}(0)\right)_{2\le i,j\le 4} \tag{4.4}$$

are real;

(2) $F^*\omega_0$ *is isomorphic to Roussarie's normal form* $\Sigma^{\mathrm{e}}_{2,2,0}$ *if and only if the two nonzero eigenvalues of the matrix* (4.4) *are imaginary.*

Proof. Let $\iota = (\iota_1, \dots, \iota_4) \colon \Sigma_2(F^*\omega_0) \to \mathbb{R}^4$,

$$\iota(\overline{v}_2, \overline{v}_3, \overline{v}_4) = (\varphi(\overline{v}_2, \overline{v}_3, \overline{v}_4), \overline{v}_2, \overline{v}_3, \overline{v}_4),$$

be the inclusion map. Then we can easily check that

$$d\overline{v}_2 \wedge d\overline{v}_3 \wedge d\overline{v}_4 = \iota^*(dv_2 \wedge dv_3 \wedge dv_4).$$

Set

$$\overline{\Omega}_{\Sigma_2(F^*\omega_0)} = -d\overline{v}_2 \wedge d\overline{v}_3 \wedge d\overline{v}_4 = -\iota^*(dv_2 \wedge dv_3 \wedge dv_4).$$

Then,

$$\Omega = dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4 \qquad \text{and} \qquad \overline{\Omega}_{\Sigma_2(F^*\omega_0)} \wedge df = 2\overline{\Omega}_{\Sigma_2(F^*\omega_0)} \wedge d\Delta$$

define the same orientation on $\mathbb{R}^4$. Recall that the function f was defined by the equality

$$(F^*\omega_0)^2 = f\Omega$$

and also recall that

$$F^*\omega_0 = dv_1 \wedge dv_2 + \Delta\, dv_3 \wedge dv_4 - \sum_{i=1,2} \frac{\partial f_{2n}}{\partial v_i}\, dv_i \wedge dv_3, \qquad (F^*\omega_0)^2 = 2\Delta\Omega.$$

Now we seek the vector field X on $\Sigma_2(F^*\omega_0)$ such that

$$F^*\omega_{0|\Sigma_2(F^*\omega_0)} = i(X)(\overline{\Omega}_{\Sigma_2(F^*\omega_0)}). \tag{4.5}$$

Letting

$$X=\sum_{i=2}^{4}\alpha_i(\overline{v}_2,\overline{v}_3,\overline{v}_4)\frac{\partial}{\partial\overline{v}_i},$$

we solve equation (4.5). Recall that $\overline{\Omega}_{\Sigma_2(F^*\omega_0)}=-d\overline{v}_2\wedge d\overline{v}_3\wedge d\overline{v}_4$. Then we have

$$\begin{aligned}
-\frac{\partial\varphi}{\partial\overline{v}_3}-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\overline{v}_2}-\frac{\partial f_4}{\partial v_2}&=F^*\omega_{0|\Sigma_2(F^*\omega_0)}\biggl(\frac{\partial}{\partial\overline{v}_2},\frac{\partial}{\partial\overline{v}_3}\biggr)=i(X)(\overline{\Omega}_{\Sigma_2(F^*\omega_0)})\biggl(\frac{\partial}{\partial\overline{v}_2},\frac{\partial}{\partial\overline{v}_3}\biggr)\\
&=-d\overline{v}_2\wedge d\overline{v}_3\wedge d\overline{v}_4\biggl(\sum_{i=2}^{4}\alpha_i\frac{\partial}{\partial\overline{v}_i},\frac{\partial}{\partial\overline{v}_2},\frac{\partial}{\partial\overline{v}_3}\biggr)=-\alpha_4,\\
-\frac{\partial\varphi}{\partial\overline{v}_4}&=F^*\omega_{0|\Sigma_2(F^*\omega_0)}\biggl(\frac{\partial}{\partial\overline{v}_2},\frac{\partial}{\partial\overline{v}_4}\biggr)\\
&=-d\overline{v}_2\wedge d\overline{v}_3\wedge d\overline{v}_4\biggl(\sum_{i=2}^{4}\alpha_i\frac{\partial}{\partial\overline{v}_i},\frac{\partial}{\partial\overline{v}_2},\frac{\partial}{\partial\overline{v}_4}\biggr)=\alpha_3,\\
\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\overline{v}_4}&=F^*\omega_{0|\Sigma_2(F^*\omega_0)}\biggl(\frac{\partial}{\partial\overline{v}_3},\frac{\partial}{\partial\overline{v}_4}\biggr)\\
&=-d\overline{v}_2\wedge d\overline{v}_3\wedge d\overline{v}_4\biggl(\sum_{i=2}^{4}\alpha_i\frac{\partial}{\partial\overline{v}_i},\frac{\partial}{\partial\overline{v}_3},\frac{\partial}{\partial\overline{v}_4}\biggr)=-\alpha_2.
\end{aligned}$$

Now we consider the Jacobian matrix

$$\biggl(\frac{\partial\alpha_i}{\partial\overline{v}_j}(0)\biggr)_{2\le i,j\le 4}\tag{4.6}$$

at 0 of the coefficients

$$\biggl(\alpha_2=-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\overline{v}_4},\ \alpha_3=-\frac{\partial\varphi}{\partial\overline{v}_4},\ \alpha_4=\frac{\partial\varphi}{\partial\overline{v}_3}-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\overline{v}_2}-\frac{\partial f_4}{\partial v_2}\biggr)$$

of the vector field X. According to Roussarie's theorem, we see that

$$\operatorname{rank}\biggl(\frac{\partial\alpha_i}{\partial\overline{v}_j}(0)\biggr)_{2\le i,j\le 4}=2,$$

which implies (4.3), and we see that

(1) $F^*\omega_0$ is isomorphic to Roussarie's normal form $\Sigma^{\mathrm{h}}_{2,2,0}$ if and only if the two nonzero eigenvalues of (4.6) are real;

(2) $F^*\omega_0$ is isomorphic to Roussarie's normal form $\Sigma^{\mathrm{e}}_{2,2,0}$ if and only if the two nonzero eigenvalues of (4.6) are imaginary.

This completes the proof of Theorem 4.6. □

5. CONDITIONS FOR A_k TYPE SINGULARITIES

In this section we apply the results of the previous sections to various examples containing A_k map-germs. Let F be a map-germ of the form (2.2) such that $d\Delta(0)\neq 0$. Then $F^*\omega_0$ is isomorphic to Martinet's normal form of $\Sigma_{2,0}$ singularities of closed 2-forms

$$\sum_{i=1}^{n-1}dv_{2i-1}\wedge dv_{2i}+v_{2n-1}\,dv_{2n-1}\wedge dv_{2n}$$

if and only if

$$(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0,0).$$

Let F be a fold map-germ:

$$F = (f_1, \dots, f_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0),$$

$$f_i(v) = v_i \quad (i \le 2n-1), \qquad f_{2n}(v) = v_{2n}^2.$$

Then

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + 2v_{2n}\, dv_{2n-1} \wedge dv_{2n}.$$

The above form is obviously isomorphic to Martinet's normal form $\Sigma_{2,0}$ given in Theorem 4.4:

$$\sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + v_{2n-1}\, dv_{2n-1} \wedge dv_{2n}.$$

Since

$$\Delta = 2v_{2n}, \qquad e_1(\Delta) = 0, \qquad e_2(\Delta) = 2$$

and

$$(e_1(\Delta)(0), e_2(\Delta)(0)) = (0,2) \neq (0,0),$$

$F^*\omega_0$ satisfies the condition given in Theorem 4.4 for it to be isomorphic to Martinet's normal form $\Sigma_{2,0}$.

Proposition 5.1 (A_k map-germs, $k \ge 2$). *Let $F = (f_1, \dots, f_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be an A_k map-germ of the form*

$$f_i(v) = v_i \qquad (i \le 2n-1),$$

$$f_{2n}(v) = v_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(v_1, \dots, v_{2n-1}) v_{2n}^i + b(v_1, \dots, v_{2n-1}) \qquad (k \ge 2).$$

In particular, when $n = 1$, let $F = (f_1, f_2)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a cusp map-germ:

$$f_1(v) = v_1, \qquad f_2(v) = v_2^3 + v_1 v_2. \tag{5.1}$$

Then

(1) *$F^*\omega_0$ is isomorphic to the above Martinet's normal form if and only if*

$$e_1(\Delta)(0) \neq 0, \tag{5.2}$$

or equivalently, if and only if

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) + \sum_{i=1}^{n-1} \left(-\frac{\partial a_1}{\partial v_{2i}}(0) \frac{\partial b}{\partial v_{2i-1}}(0) + \frac{\partial a_1}{\partial v_{2i-1}}(0) \frac{\partial b}{\partial v_{2i}}(0) \right) \neq 0; \tag{5.3}$$

(2) *in particular, if $b = 0$, $F^*\omega_0$ is isomorphic to Martinet's normal form if and only if*

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) \neq 0;$$

(3) *if $n = 1$, then, for the cusp map-germ* (5.1), *$F^*\omega_0$ is isomorphic to Martinet's normal form.*

Proof. Let us prove (1). Since $k \geq 2$, $e_2(\Delta)(0) = 0$. So,

$$(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0,0) \qquad \text{if and only if} \qquad e_1(\Delta)(0) \neq 0.$$

Thus, $F^*\omega_0$ is isomorphic to the above Martinet's normal form if and only if condition (5.2) holds, or equivalently, if and only if (5.3) holds.

Assertions (2) and (3) follow easily from assertion (1). □

Example 5.1. Consider the following two map-germs:

$$F_1 = (f_1, \dots, f_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0),$$
$$f_i(v) = v_i \quad (i \leq 2n-1), \qquad f_{2n}(v) = v_{2n}^3 + v_{2n-1}v_{2n},$$

and

$$F_2 = (f_1, \dots, f_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0),$$
$$f_i(v) = v_i \quad (i \leq 2n-1), \qquad f_{2n}(v) = v_{2n}^3 + v_k v_{2n}$$

(for some fixed k, $k < 2n-1$).

Then

(1) $F_1^*\omega_0$ is isomorphic to Martinet's normal form, since

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) = \frac{\partial v_{2n-1}}{\partial v_{2n-1}}(0) = 1 \neq 0;$$

(2) $F_2^*\omega_0$ is not isomorphic to Martinet's normal form, since

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) = \frac{\partial v_i}{\partial v_{2n-1}}(0) = 0.$$

Example 5.2. We revise F_2 in Example 5.1 adding the term b as follows:

$$F_3 = (f_1, \dots, f_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0),$$
$$f_i(v) = v_i \quad (i \leq 2n-1), \qquad f_{2n}(v) = v_{2n}^3 + v_{2k-1}v_{2n} + v_{2k} \quad (\text{or} \ \ v_{2n}^3 + v_{2k}v_{2n} + v_{2k-1})$$

(for some fixed k, $k < n$).

Then $F_3^*\omega_0$ is isomorphic to Martinet's normal form, since

$$\begin{aligned} e_1(\Delta)(0) &= \frac{\partial a_1}{\partial v_{2n-1}}(0) + \sum_{i=1}^{n-1}\biggl(-\frac{\partial a_1}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0) + \frac{\partial a_1}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0)\biggr) \\ &= -\frac{\partial a_1}{\partial v_{2k}}(0)\frac{\partial b}{\partial v_{2k-1}}(0) + \frac{\partial a_1}{\partial v_{2k-1}}(0)\frac{\partial b}{\partial v_{2k}}(0) = \pm 1 \neq 0. \end{aligned}$$

Example 5.3. Let

$$F_4 = (f_1, \dots, f_{2n})\colon (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0),$$
$$f_i(v) = v_i \quad (i \leq 2n-1), \qquad f_{2n}(v) = v_{2n-1}v_{2n}.$$

Then, although F_4 is very degenerate as a map-germ, $F_4^*\omega_0$ is stable as a closed 2-form and isomorphic to Martinet's normal form, since $\Delta = v_{2n-1}$ and

$$e_1(\Delta)(0) = \frac{\partial v_{2n-1}}{\partial v_{2n-1}}(0) + \sum_{i=1}^{n-1}\biggl(-\frac{\partial v_{2n-1}}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0) + \frac{\partial v_{2n-1}}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0)\biggr) = 1 \neq 0.$$

Since the classification of $\Sigma_{2,2,0}$ singularities of closed 2-forms is completed only for $n=2$, we consider only the case where $n=2$. In this case, we consider the introductory pre-normal form of type (2.2). Let us suppose that $F^*\omega_0$ is not isomorphic to Martinet's normal form of $\Sigma_{2,0}$ type singularities, i.e., suppose that

$$e_1(\Delta)(0)=\frac{\partial a_1}{\partial v_3}(0)-\frac{\partial a_1}{\partial v_2}(0)\frac{\partial b}{\partial v_1}(0)+\frac{\partial a_1}{\partial v_1}(0)\frac{\partial b}{\partial v_2}(0)=0.$$

Then $F^*\omega_0$ is isomorphic to Roussarie's $\Sigma_{2,2,0}$ normal form (see Theorem 4.5) if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0)\\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0)\end{pmatrix}=2.$$

Theorem 5.1. *Let $F=(f_1,\dots,f_4)\colon(\mathbb{R}^4,0)\to(\mathbb{R}^4,0)$ be an A_k map-germ with $b=0$ of the form*

$$f_i(v)=v_i\quad(i\le 3),\qquad f_4(v)=v_4^{k+1}+\sum_{i=1}^{k-1}a_i(v_1,v_2,v_3)v_4^i\quad(2\le k\le 4)$$

such that $F^*\omega_0$ is not isomorphic to Martinet's normal form of $\Sigma_{2,0}$ type singularities. Then $F^*\omega_0$ is isomorphic to Roussarie's $\Sigma_{2,2,0}$ normal forms

$$dv_1\wedge dv_2+v_3dv_2\wedge dv_3+d\left(v_1v_3+v_2v_4-\frac{v_3^3}{3}\right)\wedge dv_4\qquad(\Sigma^{\mathrm{e}}_{2,2,0})$$

or

$$dv_1\wedge dv_2+v_3dv_2\wedge d_3+d\left(v_1v_3-v_2v_4-\frac{v_3^3}{3}\right)\wedge dv_4\qquad(\Sigma^{\mathrm{h}}_{2,2,0})$$

if and only if

$$\operatorname{rank}\begin{pmatrix}\frac{\partial^2 a_1}{\partial v_3^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0)\\ 2\frac{\partial a_2}{\partial v_3}(0) & 6\end{pmatrix}=2.$$

Proof. In this case,

$$\Delta=(k+1)v_4^k+\sum_{i=1}^{k-1}ia_i(v_1,v_2,v_3)v_4^{i-1},$$

$$e_1=-\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1}+\frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2}+\frac{\partial}{\partial v_3}=-\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_2}v_4^i\right)\frac{\partial}{\partial v_1}+\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_1}v_4^i\right)\frac{\partial}{\partial v_2}+\frac{\partial}{\partial v_3},$$

$$e_2=\frac{\partial}{\partial v_4},$$

$$e_1(\Delta)=-\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_2}v_4^i\right)\left(\sum_{j=1}^{k-1}j\frac{\partial a_j}{\partial v_1}v_4^{j-1}\right)+\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_1}v_4^i\right)\left(\sum_{j=1}^{k-1}j\frac{\partial a_j}{\partial v_2}v_4^{j-1}\right)+\sum_{j=1}^{k-1}j\frac{\partial a_j}{\partial v_3}v_4^{j-1},$$

$$e_2(\Delta)=(k+1)kv_4^{k-1}+\sum_{j=2}^{k-1}j(j-1)a_jv_4^{j-2}.$$

Thus, by straightforward calculations for $k=2,3,4$ we have

$$\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0)\\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0)\end{pmatrix}=\begin{pmatrix}0 & \frac{\partial^2 a_1}{\partial v_3^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0)\\ 0 & 2\frac{\partial a_2}{\partial v_3}(0) & 6\end{pmatrix}.$$

This completes the proof of Theorem 5.1. $\square$

Example 5.4. Consider the following two cusp map-germs:

$$F_{5\pm} = (f_1, \dots, f_4)\colon (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0),$$

$$f_i(v) = v_i \quad (i \le 3), \qquad f_4(v) = v_4^3 + (v_1 \pm v_3^2)v_4,$$

and

$$F_6 = (f_1, \dots, f_4)\colon (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0),$$

$$f_i(v) = v_i \quad (i \le 3), \qquad f_4(v) = v_4^3 + v_1 v_4.$$

Using Theorem 4.4 or its corollary, one can easily check that neither $F_{5\pm}^*\omega_0$ nor $F_6^*\omega_0$ is isomorphic to Martinet's $\Sigma_{2,0}$. We see that $F_{5\pm}^*\omega_0$ is isomorphic to Roussarie's $\Sigma_{2,2,0}$ but $F_6^*\omega_0$ is not. To prove this fact, we apply Theorem 5.1 for $k = 2$. First we consider $F_{5\pm}^*\omega_0$. In this case

$$\operatorname{rank}\begin{pmatrix} \frac{\partial^2 a_1}{\partial v_3^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 2\frac{\partial a_2}{\partial v_3}(0) & 6 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \pm 2 & 0 \\ 0 & 6 \end{pmatrix} = \pm 2.$$

Therefore, by Theorem 5.1, $F_{5\pm}^*\omega_0$ is isomorphic to Roussarie's $\Sigma_{2,2,0}$. Moreover, $F_{5+}^*\omega_0$ is of type $\Sigma_{2,2,0}^{\mathrm{e}}$ and $F_{5-}^*\omega_0$ is of type $\Sigma_{2,2,0}^{\mathrm{h}}$.

Now we consider $F_6^*\omega_0$. In this case, since $f_4 = v_4^3 + v_1 v_4$,

$$\operatorname{rank}\begin{pmatrix} \frac{\partial^2 a_1}{\partial v_3^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 2\frac{\partial a_2}{\partial v_3}(0) & 6 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \ne 2.$$

Therefore, by Theorem 5.1, $F_6^*\omega_0$ is not isomorphic to Roussarie's $\Sigma_{2,2,0}$ form.

6. POISSON–LIE ALGEBRA OF HAMILTONIANS ASSOCIATED TO SINGULAR SYMPLECTIC FORMS

In this section we present the basic properties of the Poisson–Lie algebras of singular Hamiltonians determined by singular closed 2-forms.

Two germs ω and ω' of closed 2-forms on $\mathbb{R}^{2n}$ at p and q, respectively, are said to be *isomorphic* if there exists a diffeomorphism-germ $\varphi\colon (\mathbb{R}^{2n}, q) \to (\mathbb{R}^{2n}, p)$ such that $\omega' = \varphi^*\omega$.

Let ω be the germ at $0 \in \mathbb{R}^{2n}$ of a closed 2-form on $\mathbb{R}^{2n}$. For a function-germ h at $0 \in \mathbb{R}^{2n}$, the Hamiltonian vector field of h with respect to ω is the vector field $X_{\omega,h}$ *formally* defined by the equation (cf. [11, 21])

$$\omega(X_{\omega,h}, Y) = -Y(h) \qquad \text{for any vector field } Y \text{ on } \mathbb{R}^{2n}. \tag{6.1}$$

We often abbreviate $X_{\omega,h}$ as X_h.

The reason why we say "formally defined" in the above definition is that if ω is a degenerate closed 2-form, there are functions h for which the Hamiltonian vector fields $X_{\omega,h}$ are not defined on the singular point set of ω (see the example at the end of this section).

For the germ ω of a closed 2-form on $\mathbb{R}^{2n}$ at $0 \in \mathbb{R}^{2n}$, we set

$$\mathcal{H}_\omega = \{h \in \mathcal{E}_{2n} \mid X_h \text{ is smooth}\}. \tag{6.2}$$

Now, for two elements $h, k \in \mathcal{H}_\omega$, we define formally degenerate Poisson–Lie bracket $\{h, k\}_\omega$ with respect to the degenerate 2-form ω by

$$\{h, k\}_\omega = \omega(X_h, X_k) = X_k(h) = -X_h(k). \tag{6.3}$$

In the case where ω is a degenerate 2-form, it is not trivial that $\{h, k\}_\omega \in \mathcal{H}_\omega$. However, we can show that $\{h, k\}_\omega \in \mathcal{H}_\omega$ under a generic condition on ω that it has a representative closed 2-form defined on an open neighborhood U of 0, which we also denote by the same symbol ω, such that the set

$$O = \{p \in U \mid \operatorname{corank}_p \omega = 0\} \tag{6.4}$$

is open and dense in U, where $\operatorname{corank}_p \omega$ is the corank of ω at p.

Theorem 6.1. *Let ω be the germ of a closed 2-form satisfying the above generic condition. Then $\mathcal{H}_\omega$ is a Poisson–Lie algebra with the degenerate Poisson–Lie bracket $\{\cdot,\cdot\}_\omega$.*

Proof. Since the restriction $\omega_{|O}$ of ω to O is a nondegenerate 2-form on O, for any smooth function h on U the restriction $X_{h|O}$ of X_h to O is an ordinary Hamiltonian system with respect to the symplectic structure $\omega_{|O}$.

Let $h,k \in \mathcal{H}_\omega$. Then h, k, X_h and X_k are all smooth on U. Now $\{h,k\}_\omega = X_h(k)$ is smooth on O and we have

$$X_{\{h,k\}_\omega|O} = \bigl[X_{h|O}, X_{k|O}\bigr]. \tag{6.5}$$

Since $h,k \in \mathcal{H}_\omega$, X_h and X_k are smooth on U. Therefore, the right-hand side of (6.5) is extendable to the Lie bracket vector field $[X_h, X_k]$ of X_h and X_k, which is smooth on U. Thus $X_{\{h,k\}_\omega|O}$ is also extendable to a smooth vector field on U, which must be $X_{\{h,k\}_\omega}$, for O is open and dense in U. Thus $X_{\{h,k\}_\omega}$ is smooth and $\{h,k\}_\omega \in \mathcal{H}_\omega$. This completes the proof of the theorem. □

Theorem 6.2. *Let ω and ω' be the germs of closed 2-forms. If they are isomorphic and $\omega' = \varphi^*\omega$, then their associated Poisson–Lie algebras are isomorphic*:

$$\varphi^*\colon\ \mathcal{H}_\omega \cong \mathcal{H}_{\omega'}. \tag{6.6}$$

Let ω and ω' be the germs of closed 2-forms at $0 \in \mathbb{R}^{2n}$. Suppose that ω and ω' are isomorphic: $\omega' = \varphi^*\omega$ for the germ of a diffeomorphism $\varphi\colon (\mathbb{R}^{2n},0) \to (\mathbb{R}^{2n},0)$. To prove that $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega'}$ are isomorphic, we prove that the ring isomorphism

$$\varphi^*\colon\ \mathcal{E}_{2n} \to \mathcal{E}_{2n}, \qquad \varphi^*(h) = h \circ \varphi$$

induces an isomorphism

$$\varphi^*\colon\ \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$$

of Lie algebras. We prove this fact by proving the following two lemmas.

Lemma 6.1. *If $h \in \mathcal{H}_\omega$, then $\varphi^*(h) \in \mathcal{H}_{\omega'}$.*

Lemma 6.2. *Let $h,k \in \mathcal{H}_\omega$. Then $\varphi^*(\{h,k\}_\omega) = \{\varphi^*(h), \varphi^*(k)\}_{\omega'}$.*

Since $\varphi\colon (\mathbb{R}^{2n},0) \to (\mathbb{R}^{2n},0)$ is a diffeomorphism, from Lemma 6.1 we see that $\varphi^*(\mathcal{H}_\omega) \subset \mathcal{H}_{\omega'}$ and $(\varphi^{-1})^*(\mathcal{H}_{\omega'}) \subset \mathcal{H}_\omega$ and hence $\varphi^*\colon \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$ is a bijection. Since $\varphi^*\colon \mathcal{E}_{2n} \to \mathcal{E}_{2n}$ is a ring isomorphism, we see that for $h,k \in \mathcal{H}_\omega$

$$\varphi^*(h+k) = \varphi^*(h) + \varphi^*(k).$$

Then, with Lemma 6.2, we see that $\varphi^*\colon \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$ is an isomorphism of Lie algebras.

Proof of Lemma 6.1. Suppose that $\omega' = \varphi^*\omega$ for a diffeomorphism-germ φ and let $h \in \mathcal{H}_\omega$. Then we are going to show that $\varphi^*(h) = h \circ \varphi \in \mathcal{H}_{\omega'}$. By definition,

$$\mathcal{H}_\omega = \{h \in \mathcal{E}_{2n} \mid X_{\omega,h} \text{ is smooth}\}$$

and $X_{\omega,h}$ is defined by the equation

$$\omega(X_{\omega,h}, Y) = -Y(h)$$

for any vector field Y on $\mathbb{R}^{2n}$.

We are going to prove that if $X_{\omega,h}$ is smooth then $X_{\omega',h\circ\varphi}$ is also smooth. We prove this using local coordinates. Let $(u_1,u_2,\dots,u_{2n})$ be local coordinates in a neighborhood of $0\in\mathbb{R}^{2n}$ and let $\varphi=(\varphi_1,\dots,\varphi_{2n})\colon(\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$. Since $X_{\omega,h}$ and $X_{\omega',h\circ\varphi}$ are vector fields, they are formally of the form

$$X_{\omega,h}=\sum_{i=1}^{2n}a_i(u)\frac{\partial}{\partial u_i},\qquad X_{\omega',h\circ\varphi}=\sum_{i=1}^{2n}b_i(u)\frac{\partial}{\partial u_i}.$$

Since $\omega'=\varphi^*\omega$, we have

$$\omega'\biggl(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\biggr)=\sum_{k=1}^{2n}\sum_{\ell=1}^{2n}\frac{\partial\varphi_k}{\partial u_i}\omega\biggl(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\biggr)\frac{\partial\varphi_\ell}{\partial u_j}. \tag{6.7}$$

Therefore,

$$\omega'\biggl(X_{\omega',h\circ\varphi},\frac{\partial}{\partial u_j}\biggr)=\omega'\biggl(\sum_{i=1}^{2n}b_i(u)\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\biggr)=\sum_{i=1}^{2n}b_i(u)\sum_{k=1}^{2n}\sum_{\ell=1}^{2n}\frac{\partial\varphi_k}{\partial u_i}\omega\biggl(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\biggr)\frac{\partial\varphi_\ell}{\partial u_j}.$$

On the other hand, we have

$$\begin{aligned}\omega'\biggl(X_{\omega',h\circ\varphi},\frac{\partial}{\partial u_j}\biggr)&=-\frac{\partial}{\partial u_j}(h\circ\varphi)=-\sum_{m=1}^{2n}\frac{\partial h}{\partial u_m}(\varphi(u))\frac{\partial\varphi_m}{\partial u_j}\\&=\sum_{m=1}^{2n}\omega\biggl(X_{\omega,h}(\varphi(u)),\frac{\partial}{\partial u_m}\biggr)\frac{\partial\varphi_m}{\partial u_j}=\sum_{m=1}^{2n}\sum_{p=1}^{2n}a_p(\varphi(u))\,\omega\biggl(\frac{\partial}{\partial u_p},\frac{\partial}{\partial u_m}\biggr)\frac{\partial\varphi_m}{\partial u_j},\end{aligned}$$

where the first equality holds by the definition of $X_{\omega',h\circ\varphi}$, and the third, by the definition of $X_{\omega,h}$. Thus we have

$$\begin{aligned}\sum_{i=1}^{2n}b_i(u)\sum_{k=1}^{2n}\sum_{\ell=1}^{2n}\frac{\partial\varphi_k}{\partial u_i}\,\omega\biggl(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\biggr)\frac{\partial\varphi_\ell}{\partial u_j}&=\omega'\biggl(X_{\omega',h\circ\varphi},\frac{\partial}{\partial u_j}\biggr)\\&=\sum_{m=1}^{2n}\sum_{p=1}^{2n}a_p(\varphi(u))\,\omega\biggl(\frac{\partial}{\partial u_p},\frac{\partial}{\partial u_m}\biggr)\frac{\partial\varphi_m}{\partial u_j}.\end{aligned}$$

Expressing this equality in matrices, we have

$$\begin{aligned}(b_1(u),\dots,b_{2n}(u))\,{}^{\mathrm t}\biggl(\frac{\partial\varphi_k}{\partial u_i}(u)\biggr)\biggl(\omega\biggl(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\biggr)(\varphi(u))\biggr)\biggl(\frac{\partial\varphi_\ell}{\partial u_j}(u)\biggr)&\\=\bigl(a_1(\varphi(u)),\dots,a_{2n}(\varphi(u))\bigr)\biggl(\omega\biggl(\frac{\partial}{\partial u_p},\frac{\partial}{\partial u_m}\biggr)(\varphi(u))\biggr)\biggl(\frac{\partial\varphi_m}{\partial u_j}(u)\biggr)&\end{aligned}$$

and hence

$$(b_1(u),\dots,b_{2n}(u))\,{}^{\mathrm t}\biggl(\frac{\partial\varphi_k}{\partial u_i}(u)\biggr)=\bigl(a_1(\varphi(u)),\dots,a_{2n}(\varphi(u))\bigr). \tag{6.8}$$

Since $X_{\omega,h}=\sum_{i=1}^{2n}a_i(u)(\partial/\partial u_i)$ is smooth, $a_1,\dots,a_{2n}$ are smooth functions and ${}^{\mathrm t}(\partial\varphi_k/\partial u_i)$ is an invertible matrix smoothly depending on u, we see that $b_1,\dots,b_{2n}$ are smooth functions. Thus $X_{\omega',h\circ\varphi}=\sum_{i=1}^{2n}b_i(u)(\partial/\partial u_i)$ is smooth and $\varphi^*(h)=h\circ\varphi\in\mathcal{H}_{\omega'}$. $\square$

Proof of Lemma 6.2. By definition,

$$\{h,k\}_\omega(u)=\omega(X_{\omega,h},X_{\omega,k})(u),\qquad\{h\circ\varphi,k\circ\varphi\}_{\omega'}(u)=\omega'(X_{\omega',h\circ\varphi},X_{\omega',k\circ\varphi})(u).$$

We express $X_{\omega,h}$, $X_{\omega,k}$, $X_{\omega',h\circ\varphi}$ and $X_{\omega',k\circ\varphi}$ again using local coordinates $u_1,\dots,u_{2n}$:

$$X_{\omega,h}=\sum_{i=1}^{2n}a_i(u)\frac{\partial}{\partial u_i},\qquad X_{\omega',h\circ\varphi}=\sum_{i=1}^{2n}\alpha_i(u)\frac{\partial}{\partial u_i},$$
$$X_{\omega,k}=\sum_{i=1}^{2n}b_i(u)\frac{\partial}{\partial u_i},\qquad X_{\omega',k\circ\varphi}=\sum_{i=1}^{2n}\beta_i(u)\frac{\partial}{\partial u_i}.$$

Then from (6.8) we have

$$(\alpha_1(u),\dots,\alpha_{2n}(u))\,{}^{\mathrm t}\!\left(\frac{\partial\varphi_k}{\partial u_i}(u)\right)=\bigl(a_1(\varphi(u)),\dots,a_{2n}(\varphi(u))\bigr),$$
$$(\beta_1(u),\dots,\beta_{2n}(u))\,{}^{\mathrm t}\!\left(\frac{\partial\varphi_k}{\partial u_i}(u)\right)=\bigl(b_1(\varphi(u)),\dots,b_{2n}(\varphi(u))\bigr).$$

Thus

$$\begin{aligned}\{h\circ\varphi,k\circ\varphi\}_{\omega'}(u)&=\omega'(X_{\omega',h\circ\varphi},X_{\omega',k\circ\varphi})(u)=\omega'\left(\sum_{i=1}^{2n}\alpha_i(u)\frac{\partial}{\partial u_i},\sum_{i=1}^{2n}\beta_i(u)\frac{\partial}{\partial u_i}\right)\\&=\sum_{i=1}^{2n}\sum_{j=1}^{2n}\alpha_i(u)\beta_i(u)\,\omega'\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)\\&=(\alpha_1(u),\dots,\alpha_{2n}(u))\left(\omega'\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)(u)\right){}^{\mathrm t}(\beta_1(u),\dots,\beta_{2n}(u)).\end{aligned}$$

Since $\omega'=\varphi^*\omega$, from the proof of Lemma 6.1 we have

$$\omega'\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)(u)=\sum_{k=1}^{2n}\sum_{\ell=1}^{2n}\frac{\partial\varphi_k}{\partial u_i}(u)\,\omega\left(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\right)(u)\,\frac{\partial\varphi_\ell}{\partial u_j}(u)$$

and hence

$$\left(\omega'\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)(u)\right)={}^{\mathrm t}\!\left(\frac{\partial\varphi_k}{\partial u_i}(u)\right)\left(\omega\left(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\right)(u)\right)\left(\frac{\partial\varphi_\ell}{\partial u_j}(u)\right).$$

Thus

$$\begin{aligned}&\{h\circ\varphi,k\circ\varphi\}_{\omega'}(u)\\&\quad=(\alpha_1(u),\dots,\alpha_{2n}(u))\,{}^{\mathrm t}\!\left(\frac{\partial\varphi_k}{\partial u_i}(u)\right)\left(\omega\left(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\right)(u)\right)\left(\frac{\partial\varphi_\ell}{\partial u_j}(u)\right){}^{\mathrm t}(\beta_1(u),\dots,\beta_{2n}(u))\\&\quad=\bigl(a_1(\varphi(u)),\dots,a_{2n}(\varphi(u))\bigr)\left(\omega\left(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell}\right)(u)\right){}^{\mathrm t}\bigl(b_1(\varphi(u)),\dots,b_{2n}(\varphi(u))\bigr)\\&\quad=\omega(X_{\omega,h},X_{\omega,k})(\varphi(u))=\{h,k\}_\omega(\varphi(u)).\end{aligned}$$

And we have

$$\{\varphi^*h,\varphi^*k\}_{\omega'}=\varphi^*\{h,k\}_\omega.$$

This completes the proof of Lemma 6.2. ☐

Example 6.1. Consider the closed 2-form

$$\omega=u_1\,du_1\wedge du_2\qquad\text{on }\mathbb{R}^2$$

and a function $h = u_2$. Then X_h is defined by

$$\omega\left(X_h, \frac{\partial}{\partial u_i}\right) = -\frac{\partial h}{\partial u_i}.$$

Since X_h has the form $X_h = a_1(u)(\partial/\partial u_1) + a_2(u)(\partial/\partial u_2)$, the equation becomes

$$u_1\, du_1 \wedge du_2 \left(a_1(u)\frac{\partial}{\partial u_1} + a_2(u)\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_i}\right) = -\frac{\partial h}{\partial u_i}, \qquad i = 1, 2.$$

Then we have

$$-u_1 a_2(u) = -\frac{\partial h}{\partial u_1} = 0, \qquad u_1 a_1(u) = -\frac{\partial h}{\partial u_2} = -1.$$

Since $u_1(0) = 0$, there are no functions $a_1(u)$ such that $u_1 a_1 = -1$. In this case, X_h is not defined on the set $\{u_1 = 0\}$, which is the singular point set of ω.

7. POISSON–LIE ALGEBRAS FOR $\Sigma_{2,0}$, $\Sigma^{\mathrm{e}}_{2,2,0}$ AND $\Sigma^{\mathrm{h}}_{2,2,0}$ STABLE SINGULARITIES

In this section we will characterize properties of the Poisson–Lie algebra for the singular symplectic structures of Martinet's and Roussarie's forms.

Proposition 7.1. *Let $\omega_{2,0}$ denote Martinet's normal form*:

$$\omega_{2,0} = v_1\, dv_1 \wedge dv_2 + dv_3 \wedge dv_4 + \ldots + dv_{2n-1} \wedge dv_{2n}. \tag{7.1}$$

Then

$$\mathcal{H}_{\omega_{2,0}} = \langle v_1^2 \rangle_{\mathcal{E}_{v_1,\ldots,v_{2n}}} + \mathcal{E}_{v_3,\ldots,v_{2n}}. \tag{7.2}$$

Proof. In what follows, let $\partial_i = \partial/\partial v_i$. Then for $1 \le i \le j \le 2n$

$$\omega_{2,0}(\partial_i, \partial_j) = \begin{cases} v_1 & \text{for } i = 1,\ j = 2, \\ 1 & \text{for } i = 2k-1,\ j = 2k,\ 2 \le k \le n, \\ 0 & \text{otherwise} \end{cases}$$

and we have

$$\omega_{2,0}^{-1}(\partial_i, \partial_j) = \begin{cases} -\dfrac{1}{v_1} & \text{for } i = 1,\ j = 2, \\ -1 & \text{for } i = 2k-1,\ j = 2k,\ 2 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \in \mathcal{H}_{\omega_{2,0}}$ if $\sum_j \omega_{2,0}^{-1}(\partial_i, \partial_j)(\partial h/\partial v_j)$ is smooth for $1 \ge i \le 2n$. But this implies

$$\frac{\partial h}{\partial v_1}, \frac{\partial h}{\partial v_2} \in \langle v_1 \rangle_{\mathcal{E}_v}.$$

So we express h in the form

$$h(v) = v_1^2 \alpha(v) + v_1 \beta(v_2, \ldots, v_{2n}) + \gamma(v_2, \ldots, v_{2n}).$$

Then $\partial h/\partial v_1, \partial h/\partial v_2 \in \langle v_1 \rangle_{\mathcal{E}_v}$ if and only if

$$\beta(v_2, \ldots, v_{2n}), \frac{\partial \gamma}{\partial v_2}(v_2, \ldots, v_{2n}) \in \langle v_1 \rangle_{\mathcal{E}_v},$$

which holds if and only if

$$\beta(v_2, \ldots, v_{2n}) = 0, \qquad \frac{\partial \gamma}{\partial v_2}(v_2, \ldots, v_{2n}) = 0,$$

which holds if and only if h has the form

$$h(v) = v_1^2\alpha(v) + \gamma(v_3, \dots, v_{2n}).$$

Therefore, $h \in \mathcal{H}_{\omega_{2,0}}$ if and only if $h \in \langle v_1^2\rangle_{\mathcal{E}_v} + \mathcal{E}_{v_3,\dots,v_{2n}}$. $\square$

For comparison with the general calculations we continue with an example and Roussarie's elliptic and hyperbolic normal forms.

Example 7.1 ($\Sigma_{2,2,0}$-type cusps)**.** We consider the following two cusps $F_{5\pm}$:

$$F_{5\pm} = (f_1, \dots, f_4)\colon (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0),$$
$$f_i(v) = v_i \quad (i \le 3), \qquad f_4(v) = v_4^3 + (v_1 \pm v_3^2)v_4.$$

Then $F_{5+}^*\omega_0$ is of type $\Sigma^{\mathrm{e}}_{2,2,0}$ and $F_{5-}^*\omega_0$ is of type $\Sigma^{\mathrm{h}}_{2,2,0}$, and

$$F_{5\pm}^*\omega_0 = dv_1 \wedge dv_2 - v_4\, dv_1 \wedge dv_3 + (3v_4^2 + v_1 \pm v_3^2)\, dv_3 \wedge dv_4. \tag{7.3}$$

Let ω_{e} and ω_{h} denote Roussarie's elliptic and hyperbolic normal forms, respectively:

$$\omega_{\mathrm{e}} = dv_1 \wedge dv_2 + v_3\, dv_1 \wedge dv_4 + v_3\, dv_2 \wedge dv_3 + v_4\, dv_2 \wedge dv_4 + (v_1 - v_3^2)\, dv_3 \wedge dv_4, \tag{7.4}$$

$$\omega_{\mathrm{h}} = dv_1 \wedge dv_2 + v_3\, dv_1 \wedge dv_4 + v_3\, dv_2 \wedge dv_3 - v_4\, dv_2 \wedge dv_4 + (v_1 - v_3^2)\, dv_3 \wedge dv_4. \tag{7.5}$$

In what follows, let $\partial_i = \partial/\partial v_i$. Then from (7.4), (7.5) and (7.3) we have

$$(\omega_{\mathrm{e}}(\partial_i, \partial_j))^{-1} = \frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & v_4 & -v_3 \\ v_1 - v_3^2 & 0 & -v_3 & 0 \\ -v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix},$$

$$(\omega_{\mathrm{h}}(\partial_i, \partial_j))^{-1} = \frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & -v_4 & -v_3 \\ v_1 - v_3^2 & 0 & -v_3 & 0 \\ v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix},$$

$$\bigl(F_{5\pm}^*\omega_0(\partial_i, \partial_j)\bigr)^{-1} = \frac{1}{v_1 \pm v_3^2 + 3v_4^2}\begin{pmatrix} 0 & -(v_1 \pm v_3^2 + 3v_4^2) & 0 & 0 \\ v_1 \pm v_3^2 + 3v_4^2 & 0 & 0 & -v_4 \\ 0 & 0 & 0 & -1 \\ 0 & v_4 & 1 & 0 \end{pmatrix}.$$

We also get

$$\det(\omega_{\mathrm{e}}(\partial_i, \partial_j)) = \det(\omega_{\mathrm{h}}(\partial_i, \partial_j)) = v_1^2, \qquad \det\bigl(F_{5\pm}^*\omega_0(\partial_i, \partial_j)\bigr) = \bigl(v_1 \pm v_3^2 + 3v_4^2\bigr)^2.$$

Now we provide implicit formulas for the Poisson–Lie algebras $\mathcal{H}_{\omega_{\mathrm{e}}}$, $\mathcal{H}_{\omega_{\mathrm{h}}}$ and $\mathcal{H}_{F_{5\pm}^*\omega_0}$, associated to Roussarie's hyperbolic and elliptic normal forms ω_{e} and ω_{h} as well as to $\Sigma_{2,2,0}$-type cusp example. By straightforward calculations we get

Proposition 7.2 (first implicit formula)**.** 1. *Let $h \in \mathcal{E}_v$. Then $h \in \mathcal{H}_{\omega_{\mathrm{e}}}$ if and only if h satisfies the following conditions*:

$$-v_4\frac{\partial h}{\partial v_1} + v_3\frac{\partial h}{\partial v_2} - \frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \qquad v_3\frac{\partial h}{\partial v_1} + \frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{7.6}$$

2. *Let $h \in \mathcal{E}_v$. Then $h \in \mathcal{H}_{\omega_{\mathrm{h}}}$ if and only if h satisfies the following conditions*:

$$v_4\frac{\partial h}{\partial v_1} + v_3\frac{\partial h}{\partial v_2} - \frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \qquad v_3\frac{\partial h}{\partial v_1} + \frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{7.7}$$

3. *Let* $h \in \mathcal{E}_v$. *Then* $h \in \mathcal{H}_{F^*_{5\pm}\omega_0}$ *if and only if* h *satisfies the following conditions*:

$$\frac{\partial h}{\partial v_4} \in \langle v_1 \pm v_3^2 + 3v_4^2\rangle_{\mathcal{E}_v}, \qquad v_4\frac{\partial h}{\partial v_2} + \frac{\partial h}{\partial v_3} \in \langle v_1 \pm v_3^2 + 3v_4^2\rangle_{\mathcal{E}_v}.$$

Next, for $\mathcal{H}_{\omega_e}$ and $\mathcal{H}_{\omega_h}$ we get less implicit differential algebraic formulas. Expressing h in the form

$$h = v_1^2\alpha(v) + v_1\beta(v_2, v_3, v_4) + \gamma(v_2, v_3, v_4), \tag{7.8}$$

we have

Proposition 7.3 (second implicit formula)**.**

$$\mathcal{H}_{\omega_e} = \langle v_1^2\rangle_{\mathcal{E}_v} + \left\{ v_1\beta + \gamma \;\middle|\; \beta, \gamma \in \mathcal{E}_{v_2,v_3,v_4} \text{ satisfying the equations} \right.$$
$$-v_4\beta(v_2,v_3,v_4) + v_3\frac{\partial\gamma}{\partial v_2}(v_2,v_3,v_4) - \frac{\partial\gamma}{\partial v_4}(v_2,v_3,v_4) = 0,$$
$$\left. v_3\beta(v_2,v_3,v_4) + \frac{\partial\gamma}{\partial v_3}(v_2,v_3,v_4) = 0 \right\}$$

and

$$\mathcal{H}_{\omega_h} = \langle v_1^2\rangle_{\mathcal{E}_v} + \left\{ v_1\beta + \gamma \;\middle|\; \beta, \gamma \in \mathcal{E}_{v_2,v_3,v_4} \text{ satisfying the equations} \right.$$
$$v_4\beta(v_2,v_3,v_4) + v_3\frac{\partial\gamma}{\partial v_2}(v_2,v_3,v_4) - \frac{\partial\gamma}{\partial v_4}(v_2,v_3,v_4) = 0,$$
$$\left. v_3\beta(v_2,v_3,v_4) + \frac{\partial\gamma}{\partial v_3}(v_2,v_3,v_4) = 0 \right\}.$$

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