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SOLVABLE SUBMANIFOLDS OF TANGENT BUNDLE AND J. MATHER GENERIC LINEAR EQUATIONS*

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Abstract. Using J. Mather results on solutions of generic linear equations the smooth solvability of implicit differential systems is investigated. Implicit Hamiltonian systems are considered and algebraic version of J. Mather theorem was applied in this case. For the generalized Hamiltonian systems defined by P.A.M. Dirac on smooth constraints we find the corresponding Poisson-Lie algebras as a basic symplectic invariants of submanifolds in the symplectic space.

Key words. Symplectic manifold, singularities, Hamiltonian systems, Poisson-Lie algebras.

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1. Introduction. Let M be a smooth submanifold of $T\mathbb{R}^n$, $M \subset T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$, $dimM = m$. M is considered as a first order differential system. If M is transversal to the fibers of the tangent bundle projection $\pi : T\mathbb{R}^n \to \mathbb{R}^n$ then M is locally solvable at each point of M. A point $(x, \dot{x}) \in M \subset T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$ is called a *solvable point* of M if there exists a smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that its prolongation $\dot{\gamma} : \mathbb{R} \to T\mathbb{R}^n : t \mapsto \dot{\gamma}(t)$ belongs to M, $im(\dot{\gamma}) \subset M$.

M is called *solvable* if M consists only of solvable points. If additionally the integral curve γ depends smoothly on initial conditions in a neighborhood of every point of M then we say that M is *smoothly solvable.* If $\pi\mid_M$ is a diffeomorphism then M is a smoothly solvable vector field on $\mathbb{R}^n$. If M is not transversal to the fibers of π, then M may not be solvable in critical points of $\pi\mid_M$, and this is common property for typical position of M. In this case M is also called the system of implicit differential equations. Now the natural question we ask is to find a necessary and sufficient condition for a manifold $M \subset T\mathbb{R}^n$ to be smoothly solvable. The natural necessary condition for a point $(x, \dot{x}) \in M$ to be solvable is that

$$\dot{x} \in d(\pi\mid_M)_{(x,\dot{x})}(T_{(x,\dot{x})}M), \quad \pi(x, \dot{x}) = x.$$

In what follows we call this condition the *tangential solvability condition.*

We can ask whether this condition is also a sufficient condition for a submanifold M to be solvable. Although the answer for this question is negative, there is a wide class of submanifolds of $T\mathbb{R}^n$ for which the tangential solvability condition is also sufficient.

The solvability we consider is a local property. Thus we suppose M to be the image of an embedding $M = F(U)$, $F = (f, \dot{f}) : U \to T\mathbb{R}^m$ of an open set U of $\mathbb{R}^m$. Then we have that an implicit differential equation $M = F(U)$ of $T\mathbb{R}^m$, where F is an embedding, is *smoothly solvable* if there exists a smooth tangent vector field X on U such that $(f(u), \dot{f}(u)) = df(X(u))$, for all $u \in U$.

If an implicit differential equation $M = F(U)$ of $T\mathbb{R}^m$ is smoothly solvable with a smooth vector field X on U, then every point $(x_0, \dot{x}_0) \in M$ is a solvable point of M.

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Moreover, integral curves of the vector field X give a family of general solutions of M smoothly depending on initial conditions. If the smooth vector field X on U has the form $X(u) = \sum_{i=1}^{m} a_i(u)\frac{\partial}{\partial u_i}|_u$, where $a_i(u)$ are smooth and Jf is a Jacobian matrix of f. Then we immediately have that an implicit differential equation $M = F(U) \subset T\mathbb{R}^m$ given by an embedding $F = (f, \dot{f})$ is smoothly solvable if and only if the linear equation $\dot{f}(u) = Jf(u)a(u)$ has a smooth solution $a(u) = (a_1(u), \cdots, a_m(u))$. This condition fulfilled to each $u \in U$ is a tangential solvability condition for $F(U)$.

Now solvability of implicit differential equations becomes equivalent to a smooth solvability of linear algebraic equations. Using the classical result by J. Mather [9] we get the basic solvability result. We prove that if $J(f_1, \cdots, f_m) : U \to M(m)$ is transversal to $\Sigma_r(m)$ at the origin 0, then an implicit differential equation $M = F(U) \subset T\mathbb{R}^m$ is smoothly solvable in a neighborhood of $(f(0), \dot{f}(0))$. Here $M(m)$ denotes the set of all $m \times m$ real matrices and $\Sigma_r(m)$ denotes the set of all $m \times m$ real matrices with rank r. The more general algebraic version of this result is also proved. The solvability results in general case are applied to implicit Hamiltonian system which is an isotropic embedding $F : \mathbb{R}^{2n} \supset U \to T\mathbb{R}^{2n}$ into the tangent bundle $T\mathbb{R}^{2n}$ endowed with a symplectic structure $\dot{\omega}$ defined by the canonical flat morphism between tangent and co-tangent bundles of the symplectic space $(\mathbb{R}^{2n}, \omega)$, (see [13]). The solvability properties of $F(U)$ were partially investigated in [5]. In this paper we extend the notion of implicit Hamiltonian system allowing f to be singular. In this case all the properties of the implicit Hamiltonian system are defined by its parametrization F and we will call F a Hamiltonian mapping if it is isotropic, after $F^*\dot{\omega} = 0$, solvable and $F^*\dot{\theta} = -dh$ for some smooth function h (called the generating function of F). To each F we associate $f = \pi \circ F$, where $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a tangent bundle projection and look at F as a vector field along f. We investigate the space $\mathcal{R}_f$ of all generating functions of F for fixed singular f (all vector fields along f). Thus for the corank 1 case of f the generating function h for isotropic F along f, or more precisely its derivative $\partial_e h$ belongs to the ideal generated by the determinant of a Jacobian matrix $\Delta_f = det(Jf)$, where e spans the kernel of the Jacobian matrix at singular point.

The purpose of this paper is twofold: First we provide necessary and sufficient conditions of solvability for various classes of implicit systems and search for generic peculiarities of non-solvable points. Second we formulate and investigate the problem of solvability for implicit Hamiltonian systems as a specialized class of implicit systems.

In Sections 2 we formulate the solvability problem, and prove the first part of the main results of the paper: the smooth solvability conditions for the generic (in the sense of J. Mather) differential systems (Theorem 2.2). In Section 3 we extend J. Mather's results weakening his transversality condition to algebraic conditions on determinants. This result is formulated in Theorem 3.1. and Theorem 3.2. Section 4 is devoted to the non-solvable points of differential systems. We prove here (Theorem 4.1, Theorem 4.2) that the non-solvable points are isolated generically in the sense of J.C. Tougeron. Specialization of our differential systems to isotropic ones, was done in Section 5. In this case we applied the main results of the previous sections and obtained the necessary and sufficient conditions for solvability of implicit Hamiltonian systems.

2. Solvable submanifolds of tangent bundle. To come closer to our problem we explain some sufficient conditions for a submanifold M of $T\mathbb{R}^n$ to be solvable. One of them plays an important role in the proofs of our main theorems. At first we have the two immediate Lemmas.

LEMMA 2.1. *Let $M \subset T\mathbb{R}^n$ be a submanifold satisfying tangential solvability condition.*

1) *If $\pi\mid_M: M \to \mathbb{R}^n$ is either an embedding or a submersion, then M is solvable.*

2) *If $\pi(M)$ is a submanifold of $\mathbb{R}^n$ and $\pi\mid_M: M \to \pi(M)$ is a submersion, then M is solvable.*

This is a local problem so we may suppose that the manifold M under consideration is the image of an embedding of $\mathbb{R}^m$.

$$M = F(\mathbb{R}^m), \qquad F = (f, \dot{f}) : \mathbb{R}^m \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \quad \text{is an embedding.}$$

Let $Jf(u)$ denote the Jacobian matrix of $f = (f_1, \ldots, f_n)$ at $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. Since $d(\pi\mid_M)(T_{(x,\dot{x})}M) = Jf(u)(\mathbb{R}^m)$ we have immediately

LEMMA 2.2. *Let $M = (f, \dot{f})(\mathbb{R}^m) \subset T\mathbb{R}^n$ be as above. Then*

1) *M satisfies the tangential solvability condition if and only if*

$$\dot{f}(u) \in Jf(u)(\mathbb{R}^m) \quad \text{for every} \quad u \in \mathbb{R}^m.$$

2) *M is smoothly solvable if there exists a smooth vector field X on $\mathbb{R}^m$ such that*

$$F(u) = df(X(u)), \quad \forall u \in \mathbb{R}^m \tag{1}$$

COROLLARY 2.1. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth mapping with*

$$rankdf_u \geq \frac{m}{2}, \qquad \forall u \in \mathbb{R}^m.$$

Then there exists (always) a smooth mapping $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ such that $(f, \dot{f}) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$ is an embedding and $M = (f, \dot{f})(\mathbb{R}^m)$ is a smoothly solvable manifold.

Proof. Since

$$r = \quad \mathrm{rank} df_u : T_u\mathbb{R}^m \to T_{f(u)}\mathbb{R}^n \geq \frac{m}{2}, \qquad \forall u \in \mathbb{R}^m,$$

we may choose local coordinates such that f is of the form

$$f(u_1, \ldots, u_m) = (u_1, \ldots, u_r, f_{r+1}(u_1, \ldots, u_m), \ldots, f_n(u_1, \ldots, u_m)).$$

Let $a_1(u), \ldots, a_n(u)$ be the smooth functions chosen in the following form

$$a_1(u) = u_{r+1}, \ldots, a_{m-r}(u) = u_m \qquad \text{and} \qquad a_{m-r+1}(u), \ldots, a_n(u) \quad \text{being arbitrary.}$$

And consider a mapping $\dot{f} = (g_1, \ldots, g_n) : \mathbb{R}^m \to \mathbb{R}^n$ defined by equation (1), where $X(u) = \sum_{i=1}^m a_i(u) \frac{\partial}{\partial u_i}$. Then $(f, \dot{f}) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$ is an embedding and from Lemma 2.2 we see that $M = (f, \dot{f})(\mathbb{R}^m)$ is a smoothly solvable manifold. □

REMARK 2.1. Now we came to the natural question: *Does some pathology of the critical point set on M imply the non solvability ?* To some extent the answer is positive. We may formulate the following supposition: if $f : \mathbb{R}^m \to \mathbb{R}^n$ is stable (cf. [10]), then for any $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ such that $(f, \dot{f}) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$ is an embedding and $M = (f, \dot{f})(\mathbb{R}^m)$ satisfies tangential solvability condition, M has to be solvable.

Therefore if this conjecture is true, then it implies that if M is not solvable, then f is not stable, and therefore $\pi\,|_M\colon M \to \mathbb{R}^n$ is not a stable map. Here let us remind that the two maps $f : \mathbb{R}^m \to \mathbb{R}^n$ and $\pi\,|_M\colon M \to \mathbb{R}^n$ are right equivalent in J. Mather's sense, since $(f, \dot{f}) : \mathbb{R}^m \to M = (f, \dot{f})(\mathbb{R}^m)$ is a diffeomorphism. Thus the singular point set is not the one of a stable map. On the other hand, the Corollary 2.1 says that if

$$\operatorname{rank}(d\pi_1\,|_M) : T_{(x,\dot{x})}(M) \to T_x(\mathbb{R}^n) \geq \frac{m}{2},$$

then no matter how much pathological its singularity is, with a good partner $\dot{f}$, $M = (f, \dot{f})(\mathbb{R}^m)$ becomes solvable.

Now we prove that the tangential solvability condition is also sufficient for a class of manifolds, which we call the *generic manifolds* and which is much wider than the class of manifolds $M \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ such that the projections $\pi\,|_M\colon M \to \mathbb{R}^n$ are $C^\infty-$stable.

We saw in Lemma 2.1 that if the linear equation (1) has a smooth solution, then the manifold M is solvable. For a condition that a linear equation (1) has a smooth solution, we have the theorem of J. Mather ([9]). Combining Lemma 2.2 and J. Mather's theorem we obtain a sufficient condition for a manifold to be solvable.

Let $J^1(M, \mathbb{R}^n)$ be the 1 jet bundle of 1 jets of maps f of M into $\mathbb{R}^n$, and let $S_r(M, \mathbb{R}^n)$ be the subset of all 1 jets $j^1f(u) \in J^1(M, \mathbb{R}^n)$ such that the rank of the differential $df_u : T_u(M) \to T_{f(u)}\mathbb{R}^n$ at $u \in M$ is r. Then $S_r(M, \mathbb{R}^n)$ is a submanifold of $J^1(M, \mathbb{R}^n)$ of codimension $(n - r) \times (m - r)$, where $m = \dim M$.

Definition 2.1 (following J. Mather [9]). *A manifold $M \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is said to be a generic submanifold of $T\mathbb{R}^n$ if the 1 jet extension $j^1(\pi\,|_M) : M \to J^1(M, \mathbb{R}^n)$ is transversal to all $S_r(M, \mathbb{R}^n)$, $r = 0, 1, \ldots, min(m, n)$.*

Theorem 2.1. *A generic submanifold M of $T\mathbb{R}^n$ is solvable if and only if it satisfies tangential solvability condition. Thus tangential solvability condition holds sufficient for generic submanifolds of $T\mathbb{R}^n$.*

Remark that the generic submanifolds form a very wide class in the set of all submanifolds of $T\mathbb{R}^n$. This can be seen as follows. We suppose that the manifold M under consideration is the image of an embedding of $\mathbb{R}^m$ into $T\mathbb{R}^n$:

$$M = (f, \dot{f})(\mathbb{R}^m), \qquad (f, \dot{f}) : \mathbb{R}^m \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \quad \text{an embedding.}$$

Then, the 1 jet extension $j^1(\pi\,|_M) : M \to J^1(M, \mathbb{R}^n)$ is transversal to $S_r(M, \mathbb{R}^n)$ if and only if the 1 jet extension $j^1f : \mathbb{R}^m \to J^1(\mathbb{R}^m, \mathbb{R}^n)$ is transversal to $S_r(\mathbb{R}^m, \mathbb{R}^n)$. By Thom's transversality theorem, the set of smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $j^1f : \mathbb{R}^m \to J^1(\mathbb{R}^m, \mathbb{R}^n)$ is transversal to all $S_r(\mathbb{R}^m, \mathbb{R}^n)$, $r = 0, 1, \ldots, \min(m, n)$ form an open dense set in the set of smooth maps of $\mathbb{R}^m$ into $\mathbb{R}^n$. Thus generic submanifolds form a very wide class in the set of all submanifolds of $T\mathbb{R}^n$.

Now we reformulate our problem in more accessible terms.

Definition 2.2. *A smooth map $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ is said to be tangent to a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ if the pair $(f, \dot{f}) : \mathbb{R}^m \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ forms an embedding and the linear equation (1) has a solution $(a_1(x), a_2(x), \ldots, a_n(x))$ for every point $x \in \mathbb{R}^m$.*

We see that the tangential solvability condition becomes exactly the tangency of the smooth map $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ to the smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$, defining the embedding $(f, \dot{f}) : \mathbb{R}^m \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ of M. If we treat $\dot{f}$ as a vector field $\dot{f}(u) = \sum_{j=1}^{n} g_j(u)\frac{\partial}{\partial x_j}$ then the tangency condition means that the equation $(f_* X)(u) = \dot{f}(u)$ has a solution $X(u) = \sum_{i=1}^{m} X_i(u)\frac{\partial}{\partial u_i}$ for any $u \in \mathbb{R}^m$.

Now the problem we came to solve is to find conditions to be posed on a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ so that the image $M = (f, \dot{f})(\mathbb{R}^m)$ is a solvable manifold for every smooth map $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ tangent to f. We can also reformulate this problem in other words as follows:
Find conditions to be posed on a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ so that the linear equation (1) has a smooth solution $(a_1(u), a_2(u), \dots, a_m(u))$ for every smooth map $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ tangent to f.

The result of J. Mather [9] concerning solutions of linear equation helps to give a good answer to this problem. In the rest of this section, we will follow the notation from [9]. Consider an $n \times m$ matrix $A(u)$ of smooth function-germs : $(\mathbb{R}^k, 0) \to \mathbb{R}$

$$A(u) = \begin{pmatrix} a_{11}(u) & a_{12}(u) & \dots & a_{1m}(u) \\ a_{21}(u) & a_{22}(u) & \dots & a_{2m}(u) \\ \dots & \dots & \dots & \dots \\ a_{n1}(u) & a_{n2}(u) & \dots & a_{nm}(u) \end{pmatrix},$$

and a column vector

$$g(u) = \begin{pmatrix} g_1(u) \\ \vdots \\ g_n(u) \end{pmatrix},$$

where $a_{ij}(u)$ and g_i are smooth function-germs $\mathbb{R}^k \to \mathbb{R}$. Consider the linear equation

$$A(u)b = g(u), \tag{2}$$

where b is a column vector of length m.

Let $E(n, m)$ denote the space of $n \times m$ matrices of real numbers. Let S_r denote the subset of $E(n, m)$ consisting of matrices of rank r. Then S_r is a submanifold of codimension $(n - r) \times (m - r)$ in $E(n, m)$.

Definition 2.3. *The matrix $A(u)$ of smooth function-germs $a_{i,j}(u)$ is said to be generic if $A : (\mathbb{R}^k, 0) \to E(n, m)$ is transversal to all $S_r, r = 0, 1, \dots, min(n, m)$.*

J. Mather's Theorem ([9]). Let $A : (\mathbb{R}^k, 0) \to E(n, m)$ be smooth map-germ and let $g : (\mathbb{R}^k, 0) \to \mathbb{R}^n$ be a smooth map-germ such that the linear equation (2) has a solution $b(u)$ for every $u \in \mathbb{R}^k$ close to the origin $0 \in \mathbb{R}^k$. If the map-germ $A : (\mathbb{R}^k, 0) \to E(n, m)$ is generic, i.e. transversal to all S_r, $r = 0, 1, \dots, min(n, m)$ at every point $u \in \mathbb{R}^k$ close to the origin $0 \in \mathbb{R}^k$, then the equation (2) has a local smooth solution $b(u)$ defined in a neighborhood of the origin $0 \in \mathbb{R}^k$.

We reformulate theorem 2.1 in the local terms. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth mapping. We may suppose that $f(0) = 0$ without loss of generality. Denote by $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ its map-germ at $0 \in \mathbb{R}^m$. Consider the Jacobian map-germ,

$$Jf : (\mathbb{R}^m, 0) \to E(n, m) \quad \text{given by} \quad Jf(u) = \text{the jacobian matrix at } u \in \mathbb{R}^m$$

Now the statement of the Theorem 2.1 becomes of the following form.

THEOREM 2.2. *Let the rank of $Jf(0)$ be equal to r. If $Jf : (\mathbb{R}^m, 0) \to E(n,m)$ is transversal to S_r at $0 \in \mathbb{R}^m$, then, for any smooth map-germ $\dot{f} : (\mathbb{R}^m, 0) \to \mathbb{R}^n$ tangent to $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, the germ of the manifold*

$$M = (f, \dot{f})(\mathbb{R}^m) \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$$

at $(0, \dot{f}(0))$ is solvable.

Proof. Let the rank of $Jf(0)$ be equal to r. Suppose that $Jf : (\mathbb{R}^m, 0) \to E(n,m)$ is transversal to S_r at $0 \in \mathbb{R}^m$. Since the stratification

$$\{S_i\}_{i=0}^{min\{m,n\}}$$

of $E(n,m)$ satisfies Whitney's condition (a), this implies that $Jf : (\mathbb{R}^m, 0) \to E(n,m)$ is transversal to all $S_i(\mathbb{R}^m, \mathbb{R}^n)$ at every point near the origin $0 \in \mathbb{R}^m$. Thus from J. Mather's Theorem, for any smooth map-germ $\dot{f} : (\mathbb{R}^m, 0) \to \mathbb{R}^n$ tangent to $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, the linear equation $Jf(u)b = g(u)$ has a smooth solution $b(u)$ defined in a neighborhood of the origin $0 \in \mathbb{R}^m$. Therefore, from Lemma 2.2, the germ of the manifold $(M = (f, \dot{f})(\mathbb{R}^m), (0, \dot{f}(0)))$ is solvable. □

COROLLARY 2.2. *Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ be a C^∞ stable map-germ. Then for any smooth map-germ $\dot{f} : (\mathbb{R}^m, 0) \to \mathbb{R}^n$ tangent to $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, the manifold germ $(M = (f, \dot{f})(\mathbb{R}^m), (f(0) = 0, \dot{f}(0)))$ is solvable.*

Proof. Since stable maps are transversal to every $\mathcal{A}-$orbit in the jet spaces (cf. [8, 10]), so they are also transversal to all S_r, $r = 0, 1, \ldots, min(m, n)$. Thus from the Theorem 2.1, the germ of the manifold $M = (f, \dot{f})(\mathbb{R}^m)$ at $(0, \dot{f}(0))$ is solvable. □

3. Algebraic version of J. Mather theorem. J. Mather's theorem is stated in geometric terms using the concept of transversality. However if we look more precisely into his proof of the theorem we obtain more weak algebraic condition for the linear equation (2) to have a smooth solution. In order to make our argument simpler we consider our problem in the real analytic category rather than in the C^∞ category.

Consider a $n \times m$ matrix $A(x)$ of real analytic function-germs defined at the origin of $\mathbb{R}^k$: and a column vector $g(x)$ of n real analytic function-germs defined at the origin of $\mathbb{R}^k$.

Consider the linear equation

$$A(x)b = g(x). \tag{3}$$

Let $\mathcal{O}_k$ denote the ring of germs at $0 \in \mathbb{R}^k$ of real analytic functions of k variables. First we consider the case $m = n$.

PROPOSITION 3.1. *Let $m = n$. If the ideal $< \det A(x) >$ in $\mathcal{O}_k$ generated by the determinant of the matrix $A(x)$ has property of zeroes, (i.e. if any function $h(x)$ vanish on the variety defined by this ideal then $h(x)$ belongs to the ideal), then the linear equation (3) has a real analytic solution $b(x)$.*

Proof (cf. J. Mather [9], p.190). Let $\widetilde{A}(x)$ denote the cofactor matrix of $A(x)$. Then we have

$$\widetilde{A}(x)A(x) = A(x)\widetilde{A}(x) = \det A(x) I_m,$$

where I_m is the identity matrix of size m. Then the equation (3) is equivalent to

$$\det A(x)b = \widetilde{A}(x)g(x).$$

Therefore the component functions $\widetilde{g}_1(x), \ldots, \widetilde{g}_m(x)$ of the column vector $\widetilde{A}(x)g(x)$ vanish on the variety $\{\det A(x) = 0\}$. The hypothesis of Proposition 3.1 implies that $\det A(x)$ divides the component functions $\widetilde{g}_1(x), \ldots, \widetilde{g}_m(x)$ and

$$b(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{pmatrix} = \frac{1}{\det A(x)}\widetilde{A}(x)g(x) = \begin{pmatrix} \widetilde{g}_1(x)/\det A(x) \\ \vdots \\ \widetilde{g}_m(x)/\det A(x) \end{pmatrix},$$

is a real analytic solution of the equation (3). □

Remark 3.1. In fact one can distinguish the three possibilities for the analytic case and $m = n$. Locally, near $0 \in \mathbb{R}^m$ we can write $detA(x) = \phi_0(x)\phi_1(x)^{k_1} \ldots \phi_r(x)^{k_r}$, where $\phi_0(0) \neq 0$ and $\phi_j(0) = 0$. Now we have the three possibilities:

(i) All the irreducible components of $C = \{x : detA(x) = 0\}$ defined by ϕ_j are reduced and of codimension 1. In this case the system is solvable.

(ii) There is a component of codimension ≥ 2. In this case we give an example that fulfills tangential solvability condition but is not solvable; $F = (f, \dot{f}) : \mathbb{R}^2 \to \mathbb{R}^2, f(u, v) = (u, u^2v + v^3), \dot{f}(u, v) = (1 + v, u)$. We see that $M = (f, \dot{f})(\mathbb{R}^2)$ satisfies tangential solvability condition $\dot{f}(u, v) \in Jf(u, v)(\mathbb{R}^2)$, but M is not solvable at $(f, \dot{f})(0, 0) = (1, 0) \in M$. It tends to the Bogdanov-Takens singularity (cf. [2]) with the unique separatrix and solution which can not be analytic.

(iii) There is a non-reduced component of codimension 1. In this nonsolvable case one can construct the following example: $f(x_1, x_2) = (x_1, x_1^2x_2 - \frac{2}{3}x_1x_2^3 + \frac{1}{5}x_2^5)$, $\quad g(x_1, x_2) = x_2\frac{\partial}{\partial p_1} + (x_1 - x_2^2 + \frac{4}{3}x_2^4)\frac{\partial}{\partial p_2}$. We have $detA(x) = (x_1 - x_2^2)^2$ and the equation $Ab = g$ is reduced to $(x_1 - x_2^2)^2\dot{x}_2 = (x_1 - x_2^2)(1 - 2x_2^2)$. Thus the tangential solvability condition is satisfied. The phase portrait is given by the following equation $\frac{dx_1}{dx_2} = x_1(x_1 - x_2^2)/(1 - 2x_2^2)$. Near $x = (0, 0)$ the phase curves $x_1 = \psi(x_2)$ through the initial points $(x_{2,0}^2, x_{2,0}) \neq (0, 0)$, (i.e. on $C = \{x_1 - x_2^2 = 0\}$) are smooth and transversal to C. Hence the equation $(x_1 - x_2^2)\dot{x}_2 = 1 - 2x_2^2$ with $x_1 = \psi(x_2) = \alpha x_2 + \ldots, \alpha \neq 0$ takes the form $(x_2 - x_{2,0})\dot{x}_2 = c + \ldots$, with some constant $c \neq 0$. So we have $(x_2 - x_{2,0})(t) \sim const\sqrt{t}$.

If we go back to the original problem of solvability, then we have immediately the following result.

Theorem 3.1. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a real analytic map and $\dot{f} : \mathbb{R}^m \to \mathbb{R}^m$ be a real analytic map tangent to f. If the ideal $< \det Jf(x) >$ has property of zeros, then the germ at $(f(0), \dot{f}(0))$ of $M = (f, \dot{f})(\mathbb{R}^m) \subset T\mathbb{R}^m$ is solvable.*

Now we consider the case where $m < n$. Suppose $m < n$. We consider an $n \times m$ matrix $A(x)$ of real analytic function-germs defined at the origin of $\mathbb{R}^k$ and a column vector $g(x)$ of n real analytic function-germs defined at the origin of $\mathbb{R}^k$. Consider the equation

$$A(x)b = g(x), \tag{4}$$

where b is a column vector of length m.

Suppose that the rank of the matrix $A(0)$ at the origin $0 \in \mathbb{R}^k$ is r. Then we may suppose that the matrix $A(x)$ be of the form

$$A(x) = \begin{pmatrix} I_r & O \\ O & A'(x) \end{pmatrix}, \quad \text{where } I_r \text{ is an identity matrix of size } r$$

$$\text{and} \quad A'(x) = \begin{pmatrix} a_{r+1,r+1}(x) & a_{r+1,r+2}(x) & \cdots & a_{r+1,m}(x) \\ a_{r+2,r+1}(x) & a_{r+2,r+2}(x) & \cdots & a_{r+2,m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,r+1}(x) & a_{n,r+2}(x) & \cdots & a_{n,m}(x) \end{pmatrix}.$$

Consider the set

$$\mathcal{I} = \{I = (i_1, i_2, \ldots, i_{m-r}) \quad | \quad i_j \in \mathbf{N}, \quad r < i_1 < i_2 < \ldots < i_{m-r} \leq n\},$$

and for each element $I = (i_1, i_2, \ldots, i_{m-r}) \in \mathcal{I}$ consider the $(m-r) \times (m-r)$ minor of $A'(x)$:

$$\det_I A(x) = \det \begin{pmatrix} a_{i_1,r+1}(x) & a_{i_1,r+2}(x) & \cdots & a_{i_1,m}(x) \\ a_{i_2,r+1}(x) & a_{i_2,r+2}(x) & \cdots & a_{i_2,m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_{m-r},r+1}(x) & a_{i_{m-r},r+2}(x) & \cdots & a_{i_{m-r},m}(x) \end{pmatrix}.$$

Proposition 3.2. *If the equation $A(x)b = g(x)$has a solution $b(x)$ for every $x \in \mathbb{R}^k$ and if for every element $I \in \mathcal{I}$ the ideal $< \det_I A(x) >$ in $\mathcal{O}_k$ generated by the $(m-r) \times (m-r)$ minor $\det_I A(x)$ of the matrix $A(x)$ has property of zeroes, then the linear equation $A(x)b = g(x)$ has a real analytic solution $b(x)$.*

Proof. Although the proof of this proposition is almost the same as the argument in §7(pp.191-192) of J. Mather [9], we recall it here.

First we investigate the equation (4) at points $x \in \mathbb{R}^k$ where rank $A(x) = m = \min(m, n)$. Set

$$\Omega = \{x \in \mathbb{R}^k \mid \quad \text{rank } A(x) = m\},$$
$$\Sigma = \mathbb{R}^k - \Omega = \{x \in \mathbb{R}^k \mid \quad \text{rank } A(x) < m\}.$$

Lemma 3.1.

1) *Ω is an open dense set of $\mathbb{R}^k$.*

2) *For every point $x \in \Omega$, the equation (4) has a unique solution, which we denote by $b_\Omega(x)$.*

3) *The unique solution $b_\Omega(x)$, $x \in \Omega$, is analytic in Ω.*

Proof. 1) Is trivial.

2) Since rank $A(x) = m = \min(m, n)$, if there exists a solution $b(x)$, which follows from the assumption of the Lemma, then $b(x)$ is a unique solution of (4).

3) First note that equation (1) is of the form

$$\begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & A'(x) \end{pmatrix} \begin{pmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{pmatrix}.$$

Since $x \in \Omega$, there exists $I = (i_1, i_2, \ldots, i_{m-r}) \in \mathcal{I}$ such that the sub matrix $A'_I(x)$ has rank $m - r$, where

$$A'_I(x) = \left(\begin{array}{cccc} a_{i_1,r+1}(x) & a_{i_1,r+2}(x) & \cdots & a_{i_1,m}(x) \\ a_{i_2,r+1}(x) & a_{i_2,r+2}(x) & \cdots & a_{i_2,m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_{m-r},r+1}(x) & a_{i_{m-r},r+2}(x) & \cdots & a_{i_{m-r},m}(x) \end{array} \right).$$

Set

$$A_I(x) = \left(\begin{array}{cc} I_r & O \\ O & A'_I(x) \end{array} \right),$$

and consider the sub equation

$$g_I(x) = A_I(x)b,$$

$$\text{where} \quad g_I(x) =^t (g_1(x), \ldots, g_r(x), g_{i_1}(x), \ldots, g_{i_{m-r}}(x)).$$

Since $b_\Omega(x)$ is a solution of (4), $b_\Omega(x)$ is also a solution of the sub equation $g_I(x) = A_I(x)b$, which is analytic since $g_I(x)$ is analytic and $\det A_I(x) \neq 0$. □

Now consider the solution at points $x \in \Sigma$ where $\mathrm{rank} A(x) < m$. Take any $I = (i_1, i_2, \ldots, i_{m-r}) \in \mathcal{I}$ and consider sub equation

$$g_I(x) = A_I(x)b,$$

where $A_I(x)$ and $g_I(x)$ are defined in the same way as those in the proof of Lemma 3.1. Then, from the assumption of Proposition 3.2, the ideal $< \det A_I(x) >$ satisfies the condition posed in Proposition 3.1 that it has property of zeros. Therefore the sub equation $g_I(x) = A_I(x)b$ has an analytic solution $b_I(x)$.

Now consider the set

$$\Omega_I = \{x \in \mathbb{R}^k \mid \det\, A_I(x) \neq 0\} \quad \subset \quad \Omega.$$

Note that Ω_I is also open and dense in $\mathbb{R}^k$.

We compare the restriction $(b_I\,|_{\Omega_I})(x)$ of $b_I(x)$ to Ω_I with the restriction $(b_\Omega\,|_{\Omega_I})(x)$ of $b_\Omega(x)$ to Ω_I. Since both of $(b_I\,|_{\Omega_I})(x)$ and $(b_\Omega\,|_{\Omega_I})(x)$ are solutions of the sub equation

$$g_I(x) = A_I(x)b, \qquad x \in \Omega_I,$$

and the solution of the sub equation $g_I(x) = A_I(x)b$ is unique on the region $\Omega_I \subset \Omega$, we have

$$b_I(x) = b_\Omega(x) \qquad \forall x \in \Omega_I.$$

Now take any other $J \in \mathcal{I},\ J \neq I$ and consider the sub equation

$$g_J(x) = A_J(x)b.$$

With the same reason, the sub equation $g_J(x) = A_J(x)b$ has an analytic solution $b_J(x)$ and we have

$$b_J(x) = b_\Omega(x) \qquad \forall x \in \Omega_J.$$

Hence we have

$$b_J(x) = b_I(x) = b_\Omega(x) \qquad \forall x \in \Omega_J \cap \Omega_I.$$

Thus $b_J(x)$ and $b_I(x)$ coincide on an open dense subset $\Omega_J \cap \Omega_I$ of $\mathbb{R}^k$ and they are analytic. Therefore they coincide on the whole $\mathbb{R}^k$. Set

$$b(x) = b_I(x) \qquad \forall I \in \mathcal{I}.$$

Then $b(x)$ is an analytic solution of all the sub equations$g_I(x) = A_I(x)b$, and it is an analytic solution of the whole equation $g(x) = A(x)b$. Thus we proved the Proposition 3.2. □

Now let's go back to the original problem of solvability. We have immediately.

THEOREM 3.2. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $m < n$, be a real analytic map and $\dot{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a real analytic map tangent to f. If for every element $I \in \mathcal{I}$, $< \det_I Jf(x) >$ has property of zeros, then the germ at $(f(0), \dot{f}(0))$ of $M = (f, \dot{f})(\mathbb{R}^m) \subset T\mathbb{R}^m$ is solvable.*

4. Non-solvable points are isolated generically in Tougeron's sense. In this section we will investigate the non solvable points. We prove that *non-transversal points*, i.e. the points where $Jf : \mathbb{R}^m \to E(n, m)$ is not transversal to S_r, are generically isolated, which implies that *non-solvable points are generically isolated.* Here, in this section, the *genericity* notion is in the strongest sense introduced by J.P. Tougeron [12].

DEFINITION 4.1. We say that the non-transversal points of $M = (f, \dot{f})(\mathbb{R}^m)$ are generically isolated if there exists an ∞ codimensional subset $\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n)$ of the space $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ of C^∞ mappings of $\mathbb{R}^m$ into $\mathbb{R}^n$ such that for any $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n) - \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n)$ non-transversal points are isolated.

Let us fix the notation. Set

$$\begin{aligned} C^\infty(\mathbb{R}^m, T\mathbb{R}^n) &= \{(f, \dot{f}) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n \mid C^\infty \text{ mapping}\} \\ &= C^\infty(\mathbb{R}^m, \mathbb{R}^n) \times C^\infty(\mathbb{R}^m, \mathbb{R}^n) \\ T(\mathbb{R}^m, T\mathbb{R}^n) &= \{(f, \dot{f}) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n \mid \ \dot{f} \text{ is tangent to } f\}, \end{aligned}$$

where all these spaces are endowed with the Whitney C^∞ topology. Recall that $\dot{f}$ is tangent to f if and only if $(f, \dot{f})$ is an embedding and $M = (f, \dot{f})(\mathbb{R}^m)$ satisfies the tangential solvability condition (see Def. 2.2).

DEFINITION 4.2. *Let $(f, \dot{f}) \in T(\mathbb{R}^m, T\mathbb{R}^n)$. A point $(x, \dot{x}) = (f(u), \dot{f}(u)) \in M = (f, \dot{f})(\mathbb{R}^m)$ is a transversal point of M if $Jf : \mathbb{R}^m \to E(n, m)$ is transversal to S_r at $u \in \mathbb{R}^m$ for $r = \text{rank} Jf(u)$. Equivalently $j^1(\pi_1 |_M) : M \to J^1(M, \mathbb{R}^n)$ is transversal to*

$$S_r(M, \mathbb{R}^n) = \{j^1 h(v) \in J^1(M, \mathbb{R}^n) \mid \text{rank} Jh(v) = r\},$$

where $\pi_1 : T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection defined by $\pi_1(x, \dot{x}) = x$.

We are interested in the structure of the non-generic (in the sense of the previous sections) points and to investigate how they are located in M. We prove the following main theorems.

THEOREM 4.1. *There exists an ∞ codimensional subset $\Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$ of $T(\mathbb{R}^m, T\mathbb{R}^n)$ such that for any embedding*

$$(f, \dot{f}) \in T(\mathbb{R}^m, T\mathbb{R}^n) - \Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$$

non-transversal points of $M = (f, \dot{f})(\mathbb{R}^m)$ are isolated.

Combining this theorem with J. Mather's theorem, we obtain

THEOREM 4.2. *Let $\Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$ be the ∞ codimensional subset of $T(\mathbb{R}^m, T\mathbb{R}^n)$ given in Theorem 4.1. Then for any*

$$(f, \dot{f}) \in T(\mathbb{R}^m, T\mathbb{R}^n) - \Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$$

non-solvable points of $M = (f, \dot{f})(\mathbb{R}^m)$ are isolated.

REMARK 4.1. Let us define the set

$$\Sigma_{tr} T(\mathbb{R}^m, T\mathbb{R}^n) = \{(f, \dot{f}) \in T(\mathbb{R}^m, T\mathbb{R}^n) \mid Jf : U \to E(n, m) \text{ is not transversal to } \; S_r \; \text{ for some } \; r\}.$$

Then we see

$$\Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n) \quad \subset \quad \Sigma_{tr} T(\mathbb{R}^m, T\mathbb{R}^n),$$

$$\mathrm{codim} \Sigma_{tr} T(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{in} \quad T(\mathbb{R}^m, T\mathbb{R}^n) \quad < \quad \infty,$$

$$\mathrm{codim} \Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{in} \quad T(\mathbb{R}^m, T\mathbb{R}^n) \quad = \quad \infty.$$

This means that even if $M = (f, \dot{f})(\mathbb{R}^m)$ is not a generic submanifold in the sense of previous sections, non-transversal points and non-solvable points of $M = (f, \dot{f})(\mathbb{R}^m)$ are isolated points for a very generic $(f, \dot{f})$, i.e. for any $(f, \dot{f}) \in \Sigma_{tr} T(\mathbb{R}^m, T\mathbb{R}^n) - \Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$.

For the strong genericity result we need an appropriate transversality theorem. Let us denote

$$J^r(m, n) = \{j^r f(0) \mid f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n) \quad \text{and} \quad f(0) = 0\}$$
$$J^r(\mathbb{R}^m, \mathbb{R}^n) = \{j^r f(u) \mid f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)), \quad u \in \mathbb{R}^m\}.$$

For positive integers $s > r > 0$, we define the canonical projections

$$\pi^s_r : J^s(m, n) \to J^r(m, n), \quad \pi^s_r(j^s f(0)) = j^r f(0).$$

THEOREM 4.3 (Transversality). *Let W be a semi-algebraic subset of $J^r(m, n)$ and let $X_1, \ldots, X_\ell$ be a finite number of semi-algebraic submanifolds of $J^k(\mathbb{R}^m, \mathbb{R}^n)$. Then there exists a closed semi-algebraic subset*

$$\Sigma_W \quad \textit{of} \quad (\pi^{r+k+1}_r)^{-1}(W) \quad \textit{having codimension} \quad \geq 1$$

such that for any C^∞ mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with

$$j^{r+k+1}(0) \in (\pi^{r+k+1}_r)^{-1}(W) - \Sigma_W,$$

there exists a neighborhood U of the origin 0 of $\mathbb{R}^m$ such that $j^r f : U - \{0\} \to J^k(\mathbb{R}^m, \mathbb{R}^n)$ is transversal to $X_1, \ldots, X_\ell$ at every point $u \in U$ except at the origin $0 \in \mathbb{R}^m$

For the proof we refer to the more general version of Theorem 4.3 which is proved in [4] (Theorem 1 in [4], p. 229).

Let us identify $J^1(m,n)$ with $E(n,m)$ and set

$$\begin{aligned} S_r(\mathbb{R}^m, \mathbb{R}^n) &= \{j^1 f(u) \in J^1(\mathbb{R}^m, \mathbb{R}^n) \mid \operatorname{rank} Jf(u) = r\} \\ &= \mathbb{R}^m \times \mathbb{R}^n \times S_r \subset \mathbb{R}^m \times \mathbb{R}^n \times J^1(m,n) = J^1(\mathbb{R}^m, \mathbb{R}^n). \end{aligned}$$

For $(u_0, x_0) \in \mathbb{R}^m \times \mathbb{R}^n$, set

$$C^\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0) = \{f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)) \mid f(u_0) = x_0\}.$$

We prove Main Theorem 4.1 using the following result

THEOREM 4.4. *There exists an infinite codimensional subset $\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0)$ of $C^\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0)$ such that for any C^∞ mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with*

$$f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0) - \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0)$$

there exists a neighborhood $V(u_0)$ of u_0 in $\mathbb{R}^m$ such that $j^1 f : V(u_0) - \{u_0\} \to J^1(\mathbb{R}^m, \mathbb{R}^n)$ is transversal to $S_r(\mathbb{R}^m, \mathbb{R}^n)$ for all $r = 0, \ldots, \min(m,n)$ at every point $u \in V(u_0) - \{u_0\}$ except at u_0.

Proof of Theorem 4.1. Set

$$\begin{aligned} \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n) &= \cup_{(u,x) \in \mathbb{R}^m \times \mathbb{R}^n} \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; u, x) \subset C^\infty(\mathbb{R}^m, \mathbb{R}^n) \\ \Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) &= \{(f, \dot{f}) \in C^\infty(\mathbb{R}^m, T\mathbb{R}^n) \mid f \in \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n)\} \\ &= \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n) \times C^\infty(\mathbb{R}^m, \mathbb{R}^n) \\ \Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n) &= \Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) \cap T(\mathbb{R}^m, T\mathbb{R}^n), \end{aligned}$$

where we denote $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and

$$C^\infty(\mathbb{R}^m, T\mathbb{R}^n) = C^\infty(\mathbb{R}^m, \mathbb{R}^n) \times C^\infty(\mathbb{R}^m, \mathbb{R}^n).$$

We claim that $\Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$ is the required one. It is easy to see that it satisfies the condition:
if $(f, \dot{f}) \in T(\mathbb{R}^m, T\mathbb{R}^n) - \Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n)$, then the non-transversal points of $M = (f, \dot{f})(\mathbb{R}^m)$ are isolated.

Now we investigate the codimension of this set. Since $\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0)$ is ∞ codimensional in $C^\infty(\mathbb{R}^m, \mathbb{R}^n; u_0, x_0)$, so is $\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n)$ in $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$. Therefore

$$\operatorname{codim} \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n) \times C^\infty(\mathbb{R}^m, \mathbb{R}^n) \quad \text{in} \quad C^\infty(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{is equal to} \quad \infty.$$

Define

$$T_0(\mathbb{R}^m, T\mathbb{R}^n) = \{(f, \dot{f}) \in C^\infty(\mathbb{R}^m, T\mathbb{R}^n) \mid \dot{f}(u) \in Jf(u)(\mathbb{R}^m), \quad \forall u \in \mathbb{R}^m\},$$

$$Emb(\mathbb{R}^m, T\mathbb{R}^n) = \{(f, \dot{f}) \in C^\infty(\mathbb{R}^m, T\mathbb{R}^n) \mid (f, \dot{f}) \text{is an embedding}\}.$$

Then we have

$$T(\mathbb{R}^m, T\mathbb{R}^n) = Emb(\mathbb{R}^m, T\mathbb{R}^n) \cap T_0(\mathbb{R}^m, T\mathbb{R}^n).$$

Since

$$\Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) = \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n) \times C^\infty(\mathbb{R}^m, \mathbb{R}^n)$$

and

$$\mathrm{codim}\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n) \times C^\infty(\mathbb{R}^m, \mathbb{R}^n) \quad \text{in} \quad C^\infty(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{is equal to} \quad \infty,$$

we have

$$\mathrm{codim}\Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) \cap T_0(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{in} \quad T_0(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{is equal to} \quad \infty.$$

Since

$$Emb(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{is open in} \quad C^\infty(\mathbb{R}^m, T\mathbb{R}^n),$$

we see that

$$\mathrm{codim}\Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) \cap T_0(\mathbb{R}^m, T\mathbb{R}^n) \cap Emb(\mathbb{R}^m, T\mathbb{R}^n)$$

$$\text{in} \quad T_0(\mathbb{R}^m, T\mathbb{R}^n) \cap Emb(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{is equal to} \quad \infty.$$

Since

$$\begin{aligned}\Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n) &= \Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) \cap T(\mathbb{R}^m, T\mathbb{R}^n) \\ &= \Sigma_\infty(\mathbb{R}^m, T\mathbb{R}^n) \cap T_0(\mathbb{R}^m, T\mathbb{R}^n) \cap Emb(\mathbb{R}^m, T\mathbb{R}^n), \\ T(\mathbb{R}^m, T\mathbb{R}^n) &= T_0(\mathbb{R}^m, T\mathbb{R}^n) \cap Emb(\mathbb{R}^m, T\mathbb{R}^n)\end{aligned}$$

we see that

$$\mathrm{codim}\Sigma_\infty T(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{in} \quad T(\mathbb{R}^m, T\mathbb{R}^n) \quad \text{is equal to} \quad \infty.$$

This completes the proof of Theorem 4.1.

To prove Theorem 4.4 it suffices to prove it for $(u_0, x_0) = (0, 0)$, so we reformulate the theorem in an accessible way:

THEOREM 4.4. There exists an infinite codimensional subset $\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0)$ of $C^\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0)$ such that for any C^∞ mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with

$$f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0) - \Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0)$$

there exists a neighborhood $V(0)$ of 0 in $\mathbb{R}^m$ such that $j^1 f : V(0) - \{0\} \to J^1(\mathbb{R}^m, \mathbb{R}^n)$ is transversal to $S_r(\mathbb{R}^m, \mathbb{R}^n)$ for $\forall r = 0, \ldots, \min(m, n)$ at every point $u \in V(0) - \{0\}$ except at the origin 0.

Proof. First we apply Theorem 4.3 to

$$W = W_1 = J^1(m, n) \quad \text{and}$$

$$X_r = S_r(\mathbb{R}^m, \mathbb{R}^n) \subset J^1(\mathbb{R}^m, \mathbb{R}^n), \quad r = 0, \ldots, \min(m, n).$$

Then there exists a closed semi-algebraic subset

$$\Sigma_1 \quad \text{of} \quad (\pi_1^{1+1+1})^{-1}(W) = J^3(m,n) \quad \text{having codimension} \quad \geq 1$$

such that for any C^∞ mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with

$$j^3(0) \in J^3(m,n) - \Sigma_1,$$

there exists a neighborhood U of the origin 0 of $\mathbb{R}^m$ such that

$$j^1 f : U - \{0\} \to J^1(\mathbb{R}^m, \mathbb{R}^n) \quad \text{is transversal to} \quad S_r(\mathbb{R}^m, \mathbb{R}^n), \quad \forall r.$$

Next we apply Theorem 4.3 to

$$W = \Sigma_1 \subset J^3(m,n) \quad \text{and}$$

$$X_r = S_r(\mathbb{R}^m, \mathbb{R}^n) \subset J^1(\mathbb{R}^m, \mathbb{R}^n), \quad r = 0, \dots, \min(m,n).$$

Then, again there exists a closed semi-algebraic subset

$$\Sigma_2 \quad \text{of} \quad (\pi_3^{3+1+1})^{-1}(\Sigma_1) \subset J^5(m,n)$$

such that

$$\text{codimension of} \quad \Sigma_2 \quad \text{in} \quad (\pi_3^5)^{-1}(\Sigma_1) \quad \geq 1$$

$$\text{codimension of} \quad \Sigma_2 \quad \text{in} \quad J^5(m,n) \geq 2$$

and such that for any C^∞ mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with

$$j^5(0) \in J^5(m,n) - \Sigma_2,$$

there exists a neighborhood U of the origin 0 of $\mathbb{R}^m$ such that

$$j^1 f : U - \{0\} \to J^1(\mathbb{R}^m, \mathbb{R}^n) \quad \text{is transversal to} \quad S_r(\mathbb{R}^m, \mathbb{R}^n), \quad \forall r = 0.$$

In this way, we can prove inductively that there exists a closed semi-algebraic subset

$$\Sigma_k \quad \text{of} \quad J^{2k+1}(m,n) \quad \text{with codimension} \geq k$$

such that for any C^∞ mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with

$$j^{2k+1}(0) \in J^{2k+1}(m,n) - \Sigma_k,$$

there exists a neighborhood U of the origin 0 of $\mathbb{R}^m$ such that

$$j^1 f : U - \{0\} \to J^1(\mathbb{R}^m, \mathbb{R}^n) \quad \text{is transversal to} \quad S_r(\mathbb{R}^m, \mathbb{R}^n), \quad \forall r.$$

Set

$$\Sigma_k(\mathbb{R}^m, \mathbb{R}^n; 0, 0) = \{f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0) \mid j^{2k+1}(0) \in \Sigma_k\},$$

$$\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0) = \cap_{k=1}^\infty \Sigma_k(\mathbb{R}^m, \mathbb{R}^n; 0, 0).$$

Then $\Sigma_\infty(\mathbb{R}^m, \mathbb{R}^n; 0, 0)$ satisfies the condition required in Theorem 4.4.
This completes the proof of Theorem 4.4 and hence also the proof of Main Theorem 4.1. □

5. Solvability of isotropic mappings. Now we apply J. Mather's Theorem to implicit Hamiltonian systems in symplectic space, i.e. $M = F(\mathbb{R}^{2n})$, where $F : \mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ with $T\mathbb{R}^{2n}$ endowed with the induced symplectic structure. Let $(\mathbb{R}^{2n}, \omega)$ be a symplectic space with $\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$ in canonical Darboux coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Let θ be the Liouville 1-form on the cotangent bundle $T^*\mathbb{R}^{2n}$. Then $d\theta$ is a standard symplectic structure on $T^*\mathbb{R}^{2n}$. Let $\beta : T\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ be the canonical bundle map defined by ω, $\beta : T\mathbb{R}^{2n} \ni v \mapsto \omega(v, \cdot) \in T^*\mathbb{R}^{2n}$. Then we define the canonical symplectic structure $\dot{\omega}$ on $T\mathbb{R}^{2n}$,

$$\dot{\omega} = \beta^* d\theta = d(\beta^*\theta) = \sum_{i=1}^{n} (d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i),$$

where $(x, y, \dot{x}, \dot{y})$ are local coordinates on $T\mathbb{R}^{2n}$ and $\beta^*\theta = \sum_{i=1}^{n} (\dot{y}_i dx_i - \dot{x}_i dy_i)$. Let $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ be a smooth map-germ. We say that F is isotropic if $F^*\dot{\omega} = 0$. If we assume that $F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is an isotropic map-germ, then the germ of a differential of 1-form $(\beta \circ F)^*\theta$ vanishes, $d(\beta \circ F)^*\theta = F^*\beta^* d\theta = F^*\dot{\omega} = 0$. Thus $(\beta \circ F)^*\theta$ is a germ of a closed 1-form. And there exists a smooth function-germ $h : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ such that $(\beta \circ F)^*\theta = -dh$. For each smooth isotropic map-germ F the function-germ h is uniquely defined up to an additive constant.

Let $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ denote coordinates of the source space. For local coordinates in the neighborhood $(U, 0) \subset \mathbb{R}^{2n}$ we define $F = (f, g, \dot{f}, \dot{g}) : (U, 0) \to T\mathbb{R}^{2n}$, and $\bar{F} = \pi \circ F = (f, g) : (U, 0) \to \mathbb{R}^{2n}$, where π denotes the canonical projection, $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. In general, F can be regarded as a vector field along $\bar{F}$, i.e. a section of an induced fiber bundle $\bar{F}^*T\mathbb{R}^{2n}$. By $\mathcal{E}_U$ ($\mathcal{E}_{\mathbb{R}^{2n}}$-respectively) we denote the $\mathbb{R}$- algebra of smooth function germs at 0 on U (and on "the target space" $\mathbb{R}^{2n}$ respectively). To each isotropic map-germ F along $\bar{F}$ there exists a unique h belonging to the maximal ideal $\mathbf{m}_U$ of $\mathcal{E}_U$, $h \in \mathbf{m}_U$ which is a generating function-germ for F.

Let $F : (U, 0) \to T\mathbb{R}^{2n}$ and $G : (U, 0) \to T\mathbb{R}^{2n}$ be two isotropic map-germs along $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ and $\bar{G} : (U, 0) \to \mathbb{R}^{2n}$ respectively. Now we introduce the natural equivalence group acting on isotropic mappings through a natural lifting of diffeomorphic or symplectic equivalences of $\bar{F}$ and $\bar{G}$. The C^∞ map-germs $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ and $\bar{G} : (U, 0) \to \mathbb{R}^{2n}$ are said to be symplectomorphic or symplectically equivalent if there exist a diffeomorphism-germ $\varphi : (U, 0) \to (U, 0)$ and a symplectomorphism-germ $\Phi : (T\mathbb{R}^{2n}, 0 \to (T\mathbb{R}^{2n}, 0)$ such that $\bar{G} = \Phi \circ \bar{F} \circ \varphi$. First we recall the standard equivalence of Lagrange projections. Let $F : (U, 0) \to T\mathbb{R}^{2n}$ and $G : (U, 0) \to T\mathbb{R}^{2n}$ be two isotropic map-germs. We say that F and G are *Lagrangian equivalent* (*L-equivalent* [1]) if there exist a diffeomorphism-germ $\varphi : (U, 0) \to (U, 0)$, and a symplectomorphism-germ $\Psi : (T\mathbb{R}^{2n}, 0) \to (T\mathbb{R}^{2n}, 0)$, $\Psi^*\dot{\omega} = \dot{\omega}$, preserving the fibering π such that $G = \Psi \circ F \circ \varphi$.

Let $F : (U, 0) \to T\mathbb{R}^{2n}$ and $G : (U, 0) \to T\mathbb{R}^{2n}$ be two isotropic map-germs along $\bar{F} : (U, 0) \to \mathbb{R}^{2n}$ and $\bar{G} : (U, 0) \to \mathbb{R}^{2n}$ respectively. We say that F and G are *L-symplectic equivalent* if there exist a diffeomorphism-germ $\varphi : (U, 0) \to (U, 0)$, and a symplectomorphism-germ $\Psi : (T\mathbb{R}^{2n}, 0) \to (T\mathbb{R}^{2n}, 0)$, $\Psi^*\dot{\omega} = \dot{\omega}$, preserving the fibering π and a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$, $\Phi^*\omega = \omega$, $\pi \circ \Psi = \Phi \circ \pi$, such that $G = \Psi \circ F \circ \varphi$ and $\bar{G} = \Phi \circ \bar{F} \circ \varphi$. In this case $\bar{F}$ and $\bar{G}$ are naturally *symplectomorphic.*

To $\bar{F}$ we associate a symplectically invariant algebra $\mathcal{R}_{\bar{F}}$ of all generating function-germs, $\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : h \text{ generates an isotropic map-germ along } \bar{F}\}$. It is easy to check that if $\bar{F}$ and $\bar{G}$ are symplectomorphic, $\bar{G} = \Phi \circ \bar{F} \circ \varphi$, then we have an isomorphism $\varphi^* : \mathcal{R}_{\bar{F}} \to \mathcal{R}_{\bar{G}}$. And If $\bar{F}$ has a maximal rank, then $\mathcal{R}_{\bar{F}} = \mathcal{E}_U$. It seems that if $\bar{F}$ and $\bar{G}$ are symplectomorphic, then for $h \in \mathcal{R}_{\bar{F}}$, the isotropic map-germ F generated by h and the isotropic map-germ G generated by $\varphi^*(h)$ are $L-$symplectic equivalent, $G = \Psi \circ F \circ \varphi$. In this case $\Psi : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ is a symplectic lifting of the symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. The aim of this section is to study the case when $\bar{F}$ does not have maximal rank and establish the structure of $\mathcal{R}_{\bar{F}}$. In the rest of this section we study isotropic mappings with $\bar{F}$ of corank 1.

Let $e \in T_0U$ span the kernel of the Jacobian matrix $J\bar{F}$ of a corank one map-germ $\bar{F}$ at zero. By $\Delta_{\bar{F}}$ we denote the determinant of $J\bar{F}$ and by ∂_e the derivation into e-direction. Let $< \Delta_{\bar{F}} >$ denotes the ideal generated by $\Delta_{\bar{F}}$ in $\mathcal{E}_U$.

Proposition 5.1 (cf. [6]). *Let F be a smooth map-germ such that $\bar{F}$ has a corank one singularity at* 0*. If F is isotropic then there exists uniquely defined function-germ $h : (U,0) \to (\mathbb{R},0)$ such that $\partial_e h \in < \Delta_{\bar{F}} >$ and $(\beta \circ F)^*\theta = -dh$. Conversely, for any smooth function-germ $h : (U,0) \to \mathbb{R}$ such that $\partial_e h \in < \Delta_{\bar{F}} >$ there is a uniquely defined isotropic map-germ $F : (U,0) \to T\mathbb{R}^{2n}$ such that $\bar{F} = \pi \circ F$ and $(\beta \circ F)^*\theta = -dh$.*

Remark 5.1. Instead of isotropic F associated to $\bar{F}$ we consider pairs $(\bar{F}, h)$ with a smooth function-germ h belonging to $\mathcal{R}_{\bar{F}}$. An algebra $\mathcal{R}_{\bar{F}}$ of all generating function-germs associated to $\bar{F}$ is represented by $\bar{F}$ in the form, $\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : dh \in \mathcal{E}_U d(\bar{F}^*\mathcal{E}_{\mathbb{R}^{2n}})\}$. Thus by Theorem 5.1 we get an algebra $\mathcal{R}_{\bar{F}}$ of all generating function-germs (which is also an $\mathcal{E}_{\mathbb{R}^{2n}}$-module) for a smooth map-germ $\bar{F}$ of corank one, $\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : \partial_e h \in < \Delta_{\bar{F}} >\}$. If $F : (U,0) \to T\mathbb{R}^{2n}$ is a smooth isotropic map-germ such that $\bar{F} = \pi \circ F : (U,0) \to \mathbb{R}^{2n}$ has corank one singular point at $(0,0)$. Then F has corank at most one at $(0,0)$. The corank of F is exactly one if and only if $\partial_e(\partial_e h/\Delta_{\bar{F}})(0,0) = 0$.

To describe $\mathcal{R}_{\bar{F}}$ in more clear way we consider the classification of corank 1 stable map-germs $\bar{F}$ up to symplectic equivalence. If $\bar{F} : (U,0) \to (\mathbb{R}^{2n},0)$ is a corank 1 stable map-germ, then $\bar{F}$ is diffeomorphically equivalent (or diffeomorphic, [8]) to one of the A_k-type normal forms $(0 < k < 2n)$,

$$(w_1, \ldots, w_{2n}) \mapsto (w_1, \ldots, w_{2n-1}, w_{2n}^{k+1} + \sum_{i=1}^{k-1} w_i w_{2n}^{k-i}), \tag{5}$$

where we use the notation $(w_1, \ldots, w_{2n}) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$.

Let $\bar{F} : (U,0) \to (\mathbb{R}^{2n},0)$ be an A_k-type singular map-germ. Then it is shown in [5] that $\bar{F}$ is symplectically equivalent to the following map-germ $w = (w_1, \ldots, w_{2n}) \mapsto (w_1, \ldots, w_{2n-1}, w_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(w) w_{2n}^{k-i})$, where $a_1(w), \ldots, a_{k-1}(w)$ are smooth function-germs such that $da_1, \ldots, da_{k-1}$ and dw_{2n} are linearly independent at the origin. In this case we can write explicitly $\mathcal{R}_{\bar{F}}$ expressed by the symplectic moduli functions,

$$\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : \frac{\partial h}{\partial w_{2n}} \in < \partial(w_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(w) w_{2n}^{k-i})/\partial w_{2n} >\}.$$

And for the simple symplectic normal form - the symplectic fold we have

Proposition 5.2. *Let $\bar{F} : (U,0) \to (\mathbb{R}^{2n},0)$ be an A_1-type singularity, i.e. fold singularity. Then $\bar{F}$ is symplectically equivalent to*

$$(u_1, \ldots, u_n, v_1, \ldots, v_n) \mapsto (u_1, \ldots, u_n, v_1 \ldots, v_{n-1}, v_n^2),$$

And the corresponding algebra, $\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : \frac{\partial h}{\partial v_n} \in < v_n >\}$.

Let us extend the solvability property, introduced for a smooth submanifold of a tangent bundle defined by an immersion mapping F, to the general smooth isotropic mapping into symplectic tangent bundle. Now we redefine the notion of smooth solvability. Let $F : (U, 0) \to T\mathbb{R}^{2n}$ be a smooth isotropic map-germ with a generating function $h : (U, 0) \to \mathbb{R}$. We say that F is smoothly solvable if there exists a smooth vector field X_h on U such that $F = d\bar{F}(X_h)$.

The geometric meaning of the solvability property is explained in the following sufficient condition, proved on the basis of J. Mather Theorem.

THEOREM 5.1. *Let* $\bar{F} = (f, g) : U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be a smooth mapping such that* $\bar{F}$ *has a corank* k *singularity at the origin* $(0, 0) \in \mathbb{R}^{2n}$ *and that the jet extension* $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ *is transversal to the corank* k *stratum* Σ^k *of* $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. *If an isotropic mapping* F *along* $\bar{F}$ *satisfies the tangential solvability condition, then* F *is smoothly solvable.*

Proof. Let $\bar{F} = (f, g) : U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a smooth mapping such that $\bar{F}$ has a corank k singularity at the origin $(0, 0) \in \mathbb{R}^{2n}$ and that the jet extension $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is transversal to the corank k stratum Σ^k of $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$.

Let $F = (f, g, \dot{f}, \dot{g})$ be an isotropic mapping along $\bar{F}$ which satisfies the tangential solvability condition:

$$\begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} \in Image\ J\bar{F}(u, v). \tag{6}$$

Since F is a smooth isotropic mapping, F is generated by a smooth function h:

$$\begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} {}^tJ\bar{F}^{-1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix}. \tag{7}$$

We know that F is smoothly solvable if and only if

$$J\bar{F}^{-1} \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} {}^tJ\bar{F}^{-1} \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix} \quad \text{is smooth,} \tag{8}$$

which, on the basis of (7) is the case if and only if

$$J\bar{F}^{-1} \begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} \quad \text{is smooth,} \tag{9}$$

which is true if and only if the linear equation

$$J\bar{F} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} \tag{10}$$

has a smooth solution $(a(u, v), b(u, v))$.

Since, from (6),

$$\begin{pmatrix} \dot{f}(u, v) \\ \dot{g}(u, v) \end{pmatrix} \in Image\ J\bar{F}(u, v) \quad \text{for every point } (u, v) \in U$$

and $j^1\bar{F} : U \to J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is transversal to the corank k stratum Σ^k of $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$, then from J. Mather's theorem [9], Equation (10) has a smooth solution and F is smoothly solvable. This completes the proof. □

6. Generalized Hamiltonian systems. Let K be a submanifold of $\mathbb{R}^{2n}$ and $h : K \to \mathbb{R}$ be a smooth function on K. The notion of generalized Hamiltonian system (generalized Hamiltonian dynamics) was introduced by P.A.M. Dirac in [3]. It is defined as a sub-bundle of $T\mathbb{R}^{2n}$ over K, being a Lagrangian submanifold L of $(T\mathbb{R}^{2n}, \dot{\omega})$, (cf. [6])

$$L = \{v \in T\mathbb{R}^{2n} : \omega(v,u) = -dh(u) \quad \forall_{u \in TK}\}. \tag{11}$$

In local coordinates which we use in the setting, the generalized Hamiltonian system (11) can be written by linear in λ generating family $F : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$,

$$F(x,y,\lambda) = \sum_{\ell=1}^{k} a_\ell(x,y)\lambda_\ell + b(x,y), \tag{12}$$

where K, being a complete intersection, is defined by an ideal $I_K =< a_1, \dots, a_k >$ having property of zeros with analytic generators $a_i, 1 \le i \le k$. K is a zero-level set of the mapping $a : (x,y) \mapsto (a_1(x,y), \dots, a_k(x,y))$, $K = \{(x,y) \in \mathbb{R}^{2n} : a_i(x,y) = 0, i = 1, \dots, k\}$, and $b(x,y)$ is an arbitrary smooth extension of the function $h : K \to \mathbb{R}$. In what follows we consider the smooth K and b identified with h, and the notion L_F for the Lagrangian variety generated by F.

Generalized Hamiltonian systems are not generic in J. Mather sense. For such systems the necessary tangential solvability condition is also sufficient. The aim of this section is to investigate conditions on subvarieties of symplectic space on which the solvable generalized Hamiltonian systems may exist. We find conditions that L_F is smoothly solvable under some properties of K and general function on K.

Let us notice that the tangential solvability condition for generalized Hamiltonian system is reformulated to be the system of equations fulfilled in the smoothly solvable points of L,

$$\{\frac{\partial F}{\partial \lambda_i}, F\}(x,y,\lambda) = 0 \text{ for } (x,y,\lambda) \in C_F. \tag{13}$$

Concerning the solvability of the generalized Hamiltonian system L_F, we have already the following basic result proved in [6]. L_F is smoothly solvable if (13) is fulfilled on $K \times \mathbb{R}^k$ which is a very strong condition expressed in the following,

Proposition 6.1 ([6]). *A generalized Hamiltonian system $L \subset (T\mathbb{R}^{2n}, \dot{\omega})$ generated by the generating family (12) is smoothly solvable if and only if*

$$\{a_i, a_\ell\} = 0 \quad \textit{and} \quad \{b, a_\ell\} = 0, \quad 1 \le i, \ell \le k,$$

$$\textit{on} \quad K = \{(x,y) \in \mathbb{R}^{2n} : a_i(x,y) = 0, \quad 1 \le i \le k\},$$

and $1 \le k \le n$. If $k = n$, then $b \equiv 0$.

Solvability property of L_F defines K to be an involutive, coisotropic submanifold of $(\mathbb{R}^{2n}, \omega)$, i.e. geometrically $T_qK \supset (T_qK)^\omega = \{u \in T_q\mathbb{R}^{2n} : \omega(u,v) = 0, \forall_{v \in T_qK}\}$, and b restricts to those functions who are constant on leaves of the characteristic foliation of coisotropic K, (cf. [11]).

Remark 6.1. If $dimK < n$ and K is isotropic, i.e. $(TK)^\omega \supset TK$, then TK is solvable submanifold of L with $b \equiv 0$. In this case L_F can not be completely solvable

Hamiltonian system. If $dimK = n$, and $TK = L_F$ is solvable with $b \equiv 0$, then K is Lagrangian.

Corollary 6.1. *Let L_F be a generalized Hamiltonian system over the submanifold $K \subset \mathbb{R}^{2n}$ and its generating family F fulfills the tangential integrability condition. Then K is a coisotropic submanifold of $(\mathbb{R}^{2n}, \omega)$ and L is smoothly solvable.*

In what follows we investigate the case when L_F is not smoothly solvable. We clarify the properties of such L_F with respect to the structure of non-solvable part of it and symplectic invariant properties of constraints. The regions of solvability on L_F may be identified by analysis of (13) under some assumptions on K.

The generalized Hamiltonian system L_F is given by an immersion

$$\phi : C_F \to L \subset (T\mathbb{R}^{2n}, \dot{\omega})$$

defined by

$$\phi(x, y, \lambda) = (x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)), \quad (x, y, \lambda) \in C_F.$$

Since $\frac{\partial F}{\partial \lambda_\ell}(x, y, \lambda) = a_\ell(x, y)$, we have $C_F = K \times \mathbb{R}^k$. Then L can be written as

$$L_F = \phi(C_F) = \{(x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)) \in T\mathbb{R}^{2n} : (x, y, \lambda) \in K \times \mathbb{R}^k\}.$$

We find conditions for a submanifold or domain of L_F to be smoothly solvable. Thus the traditionally solvable Hamiltonian system exists on a submanifold K in the case where the generating family does not satisfy the involutivity condition in Theorem 6.1, i.e. $\{a_i, a_\ell\} = 0$ and $\{b, a_\ell\} = 0$ on K, $1 \le i, \ell \le k$.

Now the solvability condition (13) becomes the system of linear equations.

$$\sum_{j=1}^{k} \{a_i, a_j\}(x, y)\lambda_j = \{b, a_i\}(x, y), \quad i = 1, \ldots, k, \tag{14}$$

where $A(x, y) = (\{a_i, a_j\}(x, y))$ is a $k \times k$ skew-symmetric matrix with the Poisson bracket $\{., .\}$ defined by ω.

We define the set,

$$\widetilde{S}_F = \{(x, y, \lambda) \in C_F : \sum_{j=1}^{k} \{a_i, a_j\}(x, y)\lambda_j = \{b, a_i\}(x, y), \quad i = 1, \ldots, k\}$$

and its corresponding subset $S_F = \phi(\widetilde{S}_F) \subset L_F$ of L_F, which is a primary solvability area of L_F.

For the implicit Hamiltonian systems, defined by singular mappings, the Poisson-Lie algebra is formed by the solvable implicit Hamiltonian systems [7]. In this section we search for Poisson-Lie algebras associated to generalized Hamiltonian systems.

Let Q be a submanifold of L_F. If $\pi|_Q \colon Q \to K$ is a diffeomorphism, then Q is smoothly solvable. We showed that $\pi|_Q \colon Q \to K$ is a diffeomorphism if and only if there exists a smooth solution $\lambda(x, y)$ of (14) such that

$$Q = \phi_F\Big(\big\{(x, y, \lambda(x, y)) \mid (x, y) \in K\big\}\Big) = \phi_F(\text{ the graph of } \lambda(x, y)).$$

Let us define

$$\{a_1, \cdots, a_k\}_K^{\perp} = \{h \in \mathcal{E}_{x,y} \mid \{h, a_i\} = 0 \quad \text{on } K\}.$$

If $h \in \{a_1, \cdots, a_k\}_K^{\perp}$, then the corresponding Hamiltonian vector field X_h is tangent to K.

THEOREM 6.1. *Equation (14) has a smooth solution defined on K if and only if*

$$b \in \langle a_1, \cdots, a_k \rangle_{\mathcal{E}_{x,y}} + \{a_1, \cdots, a_k\}_K^{\perp}.$$

Proof. Suppose that (14) has a smooth solution $\lambda(x, y)$ defined on K;

$$\Big(\{a_\ell, a_m\}(x, y)\Big) \begin{pmatrix} \lambda_1(x, y) \\ \vdots \\ \lambda_k(x, y) \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix}, \quad (x, y) \in K.$$

Let's consider a function $h(x, y) = b(x, y) - \sum_{m=1}^{k} \lambda_m(x, y) a_m(x, y)$. Then

$$\begin{pmatrix} \{h, a_1\}(x, y) \\ \vdots \\ \{h, a_k\}(x, y) \end{pmatrix} = \begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix} - \Big(\{a_\ell, a_m\}(x, y)\Big) \begin{pmatrix} \lambda_1(x, y) \\ \vdots \\ \lambda_k(x, y) \end{pmatrix}$$

is vanishing on K. In the above calculations we have $\{a_\ell, \lambda_m\}(x, y) a_m(x, y) = 0$ on K. Thus $h \in \{a_1, \cdots, a_k\}_K^{\perp}$ and $b(x, y) = \sum_{m=1}^{k} \lambda_m(x, y) a_m(x, y) + h(x, y)$. Hence

$$b \in \langle a_1, \cdots, a_k \rangle_{\mathcal{E}_{x,y}} + \{a_1, \cdots, a_k\}_K^{\perp}.$$

Conversely suppose that $b \in \langle a_1, \cdots, a_k \rangle_{\mathcal{E}_{x,y}} + \{a_1, \cdots, a_k\}_K^{\perp}$. Then $b(x, y)$ has the form

$$b(x, y) = \sum_{m=1}^{k} \mu_m(x, y) a_m(x, y) + h(x, y), \quad \mu_m \in \mathcal{E}_{x,y}, \quad h \in \{a_1, \cdots, a_k\}_K^{\perp}.$$

Then

$$\begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix} = -\Big(\{a_\ell, a_m\}(x, y)\Big) \begin{pmatrix} \mu_1(x, y) \\ \vdots \\ \mu_k(x, y) \end{pmatrix} +$$
$$\begin{pmatrix} \{h, a_1\}(x, y) \\ \vdots \\ \{h, a_k\}(x, y) \end{pmatrix} = -\Big(\{a_\ell, a_m\}(x, y)\Big) \begin{pmatrix} \mu_1(x, y) \\ \vdots \\ \mu_k(x, y) \end{pmatrix}$$

on K since $h \in \{a_1, \cdots, a_k\}_K^{\perp}$. Thus $-\mu(x, y) = -(\mu_1(x, y), \cdots, \mu_k(x, y)$ is a smooth solution of (14) defined on K. □

Now we introduce the following notation:
$\mathcal{S}_{a,b} = \{\lambda(x, y) \mid$ a smooth solution of (14) defined on $K\}$,
$F_{a,b,\lambda}(x, y) = \sum_{i=1}^{k} a_i(x, y) \lambda_i(x, y) + b(x, y), \quad \lambda = (\lambda_1, \cdots, \lambda_k) \in \mathcal{S}_{a,b}$,

$\mathcal{H}_{a,K} = \{F_{a,b,\lambda}(x,y) \mid \lambda(x,y) \in \mathcal{S}_{a,b}, \quad b \in \langle a_1, \cdots, a_k \rangle_{\mathcal{E}_{x,y}} + \{a_1, \cdots, a_k\}^{\perp}_{K}\}$,
$M_{F_{a,b,\lambda}} = \phi_F\Big(\big\{(x, y, \lambda(x,y)) \mid (x,y) \in K\big\}\Big), \quad \lambda = (\lambda_1, \cdots, \lambda_k) \in \mathcal{S}_{a,b}$.

PROPOSITION 6.2. *If $F_{a,b,\lambda} \in \mathcal{H}_{a,K}$, then the Hamiltonian vector field $X_{F_{a,b,\lambda}}$ is tangent to K and $M_{F_{a,b,\lambda}}$ is smoothly solvable.*

Proof. Let $F_{a,b,\lambda} \in \mathcal{H}_{a,K}$. $\lambda(x,y)$ is a smooth solution of (14) defined on K and $F_{a,b,\lambda}$ has the form

$$F_{a,b,\lambda}(x,y) = \sum_{m=1}^{k} a_m(x,y)\lambda_m(x,y) + b(x,y).$$

Since $\lambda(x,y)$ is a smooth solution of (14) defined on K, we have

$$\begin{aligned}\{F_{a,b,\lambda}, a_\ell\}(x,y) &= \sum_{m=1}^{k} \{a_m, a_\ell\}(x,y)\lambda_m(x,y) + \{b, a_\ell\}(x,y) \\ &= -\sum_{m=1}^{k} \{a_\ell, a_m\}(x,y)\lambda_m(x,y) + \{b, a_\ell\}(x,y) = 0\end{aligned}$$

on K. Thus $\{F_{a,b,\lambda}, a_\ell\}(x,y) = 0$ on K. Hence $X_{F_{a,b,\lambda}}$ is tangent to K and $M_{F_{a,b,\lambda}}$ is smoothly solvable. □

THEOREM 6.2.
1) $\mathcal{H}_{a,K} = \{a_1, \cdots, a_k\}^{\perp}_{K}$.
2) $\mathcal{H}_{a,K} = \{a_1, \cdots, a_k\}^{\perp}_{K}$ *is a Poisson algebra with respect to ω;*

$$\textit{if } F_{a,b,\lambda}, F_{a,b',\lambda'} \in \mathcal{H}_{a,K}, \textit{ then } \{F_{a,b,\lambda}, F_{a,b',\lambda'}\} \in \mathcal{H}_{a,K}, \ \textit{ and equivalently}$$

$$\textit{if } h, h' \in \{a_1, \cdots, a_k\}^{\perp}_{K}, \textit{ then } \{h, h'\} \in \{a_1, \cdots, a_k\}^{\perp}_{K}.$$

Proof. 1) Let $F_{a,b,\lambda} \in \mathcal{H}_{a,K}$. Then as seen on the last line of the proof of Proposition 6.2, we have $\{F_{a,b,\lambda}, a_\ell\}(x,y) = 0$ on K for $1 \leq \ell \leq k$. Therefore $F_{a,b,\lambda} \in \{a_1, \cdots, a_k\}^{\perp}_{K}$ and $\mathcal{H}_{a,K} \subset \{a_1, \cdots, a_k\}^{\perp}_{K}$.

Conversely let $h \in \{a_1, \cdots, a_k\}^{\perp}_{K}$. For any k-tuple $\lambda_1, \cdots, \lambda_k \in \mathcal{E}_{x,y}$ set

$$b(x,y) = \sum_{m=1}^{k} -a_m(x,y)\lambda_m(x,y) + h(x,y). \tag{15}$$

Then we see that $\lambda(x,y) = (\lambda_1(x,y), \cdots, \lambda_k(x,y))$ is a smooth solution of (14) defined on K and that $F_{a,b,\lambda} = h$. Thus $h \in \mathcal{H}_{a,K}$.

$$\begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix} = -\Big(\{a_m, a_\ell\}(x,y)\Big) \begin{pmatrix} \lambda_1(x,y) \\ \vdots \\ \lambda_k(x,y) \end{pmatrix} +$$
$$\begin{pmatrix} \{h, a_1\}(x,y) \\ \vdots \\ \{h, a_k\}(x,y) \end{pmatrix} = \Big(\{a_\ell, a_m\}(x,y)\Big) \begin{pmatrix} \lambda_1(x,y) \\ \vdots \\ \lambda_k(x,y) \end{pmatrix}$$

on K since $h \in \{a_1, \cdots, a_k\}_K^{\perp}$. Thus $\lambda(x,y)$ is a smooth solution of (14) and $F_{a,b,\lambda}(x,y) \in \mathcal{H}_{a,K}$. Then by the definition of $F_{a,b,\lambda}(x,y)$ and (15) we have,

$$F_{a,b,\lambda}(x,y) = \sum_{m=1}^{k} a_m(x,y)\lambda_m(x,y) + b(x,y) = h(x,y).$$

Thus $h(x,y) \in \mathcal{H}_{a,K}$ and $\{a_1, \cdots, a_k\}_K^{\perp} \subset \mathcal{H}_{a,K}$. This complete the proof of 1).

2) Suppose that $h, h' \in \{a_1, \cdots, a_k\}_K^{\perp}$. Then the Hamiltonian vector fields X_h and $X_{h'}$ are both tangent to K. Then $X_{\{h,h'\}} = [X_h, X_{h'}]$ is also tangent to K. Thus $\{h, h'\} \in \{a_1, \cdots, a_k\}_K^{\perp}$. □

Definition 6.1. *We say that two map-germs $(a_1, \cdots, a_k)$ and $(\bar{a}_1, \cdots, \bar{a}_k)$ are* **symplectic$\mathcal{K}$-equivalent** *if there exist a symplectic diffeomorphism germ $\varphi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ and a family of regular matrices $G(x,y) \in Gl(k, \mathbb{R})$ smoothly depending on (x,y) such that*

$$\begin{pmatrix} \bar{a}_1(x,y) \\ \vdots \\ \bar{a}_k(x,y) \end{pmatrix} = G(x,y) \begin{pmatrix} a_1 \circ \varphi(x,y) \\ \vdots \\ a_k \circ \varphi(x,y) \end{pmatrix}. \tag{16}$$

Proposition 6.3. *Suppose that $(a_1, \cdots, a_k)$ and $(\bar{a}_1, \cdots, \bar{a}_k)$ are symplectic $\mathcal{K}$-equivalent. Then*

$$\{a_1, \cdots, a_k\}_K^{\perp} \cong \{\bar{a}_1, \cdots, \bar{a}_k\}_{\varphi^{-1}(K)}^{\perp}$$

as Poisson algebras.

Proof. If their symplectic $\mathcal{K}$-equivalence relation is given by (16), then the isomorphism is given by

$$\varphi^* : \{a_1, \cdots, a_k\}_K^{\perp} \to \{\bar{a}_1, \cdots, \bar{a}_k\}_{\varphi^{-1}(K)}^{\perp}$$

and for $h, h' \in \{a_1, \cdots, a_k\}_K^{\perp}$ we have $\{h \circ \varphi, h' \circ \varphi\} = \{h, h'\} \circ \varphi$. □

Proposition 6.4. *Let $k \leq n$. If*

$$\operatorname{rank}\Big(\{a_i, a_j\}(x,y)\Big) = 0 \quad \textit{constantly on } \mathbb{R}^{2n},$$

then $(a_1, \cdots, a_k)$ is symplectic $\mathcal{K}$-equivalent to the projection map-germ

$$p(x,y) = (y_1, \cdots, y_k)$$

and

$$\mathcal{H}_{a,K} \cong \langle y_1, \cdots, y_k \rangle^2_{\mathcal{E}_{x,y}} + \mathcal{E}_{x_{k+1}, \cdots, x_n, y}$$

Proof. Since rank $(\{a_i, a_j\}(x,y)) = 0$ constantly on $\mathbb{R}^{2n}$, by Darboux Theorem there exists a symplectic coordinate systems $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ such that $a_i = \eta_i, i = 1, \ldots, k$. Hence $(a_1, \ldots, a_k)$ is symplectic $\mathcal{K}$-equivalent to the projection map-germ $p(x,y) = (y_1, \ldots, y_k)$. This completes the proof. □

EXAMPLE 6.1. *Let* $k = 2r$. *Let* $a = (a_1, \cdots, a_k) : (\mathbb{R}^{2n}, (0,0)) \to (\mathbb{R}^k, 0)$.

$$\operatorname{rank}\Big(\{a_i, a_j\}(0,0)\Big) = k.$$

Then $(a_1, \cdots, a_k)$ *is symplectic* $\mathcal{K}$*-equivalent to the projection map-germ*

$$p(x,y) = (y_1, \cdots, y_r, x_1, \cdots, x_r)$$

and

$$\mathcal{H}_{a,K} \cong \langle x_1, \cdots, x_{r+s}, y_1, \cdots, y_r \rangle^2_{\mathcal{E}_{x,y}} + \mathcal{E}_{x_{r+1}, \cdots, x_n, y_{r+s+1}, \cdots, y_n}.$$

EXAMPLE 6.2. *Let* $k = 2r + s$. *If*

$$\operatorname{rank}\Big(\{a_i, a_j\}(x,y)\Big) = 2r \quad \textit{constantly on } \mathbb{R}^{2n},$$

then $(a_1, \cdots, a_k)$ *is symplectic* $\mathcal{K}$*-equivalent to the projection map-germ*

$$p(x,y) = (y_1, \cdots, y_r, x_1, \cdots, x_{r+s})$$

and

$$\mathcal{H}_{a,K} \cong \langle x_1, \cdots, x_{r+s}, y_1, \cdots, y_r \rangle^2_{\mathcal{E}_{x,y}} + \mathcal{E}_{x_{r+1}, \cdots, x_n, y_{r+s+1}, \cdots, y_n}.$$

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