Hamiltonian systems on submanifolds

T. Fukuda & S. Janeczko

Abstract.

A constraint submanifold in a symplectic space after P.A.M. Dirac is determined locally by geometric restriction of the symplectic form to the constraint. The natural symplectic invariant associated to this restriction is the space of Hamiltonian vector fields which uniquely restrict to the solvable Hamiltonian ones on a constraint. By investigation of solvability of generalized Hamiltonian systems we characterize the constraint invariants and find them explicitly in the generic cases. Moreover the Poisson-Lie algebra of a constraint is calculated with direct example of the 2-sphere in symplectic space.

§1. Introduction.

Let (M,ω) be a symplectic 2n-dimensional manifold, endowed with the nondegenerate, closed two-form ω. By the vector bundle morphism $\beta : TM \ni u \mapsto \omega(u,\cdot) \in T^*M$ we introduce the canonical symplectic structure $\dot{\omega}$ on TM, namely the pullback of the Liouville symplectic form $d\theta$ defined on the cotangent bundle T^*M, $\dot{\omega} = \beta^* d\theta$. A vector field $X : M \to TM$ is said to be Hamiltonian if the form $\omega(X,\cdot)$ is closed and exact. A function $H : M \to \mathbb{R}$ is called Hamiltonian function for X if $\omega(X,\cdot) = -dH(\cdot)$. If X is Hamiltonian, then its image $X(M) \subset TM$ is a Lagrangian submanifold of $(TM,\dot{\omega})$ generated by H (cf. [15]). In local Darboux coordinates, $M \cong \mathbb{R}^{2n}$, $\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$, and $\dot{\omega} = \beta^* d\theta = \sum_{i=1}^{n} (d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i)$, where $(q,\dot{q}) = ((x,y),(\dot{x},\dot{y}))$ are coordinates on $T\mathbb{R}^{2n} \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

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In what follows a smooth submanifold $N \subset TM$ is called *Hamiltonian system* if N is Lagrangian, i.e. $\dot{\omega} \mid_N = 0$. In this case $dim N = 2n$, and if $\tau \mid_N : N \to M$ is singular, where τ is a tangent bundle projection, we also call N an *implicit Hamiltonian system.*

Fundamental property of a differential system which we investigate in this paper is its local solvability. A point $(q, \dot{q}) \in N \subset TM$ is called a *solvable point* of N if there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = q$ such that its tangent lifting $\dot{\gamma}(t)$ belongs to N. N is called a *solvable manifold* if N consists of solvable points only. N is called *smoothly solvable* if it consists smoothly solvable points, i.e. around each $v \in N$ there exists a smooth family $\alpha : U \times (-\epsilon, \epsilon) \ni (\bar{v}, t) \mapsto M$ of smooth solutions of N such that $\dot{\alpha}_{\bar{v}}(0) = \bar{v}$.

If $\tau \mid_N$ is a diffeomorphism, then N is smoothly solvable vector field on $\mathbb{R}^{2n}$. If $\tau \mid_N$ is singular, then N may not be solvable in the critical points of $\tau \mid_N$. The simplest representative example of such manifold is given by $N = \{(q, \dot{q}) \in T\mathbb{R} : q = (\dot{q} - a)^2\}$. For $a \neq 0$, N is not solvable at $(0, a)$ and this is a singular point of $\tau \mid_N$. In general case (of any submanifold of TM) the necessary and sufficient conditions for a manifold $N \subset TM$ to be solvable are found in [4, 9].

1.1. Implicit Hamiltonian systems.

For a Hamiltonian system N let $v = (q, \dot{q}) \in N$ be a solvable point of N. Then there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ as above. Thus an immediate necessary condition for a point $v = (q, \dot{q}) \in N$ to be solvable is that (cf. [4])

$$\dot{q} \in d(\tau \mid_N)_v (T_v N), \tag{1}$$

where $d(\tau \mid_N)_v$ is the tangent mapping to $\tau \mid_N$ at v. In what follows we will call this condition *tangential solvability condition.*

We can ask whether this condition is also a sufficient condition for a submanifold N to be solvable. Although the answer for this question is negative, there is a wide class of submanifolds of $T\mathbb{R}^{2n}$ for which the tangential solvability condition is also sufficient. An example of the submanifold N for which the tangential solvability condition is fulfilled but N is not solvable is given in [4].

In this work we concentrate only on Hamiltonian systems and symplectic invariants connected to their solvability. As the solvability is a local property investigated globally on a manifold, then we will use the local coordinate systems and replace manifolds by their Euclidean representatives.

Let $N \subset (T\mathbb{R}^{2n}, \dot{\omega})$ be a Hamiltonian system. Suppose that

$$corankd(\tau\mid_N)_v = k$$

for some $v \in N$. Then there exists an open neighborhood $\mathcal{O}$ of v in $T\mathbb{R}^{2n}$ and a smooth function $F : \mathbb{R}^{2n}\times\mathbb{R}^k \ni (q,\lambda) \mapsto F(q,\lambda) \in \mathbb{R}$ defined on an open neighborhood of $(q_0, 0)$ in $\mathbb{R}^{2n}\times\mathbb{R}^k$, $q_0 = \tau(v)$ such that

$$N \cap \mathcal{O} = \{(q,\dot{q}); \exists_{\lambda\in\mathbb{R}^k}, \dot{x}_i = \frac{\partial F}{\partial y_i}(q,\lambda), \dot{y}_j = -\frac{\partial F}{\partial x_j}(q,\lambda), 0 = \frac{\partial F}{\partial \lambda_l}(q,\lambda)\},$$

where $1 \le i,j \le n, \quad 1 \le l \le k$, and

$$(2) \quad rank(\frac{\partial^2 F}{\partial x_j \partial \lambda_l}, \frac{\partial^2 F}{\partial y_i \partial \lambda_l}, \frac{\partial^2 F}{\partial \lambda_s \partial \lambda_l})(q_0, 0) = k, \quad \frac{\partial^2 F}{\partial \lambda_s \partial \lambda_r}(q_0, 0) = 0,$$

where $1 \le s, r \le k$. And F is called a *generating family* of $N \cap \mathcal{O}$.

If N is a Hamiltonian system generated by a generating family $F : \mathbb{R}^{2n}\times\mathbb{R}^k \to \mathbb{R}$, then the tangential solvability condition for N is equivalent to the existence of a solution $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^k$ of the following linear equation (cf. [4]),

$$(3) \qquad \sum_{j=1}^{k} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}(q,\lambda)\mu_j = \{\frac{\partial F}{\partial \lambda_i}, F\}(q,\lambda), \quad i = 1, \ldots, k$$

for each $(q,\dot{q}) \in N$, where $\{.,.\}$ denotes the Poisson bracket on $\mathbb{R}^{2n}$ induced by ω.

By the Cramer's rule equation (3) is equivalent to

$$\sum_{j=1}^{k} [\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}] \{\frac{\partial F}{\partial \lambda_j}, F\} \mid_{\{detH=0\}\cap C_F} \equiv 0,$$

where $[\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}]$ is a cofactor matrix of $H(x,y,\lambda) = (\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}(x,y,\lambda))$ and C_F is a critical manifold defined by

$$C_F = \{(x,y,\lambda); \frac{\partial F}{\partial \lambda_i}(x,y,\lambda) = 0, \quad 0 \le i \le k\}.$$

Smooth solvability of N is implied by the condition that the linear equation (3) has a smooth solution $(\mu_1(x,y,\lambda), \ldots, \mu_k(x,y,\lambda))$ on the critical manifold C_F.

The natural problems concerning solvability phenomena of implicit Hamiltonian systems are formulated as follows,

a) *find conditions to be posed on a smooth generating family* $F : \mathbb{R}^{2n}\times\mathbb{R}^k\to\mathbb{R}$*, so that the linear equation (3) has a smooth solution on* C_F*.*
b) *specify the insolvability area in general implicit Hamiltonian systems in particular those defined by constraints in the symplectic space.*
c) *determine the Poisson-Lie algebras induced by smooth submanifolds of symplectic space.*

Point *a*) is already considered in [4]. The point *b*) needs an extra conditions on regions of L fulfilling tangential solvability condition to be finally solvable, and the point *c*) provides constructions of function algebras which are equipped with the Poisson structure. This is the subject of research in the rest of the paper.

1.2. Solvability conditions.

Let $E_s(k,k)$ denote the space of $k\times k$ symmetric matrices of real numbers. For each integer $r \geq 0$ let S_r denote the subset of $E_s(k,k)$ consisting of all symmetric matrices of rank r. Then S_r is a submanifold of $E_s(k,k)$ of codimension $(k-r)(k-r+1)/2$. Now we have a well defined mapping of N into symmetric matrices $E_s(k,k)$. We can uniquely represent this mapping by $\hat{H} : C_F\to E_s(k,k)$,

$$\hat{H}(x,y,\lambda)\mid_{\{(x,y,\lambda)\in C_F\}} = \left(\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}(x,y,\lambda)\right)\mid_{\{(x,y,\lambda)\in C_F\}} . \tag{4}$$

Definition 1. *An implicit Hamiltonian system* $N \subset T\mathbb{R}^{2n}$*, generated by generating family* $F : \mathbb{R}^{2n}\times\mathbb{R}^k\to\mathbb{R}$ *is called generic if the map* $\hat{H} : C_F\to E_s(k,k)$ *is transversal to all* S_r*,* $\quad r = 0,\ldots,k-1$*.*

Now we can formulate the main result we will use in this paper (cf. [4, 11]).

Theorem 1. ([4]) *The generic implicit Hamiltonian system* $N \subset T\mathbb{R}^{2n}$ *is smoothly solvable if and only if it satisfies the tangential solvability condition.*

Using the results concerning solvability of general implicit differential systems in [4] we can get the corresponding results concerning solvability of implicit Hamiltonian systems. Now our function-matrix $\hat{H}\mid_{\{(x,y,\lambda)\in C_F\}}: C_F\to E_s(k,k)$ corresponds to matrix $A(x)$ in [4]. Let $\mathcal{O}_{C_F,0}$ denote the ring of germs at $0 \in C_F$ of real analytic functions on C_F. Then we get the following result.

Theorem 2. ([4]) *Let* $F : (\mathbb{R}^{2n}\times\mathbb{R}^k, 0)\to\mathbb{R}$ *be a real analytic function-germ. Suppose that the implicit Hamiltonian system* N *generated by* F

fulfills the tangential solvability condition. If the ideal $< det(\hat{H} \mid_{C_F})(x, y, \lambda) >$ *in* $\mathcal{O}_{C_F,0}$ *has the property of zeros (i.e. any function vanishing on the variety defined by this ideal belongs to it), then the germ at* $(0, 0)$ *of* N *is smoothly solvable.*

1.3. Generalized Hamiltonian systems.

Let K be a submanifold of $\mathbb{R}^{2n}$ and $h : K \to \mathbb{R}$ be a smooth function on K. The notion of generalized Hamiltonian system (generalized Hamiltonian dynamics) was introduced by P.A.M. Dirac in [2]. It is defined as a sub-bundle of $T\mathbb{R}^{2n}$ over K, being a Lagrangian submanifold L of $(T\mathbb{R}^{2n}, \dot{\omega})$,(cf. [10])

$$L = \{v \in T\mathbb{R}^{2n} : \omega(v, u) = -dh(u) \quad \forall_{u \in TK}\}. \tag{5}$$

In local coordinates which we use in the setting, the generalized Hamiltonian system (5) can be written by linear in λ as generating family $F : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$,

$$F(x, y, \lambda) = \sum_{\ell=1}^{k} a_\ell(x, y)\lambda_\ell + b(x, y), \tag{6}$$

where K, being a complete intersection, is defined by an ideal $I_K =< a_1, \ldots, a_k >$ having property of zeros with analytic generators $a_i, 1 \leq i \leq k$. K is a zero-level set of the mapping $a : (x, y) \mapsto (a_1(x, y), \ldots, a_k(x, y))$, $K = \{(x, y) \in \mathbb{R}^{2n} : a_i(x, y) = 0, i = 1, \ldots, k\}$, and $b(x, y)$ is an arbitrary smooth extension of the function $h : K \to \mathbb{R}$ and the rank condition (2) is fulfilled. In what follows we consider the smooth K and b identified with h.

Generalized Hamiltonian systems are not generic in the sense of Definition 1. For such systems the necessary tangential solvability condition is also sufficient. The aim of this paper is to investigate conditions on subvarieties of symplectic space on which the solvable generalized Hamiltonian systems may exist. We find conditions that L is smoothly solvable under some properties of K and general function on K.

Let us notice that the tangential solvability condition for generalized Hamiltonian system is reformulated after (3) as the system of equations fulfilled in the smoothly solvable points of L,

$$\{\frac{\partial F}{\partial \lambda_i}, F\}(x, y, \lambda) = 0 \text{ for } (x, y, \lambda) \in C_F. \tag{7}$$

Concerning the solvability of the generalized Hamiltonian system L, we have already the following basic result proved in [4]. L is smoothly

solvable if (7) is fulfilled on $K \times \mathbb{R}^k$ which is a very strong condition expressed in the following,

Theorem 3. ([4]) *A generalized Hamiltonian system $L \subset (T\mathbb{R}^{2n}, \dot{\omega})$ generated by the generating family (6) is smoothly solvable if and only if*

$$\{a_i, a_\ell\} = 0 \quad \textit{and} \quad \{b, a_\ell\} = 0, \quad 1 \le i, \ell \le k,$$

$$\textit{on} \quad K = \{(x, y) \in \mathbb{R}^{2n} : a_i(x, y) = 0, \quad 1 \le i \le k\},$$

and $1 \le k \le n$. If $k = n$, then $b \equiv 0$.

Solvability property of L defines K to be an involutive, coisotropic submanifold of $(\mathbb{R}^{2n}, \omega)$, i.e. geometrically $T_qK \supset (T_qK)^\omega = \{u \in T_q\mathbb{R}^{2n} : \omega(u, v) = 0, \forall_{v \in T_qK}\}$, and b restricts to those functions who are constant on leaves of the characteristic foliation of coisotropic K, (cf. [13]).

Remark 1. *If $dimK < n$ and K is isotropic, i.e. $(TK)^\omega \supset TK$, then TK is solvable submanifold of L with $b \equiv 0$. In this case L can not be completely solvable Hamiltonian system. If $dimK = n$, and $TK = L$ is solvable with $b \equiv 0$, then K is Lagrangian.*

Corollary 4. *Let L be a generalized Hamiltonian system over the submanifold $K \subset \mathbb{R}^{2n}$ and its generating family F fulfills the tangential integrability condition. Then K is a coisotropic submanifold of $(\mathbb{R}^{2n}, \omega)$ and L is smoothly solvable.*

In what follows we investigate the case when L is not smoothly solvable. We clarify the properties of such L with respect to the structure of non-solvable part of it and symplectic invariant properties of constraints. The regions of solvability on L may be identified by analysis of (7) under some assumptions on K.

§2. Solvability on even dimensional submanifolds.

The generalized Hamiltonian system L is given by an immersion

$$\phi : C_F \to L \subset (T\mathbb{R}^{2n}, \dot{\omega})$$

defined by

$$\phi(x, y, \lambda) = (x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)), \quad (x, y, \lambda) \in C_F.$$

Since $\frac{\partial F}{\partial \lambda_\ell}(x, y, \lambda) = a_\ell(x, y)$, we have $C_F = K \times \mathbb{R}^k$. Then L can be written as

$$L = \phi(C_F) = \{(x, y, \frac{\partial F}{\partial y}(x, y, \lambda), -\frac{\partial F}{\partial x}(x, y, \lambda)) \in T\mathbb{R}^{2n} : (x, y, \lambda) \in K \times \mathbb{R}^k\}.$$

We find conditions for a submanifold or domain of L to be smoothly solvable. Thus the traditionally solvable Hamiltonian system exists on a submanifold K in the case where the generating family does not satisfy the involutivity condition in Theorem 3, i.e. $\{a_i, a_\ell\} = 0 \quad \text{and} \quad \{b, a_\ell\} = 0 \quad \text{on} \quad K, \quad 1 \le i, \ell \le k$.

Consider the $k \times k$ skew-symmetric matrix $A(x,y) = (\{a_i, a_j\}(x,y))$ and the linear equation

$$\sum_{j=1}^{k} \{a_i, a_j\}(x,y)\lambda_j = \{b, a_i\}(x,y), \quad i = 1, \dots, k. \tag{8}$$

Set

$$\widetilde{S}_F = \{(x,y,\lambda) \in C_F : \sum_{j=1}^{k} \{a_i, a_j\}(x,y)\lambda_j = \{b, a_i\}(x,y), \quad i = 1, \dots, k\}$$

and $S_F = \phi(\widetilde{S}_F) \subset L$.

Comparing to the general implicit Hamiltonian systems (cf. [4]) we can easily see that the following three properties still hold in the present irregular generalized Hamiltonian case. Thus before we proceed to the more specified cases we formulate the following Lemmas.

Lemma 1.

1) $\phi : C_F \to L$ *is a diffeomorphism.*

2) The following three conditions are equivalent,

(a) *a submanifold Q of L is smoothly solvable*

(b) *there exists a smooth vector field ξ tangent to Q such that*

$$d\tau(\xi(x,y,\dot{x},\dot{y})) = \sum_{i=1}^{n} \dot{x}_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \dot{y}_i \frac{\partial}{\partial y_i},$$

(c) *there exists a smooth vector field $\widetilde{\xi}$ tangent to $\widetilde{Q} = \phi^{-1}(Q)$ such that*

$$d\widetilde{\pi}(\widetilde{\xi}(x,y,\lambda)) = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x,y,\lambda) \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,y,\lambda) \frac{\partial}{\partial y_i},$$

where $\widetilde{\pi} : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}^{2n}, \quad \widetilde{\pi}(x,y,\lambda) = (x,y)$.

Lemma 2. *1) For $(x,y,\lambda) \in C_F$, the vector field*

$$\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x,y,\lambda) \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,y,\lambda) \frac{\partial}{\partial y_i}$$

is tangent to K if and only if equation (8) is fulfilled.

2) Equivalently, for a point $(x, y, \dot{x}, \dot{y}) \in L$, the vector field

$$\sum_{i=1}^{n} \dot{x}_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \dot{y}_i \frac{\partial}{\partial y_i}$$

is tangent to K at (x, y) if and only if $(x, y, \dot{x}, \dot{y}) \in S_F$.

Proof. Since K is defined by equations $a_1(x, y) = 0, \ldots, a_k(x, y) = 0$, then the vector field

$$\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda) \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x, y, \lambda) \frac{\partial}{\partial y_i}$$

is tangent to K if and only if

$$(\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda) \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x, y, \lambda) \frac{\partial}{\partial y_i})(a_j(x, y)) = 0, \quad j = 1, \ldots, k,$$

which holds if and only if $\{F, a_j\}(x, y, \lambda) = 0, \quad j = 1, \ldots, k$. Inserting (6) the last equality holds if and only if

$$\sum_{i=1}^{k} \{a_i, a_j\}(x, y)\lambda_i + \{b, a_j\}(x, y) = 0, \quad j = 1, \ldots, k,$$

which gives an equation (8) and completes the proof of Lemma 2 □

Lemma 3. *Let $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in L$ and let*

$$(x_0, y_0, \lambda_0) = \phi^{-1}(x_0, y_0, \dot{x}_0, \dot{y}_0) \in C_F.$$

If $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ is a solvable point of L, then $\lambda_0 = (\lambda_{01}, \ldots, \lambda_{0k})$ is a solution of the linear equation $\sum_{j=1}^{k} \{a_i, a_j\}(x_0, y_0)\lambda_j = \{b, a_i\}(x_0, y_0), i = 1, \ldots, k$, which means that

$$(x_0, y_0, \lambda_0) \in \widetilde{S}_F \quad \textit{and} \quad (x_0, y_0, \dot{x}_0, \dot{y}_0) \in S_F.$$

Consequently any solvable submanifold of L is a subset of $S_F = TK \cap L$.

Proof. Since $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in L$ is a solvable point of L, there exists a smooth curve $\gamma(t) = (x(t), y(t)) \in \mathbb{R}^{2n}, \quad -\epsilon < t < \epsilon$ such that $(\gamma(t), \dot{\gamma}(t)) \in L, \quad -\epsilon < t < \epsilon$, and $(\gamma(0), \dot{\gamma}(0)) = (x_0, y_0, \dot{x}_0, \dot{y}_0)$. Let $\widetilde{\gamma} : (\epsilon, \epsilon) \to C_F$ be the curve defined by $\widetilde{\gamma}(t) = (x(t), y(t), \lambda(t))$, then $\phi(\widetilde{\gamma}(t)) = (\gamma(t), \dot{\gamma}(t))$.

Since $(x_0, y_0, \lambda_0) = \phi^{-1}(x_0, y_0, \dot{x}_0, \dot{y}_0)$, we have $\lambda(0) = \lambda_0$. Since $\widetilde{\gamma}(t) \in C_F, \quad -\epsilon < t < \epsilon$, we see that

$$\frac{d\widetilde{\gamma}}{dt}(0) = \dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y} + \frac{d\lambda}{dt}(0) \frac{\partial}{\partial \lambda}$$

is tangent to L. Since L is contained in $T\mathbb{R}^{2n} \mid_K$ and K is defined by $a_1(x, y) = 0, \ldots, a_k(x, y) = 0$, we have

$$(\dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y} + \frac{d\lambda}{dt}(0) \frac{\partial}{\partial \lambda})(a_j) = 0, \quad j = 1, \ldots, k.$$

And

$$0 = \frac{\partial F}{\partial y}(x_0, y_0, \lambda_0) \frac{\partial a_j}{\partial x}(0) - \frac{\partial F}{\partial x}(x_0, y_0, \lambda_0) \frac{\partial a_j}{\partial y}(0) \quad = \quad \{F, a_j\}(x_0, y_0, \lambda_0).$$

And using the form (6) of F we have

$$\sum_{i=1}^{k} \{a_i, a_j\}(x_0, y_0) \lambda_{0i} + \{b, a_j\}(x_0, y_0) = 0, \quad j = 1, \ldots, k.$$

Thus $\lambda_0 = (\lambda_{01}, \ldots, \lambda_{0k})$ is a solution of the system of linear equations

$$\sum_{j=1}^{k} \left(\{a_i, a_j\}(x_0, y_0)\right) \lambda_j = \{b, a_i\}(x_0, y_0), 1 \leq i \leq k.$$

This completes the proof of Lemma 3. □

First we have the following introductory result.

Proposition 4.
1) If a submanifold Q of L is a solvable submanifold of the generalized Hamiltonian system L, then it is a solvable submanifold of the tangent bundle TK of K.
2) If the linear equation (8) has a smooth solution $(\lambda_1(x, y), \cdots, \lambda_k(x, y))$ defined on K, then the image $G_\lambda = \phi(\widetilde{G}_\lambda)$ by ϕ of the graph of this solution

$$\widetilde{G}_\lambda = \{(x, y, \lambda_1(x, y), \ldots, \lambda_k(x, y)) : (x, y) \in K\}$$

is a smoothly solvable submanifold of L.

Proof. Part 1) is immediate by Lemma 3.
For part 2) suppose that the linear equation (8) has a smooth solution $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ defined on K. Consider the image

$$G_\lambda = \phi(\widetilde{G}_\lambda)$$

by ϕ of the graph $\widetilde{G}_\lambda = \{(x, y, \lambda_1(x, y), \ldots, \lambda_k(x, y)) \mid (x, y) \in K\}$ of the solution $(\lambda_1(x, y), \ldots, \lambda_k(x, y))$.

Since $\lambda(x, y)$ is a solution of the linear equation (8), from Lemma 2, we see that the vector field

$$d\widetilde{\pi}(\frac{\partial F}{\partial y}(x, y, \lambda(x, y))\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda(x, y))\frac{\partial}{\partial y})$$

is tangent to K. Since $\lambda(x, y)$ is smooth, then this vector field depends smoothly on (x, y). Since $\widetilde{\pi} \mid_{\widetilde{G}_\lambda} : \widetilde{G}_\lambda \to K$ is a diffeomorphism then there exists a smooth vector field $\widetilde{\xi}$ tangent to $\widetilde{G}_\lambda$ such that

$$d\widetilde{\pi}(\widetilde{\xi}(x, y, \lambda(x, y))) = \frac{\partial F}{\partial y}(x, y, \lambda(x, y))\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x, y, \lambda(x, y))\frac{\partial}{\partial y}.$$

Then, from Lemma 1. 2), the image $G_\lambda = \phi(\widetilde{G}_\lambda)$ is a smoothly solvable submanifold of L. This completes the proof of Proposition 4. □

Remark 2. *In Proposition 4. 1), in order to check that Q is smoothly solvable, it is enough to check that Q is a submanifold of TK and that Q is smoothly solvable as an implicit differential system, to which one can apply results of* [4].

We see that the Proposition 4. 1) is a direct consequence of Lemmas 2 and 3. Situation diametrically opposite to that in Theorem 3 is in the case if

$$\det(\{a_\ell, a_m\}(x, y)) \neq 0. \tag{9}$$

Under this condition we have

Proposition 5. *Let L be a generalized Hamiltonian system generated by a generating family (6). Suppose that*

$$k \quad \textit{is even and} \quad \det(\{a_\ell, a_m\}(x, y)) \neq 0, \quad \text{on } K.$$

Then S_F is a smoothly solvable submanifold of L and it is the maximal solvable submanifold of L in the sense that any other smoothly solvable submanifold of L is a submanifold of S_F. Moreover, the projection $\tau_{|S_F} : S_F \to K$ is a diffeomorphism and has no singular points. Consequently, S_F is a unique smoothly solvable submanifold of L such that $\tau(S_F) = K$.

Proof. Consider the $k \times k$ matrix $(\{a_\ell, a_m\}(x, y))$ and the linear equation

$$\sum_{m=1}^{k} \{a_\ell, a_m\}(x, y)\lambda_m = \{b, a_\ell\}(x, y), \quad 1 \leq \ell \leq k. \tag{10}$$

Since $\det(\{a_\ell, a_m\}(x,y)) \neq 0$ on K, the linear equation (10) has a unique smooth solution $\lambda(x,y) = (\lambda_1(x,y), \ldots, \lambda_k(x,y))$ on K. Then we have

$$\widetilde{S}_F = \{(x,y,\lambda) \in \mathbb{R}^{2n} \times \mathbb{R}^k \mid \lambda = \lambda(x,y), \quad (x,y) \in K\}.$$

Thus $\widetilde{S}_F$ is the graph of the map $\lambda : K \to \mathbb{R}^k$. Therefore the projection map $\widetilde{\pi} \mid_{\widetilde{S}_F} : \widetilde{S}_F \to K$ is a submersion and so is $\tau_{|S_F} : S_F \to K$. Moreover, from Lemma 2, S_F is an implicit differential system as a submanifold of TK. Thus S_F is a smoothly solvable implicit differential system and it is a smoothly solvable submanifold of L. Now the maximality of S_F follows from Lemma 3. This completes the proof of Proposition 5. □

Remark 3. If b is a pre-Hamiltonian function defined on K for the generalized Hamiltonian system L, then the corresponding Hamiltonian function for the solvable Hamiltonian vector field in the restricted symplectic space $(K, \omega \mid_K)$ is defined by

$$\hat{F}(x,y) = b(x,y) + \sum_{i=1}^{k} \lambda_i(x,y) a_i(x,y),$$

where $\lambda(x,y)$ is a unique smooth solution of the equation (10) and

$$\frac{\partial \hat{F}}{\partial y}(x,y)\frac{\partial}{\partial x} - \frac{\partial \hat{F}}{\partial x}(x,y)\frac{\partial}{\partial y} \mid_K$$

is a smooth section of TK.

§3. Solvability over constant rank constraints.

Let K be a submanifold of (M, ω). By $(T_qK)^\omega$ we denote the skew-orthogonal subspace to T_qK. The constant rank of matrix $A(q)$ at all points of K is related to the special cases of submanifolds of M.

K is said to be coisotropic if $(T_qK)^\omega \subset T_qK$ at each $q \in K$ is isotropic if $T_qK \subset (T_qK)^\omega$ at each $q \in K$. $detA(q)$ is vanishing on K in both these cases. K is said to be symplectic if $T_qK \cap (T_qK)^\omega = 0$ at each $q \in K$.

Let us denote the intersection $V_q = T_qK \cap (T_qK)^\omega$ and we assume $dimV_q = l$ is constant at each $q \in K$. V_q is a kernel of $A(q)$. The two form induced on the quotient space $(T_qK)^\omega / V_q$ is nondegenerated for $k = 1, \ldots, 2n-1$. $dim(T_qK)^\omega / V_q = k - l$ and there is a natural relation for the kernel dimension, $l \leq max\{k, 2n-k\}$. Obviously $k-l$ is an even number. We easily find that $rankA(q) = k - l, l \leq n, \ \ l \leq 2n - k$. The kernel $N_q = KerA(q)$ gives an intersection of skew-conormal fibre of K

with the tangent space TK. The constant rank of A along K implies that $V = \bigcup_{q\in K} V_q$ is a distribution on K, this is the characteristic distribution of $\omega\mid_K$. V is defined by the generating function

$$F(x,y,\lambda) = \sum_{i=1}^{k} \lambda_i a_i(x,y), (x,y) = q \in K.$$

$corankA(x,y) \leq 2n-k$, $rankA(x,y) \geq 2k-2n$, for $k > n$

$$V_q = \{\sum_{i=1}^{k} \lambda_i (\frac{\partial a_i}{\partial y}(x,y)\frac{\partial}{\partial x} - \frac{\partial a_i}{\partial x}(x,y)\frac{\partial}{\partial y})\},$$

where $\lambda \in KerA(x,y), (x,y) \in K$.

Proposition 6. *V is an integrable distribution of TK and it is a solvable submanifold of L with $b \equiv 0$.*

Let $\lambda^j(x,y)$, $j = 1,\dots l$ be $l-$independent smooth sections of the fibre bundle $KerA(x,y)$ over K, then we can re-define the defining generators a_i taking instead the new l functions, $c^j(x,y) \in I_K$,

$$c^j(x,y) = \sum_{i=1}^{k} \lambda_i^j(x,y) a_i(x,y), \quad j = 1,\dots,l.$$

We can easily check that

$$\{c^j, a_i\}\mid_K = 0, \quad j = 1,\dots,l, i = 1,\dots,k$$

and

$$\{c^j, c^s\}\mid_K = 0, \quad j = 1,\dots,l, s = 1,\dots,l.$$

After re-numeration of $a_1,\dots,a_k$ assume that $c^1,\dots,c^l,a_{l+1},\dots,a_k$ are independent and define K. Thus the matrix A reduces to the maximal rank sub-matrix $(\{a_i,a_j\})_{l+1\leq i,j\leq k}$.

Thus the problem reduces to the coisotropic submanifold $C \in \mathbb{R}^{2n}$ defined by $c^j = 0$ with b preserving fibers of V on C. The rest of functions a_i define K as a section of the foliation defined by integral surfaces of V.

§4. Solvable domains in generalized Hamiltonian systems.

When k is odd we have $\det A(x,y) = 0$ everywhere. As a result corresponding to Proposition 5, we have

Proposition 7. *Let $L \subset T\mathbb{R}^{2n}$ be a generalized Hamiltonian system generated by a generating family (6). Suppose that k is odd and the rank of $(\{a_\ell, a_m\}(x,y))$ is constant and equal to $k-1$. Suppose also that the linear equation (8) has a smooth solution $\lambda(x,y) = (\lambda_1(x,y), \ldots, \lambda_k(x,y))$ on K. Then*

1) S_F is a smoothly solvable submanifold of L and it is the maximal solvable submanifold in the sense that any other smoothly solvable submanifold of L is a submanifold of S_F.

2) Moreover, S_F is a line bundle over K with the submersion map $\tau_{|S_F} : S_F \to K$.

Proof. Let $L \subset T\mathbb{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family (6). Suppose that k is odd and the rank of $(\{a_i, a_j\}(x,y))$ is constant and equal to $k-1$. Suppose also that the linear equation (8) has a smooth solution $\lambda(x,y) = (\lambda_1(x,y), \ldots, \lambda_k(x,y))$ on K.

Since the matrix $(\{a_i, a_j\}(x,y))$ depends smoothly on $(x,y) \in K$ and has a constant rank $k-1$, the kernel set

$$\widetilde{K}_F = \{(x,y,\lambda) \in C_F \mid (\{a_i, a_j\}(x,y))\,\lambda = 0\}$$

is a smooth line bundle over K and we see that

$$\widetilde{S}_F = \{(x,y,\lambda(x,y)+\lambda) \mid (x,y) \in K, (x,y,\lambda) \in \widetilde{K}_F\}.$$

Therefore $\widetilde{S}_F$ is also a line bundle over K and so is $S_F = \phi(\widetilde{S}_F)$. Thus, S_F is a smooth manifold and the projection $\pi : S_F \to K$ is a submersion. From Proposition 5, $S_F = \phi(\widetilde{S}_F)$ is a smoothly solvable submanifold of L. The maximality of S_F follows from Lemma 3. This completes the proof of Proposition 7. □

The maximality of S_F, both in Proposition 5 and Proposition 7, follows from Lemma 3. Proposition 7. 1) is a direct consequence of Proposition 5. 2), Lemma 3 and the following more general theorem.

Theorem 5. *Let $L \subset T\mathbb{R}^{2n}$ be a generalized Hamiltonian system generated by a generating family (6). Let Q be a submanifold of L such that the projection $\tau\,|_Q \colon Q \to K$ is a submersion. Then Q is smoothly solvable if and only if $Q \subset S_F$.*

As a direct corollary of Theorem 5, we have the following Proposition which is a generalization of Proposition 7.

Proposition 8. *Let $L \subset T\mathbb{R}^{2n}$ be a generalized Hamiltonian system generated by (6). Suppose that the linear equation (8) has a smooth*

solution on K,

$$\lambda(x,y) = (\lambda_1(x,y), \ldots, \lambda_k(x,y)).$$

Suppose also that the kernel set

$$\widetilde{K}_F = \ker\left(\{a_i, a_j\}\right) = \{(x,y,\lambda) \in C \times \mathbb{R}^k \mid \left(\{a_i, a_j\}(x,y)\right)\lambda = 0\}$$

contains an $m-$ dimensional smooth vector subbundle $\widetilde{K}$ of the vector bundle $K \times \mathbb{R}^k$ over K. Then

$$R_F = \{(x, y, \frac{\partial F}{\partial y}(x,y,\lambda(x,y)+\lambda), -\frac{\partial F}{\partial x}(x,y,\lambda(x,y)+\lambda)) \mid (x,y,\lambda) \in \widetilde{K}\}$$

is a $(2n-k+m)$ dimensional smoothly solvable submanifold of L.

Proof. (of Theorem 5 and Proposition 8). Let $L \subset T\mathbb{R}^{2n}$ be an implicit Hamiltonian system generated by a Morse family (6). Suppose that M is a submanifold of L such that the projection $\tau\,|_M \colon M \to K$ is a submersion.

If M is smoothly solvable, then, from Lemma 3, we have $M \subset S_F$.

Conversely, suppose that $M \subset S_F$. Let

$$(x_0, y_0, \dot{x}_0, \dot{y}_0) \in M \quad \text{and} \quad (x_0, y_0, \lambda_0) = \phi^{-1}(x_0, y_0, \dot{x}_0, \dot{y}_0).$$

Since

$$(x_0, y_0, \dot{x}_0, \dot{y}_0) \in S_F \quad \text{and} \quad (x_0, y_0, \lambda_0) \in \widetilde{S}_F,$$

from the definition of S_F and from Lemma 2, the vector

$$\dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y} = \frac{\partial F}{\partial y}(x_0, y_0, \lambda_0)\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}(x_0, y_0, \lambda_0)\frac{\partial}{\partial y}$$

is tangent to K at (x_0, y_0) and smoothly depends on $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in M$. Since $\tau\,|_M \colon M \to K$ is a submersion, there exists a smooth vector field ξ tangent to M such that

$$d\tau(\xi(x_0, y_0, \dot{x}_0, \dot{y}_0)) = \dot{x}_0 \frac{\partial}{\partial x} + \dot{y}_0 \frac{\partial}{\partial y}, \quad \forall (x_0, y_0, \dot{x}_0, \dot{y}_0) \in M.$$

Thus, from Lemma 1, M is smoothly solvable. This completes the proof of Theorem 5. Now Proposition 8 is a direct corollary of Theorem 5. □

The condition in Proposition 8 that the kernel set $\widetilde{K}_F$ contains an m dimensional smooth vector subbundle is not generic condition if $m > 0$ for k even, and if $m > 1$ for k odd. Because in general if k is even, $\det\left(\{a_\ell, a_m\}(x,y)\right) \neq 0$ almost everywhere, and if k is odd, $\operatorname{rank}\left(\{a_\ell, a_m\}(x,y)\right) = k-1$ almost everywhere. Note that smooth

vector subbundles of $\widetilde{K}_F$ are not unique in general: If $\bar{K}$ is a smooth vector subbundle of $\widetilde{K}_F$, then any smooth vector subbundle of $\bar{K}$ is also a smooth vector subbundle of $\widetilde{K}_F$. For k even we define

$$K_{reg} = \{(x,y) \in K \mid \det(\{a_\ell, a_m\}(x,y)) \neq 0\},$$

for k odd we have

$$K_{k-1} = \{(x,y) \in K \mid \operatorname{rank}(\{a_\ell, a_m\}(x,y)) = k-1\}.$$

In generic situation, we have

Proposition 9. *Suppose that k is even and* $\det(\{a_\ell, a_m\}(x,y)) \neq 0$ *almost everywhere but* $\det(\{a_\ell, a_m\}(0,0)) = 0$. *Then $L_F \cap \tilde{\pi}^{-1}(K_{reg})$ is smoothly solvable implicit differential system of TK_{reg}. Moreover there exists a smoothly solvable differential system Q such that $\tilde{\pi}(Q) = K$ if and only if the linear equation (8) has a smooth solution on K. Such a smoothly solvable differential system Q is unique and it has the properties that $Q \cap \pi^{-1}(K_{reg}) = L_F \cap \tau^{-1}(K_{reg})$ and that $\tau_Q : Q \to K$ is a diffeomorphism.*

Proof. The fact that $L \cap \tau^{-1}(K_{reg})$ is a smoothly solvable implicit differential system of TK_{reg} is a direct corollary of Theorem 5. Now suppose that the linear equation (8) has a smooth solution

$$(\lambda_1(x,y), \ldots, \lambda_k(x,y))$$

on K. Then, by Proposition 4. 2), the image $G_\lambda = \phi(\widetilde{G}_\lambda)$ of the graph $\widetilde{G}_\lambda$ of the solution $(\lambda_1(x,y), \ldots, \lambda_k(x,y))$ is a smoothly solvable submanifold of L. Take G_λ as M we seek. Then, by Theorem 5, $M \cap TK_{reg} = G_\lambda \cap TK_{reg}$ and $S_F \cap TC_{reg}$ must coincide. Since K_{reg} is dense in K, the uniqueness of such M follows.

Conversely suppose that there exists a smoothly solvable differentiable system M such that $\pi(M) = K$. Then, again by Proposition 5, $M \cap TK_{reg}$ must coincide with $S_F \cap TK_{reg}$. Consider the inverse image $\widetilde{M} = \phi^{-1}(M) \subset C_F \subset K \times \mathbb{R}^k$. Since, by Proposition 5, $\widetilde{S}_F \cap (K_{reg} \times \mathbb{R}^k)$ is the graph of a smooth solution $\lambda : K_{reg} \to \mathbb{R}^k$ of the linear equation (8), $\widetilde{M} \cap (K_{reg} \times \mathbb{R}^k)$ must coincide with the graph of this smooth solution $\lambda(x,y)$, $(x,y) \in K_{reg}$. Since K_{reg} is dense in K and $\widetilde{M}$ is a smooth submanifold such that $\widetilde{\pi}(\widetilde{M}) = K$, $\lambda(x,y)$ can be extended to a smooth solution defined on K of the linear equation. Thus the linear equation has a smooth solution on K. This completes the proof of Proposition 7 □.

Remark 4. *In the case where k is odd we can have a similar result. However, when k is odd and the rank of the matrix $(\{a_i, a_j\}(0,0))$ is less than $k-1$,*

1) There is a question, in a generic situation, whether the kernel set

$$\widetilde{K}_F = \ker(\{a_i, a_j\}) = \{(x, y, \lambda) \in K \times \mathbb{R}^k \mid (\{a_i, a_j\}(x, y))\,\lambda = 0\}$$

contains or not a smooth line bundle over K appeared in Proposition 8.

2) Moreover when k is odd, we can not apply our condition for the linear equation to have a smooth solution. Since $\det(\{a_i, a_j\}(x, y)) = 0$, the product of the matrix $(\{a_i, a_j\}(x, y))$ and its cofactor matrix is always the zero matrix. Thus we can not apply our method.

Theorem 5 and Propositions 7, 5 and 8 are obtained by reducing the fibers of the bundle $\tau : L \to K$. However reducing the base space K, we obtain

Proposition 10. *Suppose that L is not smoothly solvable. Let $g_1, \ldots, g_s : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth functions such that the Jacobian matrix of the map $(a, g) = (a_1, \ldots, a_k, g_1, \ldots, g_s) : \mathbb{R}^{2n} \to \mathbb{R}^{k+s}$ has the maximal rank $k + s$. Let $C_g \subset K$ be a submanifold defined by*

$$C_g = \{(x, y) \in K \mid g_1(x, y) = \cdots = g_s(x, y) = 0\}.$$

Then $\phi(C_g \times \mathbb{R}^k)(\subset L_F)$ is smoothly solvable if and only if

$$\{a_\ell, a_m\} = \{b, a_m\} = 0, \{a_\ell, g_t\} = \{b, g_t\} = 0 \qquad \text{on} \quad C_g,$$

$$1 \le \ell, m \le k, \quad 1 \le t \le s.$$

Proof. This Proposition can be proved in the same way as it was done for Theorem 3. Let us consider the Morse family (6). Then we have

$$\frac{\partial F}{\partial \lambda_\ell}(x, y, \lambda) = a_\ell(x, y).$$

Set

$$C_{F,g} = \{(x, y, \lambda) \in C_F \mid g_1(x, y) = \cdots = g_s(x, y) = 0\} = C_g \times \mathbb{R}^k,$$
$$L_{F,g} = \phi(C_{F,g}).$$

Now $L_{F,g}$ is smoothly solvable if and only if there exists a smooth tangent vector filed ξ on $L_{F,g} = \phi(C_{F,g})$ such that

$$d\pi(\xi(x, y, \dot{x}, \dot{y})) = \sum_{i=1}^{n} \dot{x}_i \frac{\partial}{\partial x_i} + \dot{y}_i \frac{\partial}{\partial y_i}$$

where $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the projection of the tangent bundle, then there are smooth functions $\mu_\ell(x, y, \lambda), \ell = 1, \ldots, k$, such that the vector field

$$\widetilde{\xi}(x, y, \lambda) = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda)\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}(x, y, \lambda)\frac{\partial}{\partial y_i} + \sum_{\ell=1}^{k} \mu_\ell(x, y, \lambda)\frac{\partial}{\partial \lambda_\ell}$$

is tangent to $C_{F,g} = C_g \times \mathbb{R}^k$ if and only if

$$\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda)\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}(x, y, \lambda)\frac{\partial}{\partial y_i} \quad \text{is tangent to} \quad C_{F,g}.$$

Then

$$(\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial y_i})a_\ell = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le \ell \le k,$$

$$(\sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial y_i})g_t = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le t \le s,$$

and

$$\{F, a_\ell\} = \sum_{i=1}^{k} \{a_i, a_\ell\}\lambda_i + \{b, a_\ell\} = 0 \quad \text{on} \quad C_F, \quad 1 \le \ell \le k.$$

$$\{F, g_t\} = \sum_{i=1}^{k} \{a_i, g_t\}\lambda_i + \{b, g_t\} = 0 \quad \text{on} \quad C_{F,g}, \quad 1 \le t \le s.$$

Differentiating the equalities with respect to λ_i, we have

$$\{a_i, a_\ell\} = \{a_i, g_t\} = 0, \quad \text{and then} \quad \{b, a_\ell\} = \{b, g_t\} = 0,$$

$$\text{on} \quad C_{F,g}, \quad 1 \le \ell \le k, 1 \le t \le s.$$

Conversely, if

$$\{a_i, a_\ell\} = \{a_i, g_t\} = 0, \quad \text{then} \quad \{b, a_\ell\} = \{b, g_t\} = 0,$$

$$\text{on} \quad C_{F,g}, \quad 1 \le \ell \le k, 1 \le t \le s,$$

then $L_{F,g} = \phi(C_{F,g})$ is smoothly solvable. This completes the proof of Proposition 10. □

Example 1. *Consider the following function.*

$$F(x,y,\lambda)=\sum_{i=1}^{k}x_i\lambda_i+b_1(y_1,\dots,y_k)b_2(x_{k+1},\dots,x_m),$$

$$k+1\le m\le n,\quad b_1(0)=b_2(0)=0,\quad b_1,b_2\quad \textit{are not constantly}\quad 0.$$

Then

$$\{a_\ell,a_m\}=\{x_\ell,x_m\}=0,\quad 1\le\ell,m\le k.$$

However

$$\{a_\ell,b\}=\{x_\ell,b\}=-\frac{\partial b_1}{\partial y_\ell}\cdot b_2\ne 0$$

$$on\quad K=\{a_1=\dots=a_k=0\}=\{x_1=\dots=x_k=0\}.$$

Thus L itself is not smoothly solvable. Now consider functions

$$g_1(x,y)=x_{k+1},\dots,g_s(x,y)=x_{k+s}=x_m,\quad where\quad s=m-k,$$

and set

$$S=\{(x,y)\in\mathbb{R}^{2n}\mid a_1(x,y)=\dots=a_k(x,y)=g_1(x,y)=\dots=g_s(x,y)=0\}.$$

Then

$$\{a_\ell,b\}=-\frac{\partial b_1}{\partial y_\ell}b_2(x_{x+1},\dots,x_m)=0,$$

$$\{a_\ell,g_t\}=\{x_\ell,x_{k+t}\}=0,\quad \{b,g_t\}=\{b,x_{k+t}\}=0,$$

$$1\le\ell\le k,\quad 1\le t\le s=m-k,$$

$$on\quad S=\{a_1=\dots=a_k=g_1=\dots=g_s=0\}.$$

Then, by Proposition 10, $L_F\cap(S\times\mathbb{R}^{2n})$ is smoothly solvable.

§5. Poisson-Lie algebras on submanifolds.

The Poisson algebra is an associative algebra equipped with a Poisson bracket, which is a Lie bracket. Poisson structures on manifolds are the basic mathematical structures of mechanics. The representative one is the algebra of all smooth functions on the phase space under ordinary multiplication and the Lie structure induced by the Poisson bracket usually defined by the symplectic form. For the implicit Hamiltonian systems, defined by singular mappings, the Poisson-Lie algebra is formed by the solvable implicit Hamiltonian systems [6]. In this section we search for Poisson-Lie algebras associated to generalized Hamiltonian systems.

Let Q be a submanifold of L_F. If $\pi\,|_Q\colon Q \to K$ is a diffeomorphism, then Q is smoothly solvable. We showed that $\pi\,|_Q\colon Q \to K$ is a diffeomorphism if and only if there exists a smooth solution $\lambda(x,y)$ of (8) such that

$$Q = \phi_F\Big(\big\{(x,y,\lambda(x,y)) \mid (x,y) \in K\big\}\Big) = \phi_F(\text{ the graph of } \lambda(x,y)).$$

Let us define

$$\{a_1,\cdots,a_k\}_K^{\perp} = \{h \in \mathcal{E}_{x,y} \mid \{h,a_i\} = 0 \quad \text{on } K\}.$$

If $h \in \{a_1,\cdots,a_k\}_K^{\perp}$, then the corresponding Hamiltonian vector field X_h is tangent to K.

Theorem 6. *Equation (8) has a smooth solution defined on K if and only if*

$$b \in \langle a_1,\cdots,a_k\rangle_{\mathcal{E}_{x,y}} + \{a_1,\cdots,a_k\}_K^{\perp}.$$

Proof. Suppose that (8) has a smooth solution $\lambda(x,y)$ defined on K;

$$\Big(\{a_\ell,a_m\}(x,y)\Big)\begin{pmatrix}\lambda_1(x,y)\\ \vdots\\ \lambda_k(x,y)\end{pmatrix} = \begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots\\ \{b,a_k\}(x,y)\end{pmatrix}, \quad (x,y) \in K.$$

Let's consider a function $h(x,y) = b(x,y) - \sum_{m=1}^{k}\lambda_m(x,y)a_m(x,y)$. Then

$$\begin{pmatrix}\{h,a_1\}(x,y)\\ \vdots\\ \{h,a_k\}(x,y)\end{pmatrix} = \begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots\\ \{b,a_k\}(x,y)\end{pmatrix} - \Big(\{a_\ell,a_m\}(x,y)\Big)\begin{pmatrix}\lambda_1(x,y)\\ \vdots\\ \lambda_k(x,y)\end{pmatrix}$$

is vanishing on K. In the above calculations we have $\{a_\ell,\lambda_m\}(x,y)a_m(x,y) = 0$ on K. Thus $h \in \{a_1,\cdots,a_k\}_K^{\perp}$ and $b(x,y) = \sum_{m=1}^{k}\lambda_m(x,y)a_m(x,y) + h(x,y)$. Hence

$$b \in \langle a_1,\cdots,a_k\rangle_{\mathcal{E}_{x,y}} + \{a_1,\cdots,a_k\}_K^{\perp}.$$

Conversely suppose that $b \in \langle a_1,\cdots,a_k\rangle_{\mathcal{E}_{x,y}} + \{a_1,\cdots,a_k\}_K^{\perp}$. Then $b(x,y)$ has the form

$$b(x,y) = \sum_{m=1}^{k}\mu_m(x,y)a_m(x,y) + h(x,y), \quad \mu_m \in \mathcal{E}_{x,y}, \quad h \in \{a_1,\cdots,a_k\}_K^{\perp}.$$

Then

$$\begin{pmatrix} \{b,a_1\}(x,y) \\ \vdots \\ \{b,a_k\}(x,y) \end{pmatrix} = -\Big(\{a_\ell,a_m\}(x,y)\Big)\begin{pmatrix} \mu_1(x,y) \\ \vdots \\ \mu_k(x,y) \end{pmatrix} +$$
$$\begin{pmatrix} \{h,a_1\}(x,y) \\ \vdots \\ \{h,a_k\}(x,y) \end{pmatrix} = -\Big(\{a_\ell,a_m\}(x,y)\Big)\begin{pmatrix} \mu_1(x,y) \\ \vdots \\ \mu_k(x,y) \end{pmatrix}$$

on K since $h \in \{a_1,\cdots,a_k\}_K^\perp$. Thus $-\mu(x,y) = -(\mu_1(x,y),\cdots,\mu_k(x,y)$ is a smooth solution of (8) defined on K. □

Now we introduce the following notation:
$\mathcal{S}_{a,b} = \{\lambda(x,y) \mid$ a smooth solution of (8) defined on $K\}$,
$F_{a,b,\lambda}(x,y) = \sum_{i=1}^k a_i(x,y)\lambda_i(x,y) + b(x,y), \quad \lambda = (\lambda_1,\cdots,\lambda_k) \in \mathcal{S}_{a,b}$,
$\mathcal{H}_{a,K} = \{F_{a,b,\lambda}(x,y) \mid \lambda(x,y) \in \mathcal{S}_{a,b},\ b \in \langle a_1,\cdots,a_k\rangle\mathcal{E}_{x,y} + \{a_1,\cdots,a_k\}_K^\perp\}$,

$$M_{F_{a,b,\lambda}} = \phi_F\Big(\big\{(x,y,\lambda(x,y)) \mid (x,y) \in K\big\}\Big), \quad \lambda = (\lambda_1,\cdots,\lambda_k) \in \mathcal{S}_{a,b}.$$

Proposition 11. *If $F_{a,b,\lambda} \in \mathcal{H}_{a,K}$, then the Hamiltonian vector field $X_{F_{a,b,\lambda}}$ is tangent to K and $M_{F_{a,b,\lambda}}$ is smoothly solvable.*

Proof. Let $F_{a,b,\lambda} \in \mathcal{H}_{a,K}$. $\lambda(x,y)$ is a smooth solution of (8) defined on K and $F_{a,b,\lambda}$ has the form

$$F_{a,b,\lambda}(x,y) = \sum_{m=1}^k a_m(x,y)\lambda_m(x,y) + b(x,y).$$

Since $\lambda(x,y)$ is a smooth solution of (8) defined on K, we have

$$\begin{aligned} \{F_{a,b,\lambda},a_\ell\}(x,y) &= \sum_{m=1}^k \{a_m,a_\ell\}(x,y)\lambda_m(x,y) + \{b,a_\ell\}(x,y) \\ &= -\sum_{m=1}^k \{a_\ell,a_m\}(x,y)\lambda_m(x,y) + \{b,a_\ell\}(x,y) = 0 \end{aligned}$$

on K. Thus $\{F_{a,b,\lambda},a_\ell\}(x,y) = 0$ on K. Hence $X_{F_{a,b,\lambda}}$ is tangent to K and $M_{F_{a,b,\lambda}}$ is smoothly solvable. □

Theorem 7.
1) $\mathcal{H}_{a,K} = \{a_1,\cdots,a_k\}_K^\perp$.
2) $\mathcal{H}_{a,K} = \{a_1,\cdots,a_k\}_K^\perp$ is a Poisson algebra with respect to ω;

if $F_{a,b,\lambda}, F_{a,b',\lambda'} \in \mathcal{H}_{a,K}$, then $\{F_{a,b,\lambda}, F_{a,b',\lambda'}\} \in \mathcal{H}_{a,K}$, and equivalently

if $h, h' \in \{a_1, \cdots, a_k\}_K^{\perp}$, *then* $\{h, h'\} \in \{a_1, \cdots, a_k\}_K^{\perp}$.

Proof. 1) Let $F_{a,b,\lambda} \in \mathcal{H}_{a,K}$. Then as seen on the last line of the proof of Proposition 11, we have $\{F_{a,b,\lambda}, a_\ell\}(x,y) = 0$ on K for $1 \leq \ell \leq k$. Therefore $F_{a,b,\lambda} \in \{a_1, \cdots, a_k\}_K^{\perp}$ and $\mathcal{H}_{a,K} \subset \{a_1, \cdots, a_k\}_K^{\perp}$.

Conversely let $h \in \{a_1, \cdots, a_k\}_K^{\perp}$. For any k-tuple $\lambda_1, \cdots, \lambda_k \in \mathcal{E}_{x,y}$ set

$$b(x,y) = \sum_{m=1}^{k} -a_m(x,y)\lambda_m(x,y) + h(x,y). \tag{11}$$

Then we see that $\lambda(x,y) = (\lambda_1(x,y), \cdots, \lambda_k(x,y))$ is a smooth solution of (8) defined on K and that $F_{a,b,\lambda} = h$. Thus $h \in \mathcal{H}_{a,K}$.

$$\begin{pmatrix} \{b, a_1\}(x,y) \\ \vdots \\ \{b, a_k\}(x,y) \end{pmatrix} = -\Big(\{a_m, a_\ell\}(x,y)\Big) \begin{pmatrix} \lambda_1(x,y) \\ \vdots \\ \lambda_k(x,y) \end{pmatrix} +$$
$$\begin{pmatrix} \{h, a_1\}(x,y) \\ \vdots \\ \{h, a_k\}(x,y) \end{pmatrix} = \Big(\{a_\ell, a_m\}(x,y)\Big) \begin{pmatrix} \lambda_1(x,y) \\ \vdots \\ \lambda_k(x,y) \end{pmatrix}$$

on K since $h \in \{a_1, \cdots, a_k\}_K^{\perp}$. Thus $\lambda(x,y)$ is a smooth solution of (8) and $F_{a,b,\lambda}(x,y) \in \mathcal{H}_{a,K}$. Then by the definition of $F_{a,b,\lambda}(x,y)$ and (16) we have,

$$F_{a,b,\lambda}(x,y) = \sum_{m=1}^{k} a_m(x,y)\lambda_m(x,y) + b(x,y) = h(x,y).$$

Thus $h(x,y) \in \mathcal{H}_{a,K}$ and $\{a_1, \cdots, a_k\}_K^{\perp} \subset \mathcal{H}_{a,K}$. This complete the proof of 1).

2) Suppose that $h, h' \in \{a_1, \cdots, a_k\}_K^{\perp}$. Then the Hamiltonian vector fields X_h and $X_{h'}$ are both tangent to K. Then $X_{\{h,h'\}} = [X_h, X_{h'}]$ is also tangent to K. Thus $\{h, h'\} \in \{a_1, \cdots, a_k\}_K^{\perp}$. □

Definition 2. *We say that two map-germs* $(a_1, \cdots, a_k)$ *and* $(\bar{a}_1, \cdots, \bar{a}_k)$ *are* **symplectic$\mathcal{K}$-equivalent** *if there exist a symplectic diffeomorphism germ* $\varphi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ *and a family of regular matrices* $G(x,y) \in$

$Gl(k,\mathbb{R})$ smoothly depending on (x,y) such that

$$\begin{pmatrix} \bar{a}_1(x,y) \\ \vdots \\ \bar{a}_k(x,y) \end{pmatrix} = G(x,y) \begin{pmatrix} a_1 \circ \varphi(x,y) \\ \vdots \\ a_k \circ \varphi(x,y) \end{pmatrix}. \tag{12}$$

Proposition 12. *Suppose that $(a_1,\cdots,a_k)$ and $(\bar{a}_1,\cdots,\bar{a}_k)$ are symplectic $\mathcal{K}$-equivalent. Then*

$$\{a_1,\cdots,a_k\}_K^{\perp} \cong \{\bar{a}_1,\cdots,\bar{a}_k\}_{\varphi^{-1}(K)}^{\perp}$$

as Poisson algebras.

Proof. If their symplectic $\mathcal{K}$-equivalence relation is given by (12), then the isomorphism is given by

$$\varphi^* : \{a_1,\cdots,a_k\}_K^{\perp} \to \{\bar{a}_1,\cdots,\bar{a}_k\}_{\varphi^{-1}(K)}^{\perp}$$

and for $h,h' \in \{a_1,\cdots,a_k\}_K^{\perp}$ we have $\{h\circ\varphi, h'\circ\varphi\} = \{h,h'\}\circ\varphi$. □

Proposition 13. *Let $k \le n$. If*

$$\operatorname{rank}\Big(\{a_i,a_j\}(x,y)\Big) = 0 \quad \textit{constantly on } \mathbb{R}^{2n},$$

then $(a_1,\cdots,a_k)$ is symplectic $\mathcal{K}$-equivalent to the projection map-germ

$$p(x,y) = (y_1,\cdots,y_k)$$

and

$$\mathcal{H}_{a,K} \cong \langle y_1,\cdots,y_k\rangle^2_{\mathcal{E}_{x,y}} + \mathcal{E}_{x_{k+1},\cdots,x_n,y}$$

Proof. Since rank $(\{a_i,a_j\}(x,y)) = 0$ constantly on $\mathbb{R}^{2n}$, by Darboux Theorem there exists a symplectic coordinate systems $\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n$ such that $a_i = \eta_i, i = 1,\ldots,k$. Hence $(a_1,\ldots,a_k)$ is symplectic $\mathcal{K}$-equivalent to the projection map-germ $p(x,y) = (y_1,\ldots,y_k)$. This completes the proof. □

Example 2. *Let $k = 2r$. Let $a = (a_1,\cdots,a_k) : (\mathbb{R}^{2n},(0,0)) \to (\mathbb{R}^k,0)$.*

$$\operatorname{rank}\Big(\{a_i,a_j\}(0,0)\Big) = k.$$

Then $(a_1,\cdots,a_k)$ is symplectic $\mathcal{K}$-equivalent to the projection map-germ

$$p(x,y) = (y_1,\cdots,y_r,x_1,\cdots,x_r)$$

and

$$\mathcal{H}_{a,K} \cong \langle x_1,\cdots,x_{r+s},y_1,\cdots,y_r\rangle^2_{\mathcal{E}_{x,y}} + \mathcal{E}_{x_{r+1},\cdots,x_n,y_{r+s+1},\cdots,y_n}$$

Example 3. *Let $k = 2r + s$. If*

$$\operatorname{rank}\Big(\{a_i, a_j\}(x, y)\Big) = 2r \quad \textit{constantly on } \mathbb{R}^{2n},$$

then $(a_1, \cdots, a_k)$ is symplectic $\mathcal{K}$-equivalent to the projection map-germ

$$p(x, y) = (y_1, \cdots, y_r, x_1, \cdots, x_{r+s})$$

and

$$\mathcal{H}_{a,K} \cong \langle x_1, \cdots, x_{r+s}, y_1, \cdots, y_r\rangle^2_{\mathcal{E}_{x,y}} + \mathcal{E}_{x_{r+1},\cdots,x_n,y_{r+s+1},\cdots,y_n}.$$

§6. Hamiltonian vector fields on S^2.

Let K be a submanifold of $(\mathbb{R}^{2n}, \omega)$. First we will prove the following general property.

Proposition 14. *Let U be an open subset of $\mathbb{R}^{2n}$. Suppose that $\omega\mid_{T(K\cap U)}$ is a non-singular 2 form on $K \cap U$. Let $h \in C^\infty(U)$ such that X_h is tangent to $K \cap U$. Then*

$$X_h \mid_{K\cap U} = X_{\omega|_{T(K\cap U)},h|_{K\cap U}}, \tag{13}$$

where $X_{\omega_{K\cap U},f}$ denotes the Hamiltonian vector field on the symplectic manifold $(K \cap U, \omega\mid_{T(K\cap U)})$ generated by a function $f \in C^\infty(K \cap U)$.

Proof. Since the problem is a local problem, it suffices to prove the proposition in the case where U is a small open ball. Since $\omega\mid_{K\cap U}$ is non-singular on $K \cap U$, by the Darboux-Givental theorem, we may assume that

$$a_i(x, y) = y_i, \quad a_{r+i}(x, y) = x_i, \quad i \le r, \quad (x, y) \in U.$$

Then $(x_{r+1}, \cdots, x_n, y_{r+1}, \cdots, y_n)$ is a coordinate system on $K \cap U$ and

$$\omega\mid_{K\cap U} = dy_{r+1} \wedge dx_{r+1} + \cdots + dy_n \wedge dx_n.$$

Let $h \in C^\infty(U)$ such that X_h is tangent to $K \cap U$. Then

$$\begin{aligned}
X_h \mid_{K\cap U} &= \left(\sum_{i=1}^{n} \frac{\partial h}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial y_i}\right)\mid_{K\cap U} \\
&= \sum_{i=r+1}^{n} \frac{\partial h}{\partial y_i}\mid_{K\cap U} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\mid_{K\cap U} \frac{\partial}{\partial y_i}, \quad \text{since } X_h \text{ is tangent to } K \cap U, \\
&= \sum_{i=r+1}^{n} \frac{\partial h\mid_{K\cap U}}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial h\mid_{K\cap U}}{\partial x_i}\frac{\partial}{\partial y_i} \\
&= X_{\omega_{K\cap U},h|_{K\cap U}}.
\end{aligned}$$

This completes the proof of Proposition 14. □

Let us consider the two-dimensional case. If f is a smooth function on a 2-dimensional symplectic manifold M, then an integral curve of Hamiltonian vector field X_f passing through a point $p \in M$ is contained in the level set $f^{-1}(f(p)) = \{q \in M : f(q) = f(p)\}$ which is a curve in a generic situation. For a proper Morse function we have the following straightforward result.

Proposition 15. *Suppose that f is a proper Morse function on a 2-dimensional symplectic manifold M. Let $c \in f(M)$ and let Γ be a connected component of $f^{-1}(c)$.*
1) If Γ does not contain critical points of f, then Γ is the orbit of a periodic solution of X_f.
2) If Γ contains one and only one critical point p of f, then

$$\Gamma = \begin{cases} \{p\}, & \text{if } p \text{ is a critical point of } f \text{ with index } 0 \text{ or } 2, \\ \{p\} \cup W_s(p) \cup W_u(p), & \text{if } p \text{ is a critical point of } f \text{ with index } 1, \end{cases}$$

where $W_s(p)$ and $W_u(p)$ are the stable and unstable manifolds of a stationary point p with index 1 respectively.

Let us consider the sphere $K = S^2$ in the symplectic space $(\mathbb{R}^4, \omega)$. We assume $\omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$, and

$$a_1(x,y) = y_2, \quad a_2(x,y) = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 1. \tag{14}$$

Thus

$$K \;=\; S^2 = \{(x,y) \in \mathbb{R}^4 \mid y_2 = 0, \quad x_1^2 + x_2^2 + y_1^2 = 1\}.$$

We see that $\{a_1, a_2\} = 2x_2$ and a point $(x_1, x_2, y_1, 0) \in S^2$ is a singular point of the restricted symplectic structure $\omega\,|_{TS^2}$ if and only if $x_2 = 0$:

$$\Sigma(\omega\,|_{TS^2}) = S^1 = \{(x_1, x_2, y_1, 0) \mid x_2 = 0,\ x_1^2 + y_1^2 = 1\}. \tag{15}$$

Now we consider a Hamiltonian function on $\mathbb{R}^4$

$$F_b(x,y) = a_1(x,y)\lambda_1(x,y) + a_2(x,y)\lambda_2(x,y) + b(x,y) \tag{16}$$

together with

$$b(x,y) = x_2^2\left(1 + 2x_1^2\right) \tag{17}$$

and the Hamiltonian vector field X_{F_b} on $\mathbb{R}^4$

$$X_{F_b} = \sum_{i=1,2} \frac{\partial F_b}{\partial y_i}(x,y)\frac{\partial}{\partial x_i} - \frac{\partial F_b}{\partial x_i}(x,y)\frac{\partial}{\partial y_i}. \tag{18}$$

generated by $F_b(x,y)$. In (16)

$$\begin{aligned} \lambda_1(x,y) &= 4x_1y_1x_2 + 2y_2\left(1+2x_1^2\right), \\ \lambda_2(x,y) &= -(1+2x_1^2) \end{aligned} \tag{19}$$

is a unique smooth solution of the structure equation

$$\begin{pmatrix} 0 & \{a_1,a_2\} \\ \{a_2,a_1\} & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \{b,a_1\} \\ \{b,a_2\} \end{pmatrix}, \tag{20}$$

which is precisely

$$\begin{pmatrix} 0 & 2x_2 \\ -2x_2 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -2x_2\left(1+2x_1^2\right) \\ -8x_1y_1x_2^2 - 4x_2y_2\left(1+2x_1^2\right) \end{pmatrix}, \tag{21}$$

since

$$\{a_1,a_2\} = 2x_2, \quad \{b,a_1\} = -2x_2(1+2x_1^2), \quad \{b,a_2\} = -8x_1y_1x_2^2 - 4x_2y_2(1+2x_1^2).$$

Thus we have got that X_{F_b} is tangent to S^2 as well as to $\Sigma(\omega\mid_{TS^2})$. $X_{F_b}\mid_{\Sigma(\omega\mid_{TS^2})}$ has no stationary points and its integral curves move anti-clockwise in the (x_1,y_1)-plane. Therefore, in order to understand the phase portrait of $X_{F_b}\mid_{S^2}$, it suffices to understand the phase portrait of $X_{F_b}\mid_{S^2-\Sigma(\omega\mid_{TS^2})}$. Let

$$U_+ = \{(x,y)\in\mathbb{R}^4 : x_2 > 0\}, \qquad U_- = \{(x,y)\in\mathbb{R}^4 : x_2 < 0\}. \tag{22}$$

Then

$$S^2 \cap (U_+ \cup U_-) = S^2 - \Sigma(\omega\mid_{TS^2}).$$

Thus in order to understand the phase portrait of $X_{F_b}\mid_{S^2}$, it suffices to understand the phase portrait of $X_{F_b}\mid_{S^2\cap U_\pm}$. Since $\omega\mid_{T(S^2\cap U_\pm)}$ is nonsingular, by Proposition 14 , we see that

$$X_{F_b|_{S^2\cap U_\pm}} = X_{\omega|_{T(S^2\cap U_\pm)},F_b|_{S^2\cap U_\pm}}. \tag{23}$$

Now we apply Proposition 15. In our case M is $S^2\cap U_\pm$ and the function is $F_b\mid_{S^2\cap U_\pm}$. Note that $F_b\mid_{S^2\cap U_\pm}$ is a proper function. We adapt

$$(x_1,y_1), \quad (\text{with} \quad x_1^2+y_1^2<1),$$

as a coordinate system on $S^2\cap U_\pm$. Then, we see that

$$\omega\mid_{T(S^2\cap U_\pm)} = dy_1\wedge dx_1. \tag{24}$$

Since

$$F_b(x,y) = a_1(x,y)\lambda_1(x,y) + a_2(x,y)\lambda_2(x,y) + b(x,y)$$

we have

$$F_b \mid_{S^2\cap U_\pm} = b \mid_{S^2\cap U_\pm} = x_2^2(1+2x_1^2) \mid_{S^2\cap U_\pm} .$$

Therefore

$$F_b \mid_{S^2\cap U_\pm} = b \mid_{S^2\cap U_\pm} = (1-x_1^2-y_1^2)(1+2x_1^2), \tag{25}$$

where $x_2^2 = 1-x_1^2-y_1^2$ on S^2. From (23), (24) and (25) we see that the solutions of the Hamiltonian systems $X_{F_b|_{S^2\cap U_+}}$ and $X_{F_b|_{S^2\cap U_-}}$ are the same in x_1, y_1 coordinates. Hence, from now on we investigate only the case $X_{F_b|_{S^2\cap U_+}}$.

Proposition 16. *1) The function $F_b \mid_{S^2\cap U_+} = b \mid_{S^2\cap U_+}$ is a Morse function with three critical points $(x_1, y_1) = (0,0)$ and $(x_1, y_1) = (\pm\frac{1}{2}, 0)$.*
2) The point $(0,0)$ is a saddle point (index 1) and $b(0,0) = 1$. The points $(\pm\frac{1}{2}, 0)$ are the maximum points of $b \mid_{U_+}$ and $b(\pm\frac{1}{2}, 0) = 9/8$.
3) Thus the phase portrait of $X_{F_b|_{S^2\cap U_+}}$ is as follows.

$$(b_{U_+})^{-1}(c) = \begin{cases} \emptyset, & \text{for } c > \frac{9}{8}, \\ \{(\frac{1}{2},0),(-\frac{1}{2},0)\} \quad \text{the two stationary points,} & \text{for } c = \frac{9}{8}, \\ \text{the disjoint sum of two periodic orbits,} & \text{for } \frac{9}{8} > c > 1, \\ \{(0,0)\} \cup W_s(0,0) \cup W_u(0,0), & \text{for } c = 1, \\ \text{a periodic orbit,} & \text{for } 1 > c > 0, \end{cases}$$

where all the periodic solutions move anti-clockwise in (x_1, y_1)-plane, (see Fig.1).

Proof. First we search for critical points of $F_b \mid_{S^2\cap U_\pm} = b \mid_{S^2\cap U_\pm} = (1-x_1^2-y_1^2)(1+2x_1^2)$. Since

$$\begin{aligned} \frac{\partial b \mid_{S^2\cap U}}{\partial y_1}(x_1, y_1) &= -2y_1(1+2x_1^2) \\ \frac{\partial b \mid_{S^2\cap U}}{\partial x_1}(x_1, y_1) &= -2x_1((1+2x_1^2) + 4x_1(1-x_1^2-y_1^2) \\ &= 2x_1(1-4x_1^2-2y_1^2), \end{aligned}$$

the critical points of $F_b \mid_{S^2\cap U} = b \mid_{S^2\cap U}$ are

$$(x_1, y_1) = (0,0) \quad \text{and} \quad (x_1, y_1) = (\pm\frac{1}{2}, 0) \tag{26}$$

and the critical values are

$$F_b \mid_{S^2\cap U} (0,0) = 1 \quad \text{and} \quad F_b \mid_{S^2\cap U} (\pm\frac{1}{2}, 0) = \frac{9}{8}. \tag{27}$$

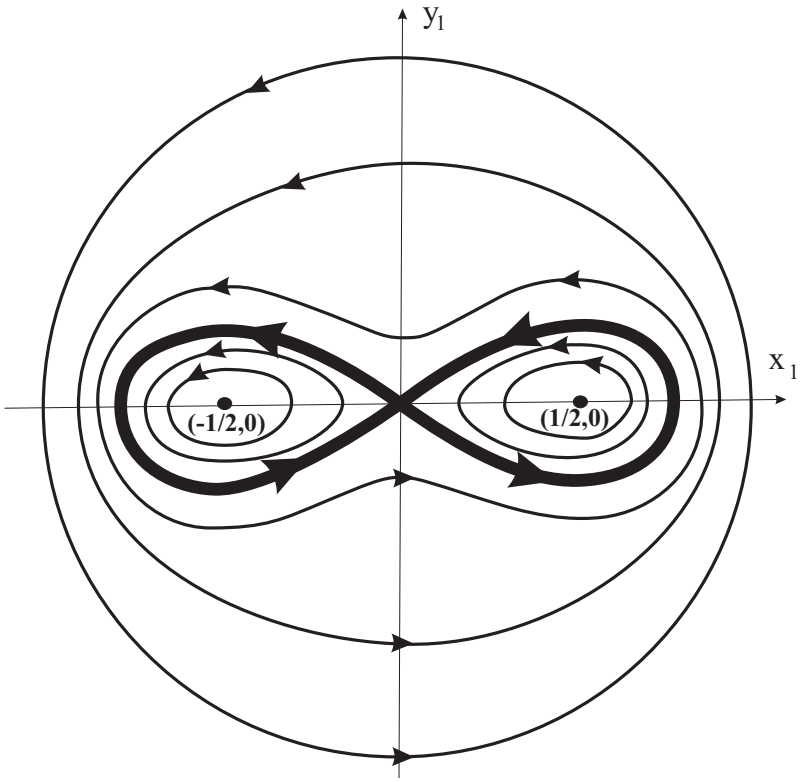

Fig. 1. Flow of $d\pi(X_{F_b}\mid_{S^2}), \pi(x_1, x_2, y_1) = (x_1, y_1)$

The Hessian matrices at the critical points are

$$(28) \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{at} \quad (0,0), \qquad \begin{pmatrix} -4 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{at} \quad (\pm\frac{1}{2}, 0).$$

Thus $(0,0)$ is a saddle point and $F_b \mid_{S^2\cap U} = b \mid_{S^2\cap U}$ takes maximal (actually maximum) values $\frac{9}{8}$ at $(\pm\frac{1}{2}, 0)$. This proves 1) and 2). Therefore the shape of the graph of $b \mid_{S^2\cap U}$ looks like an island having two mountains with the same hight and a mountain pass (a saddle point) between them. Every periodic solution inherits its orientation from those of integral curves on $\Sigma(\omega \mid_{TS^2})$ which move anti-clockwise in the (x_1, y_1)-plane. This proves 3). This completes the proof of Proposition 16. □

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(T. Fukuda) *Department of Mathematics College of Humanities and Sciences, Nihon University, Sakurajousui*
3-25-40, Setagaya-ku, 156-8550 Tokyo, Japan,
E-mail address: fukuda@math.chs.nihon-u.ac.jp

(S. Janeczko) *Instytut Matematyczny PAN, ul. Śniadeckich 8, 00-950 Warszawa, Poland*
and Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, Pl. Politechniki 1, 00-661 Warszawa, Poland
E-mail address: janeczko@mini.pw.edu.pl