



# On the Poisson algebra of a singular map

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## ABSTRACT

We construct the Poisson algebra associated to a singular mapping into symplectic space and show that this is an algebra of smooth functions generating solvable implicit Hamiltonian systems.

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## 1. Introduction

Let  $(M, \omega)$  be a smooth, symplectic  $2n$ -dimensional manifold and let  $(TM, \dot{\omega})$  be its tangent bundle endowed with a symplectic structure  $\dot{\omega}$ . This structure is defined by the Liouville form  $\theta$  using the canonical flat morphism  $\beta : TM \ni v \mapsto \omega(v, \cdot) \in T^*M$ , between tangent and cotangent bundles of the symplectic manifold  $(M, \omega)$ ,  $\dot{\omega} = \beta^*d\theta$ .

Let  $\bar{F} : \mathbb{R}^{2n} \supset (U, 0) \rightarrow (M, \omega)$  be a smooth map-germ. With this mapping we associate all smooth, isotropic map-germs  $F : (U, 0) \rightarrow (TM, \dot{\omega})$  which are vector fields along  $\bar{F}$  such that  $\bar{F} = \pi \circ F$ ,  $\pi : TM \rightarrow M$ , and  $F^*\dot{\omega} = 0$ . This is a generalization of standard Hamiltonian systems to include the implicit case (see [1,2]). For each isotropic  $F$  we have  $d(\beta \circ F)^*\theta = F^*\dot{\omega} = 0$ . Thus  $(\beta \circ F)^*\theta$  is a germ of a closed 1-form, so there exists a smooth function-germ  $h : (U, 0) \rightarrow \mathbb{R}$  which we call the generating function for  $F$ , such that  $(\beta \circ F)^*\theta = -dh$ .

We consider the space  $\mathcal{R}_{\bar{F}}$  of generating functions of isotropic map-germs  $F$  along a fixed singular map-germ  $\bar{F}$ . Among the  $h \in \mathcal{R}_{\bar{F}}$  appear such ones that the vector field  $X_h$  on  $U$  defined by the formula

$$\bar{F}^*\omega(X_h, \xi) = -\xi(h) \quad \text{for each vector field } \xi \text{ on } U,$$

is smooth. In this case the generating functions are called Hamiltonians and we denote by  $\mathcal{H}_{\bar{F}}$  the space of Hamiltonians. In this paper we investigate the spaces  $\mathcal{R}_{\bar{F}}$  and  $\mathcal{H}_{\bar{F}}$ . We show (Theorem 3.2) that the space  $\mathcal{H}_{\bar{F}}$  of Hamiltonians endowed with the brackets  $\{k, h\}_{\bar{F}^*\omega} := \bar{F}^*\omega(X_k, X_h)$  forms a Poisson algebra associated to  $\bar{F}$ , which is not extendable to  $\mathcal{R}_{\bar{F}}$ .

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For each isotropic  $F$  associated to  $\bar{F}$  we consider  $N := F(U) \subset (TM, \omega)$  as an implicit differential system, possibly with singularities of the projection  $\pi \circ F$ . We investigate the smooth solvability of such systems. The point  $(p, \dot{p}) \in N$  is called solvable if there exists a smooth curve  $\gamma_p : (-\epsilon, \epsilon) \rightarrow M$  such that  $\gamma_p(0) = p$ ,  $\gamma'_p(0) = \dot{p}$  and  $\kappa_p(t) = (\gamma_p(t), \gamma'_p(t)) \in N$  for all  $t \in (-\epsilon, \epsilon)$ ,  $\epsilon > 0$ . If  $\kappa$  is smooth in the neighborhood of  $p$  we call it smoothly solvable.  $N$  is called smoothly solvable if it consists only of smoothly solvable points. We show (Theorem 5.1) that  $F$  is smoothly solvable if and only if its generating function belongs to the Poisson algebra associated to  $\bar{F}$ .

For each symplectically invariant singularity of  $\bar{F}$  we have its Poisson algebra which is the fundamental object of singularity theory endowed with an additional structure. In Section 2 isotropic mappings into symplectic tangent bundle to the symplectic manifold are investigated through their generating functions. The Hamiltonian functions are distinguished and their Poisson algebra structure is introduced in Section 3. Solvability criteria for isotropic mappings are studied in Section 4 and the main theorem of the paper saying that the only solvable Hamiltonian systems are those generated by Hamiltonian generating functions is proved in Section 5. An investigation of Poisson algebras associated to singular mappings into a symplectic manifold and their representative examples are presented in Section 6.

## 2. Generating functions of isotropic mappings

Let  $(\mathbb{R}^{2n}, \omega)$  be a Euclidean symplectic space endowed with  $\omega = \sum_{i=1}^n dy_i \wedge dx_i$  in canonical Darboux coordinates  $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ .

Let  $\theta$  be the Liouville 1-form on the cotangent bundle  $T^*\mathbb{R}^{2n}$ . Then  $d\theta$  is the standard symplectic structure on  $T^*\mathbb{R}^{2n}$ . Let  $\beta : T\mathbb{R}^{2n} \rightarrow T^*\mathbb{R}^{2n}$  be the canonical bundle map defined through  $\omega$  by

$$\beta : T\mathbb{R}^{2n} \ni v \mapsto \omega(v, \cdot) \in T^*\mathbb{R}^{2n}.$$

Then we can define (cf. [3]) the canonical symplectic structure  $\dot{\omega}$  on  $T\mathbb{R}^{2n}$  by

$$\dot{\omega} = \beta^*d\theta = d(\beta^*\theta) = \sum_{i=1}^n (d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i),$$

where  $(x, y, \dot{x}, \dot{y})$  are local coordinates on  $T\mathbb{R}^{2n}$  and  $\beta^*\theta = \sum_{i=1}^n (\dot{y}_i dx_i - \dot{x}_i dy_i)$ .

Throughout the paper, unless otherwise stated, all objects are germs at  $0 \in \mathbb{R}^{2n}$  of smooth functions, mappings, forms etc. or their representatives on an open neighborhood of  $0$  in  $\mathbb{R}^{2n}$ .

**Definition 2.1.** Let  $F : (\mathbb{R}^{2n}, 0) \rightarrow T\mathbb{R}^{2n}$  be a smooth map-germ. We say that  $F$  is isotropic if  $F^*\dot{\omega} = 0$ .

If a map-germ  $F : (\mathbb{R}^{2n}, 0) \rightarrow T\mathbb{R}^{2n}$  is isotropic, then the germ of a differential of the 1-form  $(\beta \circ F)^*\theta$  vanishes;  $d(\beta \circ F)^*\theta = F^*\beta^*d\theta = F^*\dot{\omega} = 0$ . Thus  $(\beta \circ F)^*\theta$  is the germ of a closed 1-form and there exists a smooth function-germ  $h : (\mathbb{R}^{2n}, 0) \rightarrow \mathbb{R}$  such that

$$(\beta \circ F)^*\theta = -dh. \tag{2.1}$$

We call  $h$  a generating function for  $F$ . For each smooth isotropic map-germ  $F$ , its generating function is unique up to an additive constant.

Let  $U$  be an open neighborhood of  $0$  in the source space  $\mathbb{R}^{2n}$  and let  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  be an isotropic map-germ. Let

$$\bar{F} = \pi \circ F : (U, 0) \rightarrow T\mathbb{R}^{2n},$$

where  $\pi : T\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is the tangent bundle projection. We express  $\bar{F}$  and  $F$  in the form

$$\bar{F} = (f, g) : (U, 0) \rightarrow \mathbb{R}^{2n} \quad \text{and} \quad F = (f, g, \dot{f}, \dot{g}) : (U, 0) \rightarrow T\mathbb{R}^{2n}$$

respectively.

By  $\mathcal{E}_U$  (and by  $\mathcal{E}_{\mathbb{R}^{2n}}$  respectively) we denote the  $\mathbb{R}$ -algebra of smooth function-germs at  $0$  on  $U$  (and on the target space  $\mathbb{R}^{2n}$  respectively). Let  $\mathfrak{m}_U$  ( $\mathfrak{m}_{\mathbb{R}^{2n}}$  respectively) denote the maximal ideal in  $\mathcal{E}_U$  (in  $\mathcal{E}_{\mathbb{R}^{2n}}$  respectively).

In general  $F$  can be regarded as a vector field along  $\bar{F}$ , i.e. a section of an induced fiber bundle  $\bar{F}^*T\mathbb{R}^{2n}$ . To each isotropic map-germ  $F$  along  $\bar{F}$ , there exists a unique generating function-germ  $h \in \mathfrak{m}_U$  for  $F$ .

Now we introduce a natural equivalence group acting on isotropic mappings through a natural lifting of diffeomorphic or symplectic equivalences of  $\bar{F}$  and  $\bar{G}$ . Before introducing it, we first recall the standard equivalence relations of smooth map-germs and of Lagrange projections. Two map-germs  $\bar{F} : (U, 0) \rightarrow \mathbb{R}^{2n}$  and  $\bar{G} : (U, 0) \rightarrow \mathbb{R}^{2n}$  are symplectomorphic if there exists a diffeomorphism-germ  $\varphi : (U, 0) \rightarrow (U, 0)$  and a symplectomorphism-germ  $\Phi : (\mathbb{R}^{2n}, 0) \rightarrow (\mathbb{R}^{2n}, 0)$ ,  $\Phi^*\omega = \omega$ , such that  $\bar{G} = \Phi \circ \bar{F} \circ \varphi$ . Two isotropic mappings  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  and  $G : (U, 0) \rightarrow T\mathbb{R}^{2n}$  are Lagrangian equivalent ( $L$ -equivalent [4]) if there exist a diffeomorphism-germ  $\varphi : (U, 0) \rightarrow (U, 0)$  and a symplectomorphism-germ  $\Psi : (T\mathbb{R}^{2n}, F(0)) \rightarrow (T\mathbb{R}^{2n}, G(0))$ ,  $\Psi^*\dot{\omega} = \dot{\omega}$ , preserving the fibering  $\pi$  such that  $G = \Psi \circ F \circ \varphi$ .

**Definition 2.2.** Let  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  and  $G : (U, 0) \rightarrow T\mathbb{R}^{2n}$  be two isotropic map-germs along smooth map-germs  $\bar{F} : (U, 0) \rightarrow \mathbb{R}^{2n}$  and  $\bar{G} : (U, 0) \rightarrow \mathbb{R}^{2n}$  respectively. We say that  $F$  and  $G$  are  $L$ -symplectic equivalent if there exist a

diffeomorphism-germ  $\varphi : (U, 0) \rightarrow (U, 0)$ , a symplectomorphism-germ  $\Psi : (T\mathbb{R}^{2n}, F(0)) \rightarrow (T\mathbb{R}^{2n}, G(0))$  preserving the fibering  $\pi$  and a symplectomorphism-germ  $\Phi : (\mathbb{R}^{2n}, 0) \rightarrow (\mathbb{R}^{2n}, 0)$ ,  $\Phi^*\omega = \omega$  such that  $\pi \circ \Psi = \Phi \circ \pi$ ,  $G = \Psi \circ F \circ \varphi$  and  $\bar{G} = \Phi \circ \bar{F} \circ \varphi$ . In this case we also say that  $F$  and  $G$  are symplectomorphic or symplectically equivalent.

To  $\bar{F}$  we associate a symplectically invariant algebra  $\mathcal{R}_{\bar{F}}$  of all generating function-germs of isotropic map-germs along  $\bar{F}$ ,

$$\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : h \text{ generates an isotropic map-germ along } \bar{F}\}.$$

The algebra  $\mathcal{R}_{\bar{F}}$  is an  $\mathbb{R}$ -algebra and is an  $\mathcal{E}_{\mathbb{R}^{2n}}$ -module as well.  $\mathcal{R}_{\bar{F}}$  is a symplectic invariant, namely: If two map-germs  $\bar{F} : (U, 0) \rightarrow \mathbb{R}^{2n}$  and  $\bar{G} : (U, 0) \rightarrow \mathbb{R}^{2n}$  are symplectomorphic, i.e. if there exist a diffeomorphism-germ  $\varphi : (U, 0) \rightarrow (U, 0)$  and a symplectomorphism-germ  $\Phi : (\mathbb{R}^{2n}, 0) \rightarrow (\mathbb{R}^{2n}, 0)$  such that  $\bar{G} = \Phi \circ \bar{F} \circ \varphi$ , then

$$\varphi^* : \mathcal{R}_{\bar{F}} \rightarrow \mathcal{R}_{\bar{G}}$$

is an isomorphism of  $\mathbb{R}$ -algebras. Moreover, for  $h \in \mathcal{R}_{\bar{F}}$ , the isotropic map-germ  $F$  generated by  $h$  and the isotropic map-germ  $G$  generated by  $\varphi^*h = h \circ \varphi$  are symplectomorphic: there exists another symplectomorphism-germ  $\Psi : (T\mathbb{R}^{2n}, F(0)) \rightarrow (T\mathbb{R}^{2n}, G(0))$  such that  $\pi \circ \Psi = \Phi \circ \pi$ ,  $G = \Psi \circ F \circ \varphi$  and  $\bar{G} = \Phi \circ \bar{F} \circ \varphi$ .

It is easy to see that if  $\bar{F}$  has maximal rank then  $\mathcal{R}_{\bar{F}} = \mathcal{E}_U$ . The aim of this section is to study the case where  $\bar{F}$  does not have maximal rank at 0 and establish the structure of  $\mathcal{R}_{\bar{F}}$ . The algebra  $\mathcal{R}_{\bar{F}}$  of all generating function-germs associated to  $\bar{F}$  can be represented by  $\bar{F}$  in the following form (cf. [5]),

$$\mathcal{R}_{\bar{F}} = \{h \in \mathcal{E}_U : dh \in \mathcal{E}_U d(\bar{F}^* \mathcal{E}_{\mathbb{R}^{2n}})\}.$$

In the rest of this section we study isotropic map-germs along a corank 1 map-germ (cf. [6,7]). For a smooth map-germ  $\bar{F} = (f, g) : (U, 0) \rightarrow \mathbb{R}^{2n}$  of corank one we can choose coordinates  $(u, v)$  in  $U$  and canonical (or symplectic) local coordinates in  $(\mathbb{R}^{2n}, \omega)$  such that

$$\begin{aligned} f_i(u, v) &= u_i, & i &= 1, \dots, n \\ g_i(u, v) &= v_i, & i &= 1, \dots, n-1 \\ \frac{\partial g_n}{\partial v_n}(0, 0) &= 0. \end{aligned} \tag{2.2}$$

By straightforward calculations we have.

**Proposition 2.3.** Let  $\bar{F} = (f, g) : (U, 0) \rightarrow (\mathbb{R}^{2n}, 0)$  be a corank one map-germ of the form (2.2). Then  $\mathcal{R}_{\bar{F}}$  is given in the form

$$\mathcal{R}_{\bar{F}} = \left\{ h \in \mathcal{E}_U : \frac{\partial h}{\partial v_n} \in \langle \Delta_{\bar{F}} \rangle \right\},$$

where  $\langle \Delta_{\bar{F}} \rangle$  is the ideal in  $\mathcal{E}_U$  generated by the determinant  $\Delta_{\bar{F}} = \frac{\partial g_n}{\partial v_n}$  of the Jacobi matrix  $J\bar{F}$  of  $\bar{F}$ .

**Remark 2.4.** Let  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  be a smooth isotropic map-germ such that  $\bar{F} = \pi \circ F : (U, 0) \rightarrow \mathbb{R}^{2n}$  has a corank one singular point at  $(0, 0)$ . Then  $F$  has corank at most one at  $(0, 0)$ . The corank of  $F$  is exactly one if and only if

$$\partial_e(\partial_e h / \Delta_{\bar{F}})(0, 0) = 0,$$

where  $\partial_e$  is the derivation in the  $e$ -direction, which belongs to  $T_0U$  and spans the kernel of the Jacobi matrix  $J\bar{F}$ .

### 3. Hamiltonians of isotropic mappings

Let  $\bar{F} : \mathbb{R}^{2n} \supset U \rightarrow (\mathbb{R}^{2n}, \omega)$  be a smooth map-germ. Then  $\bar{F}$  induces a possibly degenerate 2-form  $\bar{F}^*\omega$  on  $U$ . For a smooth function  $h$  defined on  $U$  we formally define the Hamiltonian vector field  $X_h$  (which may not be smooth) on  $U$  by the equality

$$\bar{F}^*\omega(X_h, \xi) = -\xi(h) \quad \text{for each vector field } \xi \text{ on } U. \tag{3.1}$$

For smooth functions  $k, h$  defined on  $U$  we can define also the formal brackets  $\{k, h\}_{\bar{F}^*\omega}$ , by

$$\{k, h\}_{\bar{F}^*\omega} := \bar{F}^*\omega(X_k, X_h). \tag{3.2}$$

It may happen that  $X_h, X_k$  and  $\{k, h\}_{\bar{F}^*\omega}$  diverge on the singular point set of  $\bar{F}$ . However they are ordinary Hamiltonian vector fields and Poisson brackets outside the singular point set.

**Definition 3.1.** Let  $h : \mathbb{R}^{2n} \supset U \rightarrow \mathbb{R}$  be a smooth function. If  $X_h$  defined by (3.1) is smooth then  $X_h$  is called a Hamiltonian vector field and  $h$  is called the Hamiltonian function. By  $\mathcal{H}_{\bar{F}}$  we denote the space of all Hamiltonians associated to  $\bar{F}$ :

$$\mathcal{H}_{\bar{F}} = \{h \in C^\infty(U) : X_h \text{ is smooth}\}. \tag{3.3}$$

**Theorem 3.2.** Let  $\bar{F} : \mathbb{R}^{2n} \supset U \rightarrow (\mathbb{R}^{2n}, \omega)$  be a smooth map whose regular point set is dense in  $U$ . Then  $\mathcal{H}_{\bar{F}}$  is closed under the brackets  $\{\cdot, \cdot\}_{\bar{F}^*\omega}$  and the space  $(\mathcal{H}_{\bar{F}}, \{\cdot, \cdot\}_{\bar{F}^*\omega})$  is a Poisson algebra.

**Proof.** Let  $U$  be an open ball neighborhood of the origin of  $\mathbb{R}^m$ . Let  $\Delta(x_1, \dots, x_m)$  be a smooth function defined on  $U$  and let  $\Omega$  be the set  $\{x \in U \mid \Delta(x) \neq 0\}$ . Suppose that  $\Omega$  is dense in  $U$ . Let  $a(x)$  be a fractional function whose numerator is a smooth function defined on  $U$  and whose denominator is  $\Delta(x)$ :

$$a(x) = \frac{\alpha(x)}{\Delta(x)}.$$

If the restriction  $a|_{\Omega}$  to  $\Omega$  is extendable to a smooth function on  $U$ , then  $a(x)$  itself is smooth on  $U$ , i.e.  $\alpha$  is divisible by  $\Delta$ .

Let  $U$  be an open ball neighborhood of the origin  $(0, 0)$  in  $\mathbb{R}^{2n}$ . Let  $\bar{F} : \mathbb{R}^{2n} \supset U \rightarrow (\mathbb{R}^{2n}, \omega)$  be a map whose regular point set is dense in  $U$ . Let  $\Delta_{\bar{F}}(u, v)$  be the Jacobian determinant of  $\bar{F}$ .

Let  $\Omega = \{(u, v) \in U \mid \Delta_{\bar{F}}(u, v) \neq 0\}$  be the set of regular points of  $\bar{F}$  which we assume is dense in  $U$ . Then the restriction  $\bar{F}^*\omega|_{\Omega}$  to  $\Omega$  of the 2-form  $\bar{F}^*\omega$  is non-degenerate. Let  $h$  be a smooth function defined on  $U$ . Then the Hamiltonian vector field  $X_h$  is defined by the equality

$$\bar{F}^*\omega(X_h, \xi) = -\xi(h), \quad \text{for each vector field } \xi \text{ on } U.$$

Let us express  $X_h$  in the form

$$X_h = \sum_{i=1}^n \left( a_i(u, v) \frac{\partial}{\partial u_i} + b_i(u, v) \frac{\partial}{\partial v_i} \right). \tag{3.4}$$

Then, after some calculations we obtain that each coefficient  $a_i(u, v)$  or  $b_i(u, v)$  of the  $X_h$  is the sum of a smooth function, a fractional function whose numerator is a smooth function and whose denominator is  $\Delta_{\bar{F}}$  and a fractional function whose numerator is a smooth function and whose denominator is  $\Delta_{\bar{F}}^2$ . Note that denominators may vanish as well.

For any smooth function  $h$ , the restriction  $X_h|_{\Omega}$  to  $\Omega$  of the vector field  $X_h$  is always smooth. Therefore the restrictions  $a_i|_{\Omega}, b_i|_{\Omega}$ 's to  $\Omega$  of the coefficients  $a_i, b_i$ 's are also always smooth. Thus from the form of (3.4) we see that  $X_h$  is smooth if and only if  $a_i|_{\Omega}, b_i|_{\Omega}$ 's are extendable to smooth functions defined on  $U$ .

Now let  $h, k \in \mathcal{H}_{\bar{F}}$ . Then  $h, k, X_h, X_k$  are all smooth on  $U$ . Hence  $\{h, k\}_{\bar{F}^*\omega} = X_h(k)$  is smooth on  $U$ . And we have

$$X_{\{h,k\}_{\bar{F}^*\omega}}|_{\Omega} = [X_h|_{\Omega}, X_k|_{\Omega}] = X_h|_{\Omega}X_k|_{\Omega} - X_k|_{\Omega}X_h|_{\Omega}. \tag{3.5}$$

Since  $X_h$  and  $X_k$  are smooth on  $U$ , the right-hand side of (3.5) is extendable to the bracket vector field  $[X_h, X_k]$  which is smooth on  $U$ . Since the coefficients of  $X_{\{h,k\}_{\bar{F}^*\omega}}|_{\Omega}$  are extendable to the coefficients of  $[X_h, X_k]$  which are smooth on  $U$ , it follows that the coefficients of  $X_{\{h,k\}_{\bar{F}^*\omega}}$  themselves are smooth on  $U$ . Thus  $X_{\{h,k\}_{\bar{F}^*\omega}}$  is also smooth on  $U$ . Thus  $\{h, k\}_{\bar{F}^*\omega} \in \mathcal{H}_{\bar{F}}$ .  $\square$

**Definition 3.3.** The space  $(\mathcal{H}_{\bar{F}}, \{\cdot, \cdot\}_{\bar{F}^*\omega})$  endowed with

$$\{k, h\}_{\bar{F}^*\omega} := \bar{F}^*\omega(X_k, X_h), \quad h, k \in \mathcal{H}_{\bar{F}},$$

is called the Poisson algebra associated to  $\bar{F}$  (or  $\bar{F}$ -Poisson algebra) endowed with the Poisson brackets  $\{k, h\}_{\bar{F}^*\omega}$ .

### 4. Smoothly solvable isotropic mappings

A natural property of smooth dynamical systems defined by smooth vector fields is their local solvability. This notion was generalized in [8,9] to smooth submanifolds of a tangent bundle with possible singular projection into the base space as follows.

Let  $N \subset T\mathbb{R}^{2n}$  be a  $2n$  dimensional submanifold of  $T\mathbb{R}^{2n}$  defined as the image of an embedding  $F : U \rightarrow T\mathbb{R}^{2n}$ , where  $U$  is an open neighborhood of the origin of  $\mathbb{R}^{2n}$ . We consider  $N$  as an implicit differential equation. A point  $(x, y, \dot{x}, \dot{y}) \in N$  is called a solvable point of  $N$  if there exists a smooth (at least of class  $C^1$ ) curve  $\gamma_{(x,y,\dot{x},\dot{y})} : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{2n}$  such that  $\gamma_{(x,y,\dot{x},\dot{y})}(0) = (x, y), \gamma'_{(x,y,\dot{x},\dot{y})}(0) = (\dot{x}, \dot{y})$ , and

$$\kappa_{(x,y,\dot{x},\dot{y})}(t) := (\gamma_{(x,y,\dot{x},\dot{y})}(t), \gamma'_{(x,y,\dot{x},\dot{y})}(t)) \in N \tag{4.1}$$

for all  $t \in (-\epsilon, \epsilon), \epsilon > 0$ . The implicit differential equation  $N$  is said to be solvable if it consists of only solvable points. A point  $(x_0, y_0, \dot{x}_0, \dot{y}_0)$  of a solvable implicit differential equation  $N$  is said to be smoothly solvable if there exist a neighborhood  $W$  of  $(x_0, y_0, \dot{x}_0, \dot{y}_0)$  in  $N$  and a number  $\epsilon > 0$  such that a mapping  $\kappa : W \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{2n}$  defined by

$$\kappa(x, y, \dot{x}, \dot{y}, t) = \kappa_{(x,y,\dot{x},\dot{y})}(t)$$

is smooth. We say that  $N$  is smoothly solvable if it consists of only smoothly solvable points.

Conditions for smooth solvability of implicit differential systems were investigated in [9] (cf. [10]). In the case where regular points of  $\bar{F} = \pi \circ F : U \rightarrow \mathbb{R}^{2n}$  are dense in  $U$ ,  $N$  is smoothly solvable if and only if there exists a smooth vector field  $X$  on  $U$  such that

$$F = d\bar{F}(X) := (\bar{F}, J\bar{F}(X)),$$

where  $J\bar{F}$  is the Jacobian matrix of  $\bar{F}$ .

Now we adapt this property as a definition of smooth solvability of general isotropic mappings.

**Definition 4.1.** Let  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  be a smooth isotropic map-germ. We say that  $F$  is smoothly solvable if there exists a smooth vector field  $X$  on  $U$  such that

$$F = d\bar{F}(X).$$

In the next section we will investigate a relation between smooth solvability of an isotropic map-germ  $F$  and the Poisson algebra  $\mathcal{H}_{\bar{F}}$ , where  $\bar{F} = \pi \circ F$ .

A necessary condition for a smooth submanifold  $N \subset T\mathbb{R}^{2n}$  to be solvable was found in [8] (cf. [9]). If  $\pi$  be the tangent bundle projection. Then the necessary solvability condition

$$(\dot{x}, \dot{y}) \in d(\pi|_N)_{(x,y,\dot{x},\dot{y})}(T_{(x,y,\dot{x},\dot{y})}N) \tag{4.2}$$

at  $(x, y, \dot{x}, \dot{y}) \in N$  is called a *tangential solvability condition* and extended to the general smooth mapping  $F = (f, g, \dot{f}, \dot{g}) : U \rightarrow T\mathbb{R}^{2n}$  is written in the form

$$\dot{F}(u, v) \in J\bar{F}(u, v)(\mathbb{R}^{2n}), \tag{4.3}$$

where  $\dot{F}(u, v) = (\dot{f}, \dot{g})(u, v)$ .

Conditions for smooth solvability of implicit differential systems were investigated in [9] (cf. [10,6]). Now we extend the solvability property introduced for a smooth submanifold of a tangent bundle defined by an immersion mapping  $F$  to general smooth isotropic mappings into the tangent bundle.

It was shown in [9, Example 5.1] that the tangential solvability condition is not sufficient for  $N$  to be smoothly solvable. The geometric meaning of the solvability property is explained in the following sufficient condition.

**Theorem 4.2.** Let  $\bar{F} = (f, g) : U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  be a smooth mapping such that  $\bar{F}$  has a corank  $k$  singularity at the origin  $(0, 0) \in \mathbb{R}^{2n}$  and that the jet extension  $j^1\bar{F} : U \rightarrow J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$  is transversal to the corank  $k$  stratum  $\Sigma^k$  of  $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ . If an isotropic mapping  $F$  along  $\bar{F}$  satisfies the tangential solvability condition, then  $F$  is smoothly solvable.

**Proof.** Let  $F = (\bar{F}, \dot{F}) = (f, g, \dot{f}, \dot{g})$  be an isotropic mapping along  $\bar{F}$  which satisfies the tangential solvability condition (4.2). By  $\mathbb{I}_{2n}$  we denote the symplectic unit,

$$\mathbb{I}_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{4.4}$$

Since  $F$  is a smooth isotropic mapping it is generated by a smooth function  $h$ :

$$\dot{F} = \mathbb{I}_{2n} {}^t J\bar{F}^{-1} \partial h, \tag{4.5}$$

where  $\partial h = {}^t (\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v})$ .

We know that  $F$  is smoothly solvable if and only if

$$J\bar{F}^{-1} \mathbb{I}_{2n} {}^t J\bar{F}^{-1} \partial h \text{ is smooth,} \tag{4.6}$$

but on the basis of (4.5) this is the case if and only if  $J\bar{F}^{-1} \dot{F}(u, v)$  is smooth, which is true if and only if the linear equation

$$J\bar{F}A(u, v) = \dot{F}(u, v) \tag{4.7}$$

has a smooth solution  $A(u, v) = (a(u, v), b(u, v))$ .

Since, from (4.2),  $\dot{F}(u, v) \in \text{Image} J\bar{F}(u, v)$  for every point  $(u, v) \in U$  and  $j^1\bar{F} : U \rightarrow J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$  is transversal to the corank  $k$  stratum  $\Sigma^k$  of  $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ , it follows from J. Mather's theorem [11, Theorem 1], that Eq. (4.7) has a smooth solution and  $F$  is smoothly solvable.  $\square$

### 5. Lie algebra of solvable isotropic mappings

Now we are ready to show that the Poisson algebra structure of Hamiltonians associated to  $\bar{F}$  is equivalent to smooth solvability of the corresponding isotropic map-germs generated by these Hamiltonians.

**Theorem 5.1.** Let  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  be a smooth isotropic map-germ along a smooth map-germ  $\bar{F} : (U, 0) \rightarrow \mathbb{R}^{2n}$  such that the regular point set of  $\bar{F}$  is dense in  $U$ . Let  $h : (U, 0) \rightarrow \mathbb{R}$  be a generating function-germ of  $F$ . Then  $F$  is smoothly solvable if and only if  $h \in \mathcal{H}_{\bar{F}}$ , i.e.  $h$  is a Hamiltonian function.

**Proof.** Following the proof of [Theorem 3.2](#), to prove the theorem we need to show that [Eq. \(3.1\)](#) defining the Hamiltonian vector field  $X_h$  is equivalent to [Eq. \(2.1\)](#) expressed in the form

$$(\beta \circ d\bar{F}(X_h))^* \theta = -dh. \tag{5.1}$$

Once this is done, then we have  $F = d\bar{F}(X_h)$  and, by [Definition 4.1](#), we see immediately that  $F$  is smoothly solvable if and only if  $X_h$  is smooth.

Let  $X_h = \sum_{i=1}^n (a_i(u, v) \frac{\partial}{\partial u_i} + b_i(u, v) \frac{\partial}{\partial v_i})$ . Putting  $\frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i}$  into [\(3.1\)](#) instead of  $\xi$  we obtain

$$\begin{aligned} \frac{\partial h}{\partial w_i} &= -\bar{F}^* \omega \left( X_h, \frac{\partial}{\partial w_i} \right) = \sum_{j=1}^n \sum_{k=1}^n a_j(u, v) \left( -\frac{\partial f_k}{\partial w_i} \frac{\partial g_k}{\partial u_j} + \frac{\partial g_k}{\partial w_i} \frac{\partial f_k}{\partial u_j} \right) \\ &\quad + \sum_{j=1}^n \sum_{k=1}^n b_j(u, v) \left( -\frac{\partial f_k}{\partial w_i} \frac{\partial g_k}{\partial v_j} + \frac{\partial g_k}{\partial w_i} \frac{\partial f_k}{\partial v_j} \right), \end{aligned} \tag{5.2}$$

where  $(w_1, \dots, w_{2n}) = (u_1, \dots, u_n, v_1, \dots, v_n)$ . It is easy to see that [\(5.2\)](#) is equivalent in matrix form to the equation

$$\partial h = {}^t J \bar{F} \mathbb{I}_{2n} J \bar{F} A,$$

where  $A = {}^t(a, b) \in \mathbb{R}^{2n}$ . Thus [\(5.1\)](#) is smoothly invertible for  $X_h$ .  $\square$

**Remark 5.2.** Since smooth solvability of an isotropic map  $F$  generated by a smooth function  $h : U \rightarrow \mathbb{R}$  is defined by smoothness of  $X_h$ , an equivalent condition for smooth solvability of  $F$  can be given in terms of the Poisson bracket, namely:

*F is smoothly solvable or equivalently h is a Hamiltonian function on U if  $\{h, \alpha\}_{\bar{F}^* \omega}$  is smooth on U for all smooth functions  $\alpha$  defined on U.*

Smooth solvability is a structural property preserved by Poisson bracket defined on the space of Hamiltonians  $\mathcal{H}_{\bar{F}}$ . However the space of generating functions  $\mathcal{R}_{\bar{F}}$  is not preserved by the Poisson bracket  $\{\cdot, \cdot\}_{\bar{F}^* \omega}$ . As an example we consider the fold map

$$\bar{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \bar{F}(u, v) = (u, v^2/2).$$

In this case  $\mathcal{R}_{\bar{F}} = \{h : \frac{\partial h}{\partial v} \in \langle v \rangle\}$ . Taking  $h = u \in \mathcal{R}_{\bar{F}}, k = v^3 \in \mathcal{R}_{\bar{F}}$  we find  $\{h, k\}_{\bar{F}^* \omega} = -3v$  thus

$$\frac{\partial \{h, k\}_{\bar{F}^* \omega}}{\partial v} \notin \langle v \rangle \quad \text{and} \quad \{h, k\}_{\bar{F}^* \omega} \notin \mathcal{R}_{\bar{F}}.$$

Let us consider the natural subspace  $\mathcal{R}_{\bar{F}}^T$  of the space of generating functions for isotropic mappings along  $\bar{F}$  satisfying the tangential solvability condition [\(4.2\)](#).

$$\mathcal{R}_{\bar{F}}^T = \{h \in C^\infty(U) : h \in \mathcal{R}_{\bar{F}} \text{ and } F \text{ generated by } h \text{ satisfies } (4.2)\},$$

which will be called *the space of tangential generating functions*.

In the case that  $\bar{F}$  has a corank  $k$  singularity at 0 and the transversality assumption of [Theorem 4.2](#) is satisfied, then  $\mathcal{R}_{\bar{F}}^T = \mathcal{H}_{\bar{F}}$ . In general  $\mathcal{H}_{\bar{F}}$  is a proper subset of  $\mathcal{R}_{\bar{F}}^T$  and there is a natural question whether the Poisson structure  $\{\cdot, \cdot\}_{\bar{F}^* \omega}$  can be extended to  $\mathcal{R}_{\bar{F}}^T$ . By the following example we show that this is impossible.

**Example 5.3.** Let  $\bar{F} : \mathbb{R}^2 \rightarrow (\mathbb{R}^2, \omega)$  be defined by

$$\bar{F}(u, v) = \left( u, u^2 v + \frac{1}{3} v^3 \right).$$

We show that  $\mathcal{R}_{\bar{F}}^T$  is not closed under the Poisson bracket. First we calculate the Jacobian matrix of  $\bar{F}$ ,

$$J\bar{F}(u, v) = \begin{pmatrix} 1 & 0 \\ 2uv & u^2 + v^2 \end{pmatrix}, \quad J\bar{F}^{-1}(u, v) = \begin{pmatrix} 1 & 0 \\ -2uv & 1 \\ u^2 + v^2 & u^2 + v^2 \end{pmatrix}.$$

From the condition of isotropicity we have

$$\frac{\partial h}{\partial v} \in \langle \Delta_{\bar{F}} \rangle = \langle u^2 + v^2 \rangle$$

thus  $h$  has the form

$$h = (u^2 + v^2)^2 \alpha(u, v) + \beta(u).$$

Now we check the tangential solvability condition at  $(0, 0)$ ,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t J_{\bar{F}}^{-1}(u, v) \begin{pmatrix} \frac{\partial h}{\partial u} \\ \frac{\partial h}{\partial v} \end{pmatrix} \Big|_{(u,v)=(0,0)} \in \text{Image} J_{\bar{F}}(0, 0).$$

And we obtain

$$h(u, v) = (u^2 + v^2)^2 \alpha(u, v) + u^4 \beta(u) + \text{const.} \tag{5.3}$$

Thus,

$$\begin{aligned} \mathcal{R}_{\bar{F}}^T &= \{h \in C^\infty(U) \mid h(u, v) = (u^2 + v^2)^2 \alpha(u, v) + u^4 \beta(u) + \text{const.} \\ &\text{for some smooth } \alpha(u, v) \text{ and } \beta(u)\}. \end{aligned} \tag{5.4}$$

Consider the following two elements of  $\mathcal{R}_{\bar{F}}^T$

$$\begin{aligned} h(u, v) &= (u^2 + v^2)^2 + u^4, \\ k(u, v) &= (u^2 + v^2)^2 v + u^4. \end{aligned}$$

The Poisson bracket of  $h$  and  $k$  is given by

$$\{h, k\}_{\bar{F}^* \omega} = -4u(u^2 + v^2)^2 - 4u^3(u^2 + v^2) - 16u^3 v^2 + 16u^3 v.$$

And consequently

$$\{h, k\}_{\bar{F}^* \omega} \notin \mathcal{R}_{\bar{F}}^T.$$

Thus,  $\mathcal{R}_{\bar{F}}^T$  is not closed under the Poisson bracket.

We can easily see that the transversality condition of [Theorem 4.2](#) is only a sufficient condition. We can find examples of  $\bar{F}$  such that the jet extension  $j^1 \bar{F} : U \rightarrow J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$  is not transversal to the corank  $k$  stratum  $\Sigma^k$  of  $J^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$  but  $\mathcal{R}_{\bar{F}}^T$  is closed under the Poisson bracket. In fact we can take

$$\bar{F} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \quad \bar{F}(u, v) = \left( u, \frac{1}{k+1} v^{k+1} \right).$$

We see that  $\bar{F}$  has corank 1 at  $(u, 0)$  but  $j^1 \bar{F}$  is not transversal to the corank 1 stratum in the jet space for  $k \geq 2$ . Then by straightforward calculations we can show also that  $\mathcal{R}_{\bar{F}}^T$  is closed under the Poisson bracket. Moreover in this example we have  $\mathcal{R}_{\bar{F}}^T = \mathcal{H}_{\bar{F}}$ . Then the natural question arises: Is there any smooth mapping  $\bar{F}$  such that  $\mathcal{R}_{\bar{F}}^T$  is closed under the Poisson bracket but  $\mathcal{R}_{\bar{F}}^T \neq \mathcal{H}_{\bar{F}}$ . We conjecture that

$$\mathcal{R}_{\bar{F}}^T = \mathcal{H}_{\bar{F}} \text{ holds always if } \mathcal{R}_{\bar{F}}^T \text{ is closed under the Poisson bracket.}$$

### 6. Examples of Poisson algebras associated to $\bar{F}$

Now we find conditions for a function  $h$  to be an element of the Poisson algebra associated to  $\bar{F}$  which has a corank 1 singularity at the origin  $(0, 0) \in U \subset \mathbb{R}^{2n}$ .

**Proposition 6.1.** *Let  $F : (U, 0) \rightarrow T\mathbb{R}^{2n}$  be a smooth isotropic map-germ such that  $\bar{F} = \pi \circ F$  has a corank 1 singularity at  $(0, 0) \in U \subset \mathbb{R}^{2n}$  expressed in local coordinates  $(u, v)$  defined in (2.2). Let  $h : (U, 0) \rightarrow \mathbb{R}$  be a smooth generating function-germ for  $F$  defined on  $U$ . Then  $F$  is smoothly solvable if and only if*

$$\frac{\partial h}{\partial v_n} \in \langle \Delta_{\bar{F}} \rangle, \tag{6.1}$$

and

$$\sum_{i=1}^{n-1} \left( \frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial u_i} - \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_i} \right) - \frac{\partial h}{\partial u_n} \in \langle \Delta_{\bar{F}} \rangle. \tag{6.2}$$

**Proof.** From (3.1), taking  $X_h = \sum_{i=1}^n (a_i(u, v) \frac{\partial}{\partial u_i} + b_i(u, v) \frac{\partial}{\partial v_i})$  for the local form of  $\bar{F}$  given by (2.2), we calculate the coefficients of  $X_h$ ,

$$\begin{aligned} a_i &= \frac{\partial h}{\partial v_i} - \frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial v_n} / \Delta_{\bar{F}}, \quad i = 1, \dots, n-1, \\ a_n &= \frac{\partial h}{\partial v_n} / \Delta_{\bar{F}}, \\ b_i &= -\frac{\partial h}{\partial u_i} + \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_n} / \Delta_{\bar{F}}, \quad i = 1, \dots, n-1, \\ b_n &= \frac{1}{\Delta_{\bar{F}}} \left( -\frac{\partial h}{\partial u_n} + \sum_{i=1}^{n-1} \frac{\partial g_n}{\partial v_i} \frac{\partial h}{\partial u_i} - \sum_{i=1}^{n-1} \frac{\partial g_n}{\partial u_i} \frac{\partial h}{\partial v_i} \right) \end{aligned} \tag{6.3}$$

which are smooth if and only if (6.1) and (6.2) are satisfied.  $\square$

For the symplectic fold singularity we have immediately.

**Corollary 6.2.** Let  $\bar{F} : (U, 0) \rightarrow \mathbb{R}^{2n}$  be an  $A_1$ -type map-germ. Then  $\bar{F}$  is symplectically equivalent to the normal form of fold map-germ,

$$\bar{F}_0 : (u_1, \dots, u_n, v_1, \dots, v_n) \mapsto (u_1, \dots, u_n, v_1, \dots, v_{n-1}, v_n^2), \tag{6.4}$$

and  $\mathcal{H}_{\bar{F}}$  is isomorphic to  $\mathcal{H}_{\bar{F}_0}$  as a Poisson algebra, where

$$\mathcal{H}_{\bar{F}_0} = \left\{ h : \frac{\partial h}{\partial v_n}, \frac{\partial h}{\partial u_n} \in \langle \Delta_{\bar{F}_0} \rangle = \langle v_n \rangle \right\}$$

and for  $h, k \in \mathcal{H}_{\bar{F}_0}$

$$\{h, k\}_{\bar{F}^* \tilde{\omega}} = \sum_{i=1}^{n-1} \left( \frac{\partial h}{\partial v_i} \frac{\partial k}{\partial u_i} - \frac{\partial k}{\partial v_i} \frac{\partial h}{\partial u_i} \right) + \frac{1}{2v_n} \left( \frac{\partial h}{\partial v_n} \frac{\partial k}{\partial u_n} - \frac{\partial k}{\partial v_n} \frac{\partial h}{\partial u_n} \right). \tag{6.5}$$

**Remark 6.3.** The space of Hamiltonian functions  $\mathcal{H}_{\bar{F}}$  and its corresponding space of smoothly solvable isotropic mappings along  $\bar{F}$  are symplectically invariant Lie algebras.  $\mathcal{H}_{\bar{F}}$  is an  $\mathbb{R}$ -subalgebra of the  $\mathbb{R}$ -algebra  $\mathcal{R}_{\bar{F}}$  which is an  $\mathcal{E}_{\mathbb{R}^{2n}}$ -submodule of  $\mathcal{E}_U$ ,

$$\mathcal{H}_{\bar{F}} \subset \mathcal{R}_{\bar{F}} \subset \mathcal{E}_U.$$

The natural ideals of  $\mathcal{H}_{\bar{F}}$  are those generated by powers of the Jacobian determinant. Let  $\Delta_{\bar{F}}$  denote the Jacobian determinant  $\det J\bar{F}$  and let  $\tilde{J}\bar{F}$  denote the cofactor matrix of  $J\bar{F}$ . Then we have

$$J\bar{F}^{-1} = \frac{1}{\Delta_{\bar{F}}} \tilde{J}\bar{F}.$$

Therefore  $h$  belongs to  $\mathcal{H}_{\bar{F}}$  if and only if

$$\frac{1}{\Delta_{\bar{F}}^2} \tilde{J}\bar{F} \mathbb{I}_{2n} {}^t \tilde{J}\bar{F} \partial h$$

is smooth. Thus if  $h$  belongs to the ideal  $\langle \Delta_{\bar{F}}^3 \rangle$ , then  $h \in \mathcal{H}_{\bar{F}}$ . Now we have the following lemma from linear algebra.

**Lemma 6.4.** Let  $A = (a_{ij})$  be a square matrix of size  $2n$  and let  $\tilde{A}$  denote its cofactor matrix, Let

$$B = (b_{k\ell}) = \tilde{A} \mathbb{I}_{2n} {}^t \tilde{A}, \quad \text{where } {}^t \tilde{A} \text{ denotes the transpose of } \tilde{A}.$$

Then we have

$$b_{k\ell} \in \langle \det A \rangle_{\mathbb{R}[a_{11}, a_{12}, \dots, a_{nn}]}. \tag{6.6}$$

In other words,  $\det A$  divides every entry  $b_{k\ell}$  of the matrix  $\tilde{A} \mathbb{I}_{2n} {}^t \tilde{A}$  as polynomials in the variables  $a_{11}, a_{12}, \dots, a_{nn}$ .

Using this lemma we can prove a stronger result on the structure of  $\mathcal{H}_{\bar{F}}$ .

**Proposition 6.5.** Let  $\bar{F} = (f, g) : (U, 0) \rightarrow \mathbb{R}^{2n}$  be a smooth map-germ. Then the following hold:

- (1)  $\langle \Delta_{\bar{F}}^2 \rangle \subset \mathcal{H}_{\bar{F}}$
- (2) For  $\ell \geq 3$ ,  $\langle \Delta_{\bar{F}}^\ell \rangle$  is a Lie subalgebra of  $\mathcal{H}_{\bar{F}}$ .
- (3) For  $\ell \geq 3$ ,  $\langle \Delta_{\bar{F}}^\ell \rangle$  is an ideal of  $\langle \Delta_{\bar{F}}^3 \rangle$ .



**Proof.**  $h$  belongs to  $\mathcal{H}_{\tilde{F}}$  if and only if  $\frac{1}{\Delta_{\tilde{F}}^2} \tilde{J}\tilde{F}\mathbb{I}_{2n} {}^t\tilde{J}\tilde{F}\partial h$  is smooth. Now applying Lemma 6.4 to  $A = \tilde{J}\tilde{F}$ , we see that every entry of  $\tilde{J}\tilde{F}\mathbb{I}_{2n} {}^t\tilde{J}\tilde{F}$  is an element of  $\langle \Delta_{\tilde{F}} \rangle$ . Therefore if  $h \in \langle \Delta_{\tilde{F}}^2 \rangle$ , then  $h \in \mathcal{H}_{\tilde{F}}$ . Thus  $\langle \Delta_{\tilde{F}}^2 \rangle \subset \mathcal{H}_{\tilde{F}}$ . This proves (1).

Let  $\ell \geq 3$  and let  $h, k \in \langle \Delta_{\tilde{F}}^\ell \rangle$ . From Definition 3.3

$$\{h, k\}_{\tilde{F}^*\tilde{\omega}} = \partial k \tilde{F}^{-1} \mathbb{I}_{2n} {}^t \tilde{J} \tilde{F}^{-1} \partial h.$$

Since  $h, k \in \langle \Delta_{\tilde{F}}^\ell \rangle$ , then

$$\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}, \frac{\partial k}{\partial u}, \frac{\partial k}{\partial v} \in \langle \Delta_{\tilde{F}}^{\ell-1} \rangle$$

and on the basis of Lemma 6.4

$$\{h, k\}_{\tilde{F}^*\tilde{\omega}} \in \langle \Delta_{\tilde{F}}^{2\ell-2-1} \rangle.$$

Since  $\ell \geq 3$ ,  $2\ell - 2 - 1 \geq \ell$ . This proves (2). (3) can be proved in a similar way.  $\square$

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